Asymptotic Design and Analysis of Multistage Detectors and Multistage Channel Estimators for Multipath Fading Channels

Laura Cottatellucci and Ralf Müller
Telecommunications Research Centre Vienna (ftw.)
Tech Gate Vienna, Donau–City Str. 1/III
A–1220 Vienna, Austria
e-mail: {cottatellucci, mueller}@ftw.at

Abstract — **The impact of channel estimation errors on the performance of the LMMSE multistage detector for large multiuser systems with random spreading sequences is analyzed. The receiver consists of a multistage filter for channel estimation and an LMMSE multistage detector that takes into account the imperfect knowledge of the channels. The asymptotic performance of the whole receiver is provided.**

I. Analysis and Results

Let us consider a synchronous CDMA system with spreading factor N and K users. We assume multipath fading channel with known channel length L and additive noise. The equivalent baseband signal at the chip matched filter output is given by

$$\mathbf{y}(m) = \mathbf{S}(m)\mathbf{A}(m)\mathbf{b}(m) + \mathbf{n}(m) \tag{1}$$

or equivalently by

$$\mathbf{y}(m) = \mathbf{S}(m)\mathbf{B}(m)\mathbf{a} + \mathbf{n}(m). \tag{2}$$

$\mathbf{y}(m)$ is the N-dimensional received signal vector. $\mathbf{b}(m)$ is the K-dimensional vector of transmitted symbols in a finite alphabet in $\mathbb{C}$ such that $\mathrm{E}\{\mathbf{b}(m)\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{b}(s)\mathbf{b}^H(t)\} = \mathbf{I}_K\delta_{st}$. $\mathbf{B}(m)$ is the $KL \times KL$ diagonal matrix obtained by $\mathbf{B}(m) = \mathrm{diag}\,(\mathbf{b}(m) \otimes \mathbf{I}_L)$. $\mathbf{A}$ is the $KL \times K$ block diagonal matrix of fading channel amplitudes with the k-th block, $\mathbf{a}_k$, of size $L \times 1$. $\mathbf{a} = \left(\mathbf{a}_1^T, \ldots, \mathbf{a}_K^T\right)^T$ is the KL-dimensional vector of the fading channel amplitudes. The entries in $\mathbf{a}$ are realizations of independent circular symmetric complex zero mean Gaussian variables. The spreading matrix $\mathbf{S}$ is an $N \times KL$ block random matrix in $\mathbb{C}$ with K blocks $\mathbf{S}_k = \left(\mathbf{s}_{(k-1)L+1}, \ldots, \mathbf{s}_{kL}\right)$ of size $N \times L$. The elements in $\mathbf{s}_{(k-1)L+1}$ are independent and Gaussian distributed with zero mean and variance $\frac{1}{N}$. They are also independent from block to block. Within a block the vector $\mathbf{s}_{(k-1)L+s}$ is a cyclically shifted version of $\mathbf{s}_{(k-1)L+1}$ by $s-1$ positions, since we assume that the intersymbol interference can be neglected following [1, 2]. $\mathbf{n}(m)$ is the complex Gaussian noise vector with $\mathrm{E}\{\mathbf{n}(m)\} = \mathbf{0}$ and $\mathrm{E}\{\mathbf{n}(l)\mathbf{n}(m)\} = \sigma^2\mathbf{I}_N\delta_{lm}$. $\beta = \frac{K}{N}$ is the system load. The training data and the channel covariance matrix $\mathbf{C}_{\mathcal{P}} = \mathrm{E}\{\mathbf{a}\mathbf{a}^H\}$ are known by the channel estimator. To estimate the channel coefficients over an estimation window of τ symbols we refer to the model $\mathbf{Y} = \tau^{\frac{1}{2}}\overline{\mathbf{S}}\mathbf{a} + \mathbf{N}$ where $\mathbf{Y} = (\mathbf{y}^T(1), \ldots, \mathbf{y}^T(\tau))^T$, $\mathbf{N} = (\mathbf{n}^T(1), \ldots, \mathbf{n}^T(\tau))^T$ and $\overline{\mathbf{S}} = \tau^{-\frac{1}{2}}((\mathbf{S}(1)\mathbf{B}(1))^T, \ldots, (\mathbf{S}(\tau)\mathbf{B}(\tau))^T)^T$. $\overline{\mathbf{H}} = \overline{\mathbf{S}}\mathbf{A}$ and $\overline{\mathbf{R}} = \overline{\mathbf{H}}^H\overline{\mathbf{H}}$. The individual LMMSE channel estimator of rank M_C is the linear estimator $\mathbf{L} = \sum_{m=0}^{M_C-1} \tau^{\frac{2m+1}{2}}\mathbf{W}_m\mathbf{C}_{\mathcal{P}}^{\frac{1}{2}}\overline{\mathbf{R}}^m\overline{\mathbf{H}}^H$ with $\mathbf{W}_m = \mathrm{diag}(w_{1m}, \ldots, w_{KL,m})$ such that $\mathrm{E}\{\|\mathbf{L}\mathbf{Y} - \mathbf{a}\|^2\}$ is minimized. $\mathbf{W}_j$ are obtained by

$$\mathbf{w}_k = \mathbf{\Psi}_k^{-1}\xi_k \tag{3}$$

where $\mathbf{w}_k = (w_{k\,0}, \ldots, w_{k\,M_C-1})^T$ and $w_{k\,m}$ is the element (k,k) of the matrix $\mathbf{W}_m$. $\mathbf{\Psi}_k$ is an $M_C \times M_C$ matrix and ξ_k is an M_C-dimensional vector. Their elements are $(\mathbf{\Psi}_k)_{rs} = \tau^{r+s}(\overline{\mathbf{R}}^{r+s})_{kk} + \sigma^2\tau^{r+s-1}(\overline{\mathbf{R}}^{r+s-1})_{kk}$ and $(\xi_k)_r = \tau^r(\overline{\mathbf{R}}^r)_{kk}$ respectively. As shown in [1], $(\overline{\mathbf{R}}^r)_{kk}$ converges to a deterministic limit $\overline{R}^r_{kk',\infty}$ as $N, K \to \infty$ with β constant. Then, the asymptotic weights of the individual LMMSE channel estimator are deterministic and satisfy the equation $\mathbf{w}_k^\infty = (\mathbf{\Psi}_k^\infty)^{-1}\xi_k^\infty$ with the elements of $\mathbf{\Psi}_k^\infty$ and ξ_k^∞ given by $(\mathbf{\Psi}_k^\infty)_{lm} = \tau^{l+m}\overline{R}^{l+m}_{kk,\infty} + \sigma^2\tau^{l+m-1}\overline{R}^{l+m-1}_{kk,\infty}$ and $(\xi_k^\infty)_m = \overline{R}^m_{kk,\infty}$ respectively. $\overline{R}^l_{kk,\infty}$ can be determined using the following algorithm:

First step: Let $\overline{R}^1_{kk,\infty} = \sigma^2_{\mathcal{P},k}$ and $m^1_{\overline{\mathbf{R}}} = m^1_{\mathbf{C}_{\mathcal{P}}}$.

l-th step - a: Given $\overline{R}^l_{kk,\infty} = \sigma^2_{\mathcal{P},k}\sum_{s=0}^{l-1}\overline{R}^s_{kk,\infty}m^{l-s-1}_{\overline{\mathbf{T}}}$ with $m^l_{\overline{\mathbf{T}}} = \frac{L}{\tau}\beta m^l_{\overline{\mathbf{R}}}$, develop it as a polynomial in $\sigma^2_{\mathcal{P},k}$, $\mathbb{P}(\sigma^2_{\mathcal{P},k})$.

l-th step - b: Obtain $m^l_{\overline{\mathbf{R}}}$ from $\mathbb{P}(\sigma^2_{\mathcal{P},k})$ substituting the s-th power of $\sigma^2_{\mathcal{P},k}$ by $m^s_{\mathbf{C}_{\mathcal{P}}} = \frac{1}{KL}\mathrm{E}\{\mathrm{trace}\,(\mathbf{C}^s_{\mathcal{P}})\}$.

The error covariance matrix in asymptotic conditions $\mathbf{C}_\epsilon^\infty$ is

$$(\mathbf{C}_\epsilon^\infty)_{ij} = \sigma^2_{\mathcal{P},i}\left(1 - \xi_j^{\infty\,H}(\mathbf{\Psi}_i^\infty)^{-1}\xi_j^\infty\right)\delta_{ij}. \tag{4}$$

For the data detection we assume that the channel estimator provides the data detector with channel estimates $\widehat{\mathbf{a}}$ along with the mean squared error in these estimates $\mathbf{C}_\epsilon$ [2]. With $\mathbf{D} = \widehat{\mathbf{A}}\widehat{\mathbf{A}}^H + \mathbf{C}_\epsilon$ and $\widehat{\mathbf{T}} = \mathbf{S}\mathbf{D}\mathbf{S}^H$, the individual LMMSE detector of rank M_D conditioned on the channel estimation is the linear operator $\mathbf{G} = \sum_{m=0}^{M_D-1}\mathbf{Z}_m\widehat{\mathbf{A}}^H\mathbf{S}^H\widehat{\mathbf{T}}^m$ with $\mathbf{Z}_m = \mathrm{diag}(z_{1m}, \ldots, z_{Km})$ satisfying the MSE criterion. The weights are given by

$$\mathbf{z}_k = \widehat{\mathbf{\Phi}}_k^{-1}\widehat{\mathbf{c}}_k \tag{5}$$

where $(\mathbf{z}_k)_l = (\mathbf{Z}_l)_{kk} = z_{kl}$ with $\mathbf{M}^{(n)} = \widehat{\mathbf{A}}^H\mathbf{S}^H\widehat{\mathbf{T}}^n\mathbf{S}\widehat{\mathbf{A}}$, $(\widehat{\mathbf{\Phi}}_k)_{rs} = (\mathbf{M}^{(r+s-1)})_{kk} + \sigma^2(\mathbf{M}^{(r+s-2)})_{kk}$ and $(\widehat{\mathbf{c}}_k)_s = (\mathbf{M}^{(s)})_{kk}$. As $N, K \to \infty$ with L, τ and β constant, the k-th diagonal element of the matrix $\mathbf{M}^{(r)}$ converges to a deterministic quantity $(\mathbf{M}^{(r)})_{kk,\infty}$ depending on $\widehat{\mathbf{a}}_k^H\widehat{\mathbf{a}}_k$ and $\widehat{\mathbf{a}}_k^H\mathbf{C}_{\mathcal{P}}^m\widehat{\mathbf{a}}_k$. Defined the $M_D \times M_D$ matrix $\widehat{\mathbf{\Phi}}_k^\infty$ with elements $(\widehat{\mathbf{\Phi}}_k^\infty)_{rs} = M^{(r+s)}_{kk,\infty} + \sigma^2 M^{(r+s-1)}_{kk,\infty}$ and the M_D-dimensional vector $\widehat{\mathbf{c}}_k^\infty$ with elements $(\widehat{\mathbf{c}}_k^\infty)_r = M^{(r)}_{kk,\infty}$, the asymptotic performance of the whole receiver can be determined using the results in [1]:

$$\mathrm{SINR}_k = \frac{(\widehat{\mathbf{c}}_k^\infty)^H(\widehat{\mathbf{\Phi}}_k^\infty)^{-1}\widehat{\mathbf{c}}_k^\infty}{1 - (\widehat{\mathbf{c}}_k^\infty)^H(\widehat{\mathbf{\Phi}}_k^\infty)^{-1}\widehat{\mathbf{c}}_k^\infty}. \tag{6}$$

References

[1] L. Cottatellucci and R. R. Müller, "A systematic approach to multistage detectors in multipath fading channels," *submitted to IEEE Trans. Inform. Theory.*

[2] J. Evans and D. N. C. Tse, "Large system performance of linear multiuser receivers in multipath fading channels," *IEEE Trans. Inform. Theory*, vol. 46, no. 6, pp. 2059-2078, 2000.