Presentation and Structure of Substitutes Valuations

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Abstract

We propose two different methods for presenting (gross-) substitutes valuations. Each provides short descriptions for a family of substitutes valuations. We also show that any substitutes valuation defines ultrametrics on the set of items and that substitutes valuations are closed under k-satiation.

1 Background

In [4], Kelso and Crawford introduced the notion of a *substitutes* valuation for profit-maximizing firms in a model with discrete inputs (workers). They studied the labor market generated by firms whose preferences concerning teams of workers were described by such a valuation. They showed that, when workers are substitutes, a certain matching process always converges and that the resulting matching and wages constitute a Walrasian equilibrium.

The study of substitutes valuations and their implications has been pursued by many. Several authors obtain results showing that the substitutes condition is *necessary* for important economic conclusions. Gul and Stacchetti [3] and Milgrom [6] showed that, given any valuation that is not substitutes, one can specify very simple substitutes valuations for the other agents to create an economy in which no Walrasian equilibrium exists. Similarly, Ausubel and Milgrom [1] show that if any given valuation is not substitutes, then one may specify simple substitutes valuations for the other agents such that the Vickrey outcome does not lie in the core.

Various authors have also studied the characterization of substitutes valuations. Gul and Stacchetti prove that several conditions, most importantly their *single improvement property*, are equivalent to the substitutes condition. provided a number of new characterizations of the family of substitutes valuations. Ausubel and Milgrom characterize substitutes valuations as the duals of submodular profit functions. In [2], Fujishige and Yang proposed a very powerful characterization of substitutes valuations as M#-concave functions. Other

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important results about substitutes valuations and their algorithmic properties may be found in [8, 9, 12, 5]. Most notable are the facts that substitutes valuations are closed under convolution, also called OR, and that, given prices for the items, the problem of finding a preferred bundle under those prices can be solved in polynomial time in the oracle model, that the problem of finding an optimal allocation amongst substitutes agents is solvable in polynomial time and that substitutes valuations have zero measure amongst submodular valuations. Notice, though, that no polynomial time algorithm is known that can check in the oracle model whether a valuation is substitutes. The question of how many substitutes valuations exist on n items is also a difficult open question.

These works imply that several problems of economic importance can be solved in an elegant and effective way when the agents exhibit valuations that are substitutes. In all of the mechanisms proposed, agents must describe their substitutes valuation. It is therefore important to find intuitive and concise formalisms that will enable agents to describe such valuations, guaranteeing that such valuations are indeed substitutes. In this paper two such formalisms are proposed. The first one describes a valuation as a sum of symmetric substitutes valuations defined on some hierarchy among the items. The second one describes a valuation as a compound expression of convolutions and k-satiation operations on unit-demand valuations.

Section 2 describes setting, notations and previous results. Section 3 shows that a canonical ultra-metric may be attached to any substitutes valuation. This is a motivation only for Section 4 that proposes to describe substitutes valuations by an ultra-metric, i.e. a hierarchy, and symmetric substitutes, i.e., submodular, valuations on the sets of the hierarchy. Not all substitutes valuations can be described in this way. Section 5 defines the k-satiation operation and shows that substitutes valuations are closed under k-satiation. Section 6 proposes to describe substitutes valuations as compound expressions formed by applying convolution and k-satiation to unit-demand valuations. Such descriptions may reasonably fit the values attached by a firm to any possible set of workers.

2 Setting

We consider a finite set X. A valuation on X is a real function $v : X \longrightarrow \mathcal{R}$ such that:

- **Normality** $v(\emptyset) = 0$
- **Monotonicity** if $A \subseteq B \subseteq X$ then $v(A) \leq v(B)$.

In the sequel A, B, ... will always denote subsets of X, and x, y, ... will denote elements of X. The set $A \cup \{x\}$ will be denoted $A + x$ and the set $A - \{x\}$ will be denoted $A - x$. A valuation v is *symmetric* if its value depends only on the cardinality of its argument: if $\mid A \mid = \mid B \mid$, then $v(A) = v(B)$. The marginal valuation, given a bundle A, of an element $x \in X - A$ is: $v(x \mid A) \stackrel{\text{def}}{=}$

$v(A \cup \{x\}) - v(A)$. A valuation v is *submodular* if, for any $A \subseteq B$ and for any $x \notin B$, we have $v(x \mid A) \geq v(x \mid B)$.

Definition 1 *Given a vector of real item prices $\vec{p} = (p_1 \ldots p_m)$, the* surplus *of a set of items S relative to these prices is defined as $v(S \mid \vec{p}) = v(S) - \sum_{i \in S} p_i$. A set S is a* preferred set *of v at prices $\vec{p}$ if $v(S \mid \vec{p}) = \max_T v(T \mid \vec{p})$, i.e., S maximizes the surplus. The* demand set *of v at prices $\vec{p}$ is the set of all preferred sets $D(v \mid \vec{p}) = \{S \mid v(S \mid \vec{p}) = \max_T v(T \mid \vec{p})\}$.*

The substitutes property mandates that increasing the price of an item will not decrease the demand for any other item: i.e., if an item i is in a preferred set at prices $\vec{p}$, then increasing p_j for some $j \neq i$, will still have item i in some preferred set.

Definition 2 *A valuation v is said to be* substitutes *if for any price vectors $\vec{p}$ and $\vec{q} \geq \vec{p}$ (coordinatewise comparison), if $S = \{j \in X \mid p_j = q_j\}$ and $A \in D(v \mid \vec{p})$, then there exists $A' \in D(v \mid \vec{q})$ such that $S \cap A \subseteq A'$. The class GS is set of all valuations that satisfy the substitutes property.*

Lemma 1 *If v is substitutes, then the marginal valuation $v(\cdot \mid A)$ on $X - A$ is also substitutes, for any $A \subseteq X$.*

A valuation is substitutes if and only if it satisfies the *single improvement property*. This condition states that a non-preferred set can always be improved (in terms of its surplus) by deleting at most one element from it and inserting at most one element into it.

Lemma 2 (Gul and Stacchetti(1999)) *A valuation v satisfies the substitutes property if and only if for any $\vec{p}$ and $A \notin D(v \mid \vec{p})$, there exists A' such that $\mid A' - A \mid \leq 1$ and $\mid A - A' \mid \leq 1$ and $v(A' \mid \vec{p}) > v(A \mid \vec{p})$.*

The substitutes property is stronger than submodularity.

Lemma 3 (Gul and Stacchetti(1999)) *A valuation that satisfies the substitutes property is submodular.*

It was shown in [5] that, in fact, substitutes valuations have zero measure amongst submodular valuations.

Definition 3 (Murota and Shioura [11]) *A valuation v is M#-concave if and only if for any $A, B \subseteq X$ and any $x \in A - B$*

1. *either $v(A) + v(B) \leq v(A - x) + v(B + x)$,*
2. *or there exists some $y \in B - A$ such that $v(A) + v(B) \leq v(A - x + y) + v(B - y + x)$.*

It is easy to show that any M#-concave valuation is substitutes. Fujishige and Yang [2] prove the converse: any substitutes valuation is M#-concave. The main interest of this definition is that it does not involve universal quantification over prices. This definition is price-free.

3 The ultra-metric defined by a substitutes valuation

Given any substitutes valuation v, we shall define a distance on X, and we shall show that this distance is an ultra-metric. This ultra-metric defines a hierarchy (that is, a tree of subsets) in the usual way. This hierarchy will, hopefully, some day lead to a representation result for substitutes valuations. At this point in time, it only serves as a motivation for the definition of S-valuations in Section 4.

Since, when v is substitutes, the marginal valuations $v(\cdot \mid A)$ are also substitutes, we in fact, define an ultra-metric on $X - A$ for every $A \subseteq X$.

Let $\mathcal{R}^\infty$ denote the set of nonnegative real numbers with infinity: $\mathcal{R}^\infty \stackrel{\text{def}}{=} \mathcal{R}_+ \cup \{+\infty\}$.

Definition 4 *Let v be a substitutes valuation on X. Let $\alpha_v, d_v : X \times X \longrightarrow \mathcal{R}^\infty$ be defined by:*

- $\alpha_v(x,x) \stackrel{\text{def}}{=} +\infty$, $d_v(x,x) \stackrel{\text{def}}{=} 0$ *for any $x \in X$, and*
- $\alpha_v(x,y) \stackrel{\text{def}}{=} v(x) + v(y) - v(x+y)$, $d_v(x,y) \stackrel{\text{def}}{=} 1/\alpha_v(x,y)$ *for any $x, y \in X$, $x \neq y$.*

Note that the last + in the definition of α above is *not* addition but union. When the valuation v is clear, we shall write α for α_v and d for d_v. The quantity $\alpha(x,y) = v(x) + v(y) - v(x+y)$ is nonnegative since v is submodular. It measures the deficit in *additivity* of the pair (x,y) with respect to v.

Lemma 4 *If v is substitutes, then d_v is an ultra-metric on X.*

Proof: Since v is submodular, for any $x, y \in X$ $\alpha(x,y) \geq 0$ and therefore $d(x,y) \in \mathcal{R}^\infty$. If $x \neq y$, $d(x,y) > 0$ and therefore $d(x,y) = 0$ iff $x = y$. Clearly: $d(x,y) = d(y,x)$. To prove that d is an ultra-metric, it remains to show that $d(x,y) \leq \max(d(x,z), d(z,y))$. Notice that, if x, y, z are pairwise distinct

$$d(x,y) > d(x,z) \Leftrightarrow \alpha(x,z) > \alpha(x,y) \Leftrightarrow \tag{1}$$

$$v(z) - v(x+z) > v(y) - v(x+y) \Leftrightarrow v(x \mid y) > v(x \mid z)$$

By Claim 1 of [5], then, if $x \neq z$, $d(x,y) > d(x,z)$ implies $v(y \mid x) = v(y \mid z)$ and therefore $d(y,x) = d(y,z)$. The case $x = z$ is easily dealt with. ∎

Given an ultra-metric d on X one can find a number l and an equivalence relation $\equiv$ on X such that, for any $a, b \in X$,

- if $a \not\equiv b$, then $d(a,b) = l$ and
- if $a \equiv b$, then $d(a,b) < l$.

The relation $\equiv$ defines a partition of X that provides the upper layer of a hierarchical decomposition of X. Each of the elements of the partition can then be partitioned further in a similar way, define a hierarchy as defined below in Section 4.

We do not know how to relate the ultra-metric d on X and the different ultra-metrics d_A on subsets A of X defined by the marginal valuations $v(\cdot \mid X - A)$.

4 S-presentations

We shall now describe a way of presenting substitutes valuations: we present a valuation as a sum of symmetric valuations and therefore such presentations are called S-presentations. S-presentations define S-valuations which form a strict sub-family of GS. We recall that a symmetric valuation is substitutes iff it is submodular, i.e., concave. Gul and Stacchetti [3] remarked that additively concave valuations were substitutes. We extend their remark.

Intuitively, consider an employment service. The potential employees come with a natural tree-structure, i.e., a hierarchy. For example the potential employees can be classified into: doctors, nurses and cleaners. Each of those may be further classified into: male nurses and female nurses for example, or cardiologists, oncologists and pediatricians. The value of a given team of employees for the firm may be evaluated as a (submodular) function of the number of doctors, plus a (submodular) function of the number of nurses, plus a (submodular) function of the number of cleaners minus some (supermodular) function of the total size of the team accounting for the cost of processing them.

Definition 5 *A family $\mathcal{H}$ of subsets of X is a hierarchy iff, for any $h, h' \in \mathcal{H}$ such that $h \cap h' \neq \emptyset$, either $h \subseteq h'$ or $h' \subseteq h$.*

We assume that a hierarchy on X is given and that, for every $h \in \mathcal{H}$, a *symmetric submodular valuation* on h, v_h is given. Note that v_h is substitutes. Note that $v_h(A)$ depends only on $\mid A \cap h \mid$, the cardinality of $A \cap h$. The function $A \mapsto v_h(A)$ is a GS valuation. The following lemma spells out the particular property of such valuations.

Lemma 5 *Suppose that v_h is a symmetric submodular valuation on h. Let $A, B \subseteq X$, $x \in A - B$. Then $v_h(A) + v_h(B) > v_h(A - x) + v_h(B + x)$ implies that $x \in h$ and $\mid (A - B) \cap h \mid \leq \mid (B - A) \cap h \mid$.*

Proof: First, notice that if $x \notin h$, $v_h(A) = v_h(A - x)$ and $v_h(B + x) = v_h(B)$. Assuming now that $x \in h$,

$$v_h(A) + v_h(B) > v_h(A - x) + v_h(B + x) \Longleftrightarrow v_h(x \mid A - x) > v_h(x \mid B) \Longleftrightarrow$$

$$\mid (A-x) \cap h \mid < \mid B \cap h \mid \Longleftrightarrow \mid A \cap h \mid \leq \mid B \cap h \mid \Longleftrightarrow \mid (A-B) \cap h \mid \leq \mid (B-A) \cap h \mid .$$

∎

Definition 6 *Any hierarchy $\mathcal{H}$ and symmetric submodular valuations v_h on h define a valuation v by:*

$$v(A) = \sum_{h \in \mathcal{H}} v_h(A). \tag{2}$$

A function v defined in such a way, for some hierarchy $\mathcal{H}$ and some symmetric submodular valuations v_h, is obviously a valuation and will be said to be an S-valuation.

Theorem 1 *Any S-valuation is M#-concave.*

The theorem follows immediately from the following lemma. The same result had been obtained previously by Murota [10] (see p. 141, laminar convex functions).

Lemma 6 *Let a hierarchy $\mathcal{H}$ on a finite set X, and a symmetric submodular valuation v_h on h be given for every $h \in \mathcal{H}$. Let the valuation v be defined by Equation 2 and let $A, B \subseteq X$ and $x \in X$ such that $x \in A - B$. Then,*

1. *either, for every $h \in \mathcal{H}$, $v_h(A) + v_h(B) \leq v_h(A - x) + v_h(B + x)$,*
2. *or there exists some $y \in B - A$ such that, for every $h \in \mathcal{H}$, $v_h(A) + v_h(B) \leq v_h(A - x + y) + v_h(B - y + x)$.*

Theorem 1 follows directly from Lemma 6, by Equation 2. The lemma is proved in Appendix 6.

In Theorem 1 the assumption that $\mathcal{H}$ is a hierarchy cannot be dispensed with: suppose one gets one unit for recruiting a minority and one unit for recruiting a female. If the candidates are a minority male MM (of cost 0.1), a non-minority female F (of cost 0.1) and a minority female MF (of cost 0.5) one will prefer the team $\{MM, F\}$ but if the cost of MM increases to 0.5 one will prefer the singleton $\{MF\}$, contradiction the substitutes property.

Not all GS valuations are S-valuations: a counter-example has been found but cannot be presented here for lack of space. Indeed the counter-example shows an H-valuation (see Section 6) that is not an S-valuation.

In fact the proof of Theorem 1 shows that all S-valuations satisfy two interesting properties. The first one is a strengthened form of M#-concavity: for any $A, B \subseteq X$ and any $x \in A - B$

1. either $\mid A \mid > \mid B \mid$ and $v(A) + v(B) \leq v(A - x) + v(B + x)$,
2. or there exists some $y \in B - A$ such that $v(A) + v(B) \leq v(A - x + y) + v(B - y + x)$.

We do not know whether all substitutes valuations satisfy this property.

Another property satisfied by all S-valuations is the following: for any $A, B \subseteq X$ such that $\mid A \mid > \mid B \mid$, there is some $x \in A - B$ such that $v(A) + v(B) \leq v(A - x) + v(B + x)$. We do not know whether all substitutes valuations satisfy this property.

5 Closure under k-satiation

In [3], Gul and Stacchetti noticed that the k-satiations of additively separable or additively concave valuations were substitutes (GS). We prove the more general result: the class of substitutes valuations (GS) is closed under satiation. A strengthening of the Single Improvement property will be proved first and, for this purpose, the M#-concavity characterization of substitutes valuations obtained by Fujishige and Yang in [2] will be used.

First, we need a strengthening of the single-improvement property.

Lemma 7 *Let $A, B \subseteq \Omega$ be such that $A - B \neq \emptyset$ and $B - A \neq \emptyset$. Let v be an M#-concave valuation. There are items $x \in A - B$ and $y \in B - A$ such that $v(A) + v(B) \leq v(A - x + y) + v(B - y + x)$.*

A second lemma is then needed. It says that if a bundle A of k elements is not one of the preferred bundles amongst bundles of size k, then one may drop some element of A and add some element not in A to get a bundle strictly preferred to A.

Lemma 8 *Let v be an M#-concave valuation, $\vec{p}$ a price vector and $u_{\vec{p}}$ the associated net utility function. Let $A \subseteq \Omega$ and let $k = \mid A \mid$. Assume there is some $B \subseteq \Omega$, $\mid B \mid \leq k$ such that $u_{\vec{p}}(B) > u_{\vec{p}}(A)$. Then, either one can find some $x \in A$, $y \notin A$ such that $u_{\vec{p}}(A) < u_{\vec{p}}(A - x + y)$, or one can find some $x \in A$ such that $u_{\vec{p}}(A) < u_{\vec{p}}(A - x)$.*

We may then conclude.

Theorem 2 *If v is an M#-concave valuation, its k-satiation, v_k is M#-concave, for any k.*

Proof: We show that v_k satisfies the Single Improvement property. By [2], this implies that v_k is M#-concave. Let $\vec{p}$ be a price vector. Since v_k is monotone, it is enough to consider nonnegative prices. There is therefore a preferred bundle S for $w_{\vec{p}}$ the net utility associated with v_k that is of size at most k and we have $v_k(S) = v(S)$. Let A be a bundle that is not preferred for $w_{\vec{p}}$. We must show there exists a bundle strictly preferred to A under $w_{\vec{p}}$ whose Hausdorff distance from A is at most 2.

Let $u_{\vec{p}}$ be the net utility associated with v. If the size of A is strictly less than k, then $w_{\vec{p}}(A) = u_{\vec{p}}(A)$ and, since $w_{\vec{p}}(S) \leq u_{\vec{p}}(S)$, A is not preferred for $u_{\vec{p}}$. By the Single Improvement property for v, there is therefore some bundle B, at distance at most 2 from A such that $u_{\vec{p}}(B) > u_{\vec{p}}(A)$. But B has size at most k and $u_{\vec{p}}(B) = w_{\vec{p}}(B)$.

If the size of A is equal to k, we may apply Lemma 8, to the M#-concave valuation v and we are through.

If the size of A is strictly larger than k, there exists some strict subset $A' \subset A$ of size k such that $v_k(A) = v(A')$. If there is some $x \in A - A'$ for

which $p_x > 0$, then $w_{\vec{p}}(A - x) > w_{\vec{p}}(A)$ and we have proved our claim. Otherwise $w_{\vec{p}}(A) = w_{\vec{p}}(A')$. Therefore A' is not a preferred bundle. We may apply Lemma 8 to A'. If there is some $x \in A'$ such that $w_{\vec{p}}(A' - x) > w_{\vec{p}}(A')$, then

$$w_{\vec{p}}(A - x) \geq w_{\vec{p}}(A' - x) > w_{\vec{p}}(A') = w_{\vec{p}}(A).$$

If there are $x \in A'$ and $y \notin A'$ such that $w_{\vec{p}}(A' - x + y) > w_{\vec{p}}(A')$, then, if $y \notin A$, we have:

$$w_{\vec{p}}(A - x + y) \geq w_{\vec{p}}(A' - x + y) > w_{\vec{p}}(A') = w_{\vec{p}}(A),$$

and if $y \in A$ we have

$$w_{\vec{p}}(A - x) \geq w_{\vec{p}}(A' - x + y) > w_{\vec{p}}(A') = w_{\vec{p}}(A).$$

∎

6 H-presentations

Matching algorithms such as those described in [7] assume that firms can describe their valuations for bundles of potential employees (or contracts) and that those valuations are substitutes. We shall present here a way firms may describe complex valuations that are guaranteed to be substitutes: H-presentations.

Consider a simplified version of the recruitment problem faced by a hospital. The problem is to attach a numerical value to any subset of a pool of potential employees (all employees will be called doctors for simplification). The hospital comprises a number of basic units, such as Pediatrics, Internal Medecine1, Internal Medecine2, etc ... Those basic units form a hierarchy, such as: Hospital, Building1, Building2, West wing of Building1, East wing of Building2, etc ... Each of the potential employees has a specific utility in each of the *basic* units: Dr. White has value 10 in Pediatrics, but only value 2 in Obstetrics. Each of the units in the hierarchy, including the basic units and the whole hospital have a limit on the number of doctors they can employ: not more than 132 doctors in the West wing of Building3. An allocation of doctors to the *basic* units is feasible if those limits are all satisfied. The value of an allocation of doctors to the *basic* units is the sum of the utilities of the individual doctors in the unit they are assigned to.

The value of any set of doctors is the value of the best feasible allocation of those doctors. Such a presentation of a valuation will be called an H-presentation. Notice that any description of the type described above is short, i.e. polynomial in the number of doctors and basic units. The recruitment problem, and the valuation function, may be described by an Integer Programming problem. Let D be the set of doctors, U the set of basic units and $\mathcal{H} \subseteq 2^U$ is a hierarchy. Given a^i_j the value of doctor i in unit j, and a maximal size b_h for every $h \in \mathcal{H}$,

Maximize $\sum_{i \in D, j \in U} x^i_j \, a^i_j$, under the constraints:

1. $\sum_{j \in h} \sum_{i \in D} x^i_j \leq b_h$ for every $h \in \mathcal{H}$,

2. $\sum_{j \in U} x_j^i \leq 1$ for every $i \in D$, and

3. $x_j^i \in \{0, 1\}$.

The first constraint expresses that no more than b_h doctors may work in the basic units of wing h and the second constraint expresses that a doctor may work in at most one basic unit.

Lemma 9 *The valuation defined an H-presentation is substitutes.*

This follows from the fact that, neglecting the number limitations, any basic unit defines a linear (OS in the language of [5]) valuation, therefore substitutes. The number limitations on the basic units define k-satiations of substitutes (here linear) valuations, therefore substitutes valuations, by Theorem 2. The juxtaposition of a number of units defines a valuation that is the convolution (OR in the language of [5]) of the valuations for the specific units. By Theorem 5, there, GS is closed under OR and this valuation is substitutes. Again, applying number limitations boils down to considering the k-satiation of a substitutes valuation and leaves us with a substitutes valuation.

An H-presentation defines the value of a bundle as the solution of an optimization problem: find the value of the best feasible allocation. This optimization problem is equivalent to a weighted max-flow problem: a max-flow problem in which edges have both a capacity and a weight and one wants to maximize the weight of the flow, subject to the capacity constraints. Such problems have integral optimal solutions and can be solved in polynomial time, see, for example [13] p. 191.

We have been able to find some S-valuation that can be shown not to be an H-valuation, thus separating the two classes and showing that not all substitutes valuations are H-valuations.

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A Proofs

Lemma 6 *Let a hierarchy $\mathcal{H}$ on a finite set X, and a symmetric submodular valuation v_h on h be given for every $h \in \mathcal{H}$. Let the valuation v be defined by Equation 2 and let $A, B \subseteq X$ and $x \in X$ such that $x \in A - B$. Then,*

1. *either, for every $h \in \mathcal{H}$, $v_h(A) + v_h(B) \leq v_h(A - x) + v_h(B + x)$,*
2. *or there exists some $y \in B - A$ such that, for every $h \in \mathcal{H}$, $v_h(A) + v_h(B) \leq v_h(A - x + y) + v_h(B - y + x)$.*

Proof: Without loss of generality, we may assume that there is some $h \in \mathcal{H}$ such that $v_h(A) + v_h(B) > v_h(A - x) + v_h(B + x)$. Clearly $x \in h$. Consider the elements of $\mathcal{H}$ that contain x. Since $\mathcal{H}$ is a hierarchy, those sets form a chain: $h_0 \subset \ldots \subset h_n$. Let g be the smallest h_i for which $v_h(A) + v_h(B) > v_h(A - x) + v_h(B + x)$. We have:

$$v_g(A) + v_g(B) > v_g(A - x) + v_g(B + x) \tag{3}$$

and $\forall h \in \mathcal{H}$, such that $x \in h$, $h \subset g$, we have

$$v_h(A) + v_h(B) \leq v_h(A - x) + v_h(B + x). \tag{4}$$

From (3), by Lemma 5, $x \in g$ and

$$| (A - B) \cap g | \leq | (B - A) \cap g | . \tag{5}$$

Since $x \in (A - B) \cap g$, the set $(B - A) \cap g$ is not empty. We shall take the y whose existence we assert from $(B - A) \cap g$, but not all such ys are suitable. Our choice will be in two phases. First we choose some arbitrary candidate $y \in (B - A) \cap g$ and in a further step we may decide to modify our choice. Notice, first, that, for any $y \in (B - A) \cap g$ and for any $h \in \mathcal{H}$, the following holds.

1. If $x \notin h$ and $y \notin h$, then $A - x + y \cap h = A \cap h$ and $B - y + x \cap h = B \cap h$ and therefore $v_h(A) + v_h(B) = v_h(A - x + y) + v_h(B - y + x)$.

2. If $x \in h$ and $y \in h$, then, since $x \in A - B$ and $y \in B - A$, $| (A - x + y) \cap h | = | A \cap h |$ and $| (B - y + x) \cap h | = | B \cap h |$. Therefore again, $v_h(A) + v_h(B) = v_h(A - x + y) + v_h(B - y + x)$. Note that all h_is for which $h_i \supseteq g$ are in this category, but that there may be other such h_is, for which $h_i \subset g$.

3. If $x \in h$ and $y \notin h$, then, we must have $h \subset g$ and, by (4), $v_h(A) + v_h(B) \leq v_h(A - x) + v_h(B + x)$. But $(A - x + y) \cap h = (A - x) \cap h$ and $(B - y + x) \cap h = (B + x) \cap h$ and therefore $v_h(A) + v_h(B) \leq v_h(A - x + y) + v_h(B - y + x)$.

All the above holds for any $y \in (B - A) \cap g$. To be able to deal successfully with those $h \in \mathcal{H}$ for which $x \notin h$ but $y \in h$, we shall need to be more specific with the $y \in (B - A) \cap g$ that we want to choose.

For the moment, choose any $y \in (B - A) \cap g$. The elements of $\mathcal{H}$ that contain y form a chain. This chain contains g. Let us define $\mathcal{F} = \{h \in \mathcal{H} \mid x \notin h, y \in h\}$. The elements of $\mathcal{F}$ form a chain and they are all strict subsets of g. We shall distinguish two cases.

Case 1:

$$\forall h \in \mathcal{F},\ (B - A) \cap h = (B - A) \cap g. \tag{6}$$

In other terms, all the candidate y's of g are still in every set of $\mathcal{F}$. This is the central case. In this case, the y we have chosen is suitable, and we do not need to modify our choice. In our study of case 2 we shall show that, if y does not satisfy Condition 6, one can always find some y' that does. Indeed, for any $h \in \mathcal{F}$, by Condition 6, $| (B - A) \cap h | = | (B - A) \cap g |$. By (5), $| (B - A) \cap g | \geq | (A - B) \cap g |$. Since $h \subset g$, and $x \in (A - B) \cap g$ but $x \notin h$, $| (A - B) \cap g | > | (A - B) \cap h |$. We conclude that $| (B - A) \cap h | > | (A - B) \cap h |$, therefore $| B \cap h | > | A \cap h |$, and $| (B - y) \cap h | \geq | A \cap h |$. Finally, $v_h(y \mid B - y) \leq v_h(y \mid A)$ and $v_h(A) + v_h(B) \leq v_h(A + y) + v_h(B - y)$. But, since $x \notin h$, $v_h(A + y) = v_h(A - x + y)$ and $v_h(B - y) = v_h(B - y + x)$. We conclude that, as desired, $v_h(A) + v_h(B) \leq v_h(A - x + y) + v_h(B - y + x)$.

Case 2: There is some $h \in \mathcal{F}$ such that $(B - A) \cap h \subset (B - A) \cap g$. Since $\mathcal{F}$ is a chain, there is a largest element f in $\mathcal{F}$ for which this happens. We have:

$$(B - A) \cap f \subset (B - A) \cap g \tag{7}$$

and

$$\forall h \in \mathcal{F}, \text{ if } h \supset f \text{ then } (B - A) \cap h = (B - A) \cap g. \tag{8}$$

In this case, we must change our choice of y and we shall take any y' such that $y' \in (B - A) \cap g$ but $y' \notin f$. But such a y' is an element of $(B - A) \cap g$ that falls into case 1 and we are done. ∎

Lemma 7 *Let $A, B \subseteq \Omega$ be such that $A - B \neq \emptyset$ and $B - A \neq \emptyset$. Let v be an M#-concave valuation. There are items $x \in A - B$ and $y \in B - A$ such that $v(A) + v(B) \leq v(A - x + y) + v(B - y + x)$.*

Proof: By assumption, there is some x_0 in $A - B$ and therefore, by M#-concavity, either there is some $y \in B - A$ such that

$$v(A) + v(B) \leq v(A - x_0 + y) + v(B - y + x_0),$$

and the result is proved, or we have

$$v(A) + v(B) \leq v(A - x_0) + v(B + x_0),$$

i.e., $v(x_0 \mid A - x_0) \leq v(x_0 \mid B)$. By assumption, there is some y_0 in $B - A$ and therefore, either there is some $x \in A - B$ such that $v(B) + v(A) \leq v(B - y_0 + x) + v(A - x + y_0)$, and the result is proved, or we have

$$v(B) + v(A) \leq v(B - y_0) + v(A + y_0),$$

i.e., $v(y_0 \mid B - y_0) \leq v(y_0 \mid A)$. But, then, by Lemma 3, we have:

$$v(x_0 \mid A - x_0) \leq v(x_0 \mid B) \leq v(x_0 \mid B - y_0)$$

and

$$v(y_0 \mid B - y_0) \leq v(y_0 \mid A) \leq v(y_0 \mid A - x_0).$$

Therefore

$$v(x_0 \mid A - x_0) + v(A - x_0) + v(y_0 \mid B - y_0) + v(B - y_0) \leq$$

$$v(x_0 \mid B - y_0) + v(A - x_0) + v(y_0 \mid A - x_0) + v(B - y_0)$$

and

$$v(A) + v(B) \leq v(A - x_0 + y_0) + v(B - y_0 + x_0).$$

∎

Lemma 8 *Let v be an M#-concave valuation, $\vec{p}$ a price vector and $u_{\vec{p}}$ the associated net utility function. Let $A \subseteq \Omega$ and let $k = \mid A \mid$. Assume there is some $B \subseteq \Omega$, $\mid B \mid \leq k$ such that $u_{\vec{p}}(B) > u_{\vec{p}}(A)$. Then, either one can find some $x \in A$, $y \notin A$ such that $u_{\vec{p}}(A) < u_{\vec{p}}(A - x + y)$, or one can find some $x \in A$ such that $u_{\vec{p}}(A) < u_{\vec{p}}(A - x)$.*

Proof: Let S be one of the preferred bundles among all bundles of size at most k that is closest to A for the Hausdorff distance $d(A,B) = \mid A - B \mid + \mid B - A \mid$. By assumption, $u_{\vec{p}}(A) < u_{\vec{p}}(S)$ and therefore $A \neq S$.

Let us show that $A - S$ is not empty. If it were, we would have $A \subseteq S$, but we have $\mid S \mid \leq \mid A \mid$ and $A \neq S$.

Assume now, first, that $S - A$ is not empty. We may apply Lemma 7. Let $x \in A - S$, $y \in S - A$ be such that

$$v(A) + v(S) \leq v(A - x + y) + v(S - y + x).$$

By subtracting the prices of the items concerned from both sides, we see that:

$$u_{\vec{p}}(A) + u_{\vec{p}}(S) \leq u_{\vec{p}}(A - x + y) + u_{\vec{p}}(S - y + x).$$

But $u_{\vec{p}}(S) \geq u_{\vec{p}}(S - y + x)$, $u_{\vec{p}}(S) \geq u_{\vec{p}}(A - x + y)$ and $u_{\vec{p}}(A) < u_{\vec{p}}(S)$. We conclude that

- either $u_{\vec{p}}(A) < u_{\vec{p}}(A - x + y)$ and we are through,
- or $u_{\vec{p}}(S - y + x) = u_{\vec{p}}(S)$, but this contradicts our assumption that no preferred bundle of such size is closer to A than S.

Finally, assume that $S - A$ is empty, and $S \subset A$ since $A \neq S$. Consider any $x \in A - S$:

$$u_{\vec{p}}(S) > u_{\vec{p}}(S + x)$$

since $\mid S + x \mid \leq k$ and $S + x$ is closer to A than S. Therefore:

$$p_x > v(x \mid S) \geq v(x \mid A - x)$$

by submodularity and $u_{\vec{p}}(A - x) > u_{\vec{p}}(A)$. ∎