Advanced Materials Research *Online: 2010-12-06*
ISSN: 1662-8985, Vol. 159, pp 232-235
doi:10.4028/www.scientific.net/AMR.159.232

Image Enhancement Based on Fractional Differential and Image Entropy

Yawei Liu[1, 2, a], Jianwei Li[1, b]
[1] School of Opto-electronic Engineering, Chongqing University, Chongqing, China
[2] School of Science, Chongqing Jiaotong University, Chonqing, China
[a]yaweilcn@gmail.com, [b]jw_li@cqu.edu.cn

Keywords: image enhancement; fractional differential; image entropy; mutual information

Abstract. In this paper, a new image enhancement method is proposed based on fractional differential, which can select the differential order automatically by the difference of mutual information (DMI). DMI describes the increase of mutual information in original and enhancement image. Being a measure of ascertaining the ehancement effect, it is considered getting the information balance in the images processed by different differential order. According to it, a criterion of selection differential order is put forward. Image convolutions are implemented with fractional operator in different scales, and then DMI of adjacent scales are calculated. The differential order can be selected in which the DMI is the mininum. The experimental results indicate that the proposed method is effective, and has better result compared with other methods.

Introduction

Image enhancement is a classical problem of image processing, which is almost used in all areas of image processing, such as machine vision, image understanding, pattern recognition, image coding, and image synthesis. Fractional calculus is a branch of calculus, by which the definition of calculus can be extended to fractional order [1,2,3]. Fractional calculus has only been widely used in modern engineering since Mandelbrot used it to study the Brownian motion. The numerical algorithms and time-frequency characteristics of fractional differential were analysied by Pu Y. F, and were firstly introduced in digital image processing by Pu and Yang [4,5]. Fractional differential operator has good perfomace in image processing, but the differential order is a constant selected by experience in Pu's works, which is good effect for some images, but poor for others. How to select the differential order according to the image own characteristics is still a difficult problem.

The purpose of this paper is to give a criterion to select the differential order in image enhancement by fractional differential. The article is organized as follows: next section introduces the basic theory of fractional differential, and then numerical mask of fractional differential and the definition of DMI are constructed, by which the selection criterion is proposed, to heel, some experimental analysis is arranged. Some brief conclusions are drawn in the last section.

Basic Theories on Fractional Differential

Fractional calculus is essentially non-integer order calculus, and the order may be real or complex. There are some definitions from different perspectives, but they are all not easy to understand and can not be used directly in engineering.

For $\forall v \in R$, $[v]$ is the integral part of v, $s(t) \in [a,t]$ is a signal with $m+1$ order derivatives, $a,t \in R, m \in Z$. Then definition given by *Grünwald-Letnikov (GL)*can be written as

$$ {}_a^G D_t^v s(t) \triangleq \lim_{h\to 0} s_h^{(v)}(t) \triangleq \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-v} \sum_{r=0}^{n} \begin{bmatrix} -v \\ r \end{bmatrix} s(t-rh) , \quad \begin{bmatrix} -v \\ r \end{bmatrix} = \frac{(-v)(-v+1)\cdots(-v+r-1)}{r!} \tag{1} $$

Let $f(t) \in L^2(R)$ be an one-demension signal, its Fourier transform is $\hat{f}(\omega) = \int_R f(t)e^{-i\omega t}dt$, Supposed $f(t)$ has k-order differential, $k \in Z^+$, then, its Fourier tansform of k-order differential is

$$(D_k\hat{f})(\omega) = (i\omega)^k \hat{f}(\omega) = \hat{d}_k(w)\hat{f}(\omega) \tag{2}$$

where $\hat{d}_k(w) = (i\omega)^k$, called $k-$ order multiplier, and it can be writted as the exponential form

$$\hat{d}_k(\omega) = \hat{a}_k(\omega)\exp(i\theta_k(\omega)),\quad \hat{a}_k(\omega) = |\omega|^k,\quad \theta_k(\omega) = \frac{k\pi}{2}\mathrm{sgn}(\omega) \qquad (k \in Z^+) \tag{3}$$

The integer order k in Eq. 2 can be generalized to an arbitrary order v ($v \in R^+$), and the form of fractional derivative in frequency domain can be defined as $(D_v\hat{f})(\omega) = (i\omega)^v \hat{f}(\omega) = \hat{d}_v(w)\cdot\hat{f}(\omega)$.

The exponential form of $\hat{d}_v(w)$ in frequency domain can also be obtained like Eq. 3

$$\hat{d}_v(\omega) = \hat{a}_v(\omega)\exp(i\theta_v(\omega)) = \hat{a}_v(\omega)\hat{p}_v(\omega),\ \hat{a}_v(\omega) = |\omega|^v,\ \theta_v(\omega) = \frac{v\pi}{2}\mathrm{sgn}(\omega) \tag{4}$$

In time domain, $d_v(\omega)$ can be represented as $d_v(t) = a_v(t) * p_v(t)$, here

$$a_v(t) = \frac{1}{2\pi}\int_R a_v(\omega)e^{i\omega t} = \frac{1}{\pi}\sin\frac{v\pi}{2}\frac{\Gamma(v+1)}{|t|^{v+1}}, p_v(t) = \frac{1}{2\pi}\int_R e^{i\theta_v(\omega)}e^{i\omega t}d\omega = \cos\frac{v\pi}{2}\delta(t) - \sin\frac{v\pi}{2}\frac{1}{\pi} \tag{5}$$

The characteristic curves of fractional differential on amplitude and frequency can be exhibited by Eq. 4. The differential operator has the role of enhancing high-frequency of signal, which increases rapidlly in non-linear speed with the growth of the frequency and the differential order. Meanwhile, it has the role of weaken the very lower frequency. For v-order differentional ($0 < v < 1$), suppose $\Delta \gg 1$, the component in higher frequency is strengthened where $\omega > \Delta$, but the range is much less than that of the first and second order differential. Contrary, the part of low frequency is in nonlinear manner where $1 > \omega > 0$, and the range is less than the first and second order differential.

Enhancement Algorithm

Suppose $s(t)$ is a one-dimension signal, its support domain is $\Omega = [a, t]$. Divide Ω into n equal parts by interval step $h = 1$, then $n = [\frac{t-a}{h}] = [t-a]$. By *GL* definition in Eq. 1, the v-order differential of $s(t)$ can be writted as

$${}_a^G D_t^v s(t) \triangleq \lim_{\substack{h\to 0 \\ nh=t-a}} h^{-v}\sum_{r=0}^{n}\begin{bmatrix}-v\\ r\end{bmatrix}s(t-rh) \triangleq \lim_{n\to\infty}\left\{\frac{(\frac{t-a}{n})^{-v}}{\Gamma(-v)}\sum_{r=0}^{n-1}\frac{\Gamma(r-v)}{\Gamma(r+1)}s\left(t-\frac{r(t-a)}{n}\right)\right\} \tag{6}$$

The expression in Eq. 6 can be approximately formulized and simplified approximately as

$$\begin{aligned} s^{(v)}(t) &\approx \frac{1}{\Gamma(-v)}\sum_{r=0}^{n-1}\frac{\Gamma(r-v)}{\Gamma(r+1)}s\left(t-\frac{r(t-a)}{n}\right) \\ &= s(t) + (-v)s(t-1) + \frac{(-v)(-v+1)}{2}s(t-2) + \cdots + \frac{\Gamma(-v+1)}{n!\Gamma(-v+n+1)}s(t-n) \end{aligned} \tag{7}$$

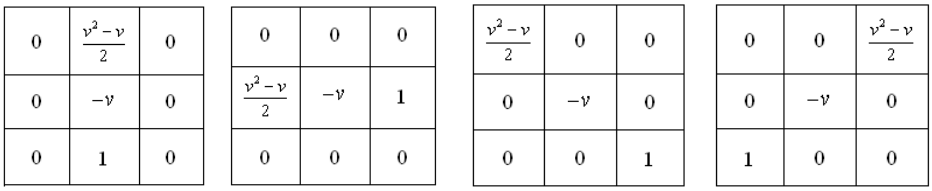


Figure 1. 3×3 two-dimension mask of fractional differential

A digital image $I(i,j)$ is essentially a two-dimension discrete signals, the support domain is $[0,w]\times[0,h]$, where w and h are the width and height of the image respectively. The fractional differential operation can be achieved by linear filtering using masks in Fig. 1. The windows size is an arbitrary odd number, and a larger window can improve the accuracy of fractional differential, but the computational complexity increases at the same time. Here the size of mask is 3×3

Let M^i and M^j be the masks in vertical and horizontal directions, M^+ and M^- be the filter mask on left and right diagonal. Then, the fractional differential of pixel (i_0,j_0) can be obtained by the convolution of image and above four masks, denoted by $G^i(i_0,j_0),G^j(i_0,j_0),G^+(i_0,j_0),G^-(i_0,j_0)$. Then gradient norm of pixel (i_0,j_0) can be defined as

$$\left\|\nabla^{(v)}I(i_0,j_0)\right\| = \left|G^i(i_0,j_0)\right| + \left|G^j(i_0,j_0)\right| + \left|G^+(i_0,j_0)\right| + \left|G^-(i_0,j_0)\right| \tag{8}$$

It is not difficult to see that the sum of mask coefficients in Fig. 1 are not equal to zero, which implies that the filter response of fractional differential is not zero in image region where the gray values of pixels are constant or changes little. This feature is necessary to enhance the texture detail, which is just the excellent feature than other operator.

The mutual information of images A and B can be defined as

$$MI(A,B) = H(A) + H(B) - H(A,B) \tag{9}$$

Where $H(A)$ and $H(B)$ are the image entropies of A and B, $H(A,B)$ is the joint entropy. Image entropy and joint entropy can be defined as fellows

$$H(I) = -\sum_{k=0,p(i)\neq 0}^{255} p(i)\log p(i) \qquad H(I,J) = -\sum_{i=0}^{255}\sum_{j=0}^{255} P(i,j)\log P(i,j) \tag{10}$$

Where $p(i)$ is the probability distribution function of image intensity, and $p(i,j)$ is the tow-dimention joint distribution function of I and J .The difference of mutual information (DMI) can also be obtained as

$$DMI_v(I) = MI(I,I^{(v+1)}) - MI(I,I^{(v)}) \tag{11}$$

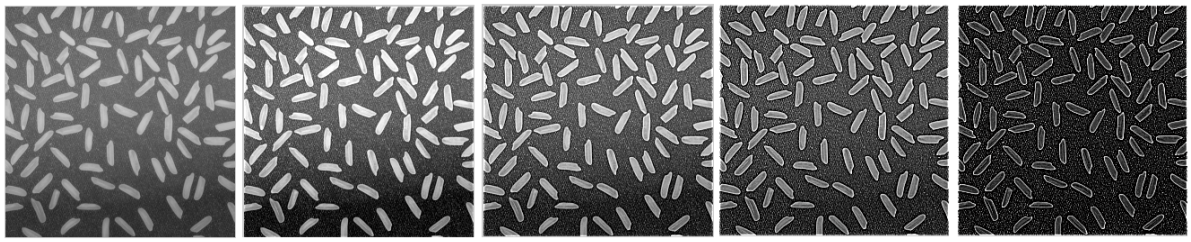
Where I and $I^{(v)}$ are the original image and the enhancement image v order differential operator.

Experimental data showed that with the increases of differential order, the MI of original and enhancement images will be mononically increasing, and finally tend to the the maximum MI(I, I). At the same time, DMI of two adjacent orders will be decreasing or drop shock, and converges to zero. DMI is considered getting the information balance in the images processed by different differential order. So a resonable selection criterion of differential order v can also be established as

$$v^* = \min_{v\in(0.1)} DMI_v(I) \tag{12}$$

Experiments

Some experiments results are given in this section, and experiments are programmed in Matlab 7, and the experimental images come from image library of Matlab.



(a) Original (b) $v = 0.2$ (c) $v = 0.4$ (d) $v = 0.6$ (e) $v = 0.8$

Figure 2. results for rice image with different differential orders

It can be seen from Fig. 2 that the enhancement effects are different when differential order changes from 0.2 to 0.8, but subjectively the effect is the best when $v = 0.4$. It also can be found that with the growth of the differential order brightness of output images become smaller, and the gray levels become narrower. So the selection of differential order is important.

(a) Original (b) $v = 0.1$ (c) $v = 0.3$ (d) $v = 0.5$ (e) $v = 0.7$

Fiure 3. results for tire image with different differential orders

Fig. 3 shows similar results, when differential order changes from 0.1 to 0.9 (interval 0.1), the enhancement effects are different. But the effect is the best when $v = 0.5$, and it is agree with the calculation result by Eq. 12 where the difference of mutual information is almost minimum.

Conclusion

Fractional differential operator has its own advantaes for image enhancement, especially in the texture detail, but the selection of differential order is based on experimence in existing works. A new method for order selection is proposed based on image entropy, by which the differential order can be selected adaptively according to the characteristics of the image itself.

References

[1] Oldham K B, Spanier J: The fractional calculus. Academic Press, New York and London (1974).

[2] Leu J S, Papamarcou A: On estimating the spectral exponent of fractional Brownian motion. IEEE Transactions on Information Theory, p. 233-244 (1995).

[3] Loverro A. in: Fractional calculus: history, definitons and applications for the engineer. www.nd.edu/msen/Teaaching/UnderRes/FracCalc.pdf.

[4] Pu Y F, Yuan X, Liao K: Five numerical algorithms of fractional calculus applied in modern singal analyzing and processing. Journal of Sichuan University. Vol. 37(5), p. 118-124(2005).

[5] Pu Y F, Wang W X, Zhou J L: Texture Details Detection of digital image by Fractional Differential and the Implementation of Fractional Differential Filter. SCIENCE IN CHINA: E. 38(12): p. 2252-2272 (2008).