REMARKS ON ORDINARY DIFFERENTIAL EQUATIONS INVOLVING DISSIPATIVE AND COMPACT OPERATORS†

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Let E be a real Banach space with norm $|\cdot|$ and let E^* be the dual space of E with the norm on E^* also denoted by $|\cdot|$. The duality mapping J from E into the class of subsets of E^* is defined by $J(x) = \{f \in E^*: f(x) = |x|^2 = |f|^2\}$ for each $x \in E$, and for each pair $(x, y) \in E \times E$ we define $\langle x, y \rangle_- = \inf\{f(y): f \in J(x)\}$. Now suppose that Ω is a convex subset of E which is locally closed (i.e., for each $z \in \Omega$ there is an $R_z > 0$ such that $\Omega \cap \{x \in E: |x-z| \leqslant R_z\}$ is closed in E). Throughout the note we suppose that B and C are functions from $[0, 1] \times \Omega$ into E which satisfy the following conditions:

(C1) B and C are continuous on $[0, 1] \times \Omega$;

(C2) $\langle x-y, B(t, x)-B(t, y)\rangle_- \leqslant 0$ for all $(t, x), (t, y) \in [0, 1] \times \Omega$;

(C3) K is a compact subset of E such that $C(t, x) \in K$ for all $(t, x) \in [0, 1] \times \Omega$;

(C4) $\liminf_{h \to 0+} d(x+h[B(t, x)+C(t, x)];\ \Omega)/h = 0$ for all $(t, x) \in [0, 1] \times \Omega$, where $d(y;\ \Omega) = \inf\{|y-x|: x \in \Omega\}$ for each $y \in E$.

Under the conditions (C1)–(C4) we consider the local existence of solutions to the initial value problem

$$u'(t) = B(t, u(t))+C(t, u(t)), \quad u(0) = z \in \Omega. \tag{IVP}$$

In general it seems to be unknown if conditions (C1)–(C4) are sufficient to ensure the existence of a local solution to (IVP) (even if Ω is open). It is the purpose of this note to add various additional conditions to (C1)–(C4) in order to guarantee that (IVP) has a solution. Our results are included in the following

THEOREM. *In addition to conditions* (C1)–(C4) *suppose that at least one of the following is satisfied*:

$(C5)_1$ *C is uniformly continuous on* $[0, 1] \times \Omega$;

$(C5)_2$ $|B(t, x)-B(t, y)| \leqslant L|x-y|$ *for all* $(t, x), (t, y) \in [0, 1] \times \Omega$ *and some* $L > 0$;

$(C5)_3$ *the dual space E^* is uniformly convex*;

$(C5)_4$ *there is an open set Λ in E such that $\Lambda \supset \Omega$, B is defined and uniformly continuous on* $[0, 1] \times \Lambda$, *and* (C2) *holds for all* $(t, x), (t, y) \in [0, 1] \times \Lambda$;

$(C5)_5$ *the function $B+C$ is uniformly continuous on* $[0, 1] \times \Omega$.

Then (IVP) *has a local solution.*

Remark. Three of the results given by the theorem follow directly from known existence theorems which employ hypotheses involving the measure of non-compactness of bounded subsets of E. If $(C5)_2$ holds, existence follows from results of Szufla **[11]** and Cellina **[1]**; if $(C5)_3$ holds, existence follows from results of Cellina **[2]**; and if

Received 29 November, 1973.

† This research was done while the author held a NATO Fellowship in Science.

$(C5)_5$ holds existence follows from results of Li [5] and Martin [7]. Since the proofs under conditions $(C5)_2$ and $(C5)_3$ are simpler in the special cases considered in this paper, we give them here.

Now let $\{\varepsilon_n\}_1^\infty$ be a sequence of positive numbers with $\lim \varepsilon_n = 0$. Using (C1), (C4) and the convexity of Ω we have from [6; Proposition 1] that there is a $T \in (0, 1]$ and an $M > 0$ such that for each positive integer n there exists a function $u_n\ [0, T] \to \Omega$ with the properties that (a) $u_n(0) = z$, (b) $|u_n(t) - u_n(s)| \leqslant M|t-s|$, and (c) $|u_n'(t) - B(t, u_n(t)) - C(t, u_n(t))| \leqslant \varepsilon_n$ for all but a countable number of $t \in [0, T]$. Moreover, by [6; Proposition 2], if the sequence $\{u_n\}_1^\infty$ converges pointwise to a function u on $[0, T]$, then u is a solution to (IVP) on $[0, T]$. Thus for the proof of the theorem we show that $\{u_n\}_1^\infty$ is uniformly Cauchy on $[0, T]$. For notational convenience let $\|g\| = \sup\{|g(t)|: t \in [0, T]\}$ for each bounded function $g: [0, T] \to E$. Also, in our proofs we use the facts that if $x, y, z \in E$ then

$$\langle x, y+z\rangle_- \leqslant \langle x, y\rangle_- + |x|\cdot|z|$$

and if $v: [0, T] \to E$ is differentiable at $t \in (0, T)$ and $p = |v|^2$, then $p_-'(t)$ exists and $p_-'(t) = 2\langle v(t), v'(t)\rangle_-$ (see Kato [3; Lemma 1.3]).

Suppose first that $(C5)_1$ holds and let $v_n(t) = C(t, u_n(t))$ for $n \geqslant 1$ and $t \in [0, T]$. By the compactness and uniform continuity of C and the equicontinuity of $\{u_n\}_1^\infty$ on $[0, T]$, it is immediate from Ascoli's theorem that $\{v_n\}_1^\infty$ has a uniformly convergent subsequence. Extracting such a subsequence and relabelling, we may assume that $\lim_{n,m\to\infty} \|v_n - v_m\| = 0$. Thus if n and m are positive integers and $p(t) = |u_n(t) - u_m(t)|^2$ for all $t \in [0, T]$, then for all but a countable number of $t \in [0, T]$ we have

$$\begin{aligned} p_-'(t) &= 2\langle u_n(t) - u_m(t), u_n'(t) - u_m'(t)\rangle_- \\ &\leqslant 2\langle u_n(t) - u_m(t), B(t, u_n(t)) + v_n(t) - B(t, u_m(t)) - v_m(t)\rangle_- + 2\|u_n - u_m\|(\varepsilon_n + \varepsilon_m) \\ &\leqslant 2\|u_n - u_m\|(\|v_n - v_m\| + \varepsilon_n + \varepsilon_m). \end{aligned}$$

Noting that $p(0) = 0$ we obtain

$$|u_n(t) - u_m(t)|^2 \leqslant 2T\|u_n - u_m\|(\|v_n - v_m\| + \varepsilon_n + \varepsilon_m) \quad \text{for all} \quad t \in [0, T].$$

Thus $\{u_n\}_1^\infty$ is uniformly Cauchy on $[0, T]$ and the existence of a solution to (IVP) follows.

If C is not uniformly continuous it does not follow in general that $\{v_n\}_1^\infty$ is equicontinuous. However, if $w_n(t) = \int_0^t C(s, u_n(s))\, ds$ for $t \in [0, T]$ and $n \geqslant 1$, then $\{w_n\}_1^\infty$ is equicontinuous. Moreover, if K_1 is the closed convex hull of K and $K_2 = \{tx: (t, x) \in [0, T] \times K_1\}$, then K_2 is compact and $w_n(t) \in K_2$ for all $t \in [0, T]$ and $n \geqslant 1$. Employing Ascoli's theorem again it may be assumed that $\{w_n\}_1^\infty$ is uniformly Cauchy on $[0, T]$. Now if $(C5)_2$ holds it follows easily that if n and m are positive integers then

$$\begin{aligned} |u_n(t) - u_m(t)| &\leqslant \left| \int_0^t B(s, u_n(s))\, ds + w_n(t) - \int_0^t B(s, u_m(s))\, ds - w_m(t) \right| + T(\varepsilon_n + \varepsilon_m) \\ &\leqslant \int_0^t L|u_n(s) - u_m(s)|\, ds + \|w_n - w_m\| + T(\varepsilon_n + \varepsilon_m) \end{aligned}$$

for all $t \in [0, T]$. Applying Gronwall's lemma shows that

$$\|u_n - u_m\| \leqslant [\|w_n - w_m\| + T(\varepsilon_n + \varepsilon_m)]\, e^{LT},$$

and the existence of a solution to (IVP) follows. If $(C5)_3$ holds, the mapping $(x, y) \to \langle x, y \rangle_-$ is uniformly continuous on bounded subsets of $E \times E$ (see Kato [3; Lemma 1.2]). Also, if $p(t) = |u_n(t) - w_n(t) - u_m(t) + w_m(t)|^2$ then

$$\begin{aligned} p_-'(t) \leqslant 2\langle u_n(t) - u_m(t) - w_n(t) + w_m(t),\, B(t, u_n(t)) - B(t, u_m(t)) \rangle_- \\ + 2\|u_n - u_m - w_n + w_m\|(\varepsilon_n + \varepsilon_m). \end{aligned}$$

Using the uniform continuity of $(x, y) \to \langle x, y \rangle_-$ and the fact that $\lim_{n, m \to \infty} \|w_n - w_m\| = 0$, it follows that

$$p_-'(t) \leqslant 2\langle u_n(t) - u_m(t),\, B(t, u_n(t)) - B(t, u_m(t)) \rangle_- + \eta_{n,m} \leqslant \eta_{n,m},$$

where $\lim_{n,m \to \infty} \eta_{n,m} = 0$. Solving this differential inequality shows that $\{u_n - w_n\}_1^\infty$ and hence $\{u_n\}_1^\infty$ is uniformly Cauchy on $[0, T]$, and we have the existence of a solution to (IVP). Now let $(C5)_4$ hold. Since Λ is open, B is uniformly continuous, and $\lim_{n,m \to \infty} \|w_n - w_m\| = 0$, we may assume that $u_n(t) - w_n(t) + w_m(t) \in \Lambda$ for all $n, m \geqslant 1$, and hence if $\beta_{n,m}(t) = B(t, u_n(t) - w_n(t) + w_m(t)) - B(t, u_n(t))$ then $\lim_{n,m \to \infty} \|\beta_{n,m}\| = 0$. Consequently, if $p(t) = |u_n(t) - w_n(t) - u_m(t) + w_m(t)|^2$,

$$\begin{aligned} p_-'(t) &\leqslant \langle u_n(t) - w_n(t) + w_m(t) - u_m(t),\, B(t, u_n(t)) - B(t, u_m(t)) \rangle_- \\ &\quad + \|u_n - u_m - w_n + w_m\|(\varepsilon_n + \varepsilon_m) \\ &\leqslant \langle u_n(t) - w_n(t) + w_m(t) - u_m(t),\, B(t, u_n(t) - w_n(t) + w_m(t)) - B(t, u_m(t)) \rangle_- \\ &\quad + \|u_n - u_m - w_n + w_m\|(\varepsilon_n + \varepsilon_m + \|\beta_{n,m}\|) \\ &\leqslant \|u_n - u_m - w_n + w_m\|(\varepsilon_n + \varepsilon_m + \|\beta_{n,m}\|). \end{aligned}$$

Again solving this differential inequality establishes existence. For a proof in the case in which $(C5)_5$ holds we refer the reader to [5] or [7].

A typical situation where these techniques may be applied is integro-differential equations of the form

$$\begin{cases} \partial u(t, s)/\partial t = f(t, s, u(t, s)) + \displaystyle\int_0^1 g(t, s, \tau, u(t, \tau))\, d\tau, \quad (t, s) \in [0, 1]^2 \\ u(0, s) = z(s), s \in [0, 1]. \end{cases}$$

If E is any of the spaces $C([0, 1], \mathbb{R})$ or $L^p([0, 1], \mathbb{R})$, $1 \leqslant p < \infty$, one may define the operators B and C on $\Omega \subset E$ by

$$[B(t, u)](s) = f(t, s, u(s)) \quad \text{for all} \quad (t, s, u) \in [0, 1]^2 \times \Omega \quad \text{and}$$

$$[C(t, u)](s) = \int_0^1 g(t, s, \tau, u(\tau))\, d\tau \quad \text{for all} \quad (t, s, u) \in [0, 1]^2 \times \Omega.$$

Various conditions insuring that B is continuous and C is completely continuous on $\Omega \subset E$ may be found in [4]. The dissipative condition (C2) is valid if $r \to f(t, s, r)$ is non-increasing on $\mathbb{R}$ for each $(t, s) \in [0, 1]^2$. If f and g are continuous our results apply in the space $E = C([0, 1], \mathbb{R})$ since C (as well as B) is uniformly continuous on bounded subsets. These results also apply when $E = L^p([0, 1], \mathbb{R})$, where $1 < p < \infty$, since E^* is uniformly convex. However, some difficulties may arise for the space $E = L^1([0, 1], \mathbb{R})$. For let $h\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy (i) $h(r) = 0$ if $r \leqslant 0$, (ii) $h(n) = -n$ and $h(n-n^{-1}) = 0$ for $n = 1, 2, \ldots$, and (iii) h is linear on $[n-1, n-n^{-1}]$ and on $[n-n^{-1}, n]$ for $n = 1, 2, \ldots$. For each $u \in L^1([0, 1], \mathbb{R})$ define

$$[Cu](s) = \int_0^1 h(u(\tau))\, d\tau \quad \text{for all} \quad s \in [0, 1].$$

Since $|h(r)| \leqslant r$ and h is continuous it is easy to see that C is completely continuous. However, C is not uniformly continuous in any neighbourhood of the origin. For let ε be a positive number and for each positive integer n define ϕ_n and ψ_n on $[0, 1]$ by $\phi_n(s) = n$ and $\psi_n(s) = n-n^{-1}$ if $s \in [0, \varepsilon n^{-1}]$, and $\phi_n(s) = \psi_n(s) = 0$ if $s \in (\varepsilon n^{-1}, 1]$. Then $\|\phi_n\|_1, \|\psi_n\|_1 \leqslant \varepsilon$, $\|\phi_n - \psi_n\|_1 = \varepsilon n^{-2}$, and $\|C\phi_n - C\psi_n\|_1 = \varepsilon$ for $n = 1, 2, \ldots$. Thus C is not uniformly continuous on $\{u \in L^1([0, 1], \mathbb{R})\colon \|u\|_1 \leqslant \varepsilon\}$ for any $\varepsilon > 0$. Note also that C (or $C-\gamma I$, where $\gamma > 0$ and I is the identity on $L^1([0, 1], \mathbb{R})$) does not satisfy a dissipative condition in any neighbourhood of the origin. Now let $k\colon \mathbb{R} \to \mathbb{R}$ be continuous and satisfy (i) $k(r) = 0$ if $r \leqslant 0$, (ii) $k(2^n) = -2^n$ and $k(r) = -2^{n-1}$ if $r \in [2^{n-1}, 2^n-1]$ and $n = 2, 3, \ldots$, (iii) $k(r) = -r$ if $r \in [0, 2]$, and (iv) k in linear in $[2^n-1, 2^n]$ for $n = 2, 3, \ldots$. Since $|k(r)| \leqslant r$ and k is non-increasing, it follows that if $[Bu](s) = k(u(s))$ for all $s \in [0, 1]$, then B is continuous and dissipative from $L^1([0, 1], \mathbb{R})$ into $L^1([0, 1], \mathbb{R})$. However, neither B nor $B+C$ is uniformly continuous in any neighbourhood of the origin. For let ε be a positive number and for each positive integer n define ϕ_n and ψ_n on $[0, 1]$ by $\phi_n(s) = 2^n$ and $\psi_n(s) = 2^n-1$ for $s \in [0, \varepsilon 2^{-n}]$ and $\phi_n(s) = \psi_n(s) = 0$ for $s \in (\varepsilon 2^{-n}, 1]$. Then $\|\phi_n\|_1, \|\psi_n\|_1 \leqslant \varepsilon$, $\|\phi_n - \psi_n\|_1 = \varepsilon 2^{-n}$, and $\|B\phi_n - B\psi_n\|_1 = \varepsilon/2$ for $n = 2, 3, \ldots$. Thus B is not uniformly continuous, and since $\lim_{n\to\infty} \|C\phi_n - C\psi_n\|_1 = 0$, we also have that $B+C$ is not uniformly continuous. Therefore, none of these results apply when Ω is an open neighbourhood of the origin in $L^1([0, 1], \mathbb{R})$. Of course these results do apply if instead of $L^1([0, 1], \mathbb{R})$ we let E be $C([0, 1], \mathbb{R})$ or $L^p([0, 1], \mathbb{R})$ with $1 < p < \infty$. It is also interesting to note that our results apply in the case in which E is $L^1([0, 1], \mathbb{R})$, if $r > 0$ and we take $\Omega = \{u \in L^1([0, 1], \mathbb{R})\colon |u(t)| \leqslant r$ for almost all $t \in [0, 1]\}$ (since C is uniformly continuous on Ω).

These techniques can also be applied under more general conditions. Let $\omega\colon [0, 1] \times [0, \infty) \to [0, \infty)$ be continuous with $\omega(t, 0) = 0$ for all $t \in [0, 1]$, and suppose that the scalar equation $\phi'(t) = \omega(t, \phi(t))$, $\phi(0) = 0$ has only the trivial solution. Then the dissipative condition (C2) on B may be replaced by

(C2)′ $\langle x-y, B(t, x)-B(t, y)\rangle_- \leqslant |x-y|\, \omega(t, |x-y|)$ for all $(t, x), (t, y) \in [0, 1] \times \Omega$.

We remark also that one may use a more general Lyapunov function than the norm in place of (C2) as well (see Martin **[8]**, Murakami [9] or Ricciardi and Tubara **[10]**). Finally we ask the following question: are conditions (C1)–(C4) sufficient to insure the local existence of solutions to (IVP)?

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