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Minimum Principle of Complementary Energy in Nonlinear Elasticity Theory of Cable Networks

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Abstract The minimum principle of complementary energy is established for cable networks involving only stress components as variables in geometrically nonlinear elasticity theory. The stress-strain relation is assumed to be piecewise linear. It is rather amazing that the complementary energy always attains minimum value at the equilibrium state irrespective of the stability of cable networks, contrary to the fact that only the stationary principles have been presented for elastic trusses and continua even in the case of stable equilibrium state. In order to show the strong duality between the minimization problems of total potential energy and complementary energy, the convex formulations of these problems are investigated based on the property of self-dual cone. The reciprocal formulations for the minimization problems of total potential energy and complementary energy are presented, which may be used to develop the force method. The existence and uniqueness of solution are also investigated for the minimization problem of complementary energy.

Keywords: complementary energy, total potential energy, cable network, large deformation, second-order cone programming, symmetric dual problem

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1 Introduction.

There exists a variety of literatures on the variational principles in elasticity theory. The principle of minimum or stationary complementary energy, however, has raised several interesting discussions [2; 1].

It is well known that the principle of minimum total potential energy is expressed by unknown displacements with the additional subsidiary conditions of compatibility between strain components and displacements. In geometrically linear elasticity theory, i.e., under assumptions of small strains and small rotations, the minimum principle of complementary energy is defined involving only stress components as variables under subsidiary conditions of equilibrium equations. Thus we see that these two principles are completely dual with each other. In the geometrically nonlinear elasticity theory, however, existence of a principle of stationary complementary energy in truly dual form has been a subject of discussions since first contribution by Hellinger [3] (See the survey by Koiter [1]). Under assumption of large displacements, the equilibrium equations with respect to the deformed configuration may contain unknown information about deformation such as rotations [4]. With the coupling between displacements and stress components, the complementary theory in finite displacement problems was regarded as useless for practical applications (See, e.g., [2, Ch. 14]). Therefore, it has been a challenge to formulate the principle of complementary energy purely in terms of stress components.

In the field of thin shell theory, there exist several papers intending to obtain complementary principles in a form at least useful for analysis. Those principles, however, still contains finite rotations as variables in the equilibrium equations [5; 7; 8; 6]. If we allow the presence of rotation variables, it is known that thin shell structures have the extremum principle of complementary energy where the zero duality gap is satisfied at the equilibrium between the total potential energy and the complementary energy [6].

For finite dimensional structures, we can refer to Libove [9] and Jennings [10] which are still classical because of the existence on unknown rigid body rotations of members. Mikkola [11] presented the principle of stationary complementary energy for trusses expressed only by force quantities. His approach is based on the Legendre transformation [12], which is well known as one of general mathematical techniques to transform variables in a variational principle. However, in the process of deriving the complementary energy function, Mikkola [11] made the assumption that the constitutive equation can be solved inversely and presented the solution without any rigorous proof. For finite displacement theory, the existence of the inversion of constitutive equation is the crucial point whether such energy principle can be really formulated explicitly [1; 13]. Besides the possibility of non-existence of the inversion, Mikkola's complementary energy function is not unique; i.e., the function cannot be determined without information of unknown axial forces at the equilibrium state. Therefore, it may not be always easy to chose the correct form of complementary energy function [11].

It should be noted that the complementary energy principles ever addressed for geometrically nonlinear theory guarantee only the stationary condition of complementary energy. This means that the minimum principle does not always hold even if the equilibrium state is stable, whereas the total potential energy is guaranteed to take the minimum value. According to the lack of minimum principle and uniqueness of complementary energy function, erroneous solutions are sometimes obtained as discussed in [9; 14; 15]. Hence, the question remains whether we can formulate the actually useful complementary energy principle.

Koiter [1] gave an extensive survey concerning three-dimensional continuum. Fraeijs de Veubeke [13]

presented the formulation involving the Piola stress tensor and the material rotation based on the polar decomposition of displacement gradients. Levinson [16] and Zubov [17] formulated the complementary principle in terms of displacement gradients and the Piola stress tensor based on a concept of the Legendre transformation. Their formulations, however, as mentioned by Fraeijs de Veubeke [13], are unsuitable for practical applications. Koiter [1] pointed out that the critical point of the approach in the context of Zubov [17] is on existence of an inversion of the stress-strain relations expressed by the Piola stress tensor and displacement gradients. He investigated the possible existence of a *multi-valued inversion* of the stress-strain relations, and presented a necessary condition for unique inversion. The stationary principle of complementary energy expressed only by the Piola stress tensor was derived explicitly for *semi-linear* isotropic material, where the Jaumann stress tensor is given as the linear function of the right extensional strain tensor. It is an interesting open question [1] what is a necessary and sufficient condition for the minimum complementary energy. The authors will give a remark in this paper that the stability of the structure is not a necessary condition and will present a more rigorous sufficient condition than Koiter's [1]. Although it has been shown that the minimum principle holds for perfectly flexible flat and curved membrane structures [1], the problem investigated is not truly geometrically nonlinear because the squares of rotation are restricted to be small and all the principal values of the stresses are assumed to be positive at any point.

In this paper, a minimum principle of complementary energy is presented for cable networks in terms of generalized stress only. Under the assumption such that all the members are not capable of transmitting compression forces, which is referred to as stress-unilateral behavior [18], it will be shown that the complementary energy function is unique and minimum principle always holds irrespective of stability of the structure. We also consider a material with the piecewise linear stress-strain law.

It should be emphasized that the basic idea presented for developing the purely complementary principle introduced herein is independent of the concepts ever addressed. We first formulate a convex optimization problem (P) such that (P) has the same solution as that of the problem of minimum potential energy. The dual problem (P^{C}) for (P) is derived by using the properties of so-called second-order cone [19], where the variables in (P) do not exist in (P^{C}). The problem (P^{C}) will be shown to coincide with the minimum principle of complementary energy expressed only by force quantities.

The second-order cone is well known as a class of the self-dual cone [20], which is used to define the Second-Order Cone Programming (SOCP) [19; 21]. SOCP is a class of convex programming including Linear Programming (LP) and convex quadratic programming, and is included in Semi-Definite Programming (SDP) [23; 24]. SOCP and SDP are known as special cases of conic LP [20; 25] which has received increasing attention for its wide fields of application [23; 22; 26] and for the development of practical algorithms referred to as the primal-dual interior-point methods [19; 24]. The SOCP formulation was presented for equilibrium analysis of linear springs [26]. The authors proposed efficient algorithms based on SDP for structural optimization under constraints on fundamental frequency [27] and linear buckling load factor [28]. Our present approach for establishing the dual problem is analogous to the formulation of the dual SDP in [29], or to the derivation of the optimality conditions for eigenvalue optimization problem presented by the authors [30]. The duality between (P) and (P^{C}) will be shown based on that of SOCPs [19].

In Section 5, the strong duality theorem will be derived between the minimization problems of total potential energy and the complementary energy formulated by the force quantities. This theorem plays a key role in validating the presented minimum principle of complementary energy;

i.e., the theorem guarantees that the optimal solutions are equivalent to the internal forces at the equilibrium state, and that the zero duality gap holds between complementary energy and total potential energy. In Section 6, the compatibility of the directions of members and obtained internal forces will be discussed in detail, because this is another crucial point in the establishment of principle of complementary energy. It will be shown in Section 7 that the complementary energy function is also obtained based on the framework of complementary work; i.e., the obtained function is guaranteed to be the *actual physical complementary energy* in the terminology of Koiter [1]. In Section 8, we will investigate the alternative formulations for possible application to the force method. Simple cable networks will be examined in Section 9 to verify the obtained results. In Sections 10 and 11, we will investigate, by using the strong duality derived in Section 5, the existence and uniqueness of the solutions to the problem for minimizing the complementary energy.

List of symbols.

$\mathcal{P}_{\rm SOCP}$, $\mathcal{D}_{\rm SOCP}$	primal and dual SOCP problems, respectively.
(Π)	minimization problem of potential energy.
$(\Pi^{\rm C})$	minimization problem of complementary energy.
$(\Pi_{\rm R})$	minimization problem of potential energy for the force method.
$(\Pi_{\rm R}^{\rm C})$	minimization problem of complementary energy for the force method.
$\Pi(\cdot)$	potential energy function.
$\Pi^{C}(\cdot)$	complementary energy function.
(P), $(\mathrm{P}^{\rm C})$	SOCP formulations of (Π) and $(\Pi^{\rm C})$, respectively.
$\Phi(\cdot)$, $\Phi^{C}(\cdot)$	objective functions of (P) and $(\mathrm{P}^{\rm C})$, respectively.
$(\mathrm{P}_{\rm R})$, $(\mathrm{P}_{\rm R}^{\rm C})$	SOCP formulations of $(\Pi_{\rm R})$ and $(\Pi_{\rm R}^{\rm C})$, respectively.
$\|\boldsymbol{p}\|$	Euclidean norm of a vector $\boldsymbol{p} \in \Re^n$ (i.e. $\|\boldsymbol{p}\| = (\boldsymbol{p}^\top \boldsymbol{p})^{1/2}$).

2 Primal and dual pair of second-order cone programs.

In this paper, all the vectors are assumed to be column vectors. To simplify the notation, however, define

$$(\boldsymbol{a}, \boldsymbol{b}) = (\boldsymbol{a}^\top, \boldsymbol{b}^\top)^\top \in \Re^{N^a + N^b}$$

for vectors $\boldsymbol{a} \in \Re^{N^a}$ and $\boldsymbol{b} \in \Re^{N^b}$. Let $\mathcal{K}(p)$ be the second-order cone in p-dimensional space defined as [21]

$$\mathcal{K}(p) = \{(x_0, \boldsymbol{x}_1) | x_0 \geq \|\boldsymbol{x}_1\|\}, \tag{1}$$

where $x_0 \in \Re$ and $\boldsymbol{x}_1 \in \Re^{p-1}$. A simple example of $p = 3$ is as illustrated in Fig.1, where $\boldsymbol{x}_1 = (x_1, x_2)$ and $\boldsymbol{x} = (x_0, \boldsymbol{x}_1)$. In this case, the second-order cone is defined as

$$\mathcal{K}(3) = \{(x_0, \boldsymbol{x}_1) | x_0 \geq \sqrt{x_1^2 + x_2^2}\}, \tag{2}$$

which coincides with the face and interior region of the circular cone in 3-dimensional space.

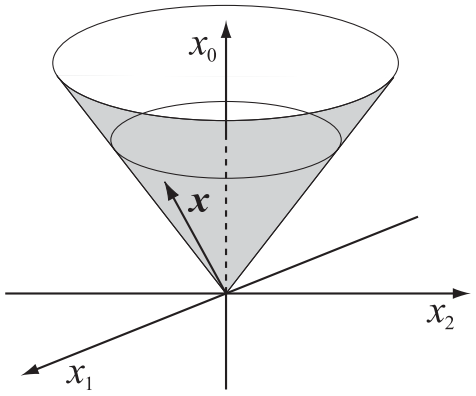


Figure 1: Second-order cone in 3-dimensional space.

$\mathcal{K}^*(p)$ denotes the dual cone of $\mathcal{K}(p)$ which is defined by $s_0 \in \Re$ and $\boldsymbol{s}_1 \in \Re^{p-1}$ as

$$\mathcal{K}^*(p) = \{(s_0, \boldsymbol{s}_1) | x_0 s_0 + \boldsymbol{x}_1^\top \boldsymbol{s}_1 \geq 0,\ \forall (x_0, \boldsymbol{x}_1) \in \mathcal{K}(p)\}.$$

Then $\mathcal{K}^*(p) = \mathcal{K}(p)$ holds, which is known as the self-duality [19; 21]. It follows that

$$(s_0, \boldsymbol{s}_1) \in \mathcal{K}(p) \Longleftrightarrow x_0 s_0 + \boldsymbol{x}_1^\top \boldsymbol{s}_1 \geq 0, \quad \forall (x_0, \boldsymbol{x}_1) \in \mathcal{K}(p) \tag{3}$$

is satisfied.

The standard form of SOCP (Second-Order Cone Programming) problem and its dual are formulated as [19]

$$\left.\begin{array}{lll} \mathcal{P}_{\rm SOCP}: & \text{Minimize} & \boldsymbol{b}^\top \boldsymbol{w} \\ & \text{subject to} & \boldsymbol{x}_i + \boldsymbol{c}_i = \boldsymbol{A}_i \boldsymbol{w},\ x_{i0} \geq \|\boldsymbol{x}_{i1}\|, \quad (i = 1, \cdots, k); \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{lll} \mathcal{D}_{\rm SOCP}: & \text{Maximize} & \displaystyle\sum_{i=1}^{k} \boldsymbol{c}_i^\top \boldsymbol{z}_i \\ & \text{subject to} & \displaystyle\sum_{i=1}^{k} \boldsymbol{A}_i^\top \boldsymbol{z}_i = \boldsymbol{b},\ z_{i0} \geq \|\boldsymbol{z}_{i1}\|, \quad (i = 1, \cdots, k). \end{array}\right\} \tag{5}$$

Here $\boldsymbol{x}_i = (x_{i0}, \boldsymbol{x}_{i1}) \in \Re^{n_i}$, $\boldsymbol{w} \in \Re^m$ and $\boldsymbol{z}_i = (z_{i0}, \boldsymbol{z}_{i1}) \in \Re^{n_i}$ are variables, $\boldsymbol{A}_i \in \Re^{n_i \times m}$ is a constant matrix, $\boldsymbol{b} \in \Re^m$ and $\boldsymbol{c}_i \in \Re^{n_i}$ are constant vectors. To simplify the notation, define $\boldsymbol{x} = (\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k) \in \Re^n$ and $\boldsymbol{z} = (\boldsymbol{z}_1, \cdots, \boldsymbol{z}_k) \in \Re^n$, where $n = \sum_{i=1}^k n_i$. We say that $(\boldsymbol{x}, \boldsymbol{w})$ and $\boldsymbol{z}$ are feasible solutions of $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$ if they satisfy all the constraints of $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$, respectively. The feasible solutions $(\boldsymbol{x}, \boldsymbol{w})$ and $\boldsymbol{z}$ satisfying $x_{i0} > \|\boldsymbol{x}_{i1}\|$ and $z_{i0} > \|\boldsymbol{z}_{i1}\|$ $(i = 1, \cdots, k)$ are called interior feasible solutions of $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$, respectively.

It is known that $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$ satisfy the following strong duality:

Theorem 2.1 (Strong duality [19; 20]). *Suppose that $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$ have interior feasible solutions. Then both of $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$ have optimal solutions which have the same optimal objective value.*

3 Minimum principle of total potential energy for cable networks.

Consider a pin-jointed cable network in three dimensional space. We use the assumptions of small strain and large rotation. The network is discretized into members which connect pin-joints and supports.

Let N^a and N^m denote the numbers of nodes and members, respectively. N^s and N^d denote the numbers of support degrees of freedom and unconstrained degrees of freedom, respectively, where $N^s + N^d = 3N^a$. The reference state and actual equilibrium state are referred to as Γ^{I} and Γ^{II}, respectively. $\boldsymbol{x}^0 \in \Re^{N^d}$ and $\boldsymbol{x}^{s0} \in \Re^{N^s}$ denote the nodal coordinates corresponding to unconstrained degrees and support degrees, respectively, at Γ^{I}. Similarly, the vectors of nodal displacements are denoted by $\boldsymbol{u} \in \Re^{N^d}$ and $\bar{\boldsymbol{u}}^s \in \Re^{N^s}$. Any kinematically admissible configuration is identified by $\boldsymbol{x}^0 + \boldsymbol{u}$ and $\boldsymbol{x}^{s0} + \bar{\boldsymbol{u}}^s$. We specify the external dead loads $\bar{\boldsymbol{f}} \in \Re^{N^d}$ for unconstrained degrees and displacements $\bar{\boldsymbol{u}}^s$ for support degrees. The initial unstressed length of the ith member is given as l_i^0.

We shall restrict ourselves to elastic materials obeying piecewise linear stress-strain relation, where the stress is a strictly increasing function of the strain. In order to identify the property of stress-strain relation rigorously, the following two classes of real-valued functions are introduced:

Definition 3.1. *$\mathcal{F}^0$ is defined as the set of all mappings $f : \Re \mapsto \Re$ satisfying the following properties:*

(i) *$f(x)$ is continuous and piecewise linear;*

(ii) *$\dfrac{df(x)}{dx}$ is positive and bounded for $x \in \Re$ and satisfies $\dfrac{df(x)}{dx} = \dfrac{df(0)}{dx}$ for $x < 0$;*

(iii) *$f(0) = 0$.*

Definition 3.2. *$\mathcal{F}^1$ is defined as the set of all mappings $f : \Re \mapsto \Re$ satisfying $\dfrac{df(x)}{dx} \in \mathcal{F}^0$ and $f(0) = 0$.*

The following is immediately obtained from Definitions 3.1 and 3.2:

Corollary 3.3. *If $f \in \mathcal{F}^1$, then $f(x)$ is*

(i) *a continuously differentiable function of x;*

(ii) *a strictly convex function;*

(iii) *a function that attains the unique minimum value $f(0) = 0$ at $x = 0$.*

Let ε_i denote the elongation that is regarded as a generalized strain of the ith member. σ_i denotes the axial force which is a generalized stress. Note that the cable member is not capable of transmitting the compression force. We assume that the relation between ε_i and σ_i satisfies the following:

Assumption 3.4 (constitutive law). *Suppose that $\sigma_i(\varepsilon_i)$ is a function of ε_i satisfying*

$$\sigma_i(\varepsilon_i) = \begin{cases} \psi_i(\varepsilon_i), & (\varepsilon_i \geq 0), \\ 0, & (-l_i^0 \leq \varepsilon_i < 0), \end{cases} \qquad \psi_i \in \mathcal{F}^0. \tag{6}$$

It follows from Definition 3.1 and Assumption 3.4 that $\psi_i(y_i)$ is a one-to-one mapping over the region $y_i \in [-\infty, +\infty]$, which implies the following:

Corollary 3.5. *There exists the inversion of $q_i = \psi_i(y_i)$, which is denoted by $y_i = \psi_i^{-1}(q_i)$, and $\psi_i^{-1} \in \mathcal{F}^0$ is satisfied.*

Note that $q_i(y_i)$ can be negative although $\sigma_i(\varepsilon_i)$ is a nonegative-valued function. Define the functions $\phi_i(y_i)$ and $\phi_i^C(q_i)$ $(i = 1, \cdots, N^m)$ as

$$\phi_i(y_i) = \int_0^{y_i} \psi_i(y_i) dy_i, \tag{7}$$

$$\phi_i^C(q_i) = \int_0^{q_i} \psi_i^{-1}(q_i) dq_i. \tag{8}$$

It follows from (6) and (7) that the stored energy (the strain energy) $w_i(\varepsilon_i)$ is obtained as

$$w_i(\varepsilon_i) = \begin{cases} \phi_i(\varepsilon_i), & (\varepsilon_i \geq 0), \\ 0, & (-l_i^0 \leq \varepsilon_i < 0). \end{cases} \tag{9}$$

It should be also remarked from (6) and (8) that $\phi_i^C(\sigma_i)$ corresponds to the complementary strain energy function. The following is immediately obtained from Definition 3.2, Assumption 3.4, (7) and (8):

Corollary 3.6. $\phi_i \in \mathcal{F}^1$, $\phi_i^C \in \mathcal{F}^1$.

Example 3.7 (linear elastic material). *For a linear elastic material obeying Hooke's law, define $\psi_i(\varepsilon_i) = k_i \varepsilon_i$, where k_i denotes the extensional stiffness of the ith member. By using* (6)*, the axial force σ_i can be written as*

$$\sigma_i(\varepsilon_i) = \begin{cases} k_i \varepsilon_i, & (\varepsilon_i \geq 0), \\ 0, & (-l_i^0 \leq \varepsilon_i < 0). \end{cases} \tag{10}$$

The inversion of $q_i = \psi_i(y_i) = k_i y_i$ is obtained as $\psi_i^{-1}(q_i) = q_i/k_i$, from which and (7) *and* (8) *it follows that*

$$\phi_i(y_i) = \frac{1}{2} k_i y_i^2, \quad \phi_i^C(q_i) = \frac{1}{2k_i} q_i^2. \tag{11}$$

From (9) *and* (11)*, the function w_i is obtained as*

$$w_i(\varepsilon_i) = \begin{cases} \dfrac{1}{2} k_i \varepsilon_i^2, & (\varepsilon_i \geq 0), \\ 0, & (-l_i^0 \leq \varepsilon_i < 0), \end{cases} \tag{12}$$

and $\phi_i^C(\sigma_i) = \sigma_i^2/(2k_i)$ is satisfied. Therefore, $w_i(\varepsilon_i)$ gives the correct strain energy function and $\phi_i^C(\sigma_i)$ denotes the complementary strain energy for the range $\sigma_i \geq 0$.

Example 3.8 (bi-linear elastic material). *Let $k_i^p > 0$ and $\bar{e}_i > 0$. Consider a bi-linear elastic material, where σ_i is defined as*

$$\sigma_i(\varepsilon_i) = \begin{cases} k_i^p(\varepsilon_i - \bar{e}_i) + k\bar{e}_i, & (\varepsilon_i \geq \bar{e}_i), \\ k_i\varepsilon_i, & (0 \leq \varepsilon_i < \bar{e}_i), \\ 0, & (-l_i^0 \leq \varepsilon_i < 0), \end{cases}$$

i.e., the extensional stiffness varies from the initial value k_i to k_i^p at $\varepsilon_i = \bar{e}_i$. From (7) *and* (8)*, ϕ_i and ϕ_i^C are obtained as*

$$\phi_i(y_i) = \begin{cases} \dfrac{1}{2}k_i^p(y_i - \bar{e}_i)^2 + k_i\bar{e}_i(y_i - \bar{e}_i) + \dfrac{1}{2}k_i\bar{e}_i^2 & (y_i \geq \bar{e}_i), \\ \dfrac{1}{2}k_i y_i^2, & (y_i < \bar{e}_i), \end{cases} \tag{13}$$

$$\phi_i^C(q_i) = \begin{cases} \dfrac{1}{2k_i^p}(q_i - k_i\bar{e}_i)^2 + \bar{e}_i(q_i - k_i\bar{e}_i) + \dfrac{1}{2}k_i\bar{e}_i^2 & (q_i \geq k_i\bar{e}_i), \\ \dfrac{1}{2k_i}q_i^2, & (q_i < k_i\bar{e}_i). \end{cases} \tag{14}$$

The strain energy function $w_i(\varepsilon_i)$ is then obtained as (9) *with* (13)*.*

The problem of minimum total potential energy is formulated as

$$\left.\begin{array}{lll} (\Pi): & \text{Minimize} & \Pi(\boldsymbol{u}) = \displaystyle\sum_{i=1}^{N^m} w_i(\varepsilon_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{u} \\ & \text{subject to} & \varepsilon_i = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i\| - l_i^0, \quad (i = 1, \cdots, N^m), \end{array}\right\} \tag{15}$$

where $\bar{\boldsymbol{x}}_i \in \Re^3$ is a constant vector defined by

$$\bar{\boldsymbol{x}}_i = \boldsymbol{B}_i^s(\boldsymbol{x}^{s0} + \bar{\boldsymbol{u}}^s), \quad (i = 1, \cdots, N^m), \tag{16}$$

which consists of the elements of $\boldsymbol{x}^{s0} + \bar{\boldsymbol{u}}^s$ if the ith member is connected to a support, otherwise $\bar{\boldsymbol{x}}_i = \boldsymbol{0}$. $\boldsymbol{B}_i \in \Re^{3\times N^d}$ and $\boldsymbol{B}_i^s \in \Re^{3\times N^s}$ $(i = 1, \cdots, N^m)$ are constant matrices determined only by the connectivity of nodes and members, and each element of these matrices is either $\{-1, 0, 1\}$. Let $\boldsymbol{\varepsilon}^{\mathrm{II}} = (\varepsilon_i^{\mathrm{II}}) \in \Re^{N^d}$ and $\boldsymbol{u}^{\mathrm{II}} \in \Re^{N^d}$, respectively, denote the vectors of member elongation and nodal displacements at Γ^{II}. From the principle of minimum total potential energy [2], $(\boldsymbol{\varepsilon}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ is the optimal solution of (Π).

Notice here that (Π) is a nonconvex optimization problem since ε_i is a nonconvex function of $\boldsymbol{u}$. This implies that the classical Lagrange dual problem of (Π) contains unknown $\boldsymbol{u}$ and does not satisfy the strong duality. Although stationary conditions for complementary energy may be derived from the Lagrangian of (Π) (See, e.g., [11] for trusses), it is not straightforward to establish the minimum principle without duality gap only in terms of the stress components. This motivates us to investigate the following problem (P):

$$\left.\begin{array}{lll} (\mathrm{P}): & \text{Minimize} & \Phi(\boldsymbol{y}, \boldsymbol{u}) = \displaystyle\sum_{i=1}^{N^m} \phi_i(y_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{u} \\ & \text{subject to} & y_i + l_i^0 \geq \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i\|, \quad (i = 1, \cdots, N^m), \end{array}\right\} \tag{17}$$

where $\boldsymbol{y} = (y_i) \in \Re^{N^m}$ and $\boldsymbol{u}$ are independent variables. Let $(\boldsymbol{y}^*, \boldsymbol{u}^*)$ denote the optimal solution of (P). The following lemma gives the relation between $(\boldsymbol{\varepsilon}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ and $(\boldsymbol{y}^*, \boldsymbol{u}^*)$:

Lemma 3.9. *$(\boldsymbol{\varepsilon}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ is an optimal solution of* (Π) *if and only if $(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}})$ is an optimal solution of* (P) *satisfying*

$$y_i^* = \begin{cases} \varepsilon_i^{\mathrm{II}} & (\varepsilon_i^{\mathrm{II}} \geq 0), \\ 0 & (-l_i^0 \leq \varepsilon_i^{\mathrm{II}} < 0). \end{cases} \tag{18}$$

Proof. See Appendix A. □

Lemma 3.9 implies that the deformation at Γ^{II} can be obtained as the solution to (P). Because of the convexity, (P) can be solved more easily than (Π) by using a nonlinear programming approach such as the interior-point method.

4 Minimum principle of complementary energy.

Let $\boldsymbol{q} = (q_i) \in \Re^{N^m}$ and $\boldsymbol{v}_i \in \Re^3$ $(i = 1, \cdots, N^m)$ denote the Lagrange multipliers for (P). For simplicity, we use the notation $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{N^m}) \in \Re^{3N^m}$. Define the Lagrangian $\Lambda(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{q}, \boldsymbol{v})$ for (P) as

$$\Lambda(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{q}, \boldsymbol{v}) = \sum_{i=1}^{N^m} \phi_i(y_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{u} - \sum_{i=1}^{N^m} q_i(y_i + l_i^0) - \sum_{i=1}^{N^m} \boldsymbol{v}_i^\top [\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i]. \tag{19}$$

Consider the following problem:

$$\text{Minimize} \quad L^P(\boldsymbol{y}, \boldsymbol{u}) = \sup\{\Lambda |\ q_i \geq \|\boldsymbol{v}_i\|,\ (i = 1, \cdots, N^m)\}. \tag{20}$$

It follows from the self-duality of the second-order cone (3) and $q_i \geq \|\boldsymbol{v}_i\|$, the inequality

$$-q_i(y_i + l_i^0) - \boldsymbol{v}_i^\top \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) \leq 0 \tag{21}$$

is satisfied if and only if $y_i + l_i^0 \geq \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u})\|$. Suppose that the constraints in (P) is not satisfied by at least one member, then we can see that the inner supremum of (20) is unbounded because the left-hand side of (21) can take arbitrary positive value. On the contrary, if all the constraints in (P) are satisfied, then Λ attains the supremum at $q_i = 0$ and $\boldsymbol{v}_i = \boldsymbol{0}$, which implies that Λ becomes equal to $\Phi(\boldsymbol{y}, \boldsymbol{u})$. Therefore (20) is equivalent to (P), which verifies that (19) is the correct Lagrangian for (P). The dual problem for (P) is defined from (20) by replacing min-sup by max-inf; i.e.,

$$\left.\begin{array}{ll} \text{Maximize} & L^D(\boldsymbol{q}, \boldsymbol{v}) = \inf\{\Lambda |\ \boldsymbol{y} \in \Re^{N^m}, \boldsymbol{u} \in \Re^{N^d}\} \\ \text{subject to} & q_i \geq \|\boldsymbol{v}_i\|, \quad (i = 1, \cdots, N^m). \end{array}\right\} \tag{22}$$

For explicit formulation of L^D, we use the following stationary conditions for Λ:

$$\frac{\partial \Lambda}{\partial y_i} = \frac{d\phi_i}{dy_i} - q_i = 0, \quad (i = 1, \cdots, N^m), \tag{23}$$

$$\frac{\partial \Lambda}{\partial \boldsymbol{u}} = -\bar{\boldsymbol{f}} - \sum_{i=1}^{N^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i = \boldsymbol{0}. \tag{24}$$

As will be shown in Lemma 4.3, q_i and $\boldsymbol{v}_i$ will be guaranteed to correspond to the axial force and internal force vector of the ith member, respectively. By substituting (16), (23) and (24) to (19), we obtain

$$L^D(\boldsymbol{q},\boldsymbol{v}) = -\sum_{i=1}^{N^m}\phi_i^C(q_i) - \sum_{i=1}^{N^m}(l_i^0 q_i + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i) + \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i, \tag{25}$$

where $\boldsymbol{h}_i^0 \in \Re^3$ is a constant vector defined as

$$\boldsymbol{h}_i^0 = \boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{B}_i^s\boldsymbol{x}^{s0} \tag{26}$$

with the same direction and length of the ith member at $\Gamma^{\rm I}$. See Appendix B for more detail of algebraic operations. By using (24) and (25) with a change of sign of the objective function to transform maximization into minimization, the dual problem (22) is reduced to

$$\left.\begin{array}{lll} (\mathrm{P^C}): & \text{Minimize} & \Phi^C(\boldsymbol{q},\boldsymbol{v}) = \displaystyle\sum_{i=1}^{N^m}\phi_i^C(q_i) + \sum_{i=1}^{N^m}(l_i^0 q_i + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m}\boldsymbol{B}_i^{\top}\boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}, \\ & & q_i \geq \|\boldsymbol{v}_i\|, \quad (i=1,\cdots,N^m). \end{array}\right\} \tag{27}$$

Let $(\boldsymbol{q}^*,\boldsymbol{v}^*)$ denote the optimal solution to $(\mathrm{P^C})$. Define the function w_i^C on $\boldsymbol{v}_i \in \Re^3$ by

$$w_i^C(\boldsymbol{v}_i) = \phi_i^C(\|\boldsymbol{v}_i\|). \tag{28}$$

Consider the following problem:

$$\left.\begin{array}{lll} (\Pi^{\rm C}): & \text{Minimize} & \Pi^C(\boldsymbol{v}) = \displaystyle\sum_{i=1}^{N^m}w_i^C(\boldsymbol{v}_i) + \sum_{i=1}^{N^m}(l_i^0\|\boldsymbol{v}_i\| + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m}\boldsymbol{B}_i^{\top}\boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}. \end{array}\right\} \tag{29}$$

Let $\boldsymbol{v}^{\rm II} = (\boldsymbol{v}_1^{\rm II},\cdots,\boldsymbol{v}_{N^m}^{\rm II})$ denote the optimal solution of $(\Pi^{\rm C})$. The following lemma guarantees that $(\mathrm{P^C})$ and $(\Pi^{\rm C})$ are equivalent:

Lemma 4.1. *$\boldsymbol{v}^*$ is an optimal solution of $(\Pi^{\rm C})$ if and only if $(\boldsymbol{q}^*,\boldsymbol{v}^*)$ is an optimal solution of $(\mathrm{P^C})$.*

Proof. It follows from Corollary 3.6 that $\phi_i^C(q_i)$ is a strictly increasing function on $q_i \geq \|\boldsymbol{v}_i\| \geq 0$. Therefore, we obtain $q_i^* = \|\boldsymbol{v}_i^*\|$, which implies that $(\mathrm{P^C})$ has the same optimizer as that of $(\Pi^{\rm C})$. □

The following theorem plays a key role in showing that $(\Pi^{\rm C})$ leads to the minimum principle of complementary energy, where the proof of the theorem will be given in Section 5:

Theorem 4.2 (strong duality). *If there exists a vector $\boldsymbol{v} = (\boldsymbol{v}_1,\cdots,\boldsymbol{v}_{N^m})$ satisfying (24), then (Π) and $(\Pi^{\rm C})$ have optimal solutions $\boldsymbol{u}^{\rm II}$ and $\boldsymbol{v}^{\rm II}$, respectively, which satisfy*

$$\Pi(\boldsymbol{u}^{\rm II}) = -\Pi^C(\boldsymbol{v}^{\rm II}). \tag{30}$$

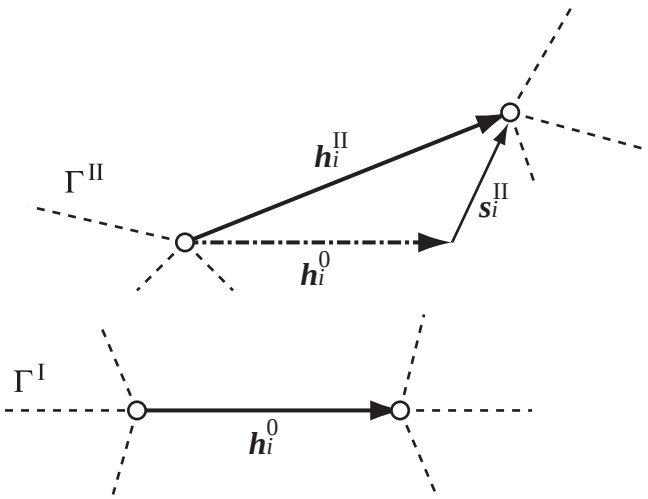


Figure 2: Relationship among $\boldsymbol{s}_i^{\rm II}$, $\boldsymbol{h}_i^{\rm II}$ and $\boldsymbol{h}_i^0$.

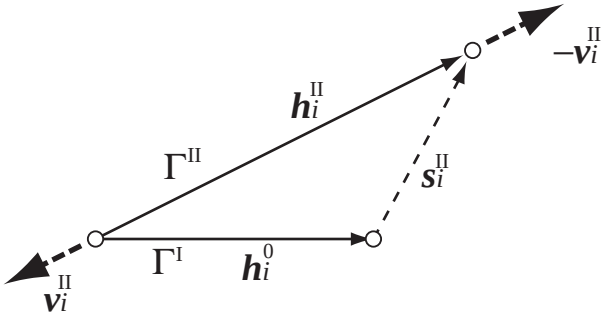


Figure 3: Physical interpretation of $\boldsymbol{v}_i^{\rm II}$.

Define $\boldsymbol{s}_i^{\rm II} \in \Re^3$ and $\boldsymbol{h}_i^{\rm II} \in \Re^3$ as $\boldsymbol{s}_i^{\rm II} = \boldsymbol{B}_i \boldsymbol{u}^{\rm II} - \boldsymbol{B}_i^s \bar{\boldsymbol{u}}^s$ and $\boldsymbol{h}_i^{\rm II} = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^{\rm II}) - \bar{\boldsymbol{x}}_i$, respectively. Then, $\boldsymbol{h}_i^{\rm II}$ has the same direction and length as that of the ith member at $\Gamma^{\rm II}$. The relationship among $\boldsymbol{s}_i^{\rm II}$, $\boldsymbol{h}_i^{\rm II}$ and $\boldsymbol{h}_i^0$ is as illustrated in Fig.2 and Table 1; i.e., $\boldsymbol{h}_i^{\rm II} = \boldsymbol{h}_i^0 + \boldsymbol{s}_i^{\rm II}$. The following is obtained from Theorem 4.2:

Lemma 4.3 (compatibility conditions). *For $i = 1, \cdots, N^m$, $\boldsymbol{v}_i^{\rm II}$ satisfies*

$$\boldsymbol{v}_i^{\rm II} = \begin{cases} -\sigma_i(\varepsilon_i^{\rm II}) \dfrac{\boldsymbol{h}_i^{\rm II}}{\|\boldsymbol{h}_i^{\rm II}\|} & (\varepsilon_i^{\rm II} > 0), \\ \boldsymbol{0} & (-l_i^0 \leq \varepsilon_i^{\rm II} \leq 0), \end{cases} \tag{31}$$

where σ_i is defined as (6).

The proof of Lemma 4.3 will be given in Section 6. Lemma 4.3 implies that $\boldsymbol{v}_i^{\rm II}$ is the internal force vector of the ith member at $\Gamma^{\rm II}$, where the relation between $\boldsymbol{v}_i^{\rm II}$ and $\boldsymbol{h}_i^{\rm II}$ is as illustrated in Fig.3. The linear constraints (24) in ($\Pi^{\rm C}$) correspond to the equilibrium equations. Hence, the

Table 1: Relation between nodal displacements and member configuration vectors.

state	nodal displacements		deformed member vectors	
	unconstrained	support	definitions	compatibility conditions
Γ^{I}	$\boldsymbol{x}^0$	$\boldsymbol{x}^{s0}$	$\boldsymbol{h}_i^0 = \boldsymbol{B}_i\boldsymbol{x}^0 - \boldsymbol{B}_i^s\boldsymbol{x}^{s0}$	$\sum_{i=1}^{N^m} \boldsymbol{H}_i^\top(\boldsymbol{h}_i^0 + \boldsymbol{B}_i^s\boldsymbol{x}^{s0}) = \boldsymbol{0}$
Γ^{II}	$\boldsymbol{u}^{\mathrm{II}}$	$\bar{\boldsymbol{u}}^s$	$\boldsymbol{s}_i^{\mathrm{II}} = \boldsymbol{B}_i\boldsymbol{u}^{\mathrm{II}} - \boldsymbol{B}_i^s\bar{\boldsymbol{u}}^s$	$\sum_{i=1}^{N^m} \boldsymbol{H}_i^\top(\boldsymbol{s}_i^{\mathrm{II}} + \boldsymbol{B}_i^s\bar{\boldsymbol{u}}^s) = \boldsymbol{0}$
Γ^{II}	$\boldsymbol{x}^0 + \boldsymbol{u}^{\mathrm{II}}$	$\boldsymbol{x}^{s0} + \bar{\boldsymbol{u}}^s$	$\boldsymbol{h}_i^{\mathrm{II}} = \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^{\mathrm{II}}) - \bar{\boldsymbol{x}}_i$	$\sum_{i=1}^{N^m} \boldsymbol{H}_i^\top(\boldsymbol{h}_i^{\mathrm{II}} + \bar{\boldsymbol{x}}_i) = \boldsymbol{0}$

complementary energy principle now can be stated such that the internal forces at Γ^{II} are obtained as the solution to (Π^{C}); i.e., $\boldsymbol{v}^{\mathrm{II}}$ can be obtained by minimizing $\Pi^C(\boldsymbol{v})$ under the constraints of equilibrium equations. It follows from the definition of $\boldsymbol{B}_i^s$ that $\sum_{i=1}^{N^m} \boldsymbol{B}_i^{s\top}\boldsymbol{v}_i$ is the vector composed of reactions. Therefore, the last term of Π^C corresponds to the complementary work done by the reactions. Notice here that the classical complementary energy function Π^C_{LIN} in the small displacement theory is written as

$$\Pi^C_{\mathrm{LIN}}(\boldsymbol{v}) = \sum_{i=1}^{N^m} w_i^C(\boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m} \boldsymbol{B}_i^{s\top}\boldsymbol{v}_i. \tag{32}$$

It can be seen that the second term in Π^C is added to Π^C_{LIN} due to the effect of large rotation. In Section 7, we will show that Π^C can be also obtained by using the framework of complementary work; i.e., Π^C is the actual physical complementary energy function. In addition, it follows from Theorem 4.2 that there exists no gap between Π and Π^C at Γ^{II}. Existence and uniqueness of the solution will be discussed in Sections 10 and 11, respectively.

It should be emphasized that (i) the presented principle is expressed purely in terms of the generalized stress components; (ii) the actual physical complementary energy function is uniquely determined; (iii) the minimum principle is satisfied for the complementary energy function. Because of these properties, in addition to the theoretical interests, the derived principle may be actually useful for practical application such as the force methods.

It has been clarified that (ii) and (iii) cannot be satisfied for elastic continua [1] or trusses [11], and the derivation of necessary conditions for (ii) or (iii) has been left open. For elastic continua, Koiter [1] presented a sufficient condition for (iii) such that the Hessian of stored energy density $w(\boldsymbol{c})$ in terms of the tensor $\boldsymbol{c}$ of displacement gradients is positive definite; i.e.,

$$\frac{\partial^2 w(\boldsymbol{c})}{\partial c_{ij}\partial c_{kl}}\delta c_{ij}\delta c_{kl} > 0 \tag{33}$$

for any kinematically admissible non-vanishing $\delta\boldsymbol{c}$, where the summation convention for repeated indices has been adopted. Accepting the conventional criterion for elastic stability based on the second variation of the energy functional, the necessary and sufficient condition for stability of

the equilibrium is known as

$$\int_V \frac{\partial^2 w(\boldsymbol{c})}{\partial c_{ij}\partial c_{kl}}\delta c_{ij}\delta c_{kl}dV > 0 \tag{34}$$

for any kinematically admissible non-vanishing $\delta\boldsymbol{c}$, where V is the entire domain of elastic body. It can be seen that (33) is a sufficient condition for (34). Koiter [1] left it as an open question whether (33) is also a necessary condition for minimum principle of complementary energy or might possibly be relaxed.

On the contrary, we have established the minimum principle (Π^{C}) for cable networks. It can be easily seen that cable networks with the constitutive law (6) may have unstable equilibrium configurations, for example, in the case where many cable members are slackening. It is obvious that the minimum principle (Π^{C}) holds even in such an unstable case. This implies that the condition (33) is not a necessary condition because (33) restricts the equilibrium to a stable state. Our sufficient condition is stated such that the equilibrium state takes the global minimum of Π^C if the constitutive law satisfies Assumption 3.4. Although our sufficient condition is restricted to the case of discrete structures, it is more rigorous than the Koiter's sufficient condition (33) because it includes even the case of unstable equilibrium where (34) is not satisfied. The second difference between these conditions exists on the differentiability of the constitutive law; i.e., the Koiter's condition (33) requires the continuous differentiability of the axial force tacitly, whereas our assumption (Assumption 3.4) permits the non-differentiability in ordinary sense. Finally, the physical meaning of (33) is not clear, whereas Assumption 3.4 is clearly related to the property of constitutive law.

Koiter [1] claims that (34) is a necessary condition, but we should emphasize that (34) is not even a necessary condition for the minimum principle of discrete structures.

5 Proof of Theorem 4.2 (strong duality between (Π) and (Π^{C})).

In order to prove Theorem 4.2, we need a series of lemmas implying the duality between (P) and (P^{C}). They are extension of or closely related to well-known results for duality theory of SOCP [19; 21] or conic LP [20; 25].

Let $\delta(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v})$ denote the duality gap between (P) and (P^{C}). Since (P^{C}) has been defined as the minimization problem, δ is obtained as the sum of the objective values of (P) and (P^{C}); i.e., $\delta(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) = \Phi(\boldsymbol{y},\boldsymbol{u}) + \Phi^C(\boldsymbol{q},\boldsymbol{v})$. The following lemma implies that the weak duality holds between (P) and (P^{C}):

Lemma 5.1 (weak duality between (P) **and** (P^{C})**).** *If* $(\boldsymbol{y},\boldsymbol{u})$ *and* $(\boldsymbol{q},\boldsymbol{v})$ *are feasible solutions of* (P) *and* (P^{C})*, respectively, then* $\Phi(\boldsymbol{y},\boldsymbol{u}) \geq -\Phi^C(\boldsymbol{q},\boldsymbol{v})$.

Proof. By using (16), (23), (24) and (76), we obtain

$$\begin{aligned}\delta(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) &= \sum_{i=1}^{N^m}(\phi_i(y_i)+\phi_i^C(q_i)) - \bar{\boldsymbol{f}}^\top\boldsymbol{u} + \sum_{i=1}^{N^m}(l_i^0 q_i + \boldsymbol{h}_i^{0\top}\boldsymbol{u}) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i \\ &= \sum_{i=1}^{N^m} q_i y_i + \Big(\sum_{i=1}^{N^m}\boldsymbol{B}_i^\top\boldsymbol{v}_i\Big)^\top\boldsymbol{u} + \sum_{i=1}^{N^m}(l_i^0 q_i + \boldsymbol{h}_i^{0\top}\boldsymbol{u}) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i \qquad (35)\\ &= \sum_{i=1}^{N^m} q_i(y_i + l_i^0) + \sum_{i=1}^{N^m}\boldsymbol{v}_i^\top[\boldsymbol{B}_i(\boldsymbol{x}^0+\boldsymbol{u}) - \bar{\boldsymbol{x}}_i].\end{aligned}$$

From the self-duality of the second-order cone (3), feasible solutions $(\boldsymbol{y},\boldsymbol{u})$ and $(\boldsymbol{q},\boldsymbol{v})$ satisfy

$$\delta_i(y_i,\boldsymbol{u},q_i,\boldsymbol{v}_i) = q_i(y_i+l_i^0) + \boldsymbol{v}_i^\top[\boldsymbol{B}_i(\boldsymbol{x}^0+\boldsymbol{u}) - \bar{\boldsymbol{x}}_i] \geq 0, \quad (i=1,\cdots,N^m). \qquad (36)$$

It follows from (35) and (36) that $\delta(\boldsymbol{y},\boldsymbol{u},\boldsymbol{q},\boldsymbol{v}) = \sum_{i=1}^{N^m}\delta_i(y_i,\boldsymbol{u},q_i,\boldsymbol{v}_i) \geq 0$. □

If feasible solutions $(\boldsymbol{y},\boldsymbol{u})$ and $(\boldsymbol{q},\boldsymbol{v})$ satisfy $\delta = 0$, then they are optimal solutions of (P) and (P$^{\rm C}$), respectively. In order to show that $\delta = 0$ is a necessary condition for global optimality, the following lemma is prepared:

Lemma 5.2. *Let $S_i = \{(t_i,\xi_i)|\ t_i \geq \phi_i(\xi_i)\}$. Then S_i can be represented by several second-order cones.*

Proof. Since any convex quadratic inequality can be rewritten by the second-order cone inequality [26], it will be shown that S_i can be represented by several convex quadratic inequalities.

Let $\alpha_i^1 = -\infty$, $\alpha_i^2 = 0$ and $\alpha_i^{n+1} = +\infty$. Because $\psi_i(\xi_i)$ satisfies Definition 3.1 (i), we can take α_i^j $(j=2,\cdots,n)$ such that $\psi_i(\xi_i)$ is a linear function of ξ_i on $\alpha_i^j \leq \xi_i \leq \alpha_i^{j+1}$ $(j=1,\cdots,n)$. Define the convex sets S_i^j as

$$S_i^1 = \{(t_i,\xi_i)|\ t_i \geq \phi_i(\xi_i),\ \xi_i \leq \alpha_i^2\}. \qquad (37)$$

$$S_i^j = \{(t_i,\xi_i)|\ t_i \geq \phi_i(\xi_i) - \phi_i(\alpha_i^j),\ \alpha_i^j \leq \xi_i \leq \alpha_i^{j+1}\}, \quad (j=2,\cdots,n), \qquad (38)$$

which satisfy $S_i = \bigcup_{j=1}^n S_i^j$. Note from (7) that $\phi_i^j(\xi_i)$ is a linear function of ξ_i if ψ_i is constant on $\alpha_i^j \leq \xi_i \leq \alpha_i^{j+1}$, otherwise $\phi_i^j(\xi_i)$ is a convex quadratic function. Since linear inequality is a special case of convex quadratic inequality, S_i^j is represented only by convex quadratic inequalities. It follows from Definition 3.1 (ii) and Corollary 3.6 that $\phi_i(\xi_i)$ is an strictly increasing function on $\xi_i \geq \alpha_i^2 = 0$. Therefore, we obtain

$$S_i = \Big\{(t_i,\xi_i)\Big|\ t_i = \sum_{j=1}^n t_i^j,\ \xi_i = \sum_{j=1}^n(\xi_i^j - \alpha_i^j),\ (t_i^j,\xi_i^j) \in S_i^j\Big\}, \qquad (39)$$

which implies that S_i can be represented only by convex quadratic inequalities. □

As an example for Lemma 5.2, consider the case of Example 3.8. By using (13), (37) and (38), we obtain

$$S_i^1 = \left\{(t_i,\xi_i)\ \middle|\ t_i \geq \frac{1}{2}k_i\xi_i^2,\ \xi_i \leq \bar{e}_i\right\},$$

$$S_i^2 = \left\{(t_i,\xi_i)\ \middle|\ t_i \geq \frac{1}{2}k_i^p(\xi_i-\bar{e}_i)^2 + k_i\bar{e}_i(\xi_i-\bar{e}_i),\ \xi_i \geq \bar{e}_i\right\}.$$

If we simply use (13), S_i is defined as

$$S_i = \left\{ (t_i, \xi_i) \;\middle|\; t_i \geq \frac{1}{2} k_i \xi_i^2,\ (\xi_i \leq \bar{e}_i), \quad t_i \geq \frac{1}{2} k_i^p (\xi_i - \bar{e}_i)^2 + k_i \bar{e}_i (\xi_i - \bar{e}_i),\ (\xi_i \geq \bar{e}_i) \right\}. \tag{40}$$

Since the definition (40) depends on the range of ξ_i, S_i has not been represented as the intersection of affine space and direct product of second-order cones. According to (39), (40) can be alternatively written as

$$S_i = \left\{ (t_i, \xi_i) \;\middle|\; t_i = t_i^1 + t_i^2,\ \xi_i = \xi_i^1 + (\xi_i^2 - \bar{e}_i),\ (t_i^1, \xi_i^1) \in S_i^1,\ (t_i^2, \xi_i^2) \in S_i^2 \right\},$$

which is represented only by linear equalities and convex quadratic inequalities.

By using Theorem 2.1 and Lemma 5.2, we obtain the following strong duality between (P) and ($\mathrm{P^C}$):

Lemma 5.3 (strong duality between (P) and ($\mathrm{P^C}$)). *Suppose that* (P) *and* ($\mathrm{P^C}$) *have interior feasible solutions. Then* (P) *and* ($\mathrm{P^C}$) *have optimal solutions* $(\boldsymbol{y}^*, \boldsymbol{u}^*)$ *and* $(\boldsymbol{q}^*, \boldsymbol{v}^*)$*, respectively, which satisfy* $\Phi(\boldsymbol{y}^*, \boldsymbol{u}^*) = -\Phi^C(\boldsymbol{q}^*, \boldsymbol{v}^*)$.

Proof. Letting $S_i^C = \{(t_i^C, q_i) |\ t_i^C \geq \phi_i^C(q_i)\}$, we introduce $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ as

$$\left.\begin{array}{lll} \overline{\mathcal{P}}: & \text{Minimize} & \displaystyle\sum_{i=1}^{N^m} t_i - \bar{\boldsymbol{f}}^\top \boldsymbol{u} \\ & \text{subject to} & y_i + l_i^0 \geq \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i\|, \\ & & (t_i, y_i) \in S_i, \quad (i = 1, \cdots, N^m); \end{array}\right\} \tag{41}$$

$$\left.\begin{array}{lll} \overline{\mathcal{D}}: & \text{Minimize} & \displaystyle\sum_{i=1}^{N^m} t_i^C + \sum_{i=1}^{N^m} (l_i^0 q_i + \boldsymbol{h}_i^{0\top} \boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top} \sum_{i=1}^{N^m} \boldsymbol{B}_i^{s\top} \boldsymbol{v}_i \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}, \\ & & (t_i^C, q_i) \in S_i^C,\ q_i \geq \|\boldsymbol{v}_i\|, \quad (i = 1, \cdots, N^m). \end{array}\right\} \tag{42}$$

Let $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}}, \overline{\boldsymbol{t}})$ and $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{v}}, \overline{\boldsymbol{t}}^C)$ denote the optimal solutions of $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$, respectively, where $\boldsymbol{t} = (t_i) \in \Re^{N^m}$ and $\boldsymbol{t}^C = (t_i^C) \in \Re^{N^m}$. Because t_i has only lower bound constraint in $\overline{\mathcal{P}}$, we obtain $\overline{t}_i = \phi_i(\overline{y}_i)$. Therefore $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}})$ is also an optimizer of $\mathcal{P}$. Similarly, it is easy to verify that $\overline{t}_i^C = \phi_i^C(\overline{q}_i)$. Consequently, we obtain $(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}}) = (\boldsymbol{y}^*, \boldsymbol{u}^*)$ and $(\overline{\boldsymbol{q}}, \overline{\boldsymbol{v}}) = (\boldsymbol{q}^*, \boldsymbol{v}^*)$.

It follows from Lemma 5.2 that S_i and S_i^C can be represented by several second-order cones. In addition, we can easily see that $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ make a pair of primal and dual SOCPs. See Appendix C for the detail. Thus $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ are embedded into $\mathcal{P}_{\mathrm{SOCP}}$ and $\mathcal{D}_{\mathrm{SOCP}}$, respectively. By applying Theorem 5.3 to $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$, these problems have optimal solutions satisfying $\Phi(\overline{\boldsymbol{y}}, \overline{\boldsymbol{u}}) = -\Phi^C(\overline{\boldsymbol{q}}, \overline{\boldsymbol{v}})$ if there exist feasible solutions of $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$. This condition is satisfied because S_i and S_i^C have interior points and (P) and ($\mathrm{P^C}$) have interior feasible solutions. □

Lemma 5.4. *If there exists a vector* $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{N^m})$ *satisfying* (24)*, then* (P) *and* ($\mathrm{P^C}$) *have optimal solutions* $(\boldsymbol{y}^*, \boldsymbol{u}^*)$ *and* $(\boldsymbol{q}^*, \boldsymbol{v}^*)$*, respectively, which satisfy* $\Phi(\boldsymbol{y}^*, \boldsymbol{u}^*) = -\Phi^C(\boldsymbol{q}^*, \boldsymbol{v}^*)$.

Proof. It suffices to show that the assumption of Lemma 5.4 is equivalent to that of Lemma 5.3. For an arbitrary $\hat{\boldsymbol{u}}$, there always exists $\hat{y}_i$ satisfying $\hat{\boldsymbol{y}}_i > \|\boldsymbol{B}_i^\top(\boldsymbol{x}^0 + \hat{\boldsymbol{u}}) - \bar{\boldsymbol{x}}_i\| - l_i^0$ if we take $\hat{y}_i$ large enough. Suppose $\hat{\boldsymbol{v}}_i$ is a solution to (24), then there exists $\hat{q}_i$ satisfying $\hat{q}_i > \|\hat{\boldsymbol{v}}_i\|$. It can be seen that such $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{u}})$ and $(\hat{\boldsymbol{q}}, \hat{\boldsymbol{v}})$, respectively, are interior feasible solutions of (P) and ($\mathrm{P^C}$). □

Theorem 4.2 is immediately obtained from Lemma 3.9, Lemma 4.1 and Lemma 5.4.

It is interesting to note that the Lagrange dual of nonlinear nonconvex programming, in general, contains some unknown variables of the primal problem, and the strong duality cannot be satisfied [31]. This is the reason from the viewpoint of duality theory why it is very difficult to establish the minimum principle of complementary energy without unknown information of deformation such as displacements, rotations, displacement gradients, etc. Although (P) and ($\mathrm{P^C}$) are nonlinear, they have no common variable and have zero duality gap as shown above. These properties play crucial roles in establishing the minimum principle of complementary energy purely in terms of generalized stresses.

6 Proof of Lemma 4.3 (compatibility conditions).

Obviously, it is necessary for a cable member that the directions of axial force and deformed member are compatible at Γ^{II}. The explicit expressions of such conditions, however, usually cause coupling of unknown stress and rotation components, which suggests impossibility of formulating complementary energy principle purely in terms of stress components. On the contrary, if the minimization problem does not have such explicit constraints, it is not obvious whether the compatibility conditions are satisfied by the solution that minimizes the complementary energy. In this section, we will prove Lemma 4.3, which guarantees that the directions of $\boldsymbol{h}_i^{\mathrm{II}}$ and $\boldsymbol{v}_i^{\mathrm{II}}$ $(i = 1, \cdots, N^m)$ obtained as the solutions of (Π) and (Π^{C}), respectively, are always compatible.

It follows from (7) and (23) that $q_i^* = \psi_i(y_i^*)$. By using (6) and Lemmas 3.9 and 4.1, we obtain

$$\|\boldsymbol{v}_i^{\mathrm{II}}\| = \sigma(\varepsilon_i^{\mathrm{II}}). \tag{43}$$

Suppose $-l_i^0 \le \varepsilon_i^{\mathrm{II}} \le 0$, then $\boldsymbol{v}_i^{\mathrm{II}} = \boldsymbol{0}$ is immediately obtained from (6) and (43).

Suppose $\varepsilon_i^{\mathrm{II}} > 0$. From Lemma 5.4, $(\boldsymbol{y}^*, \boldsymbol{u}^*)$ and $(\boldsymbol{q}^*, \boldsymbol{v}^*)$ satisfy $\delta(\boldsymbol{y}^*, \boldsymbol{u}^*, \boldsymbol{q}^*, \boldsymbol{v}^*) = 0$. It follows from (35) and (36) that the necessary and sufficient conditions for $\delta = 0$ is obtained as

$$\delta_i(y_i^*, \boldsymbol{u}^*, q_i^*, \boldsymbol{v}_i^*) = q_i^*(y_i^* + l_i^0) + \boldsymbol{v}_i^{*\top}(\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^*) - \bar{\boldsymbol{x}}_i) = 0, \quad (i = 1, \cdots, N^m). \tag{44}$$

From Lemma 3.9, there exists an optimal solution $(\boldsymbol{\varepsilon}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ satisfying

$$y_i^* + l_i^0 = \varepsilon_i^{\mathrm{II}} + l_i^0 = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}^{\mathrm{II}}) - \bar{\boldsymbol{x}}_i\| = \|\boldsymbol{h}_i^{\mathrm{II}}\|. \tag{45}$$

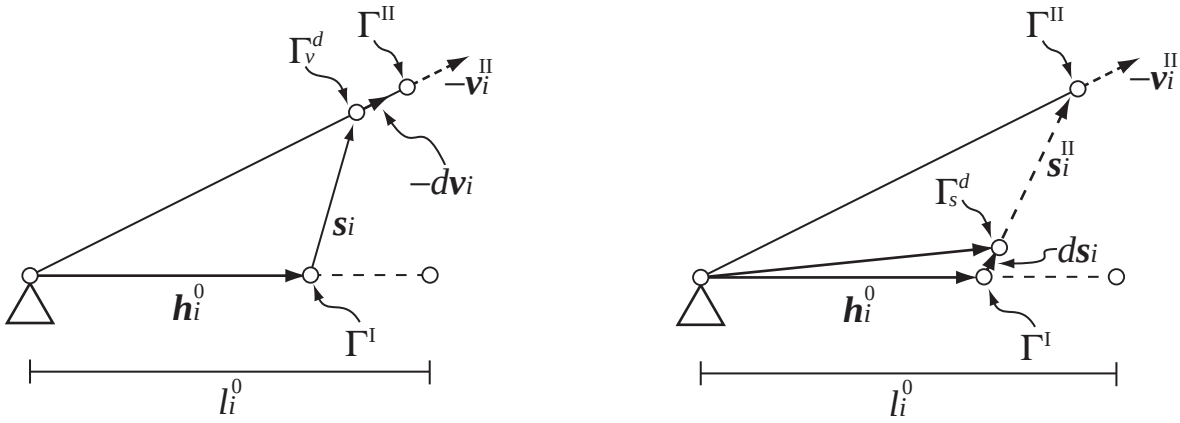
By using Lemma 4.1 (ii) and (45), (44) is reduced to

$$\|\boldsymbol{v}_i^{\mathrm{II}}\|\|\boldsymbol{h}_i^{\mathrm{II}}\| + \boldsymbol{v}_i^{\mathrm{II}\top}\boldsymbol{h}_i^{\mathrm{II}} = 0, \quad (i = 1, \cdots, N^m). \tag{46}$$

From (43) and (46), we obtain

$$\boldsymbol{v}_i^{\mathrm{II}} = -\sigma_i(\varepsilon_i^{\mathrm{II}})\frac{\boldsymbol{h}_i^{\mathrm{II}}}{\|\boldsymbol{h}_i^{\mathrm{II}}\|},$$

which completes the proof of Lemma 4.3.



(a) incremental complementary work $dW_i^C = -\boldsymbol{s}_i^\top(-d\boldsymbol{v}_i)$

(b) incremental work $dW_i = -\boldsymbol{v}_i^\top d\boldsymbol{s}_i$ $(dW_i = 0)$

Figure 4: Complementary work done by $d\boldsymbol{v}_i$ and work done by $d\boldsymbol{s}_i$.

7 Physical interpretation of the complementary energy function.

In small displacement theory, the complementary energy is obtained as the sum of complementary strain energy of all members and complementary work done at supports by reactions. It will be shown that $\Pi^C(\boldsymbol{v})$ in (Π^{C}) can be also derived by using the framework of complementary work considering large rotations of members. First, we shall investigate the complementary work done by the internal force of each member.

The complementary work done by the internal force $\boldsymbol{v}_i$ of the ith member from Γ^{I} to Γ^{II} is denoted by W_i^C $(i = 1, \cdots, N^m)$. For simplicity, Γ^{I} is given as the slackening state; i.e., $\boldsymbol{v}_i^{\mathrm{I}} = \boldsymbol{0}$ $(i = 1, \cdots, N^m)$. It is easy, however, to extend the following discussion to any configuration Γ^{I}. Eq. (31) implies that the direction of $\boldsymbol{v}_i$ is opposite to $\boldsymbol{h}_i$, which leads to $dW_i^C = \boldsymbol{s}_i^\top(-d\boldsymbol{v}_i)$. Then the complementary work done from Γ^{I} to Γ^{II} can be written as

$$dW_i^C = \int_{\Gamma^{\mathrm{I}}}^{\Gamma^{\mathrm{II}}} \boldsymbol{s}_i^\top(-d\boldsymbol{v}_i). \tag{47}$$

Since the complementary work of an elastic structure does not depend on loading history, we can choose a monotonically increasing load $\rho\boldsymbol{v}_i^{\mathrm{II}}$ from $\rho = 0$ to 1. Consequently, the relation $d\boldsymbol{v}_i = \boldsymbol{v}_i^{\mathrm{II}} d\rho$ leads to

$$W_i^C = -\int_0^1 \boldsymbol{s}_i(\rho)^\top \boldsymbol{v}_i^{\mathrm{II}} d\rho. \tag{48}$$

Since the deformed configuration $\boldsymbol{h}_i(\rho)$ always has to satisfy equilibrium while the integration in (48) is calculated, $\boldsymbol{h}_i(\rho)$ is always in the direction of $-\boldsymbol{v}_i^{\mathrm{II}}$. Consider the small perturbation of $\boldsymbol{v}_i$ from $\boldsymbol{0}$ to $d\boldsymbol{v}_i = \boldsymbol{v}_i^{\mathrm{II}} d\rho$ and the attained equilibrium state is referred to as Γ_v^d state. Then, as illustrated in Fig.4 (a), the displacement $\boldsymbol{s}_i$ at Γ_v^d from Γ^{I} is not small, which leads to the difference from the small displacement theory. For comparison purpose, consider the work W_i done by displacement $\boldsymbol{s}_i$ from $\boldsymbol{0}$ to $\boldsymbol{s}_i^{\mathrm{II}}$, where $dW_i = (-\boldsymbol{v}_i)^\top d\boldsymbol{s}_i$. Suppose that $\boldsymbol{s}_i$ is perturbed from $\boldsymbol{0}$ to $d\boldsymbol{s}_i = \boldsymbol{s}_i^{\mathrm{II}} d\rho$ and the attained equilibrium state is referred to as Γ_s^d. In this case, the

force $\boldsymbol{v}_i$ at Γ_s^d is still small even in large deformation theory. In particular, $\boldsymbol{v}_i = \boldsymbol{0}$ in the example illustrated in Fig.4 (b). Therefore, the total potential energy function in nonlinear theory is in the same form as that in small displacement theory.

It follows from $-\boldsymbol{s}_i(\rho) = -\boldsymbol{h}_i(\rho) + \boldsymbol{h}_i^0$ that

$$-\boldsymbol{s}_i(\rho)^\top \boldsymbol{v}_i^{\text{II}} = -\boldsymbol{h}_i(\rho)^\top \boldsymbol{v}_i^{\text{II}} + \boldsymbol{h}_i^{0\top} \boldsymbol{v}_i^{\text{II}}. \tag{49}$$

Because $\boldsymbol{h}_i(\rho)$ and $-\boldsymbol{v}_i$ has the same direction as illustrated in Fig.2 and $\|\boldsymbol{h}_i(\rho)\|$ is the length of the member, we obtain

$$-\boldsymbol{h}_i(\rho)^\top \boldsymbol{v}_i^{\text{II}} = \|\boldsymbol{h}_i(\rho)\|\|\boldsymbol{v}_i^{\text{II}}\| = (\varepsilon_i(\rho) + l_i^0)\|\boldsymbol{v}_i\|^{\text{II}}. \tag{50}$$

From (49) and (50), (48) is reduced to

$$\begin{aligned} W_i^C(\boldsymbol{v}_i^{\text{II}}) &= \int_0^1 \varepsilon_i(\rho)\|\boldsymbol{v}_i\|^{\text{II}} d\rho + \int_0^1 (l_i^0\|\boldsymbol{v}_i\|^{\text{II}} + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i^{\text{II}}) d\rho \\ &= w_i^C(\boldsymbol{v}_i^{\text{II}}) + l_i^0\|\boldsymbol{v}_i^{\text{II}}\| + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i^{\text{II}}, \end{aligned}$$

where (8) and (28) have been utilized. The complementary energy $\Pi^C(\boldsymbol{v})$ is obtained as the sum of the complementary work $W_i^C(\boldsymbol{v}_i)$ done in all the member by internal forces and at supports by reactions; i.e.,

$$\Pi^C(\boldsymbol{v}) = \sum_{i=1}^{N^m} W_i^C(\boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top} \sum_{i=1}^{N^m} \boldsymbol{B}_i^{s\top} \boldsymbol{v}_i.$$

In view of the derivation above, the objective function Π^C of (Π^{C}) has the physical meaning of the complementary energy, which is defined as the sum of complementary work done from Γ^{I} to a kinematically admissible deformed state.

Under the assumption of the small strain and small rotation, Γ^{I} must be given as $\|\boldsymbol{h}_i^0\| = l_i^0$, and the deformed configuration $\boldsymbol{h}_i^{\text{II}}$ as well as $-\boldsymbol{v}_i^{\text{II}}$ is assumed to have same direction as $\boldsymbol{h}_i^0$. Therefore, in (49), the approximated relation $\boldsymbol{h}_i^{0\top}\boldsymbol{v}_i^{\text{II}} = -l_i^0\|\boldsymbol{v}_i^{\text{II}}\|$ can be utilized. Hence we obtain the complementary work $W_i^C(\boldsymbol{v}_i)$ done on the ith member as $W_i^C(\boldsymbol{v}_i) = w_i^C(\boldsymbol{v}_i)$, which leads to the classical total complementary energy function Π_{LIN}^C introduced in (32).

8 Energy principles for force method.

In this section, we investigate alternative formulations for the minimum principles of potential energy (Π_{R}) and complementary energy ($\Pi_{\text{R}}^{\text{C}}$). In particular, ($\Pi_{\text{R}}^{\text{C}}$) may be used to develop the force method (flexibility method) for cable networks in the finite deformation theory. The variational principles in this type have been studied for contact problems under assumption of small displacement [33; 34; 32], and are sometimes referred to as reciprocal formulations.

Redundants are selected to reduce the indeterminate cable network to a statically determinate structure (a reduced primary structure) such that the internal force vectors $\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{N^m}$ can be determined only from the equilibrium equation. Let N^r denote degree of redundancy of the reduced primary structure. Note that $N^d + N^r = 3N^m$ is satisfied since a vector $\boldsymbol{v} = (\boldsymbol{v}_1, \cdots, \boldsymbol{v}_{N^m})$ has $3N^m$ unknown components and the equilibrium equation (24) is composed of N^d independent

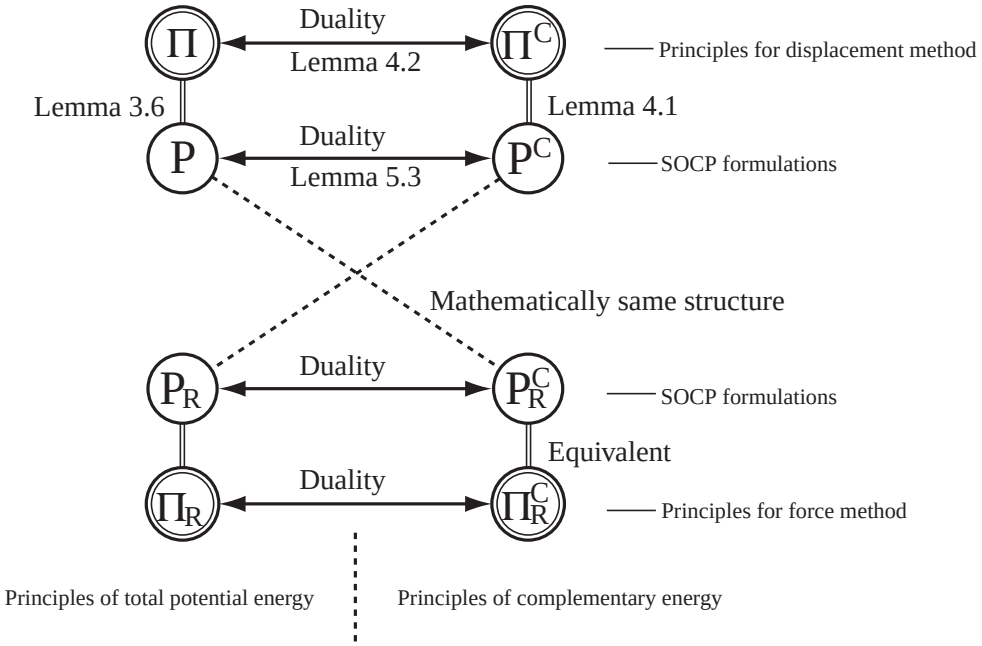


Figure 5: Schematic representation of variational principles and their convex formulations.

equations; i.e., N^r is different from the classical definition of the redundancy that is defined by $N^r = N^m - N^d$. The vector of redundants is denoted by $\boldsymbol{r} \in \Re^{N^r}$. The internal forces $\boldsymbol{v}_i$ are written in terms of $\boldsymbol{r}$ and the external loads $\bar{\boldsymbol{f}}$ as

$$\boldsymbol{v}_i = \boldsymbol{H}_i \boldsymbol{r} + \boldsymbol{H}_i^f \bar{\boldsymbol{f}}, \quad (i = 1, \cdots, N^m), \tag{51}$$

where $\boldsymbol{H}_i \in \Re^{3\times N^r}$ and $\boldsymbol{H}_i^f \in \Re^{3\times N^d}$ are constant matrices obtained only from the connectivity of nodes and members of the reduced primary structure. $\boldsymbol{r}$ and $\boldsymbol{v}$ satisfying (51) are referred to as the self-equilibrium forces.

The vectors $\boldsymbol{x}^{r0} \in \Re^{N^r}$ and $\bar{\boldsymbol{u}}^r \in \Re^{N^r}$, which make a pair doing the complementary work, are defined as

$$\boldsymbol{x}^{r0} = \sum_{i=1}^{N^m} \boldsymbol{H}_i^\top \boldsymbol{B}_i^s \boldsymbol{x}^{s0}, \quad \bar{\boldsymbol{u}}^r = \sum_{i=1}^{N^m} \boldsymbol{H}_i^\top \bar{\boldsymbol{u}}_i. \tag{52}$$

Note that the formulations (Π), ($\Pi^{\rm C}$), ($\Pi_{\rm R}$) and ($\Pi_{\rm R}^{\rm C}$) are compared as listed in Table 2. The pairs of forces and displacements doing the work are listed in the first row in Table 2, which appears in the minimization problems of total potential energy. For the minimization problems of complementary energy, the pairs generating the complementary work are listed in the second and third rows. The constraint conditions of these problems are listed in Table 2 for the purpose of comparison.

By using (51), (52) and $\sum_{i=1}^{N^m} \boldsymbol{H}_i^\top \boldsymbol{B}_i \boldsymbol{x}^0 = \boldsymbol{0}$, ($\rm P^C$) is reduced to the following convex formula-

Table 2: Comparison of displacement method and force method.

		displacement method		force method
(nodal forces, nodal displacements)	(Π)	$(\bar{\boldsymbol{f}}\ ,\quad \boldsymbol{u})$	(Π_{R})	$\Big(\bar{\boldsymbol{f}},\quad \sum_{i=1}^{N^m} \boldsymbol{H}_i^{f\top} \boldsymbol{s}_i\Big)$
(reactions, prescribed displacements)	(Π^{C})	$\Big(\sum_{i=1}^{N^m} \boldsymbol{B}_i^{s\top} \boldsymbol{v}_i,\quad \bar{\boldsymbol{u}}^s\Big)$		
(redundants, corresponding displacements)			$(\Pi_{\mathrm{R}}^{\mathrm{C}})$	$(\boldsymbol{r},\quad \bar{\boldsymbol{u}}^r)$
compatibility conditions	(Π)	$\varepsilon_i = \|\boldsymbol{B}_i(\boldsymbol{x}^0+\boldsymbol{u}) - \bar{\boldsymbol{x}}_i\| - l_i^0$	(Π_{R})	$\sum_{i=1}^{N^m} \boldsymbol{H}_i^{\top} \boldsymbol{s}_i + \bar{\boldsymbol{u}}^r = \boldsymbol{0}$
equilibrium equations	(Π^{C})	$\sum_{i=1}^{N^m} \boldsymbol{B}_i^{\top} \boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}$	$(\Pi_{\mathrm{R}}^{\mathrm{C}})$	$q_i = \|\boldsymbol{H}_i \boldsymbol{r} - \bar{\boldsymbol{f}}_i\|$

tion of the minimization of complementary energy:

$$\left.\begin{array}{ll} (\mathrm{P}_{\mathrm{R}}^{\mathrm{C}}): & \text{Minimize} \quad \Phi_R^C(\boldsymbol{q},\boldsymbol{r}) = \sum_{i=1}^{N^m} \phi_i^C(q_i) + \sum_{i=1}^{N^m} l_i^0 q_i - (\boldsymbol{x}^{r0}+\bar{\boldsymbol{u}}^r)^{\top}\boldsymbol{r} - \sum_{i=1}^{N^m} (\boldsymbol{h}_i^0 - \boldsymbol{B}_i^s \bar{u}^s)^{\top} \bar{\boldsymbol{f}}_i \\ & \text{subject to} \quad q_i \geq \|\boldsymbol{H}_i \boldsymbol{r} + \boldsymbol{H}_i^f \bar{\boldsymbol{f}}\|, \quad (i=1,\cdots,N^m), \end{array}\right\} \tag{53}$$

where the independent variables are $\boldsymbol{r}$ and $\boldsymbol{q}$. $\boldsymbol{h}_i^0$ $(i=1,\cdots .N^m)$, which identify the configuration at Γ^{I}, must be given so as to satisfy the kinematic conditions $\boldsymbol{h}_i^0 = \boldsymbol{B}_i \boldsymbol{x}^0$. Since the redundants $\boldsymbol{r}$ in the force method play the same role as displacements in the displacement method, $(\mathrm{P}_{\mathrm{R}}^{\mathrm{C}})$ has the same mathematical structure as (P) as shown in Fig.5. It follows from Corollary 3.6 that Φ_R^C is a strictly increasing function of q_i on $q_i \geq \|\boldsymbol{H}_i \boldsymbol{r} + \boldsymbol{H}_i^f \bar{\boldsymbol{f}}\| \geq 0$. Therefore, $(\mathrm{P}_{\mathrm{R}}^{\mathrm{C}})$ has the same solution as the following problem:

$$\left.\begin{array}{ll} (\Pi_{\mathrm{R}}^{\mathrm{C}}): & \text{Minimize} \quad \Pi_R^C(\boldsymbol{r}) = \sum_{i=1}^{N^m} \phi_i^C(q_i) + \sum_{i=1}^{N^m} l_i^0 q_i - (\boldsymbol{x}^{r0}+\bar{\boldsymbol{u}}^r)^{\top}\boldsymbol{r} - \sum_{i=1}^{N^m} (\boldsymbol{h}_i^0 - \boldsymbol{B}_i^s \bar{\boldsymbol{u}}^s)^{\top} \bar{\boldsymbol{f}}_i \\ & \text{subject to} \quad q_i = \|\boldsymbol{H}_i \boldsymbol{r} + \boldsymbol{H}_i^f \bar{\boldsymbol{f}}\|, \quad (i=1,\cdots,N^m), \end{array}\right\} \tag{54}$$

where the independent variables are $\boldsymbol{r}$ only. $(\Pi_{\mathrm{R}}^{\mathrm{C}})$ gives an alternative formulation of minimum principle of complementary energy. (Π) and $(\Pi_{\mathrm{R}}^{\mathrm{C}})$ have the same mathematical structures as shown in Fig.5, where both have the equality constraints as listed in Table 2.

We remind the reader that displacements $\boldsymbol{u}$ are the independent N^d variables in (Π). On the other hand, the number of the variables of $(\Pi_{\mathrm{R}}^{\mathrm{C}})$ is equal to the degrees of redundancy of the structure. This implies that, for many structures, the force method based on the principle $(\Pi_{\mathrm{R}}^{\mathrm{C}})$ may have less variables than the displacement method based on (Π). The force method, however, has disadvantage of difficulty in obtaining a reduced primary structure and there exists several schemes to select the redundants [35§ 4.11].

In order to establish an alternative formulation of the minimum principle of total potential energy, what we call (Π_{R}), consider the dual problem $(\mathrm{P}_{\mathrm{R}})$ of $(\mathrm{P}_{\mathrm{R}}^{\mathrm{C}})$. Since $(\mathrm{P}_{\mathrm{R}}^{\mathrm{C}})$ has the same

mathematical structure as (P), the problem ($\mathrm{P_R}$) can be derived in the same manner as ($\mathrm{P^C}$) as shown in Fig.5. Since the fundamental idea has been presented in Section 3, details of algebraic operations are omitted. By using (52), the dual problem for ($\mathrm{P_R^C}$) is obtained as

$$\left.\begin{array}{lll} (\mathrm{P_R}): & \text{Minimize} & \Phi_R(\boldsymbol{\eta},\boldsymbol{s}) = \displaystyle\sum_{i=1}^{N^m}\phi_i(\eta_i - l_i^0) + \sum_{i=1}^{N^m}(\boldsymbol{H}_i^f\bar{\boldsymbol{f}})^\top(\boldsymbol{s}_i - \boldsymbol{B}_i^s\bar{\boldsymbol{u}}^s) \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m}\boldsymbol{H}_i^\top\boldsymbol{s}_i + \bar{\boldsymbol{u}}^r = \boldsymbol{0}, \\ & & \eta_i \geq \|\boldsymbol{h}_i^0 + \boldsymbol{s}_i\|, \quad (i = 1,\cdots,N^m). \end{array}\right\} \tag{55}$$

Define the function $w_i^s(\boldsymbol{s}_i)$ on $\boldsymbol{s}_i \in \Re^3$ as

$$w_i^s(\boldsymbol{s}_i) = w_i(\|\boldsymbol{h}_i^0 + \boldsymbol{s}_i\| - l_i^0).$$

It follows from the convexity of Φ_R that ($\mathrm{P_R}$) is equivalent to the following problem:

$$\left.\begin{array}{lll} (\Pi_\mathrm{R}): & \text{Minimize} & \Pi_R(\boldsymbol{s}) = \displaystyle\sum_{i=1}^{N^m} w_i^s(\boldsymbol{s}_i) + \sum_{i=1}^{N^m}(\boldsymbol{H}_i^f\bar{\boldsymbol{f}})^\top(\boldsymbol{s}_i - \boldsymbol{B}_i^s\bar{\boldsymbol{u}}^s) \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m}\boldsymbol{H}_i^\top\boldsymbol{s}_i + \bar{\boldsymbol{u}}^r = \boldsymbol{0}. \end{array}\right\} \tag{56}$$

Let $\boldsymbol{s}_i^{\mathrm{II}}$ denote the optimal solution of (Π_R). By investigating the duality between ($\mathrm{P_R^C}$) and ($\mathrm{P_R}$), we obtain $\boldsymbol{s}_i^{\mathrm{II}} = \boldsymbol{B}_i\boldsymbol{u}^{\mathrm{II}} - \boldsymbol{B}_i^s\bar{\boldsymbol{u}}^s$.

Note that (Π_R) has the same mathematical structures as (Π^C) as shown in Fig.5 and Table 2. (Π_R) implies the alternative formulation of minimum principle of total potential energy; i.e., the member configurations $\hat{\boldsymbol{h}}_i$ at Γ^{II} are obtained by minimizing Π_R under the constraints of compatibility conditions.

The generalized principle is introduced by considering the Lagrange multiplier $\boldsymbol{u} \in \Re^{N^d}$ for the linear constraints of (Π^C); i.e., the Lagrangian for (Π^C) is obtained as

$$L^G(\boldsymbol{u},\boldsymbol{v}) = \sum_{i=1}^{N^m} w_i^C(\boldsymbol{v}_i) + \sum_{i=1}^{N^m}(l_i^0\|\boldsymbol{v}_i\| + \boldsymbol{h}_i^{0\top}\boldsymbol{v}_i) - \bar{\boldsymbol{u}}^{s\top}\sum_{i=1}^{N^m}\boldsymbol{B}_i^{s\top}\boldsymbol{v}_i - \boldsymbol{u}^\top\Big(\sum_{i=1}^{N^m}\boldsymbol{B}_i^\top\boldsymbol{v}_i + \bar{\boldsymbol{f}}\Big). \tag{57}$$

The equilibrium configuration is obtained from the stationary conditions of the function L^G, where the independent variables are $\boldsymbol{u}$ and $\boldsymbol{v}$ without any subsidiary conditions. Note that this principle corresponds to the Hellinger-Reissner principle derived for the continuum [2]. To the authors' knowledge, however, no principle has been proposed for structures with stress-unilateral behavior without any subsidiary conditions whether members may be in tensile or in unstressed state.

9 Examples.

9.1 A single-degree-of-freedom cable network.

Consider the simplest example of the cable network with single-degree of freedom as shown in Fig.6, where $N^m = 2$, $N^a = 3$ and $N^d = 1$. The initial unstressed lengths of two cable members are same and equal to l^0. The external force $\bar{f}$ satisfying $0 < \bar{f} < 2kl^0$ is applied at node (c).

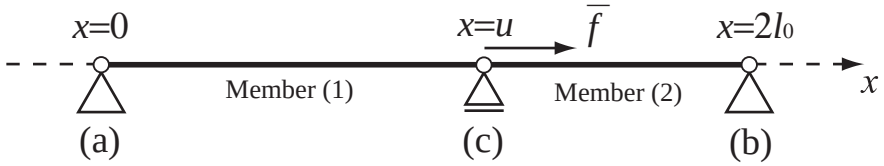


Figure 6: The deformed state of a single-degree-of-freedom cable network.

As shown in Section 4, the complementary energy function Π^C in (29) is different in the second term from $\Pi^C_{\rm LIN}$ in small displacement theory. In this example, we will illustrate that the second term is necessary in finite displacement theory even without the existence of the rotation of members. To this end, the reference state is given such that all the nodes are located at the origin; i.e., $x^0 = 0$ and $\boldsymbol{x}^{s0} = \mathbf{0} \in \Re^2$. The minimization problem of complementary energy will be solved below, and the result will be compared with the solution to minimum total potential energy.

Suppose that both members obey Hooke's law as introduced in Example 3.7 and have the same extensional stiffness k. The problems of minimum total potential energy and complementary energy are obtained as

$$\left.\begin{array}{lll} (\Pi): & \text{Minimize} & \Pi(u) = w_1(\varepsilon_1) + w_2(\varepsilon_2) - \bar{f}u \\ & \text{subject to} & \varepsilon_1 = |u| - l^0, \\ & & \varepsilon_2 = |u - 2l^0| - l^0; \end{array}\right\} \tag{58}$$

$$\left.\begin{array}{lll} (\Pi^{\rm C}): & \text{Minimize} & \Pi^C(v_1, v_2) = \dfrac{1}{2k}(v_1^2 + v_2^2) + l^0(|v_1| + |v_2|) - 2l^0 v_2 \\ & \text{subject to} & v_1 + v_2 + \bar{f} = 0, \end{array}\right\} \tag{59}$$

where $w_i(\varepsilon_i)$ is defined by (12).

From the equilibrium equation $v_1 + v_2 + \bar{f} = 0$ and the condition $\bar{f} > 0$, we obtain $v_1 \le 0$ or $v_2 \le 0$. To solve $(\Pi^{\rm C})$ analytically, we consider the following three cases:

(i) $v_1 \le 0$, $v_2 \le 0$: By using the equilibrium equation, we can eliminate v_1 from Π^C as

$$\Pi^C(v_1, v_2) = \frac{1}{k}\Big[v_2 - \Big(kl^0 - \frac{\bar{f}}{2}\Big)\Big]^2 + \frac{\bar{f}^2}{2k} - \frac{1}{k}\Big(kl^0 - \frac{\bar{f}}{2}\Big)^2 + l^0\bar{f}.$$

Since $kl^0 - \bar{f}/2 > 0$ and $v_2 \le 0$, we see that the minimum objective value is

$$\Pi^C(-\bar{f}, 0) = \frac{\bar{f}^2}{2k} + l^0\bar{f}.$$

(ii) $v_1 \ge 0$, $v_2 \le 0$: Π^C is reduced to

$$\Pi^C(v_1, v_2) = \frac{1}{k}\Big[v_2 - \Big(2kl^0 - \frac{\bar{f}}{2}\Big)\Big]^2 + \frac{\bar{f}^2}{2k} - \frac{1}{k}\Big(2kl^0 - \frac{\bar{f}}{2}\Big)^2 - l^0\bar{f}.$$

From $v_2 = -\bar{f} - v_1$ and $v_1 \ge 0$, we obtain $v_2 \le -\bar{f}$. It follow from $2kl^0 - \bar{f}/2 \le 0$ that Π^C increases strictly on $v_2 \le -\bar{f}$, which implies that the minimum objective value is

$$\Pi^C(0, -\bar{f}) = \frac{\bar{f}^2}{2k} + 3l^0\bar{f}.$$

(iii) $v_1 \le 0$, $v_2 \ge 0$: Π^C is obtained as

$$\Pi^C(v_1, v_2) = \frac{1}{k}\Big(v_2 + \frac{\bar{f}}{2}\Big)^2 + \frac{\bar{f}^2}{4k} + l^0\bar{f}.$$

The condition $-\bar{f}/2 < 0$ implies that the minimum value of Π^C is

$$\Pi^C(-\bar{f}, 0) = \frac{\bar{f}^2}{2k} + l^0\bar{f}.$$

The results of (i)–(iii) state that the solution of (Π^{C}) is $(v_1^{\mathrm{II}}, v_2^{\mathrm{II}}) = (-\bar{f}, 0)$ which is easily verified to correspond to the equilibrium state. On the other hand, the solution of (Π) is $u^{\mathrm{II}} = (\bar{f}/k) + l^0$, which is compatible to the solution of (P^{C}). At Γ^{II}, Π is written as

$$\Pi(u^{\mathrm{II}}) = \frac{1}{2}k\varepsilon_1^{\mathrm{II}2} - \bar{f}u^{\mathrm{II}} = -\frac{\bar{f}^2}{2k} - l^0\bar{f} = -\Pi^C(v_1^{\mathrm{II}}, v_2^{\mathrm{II}}),$$

which implies that there exists no duality gap between Π and Π^C.

For the purpose of comparison, consider the classical complementary energy Π^C_{LIN}. From (32), we obtain

$$\Pi^C_{\mathrm{LIN}}(v_1, v_2) = \frac{1}{2k}(v_1^2 + v_2^2) - 2l^0 v_2. \tag{60}$$

By using the equilibrium equation $v_1 + v_2 + \bar{f} = 0$, (60) is reduced to

$$\Pi^C_{\mathrm{LIN}}(v_2) = \frac{1}{k}\Big[v_2 - \Big(kl^0 - \frac{\bar{f}}{2}\Big)\Big]^2 + \frac{\bar{f}^2}{2k} - \frac{1}{k}\Big(kl^0 - \frac{\bar{f}}{2}\Big)^2,$$

which leads to the erroneous solution $v_2 = kl^0 - \bar{f}/2$.

Alternatively, consider the case where the reference state is given such that node (b) is located at $x = 2l^0$, then we obtain

$$\Pi^C(v_1, v_2) = \frac{1}{2k}(v_1^2 + v_2^2) + l^0(|v_1| + |v_2|), \quad \Pi^C_{\mathrm{LIN}}(v_1, v_2) = \frac{1}{2k}(v_1^2 + v_2^2).$$

Since Π^C_{LIN} is equal to the complementary energy function of trusses in linear elasticity theory, it will fail to obtain the correct answer. Thus, the second term of Π^C in (29) is necessary for large deformation theory of cable networks.

9.2 A geometrically nonlinear two-degree-of-freedom cable network.

Consider a cable network composed of two members as shown in Fig.7 in 2-dimensional space, where $N^m = 2$, $N^a = 3$ and $N^d = 2$. The coordinates of nodes (a) and (b) are fixed at $\bar{\boldsymbol{x}}_{\mathrm{a}} = (\bar{x}_{\mathrm{a}x}, \bar{x}_{\mathrm{a}y})$ and $\bar{\boldsymbol{x}}_{\mathrm{b}} = (\bar{x}_{\mathrm{b}x}, \bar{x}_{\mathrm{b}y})$, respectively, and the coordinates of node (c) is denoted by $\boldsymbol{u} = (u_x, u_y)$ where the external force $\bar{\boldsymbol{f}} = (\bar{f}_x, \bar{f}_y)$ is applied. The reference state is given such that nodes (a), (b) and (c) are at $\boldsymbol{x}_{\mathrm{a}}^{s0}$, $\boldsymbol{x}_{\mathrm{b}}^{s0}$ and $\boldsymbol{x}^0$, respectively, which can be given arbitrary. Then the displacements of nodes (a) and (b) are specified with $\bar{\boldsymbol{u}}_{\mathrm{a}}^s = \bar{\boldsymbol{x}}_{\mathrm{a}} - \boldsymbol{x}_{\mathrm{a}}^{s0}$ and $\bar{\boldsymbol{u}}_{\mathrm{b}}^s = \bar{\boldsymbol{x}}_{\mathrm{b}} - \boldsymbol{x}_{\mathrm{b}}^{s0}$, respectively. Suppose that two members obey Hooke's law with stiffness k_1 and k_2, respectively.

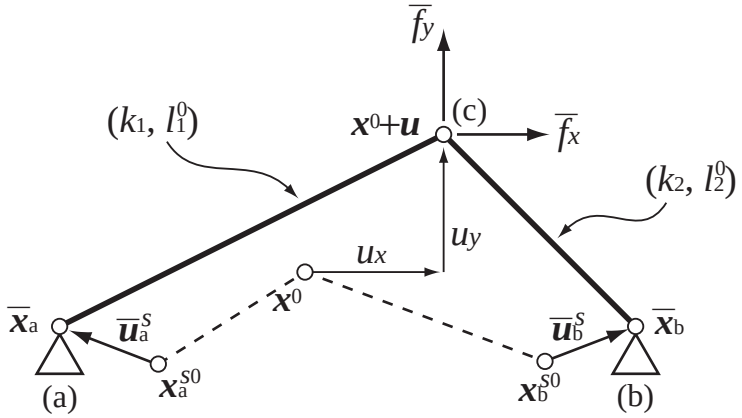


Figure 7: Two-degree-of-freedom cable network.

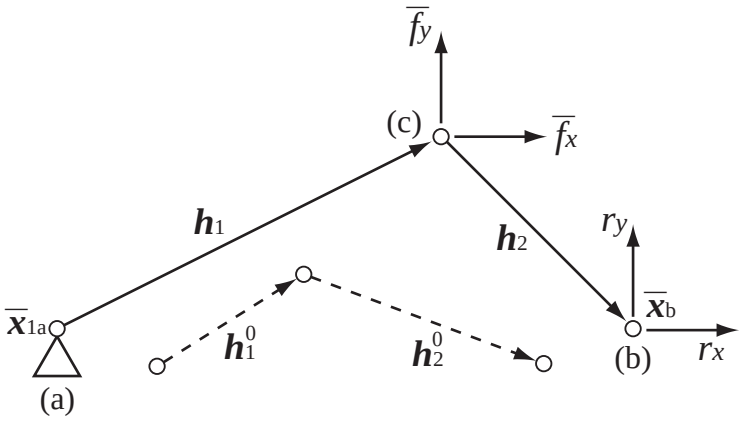
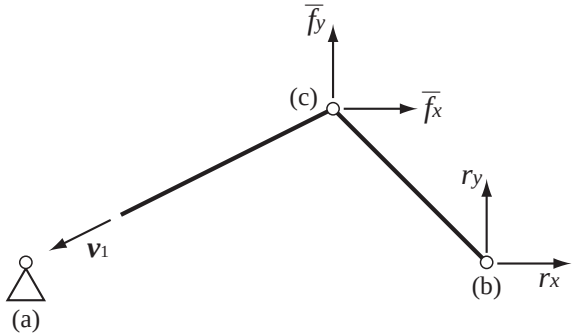
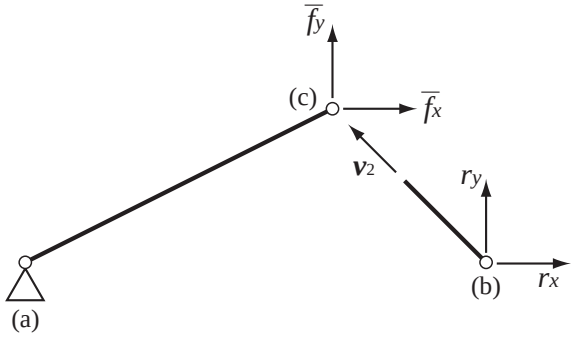


Figure 8: Reduced primary structure.



(a) member (1)



(b) member (2)

Figure 9: Free body schemes.

Let l_1^0 and l_2^0 denote the original lengths of members (1) and (2), respectively. Then, we obtain the minimization problem of the total potential energy as

$$\left.\begin{array}{lll} (\Pi): & \text{Minimize} & \displaystyle\sum_{i=1}^{2} w_i(\varepsilon_i) - \bar{\boldsymbol{f}}^\top \boldsymbol{u} \\ & \text{subject to} & \varepsilon_1 = \|(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_{\rm a}\| - l_1^0, \\ & & \varepsilon_2 = \|\bar{\boldsymbol{x}}_{\rm b} - (\boldsymbol{x}^0 + \boldsymbol{u})\| - l_2^0, \end{array}\right\} \tag{61}$$

where $w_i(\varepsilon_i)$ is defined by (12). For the consistency with the formulation (15), we should define $\bar{\boldsymbol{u}}^s = (\bar{\boldsymbol{u}}_{\rm a}^s, \bar{\boldsymbol{u}}_{\rm b}^s) \in \Re^4$, $\boldsymbol{x}^{s0} = (\boldsymbol{x}_{\rm a}^{s0}, \boldsymbol{x}_{\rm b}^{s0}) \in \Re^4$, $\boldsymbol{B}_1 = \boldsymbol{I}$ and $\boldsymbol{B}_2 = -\boldsymbol{I}$, where $\boldsymbol{I} \in \Re^{2\times 2}$ denotes the identity matrix. From $\bar{\boldsymbol{x}}_1 = \bar{\boldsymbol{x}}_{\rm a}$, $\bar{\boldsymbol{x}}_2 = -\bar{\boldsymbol{x}}_{\rm b}$ and (16), $\boldsymbol{B}_1^s$ and $\boldsymbol{B}_2^s$ are obtained as

$$\boldsymbol{B}_1^s = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B}_2^s = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It follows from Table 1 that $\boldsymbol{h}_1^0 = \boldsymbol{x}^0 - \boldsymbol{x}_{\rm a}^{s0} \in \Re^2$ and $\boldsymbol{h}_2^0 = \boldsymbol{x}_{\rm b}^{s0} - \boldsymbol{x}^0 \in \Re^2$, which correspond to the configurations of the members at $\Gamma^{\rm I}$ as illustrated in Fig.8.

Let $\boldsymbol{v}_1 = (v_{1x}, v_{1y})$ and $\boldsymbol{v}_2 = (v_{2x}, v_{2y})$ denote the internal force vectors of members (1) and (2), respectively. The minimum principle of complementary energy is derived from (29) as

$$\left.\begin{array}{lll} (\Pi^{\rm C}): & \text{Minimize} & \displaystyle\sum_{i=1}^{2} \frac{\boldsymbol{v}_i^\top \boldsymbol{v}_i}{2k_i} + (l_1^0 \|\boldsymbol{v}_1\| + \boldsymbol{h}_1^{0\top} \boldsymbol{v}_1) + (l_2^0 \|\boldsymbol{v}_2\| - \boldsymbol{h}_2^{0\top} \boldsymbol{v}_2) - \bar{\boldsymbol{u}}_{\rm a}^{s\top} \boldsymbol{v}_1 + \bar{\boldsymbol{u}}_{\rm b}^{s\top} \boldsymbol{v}_2 \\ & \text{subject to} & \boldsymbol{v}_1 - \boldsymbol{v}_2 + \bar{\boldsymbol{f}} = \boldsymbol{0}. \end{array}\right\} \tag{62}$$

Suppose that the constraints of node (b) is removed to define the reduced primary structure. The statically indeterminate forces are denoted by $\boldsymbol{r} = (r_x, r_y)$ as shown in Fig.8. Then the equilibrium equations in terms of $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{r}$ are obtained as

$$\boldsymbol{v}_1 = \boldsymbol{r} + \bar{\boldsymbol{f}}, \quad \boldsymbol{v}_2 = \boldsymbol{r}. \tag{63}$$

It follows from (63) that $\boldsymbol{H}_1 = \boldsymbol{I}$, $\boldsymbol{H}_1^f = \boldsymbol{I}$, $\boldsymbol{H}_2 = \boldsymbol{I}$ and $\boldsymbol{H}_2^f = \boldsymbol{O}$. Let $\boldsymbol{h}_1 = (h_{1x}, h_{1y})$ and $\boldsymbol{h}_2 = (h_{2x}, h_{2y})$ denote the configurations of deformed members; i.e., $\boldsymbol{h}_1 = (\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_{\rm a}$ and

$\boldsymbol{h}_2 = \bar{\boldsymbol{x}}_{\rm b} - (\boldsymbol{x}^0 + \boldsymbol{u})$. Then $\boldsymbol{s}_1$ and $\boldsymbol{s}_2$ are defined as $\boldsymbol{s}_1 = \boldsymbol{h}_1 - \boldsymbol{h}_1^0$ and $\boldsymbol{s}_2 = \boldsymbol{h}_2 - \boldsymbol{h}_2^0$, respectively. By using these matrices and vectors and from (52), we obtain $\bar{\boldsymbol{u}}^r = \bar{\boldsymbol{u}}_{\rm a}^s - \bar{\boldsymbol{u}}_{\rm b}^s$ and $\boldsymbol{x}^{r0} = \boldsymbol{x}_{\rm a}^{s0} - \boldsymbol{x}_{\rm b}^{s0}$. Then, two minimization problems for the force method are derived as

$$\left.\begin{array}{lll} (\Pi_{\rm R}^{\rm C}): & \text{Minimize} & \displaystyle\sum_{i=1}^{2} \frac{q_i^2}{2k_i} + (l_1^0 q_1 + l_2^0 q_2) - (\bar{\boldsymbol{x}}_{\rm a} - \bar{\boldsymbol{x}}_{\rm b})^\top \boldsymbol{r} - \bar{\boldsymbol{f}}^\top (\boldsymbol{h}_1^0 - \bar{\boldsymbol{u}}_{\rm a}^s) \\ & \text{subject to} & q_1 = \|\boldsymbol{r} - \bar{\boldsymbol{f}}\|, \\ & & q_2 = \|\boldsymbol{r}\|; \end{array}\right\} \tag{64}$$

$$\left.\begin{array}{lll} (\Pi_{\rm R}): & \text{Minimize} & \displaystyle\sum_{i=1}^{2} w_i^s(\boldsymbol{s}_i) - \bar{\boldsymbol{f}}^\top (\boldsymbol{s}_1 - \bar{\boldsymbol{u}}_{\rm a}^s) \\ & \text{subject to} & \boldsymbol{s}_1 + \boldsymbol{s}_2 + \bar{\boldsymbol{u}}_{\rm a}^s - \bar{\boldsymbol{u}}_{\rm b}^s = 0. \end{array}\right\} \tag{65}$$

It can be seen that the linear constraints in $(\Pi_{\rm R})$ are compatibility conditions. The number of variables of $(\Pi_{\rm R})$ is four. However, the number of independent variables is two, which is equal to that of (Π). $(\Pi_{\rm R}^{\rm C})$ has only two variables, and we can see that $(\Pi_{\rm R}^{\rm C})$ has the same mathematical structure as that of (Π). It is also interesting to note that the objective functions in (Π) and $(\Pi_{\rm R})$ include the function w_i and w_i^s, which take vanishing value for a slackening cable member. On the contrary, in the complementary principles $(\Pi^{\rm C})$ and $(\Pi_{\rm R}^{\rm C})$, there exists no term explicitly indicating such stress-unilateral behavior.

10 Existence of the solution.

We make the following assumptions:

Assumption 10.1. *Suppose that the cable network $\mathcal{C}$ satisfies the conditions such that*

(i) *there is no more than one member which has the same adjacency (there does not exist multiple edges);*

(ii) *each member connects two different nodes (no loop);*

(iii) *there exists no member connected to two pin supports;*

(iv) *there is a path between any pair of nodes (connected);*

(v) *at least one freedom of displacement with respect to each axis direction x_1, x_2 and x_3 is constrained; i.e., $\mathcal{C}$ is not a free-body with respect to each axis.*

Note that the most of actual cable networks satisfy Assumption 10.1. By applying Theorem 4.2 to (P) and $({\rm P}^{\rm C})$, the existence of solutions of the equilibrium configuration is guaranteed if there exists $\boldsymbol{v}$ satisfying (24). We shall investigate the conditions such that $\mathcal{C}$ has a solution $\boldsymbol{v}$ to (24) with respect to any external load $\bar{\boldsymbol{f}} \in \Re^{Nd}$.

Definition 10.2. *The unconstrained cable network $\mathcal{C}^*$ is obtained by removing all the constraints of displacements from $\mathcal{C}$.*

It follows that $\mathcal{C}^*$ has no support, and the number of degrees of freedom is $3N^a$. Obviously, $\mathcal{C}^*$ is uniquely determined from $\mathcal{C}$. Each node and member of $\mathcal{C}^*$ may be regarded as vertex and edge, respectively, of a graph $\mathcal{G}(\mathcal{C}^*)$ (See, e.g., [36] for basic background of graph theory). It follows from Assumption 10.1 that $\mathcal{G}(\mathcal{C}^*)$ is a connected simple directed graph. Let $\boldsymbol{D}^* \in \Re^{N^a \times N^m}$ denote the incidence matrix of $\mathcal{G}(\mathcal{C}^*)$.

Suppose $\mathcal{C}$ can be obtained by adding constraints at the nodes in the set $\mathcal{J}_u^k$ of $\mathcal{C}^*$ in the direction of x_k $(k = 1, 2, 3)$. Note that $|\mathcal{J}_u^k| = N^a - N_k^d$ where N_k^d denote the numbers of degrees of freedom of $\mathcal{C}$ in the direction x_k $(k = 1, 2, 3)$. Assumption 10.1 (v) implies $|\mathcal{J}_u^k| \geq 1$.

The equilibrium equation (24) can be reformulated into independent three sets of equations. Let $\boldsymbol{v}^k \in \Re^{N^m}$ denote the vector composed of the components of $\boldsymbol{v}$ in x_k-direction. Similarly, the external load vector $\bar{\boldsymbol{f}}$ is divided into the set of vectors $\bar{\boldsymbol{f}}^k \in \Re^{N_k^d}$. Define the matrices $\boldsymbol{D}^k \in \Re^{N_k^d \times N^m}$ by removing all the jth columns in $\mathcal{J}_u^k$ from $\boldsymbol{D}^*$, and then we see that (24) is equivalent to the following system of equations:

$$\boldsymbol{D}^k \boldsymbol{v}^k + \bar{\boldsymbol{f}}^k = \boldsymbol{0}, \quad (k = 1, 2, 3). \tag{66}$$

Lemma 10.3. *If $\mathcal{C}$ satisfies Assumption 10.1, then* $\operatorname{rank} \boldsymbol{D}^k = N_k^d$.

Proof. It follows from $|\mathcal{J}_u^k| \geq 1$ that we can choose $j_1 \in \mathcal{J}_u^k$. Letting $\boldsymbol{D}(j_1) \in \Re^{(N^a-1) \times N^m}$ denote the matrix obtained by removing the j_1th row from $\boldsymbol{D}^*$, which is referred to as a *truncated incidence matrix* and $\operatorname{rank} \boldsymbol{D}(j_1) = N^a - 1$ [37, Ch.19]. Since each row of $\boldsymbol{D}^k$ is a row of $\boldsymbol{D}(j_1)$, all rows of $\boldsymbol{D}^k$ are linearly independent. □

Lemma 10.4. *There exists $\boldsymbol{v}$ satisfying the equilibrium conditions of $\mathcal{C}$ for any $\bar{\boldsymbol{f}}$.*

Proof. As a result of linear algebra [38, Theorem 2Q], the necessary and sufficient condition for the existence of a solution $\boldsymbol{v}^k$ to the equation (66) for any $\bar{\boldsymbol{f}}^k \in \Re^{N_k^d}$ is $\operatorname{rank} \boldsymbol{D}^k = N_k^d$, which has been shown in Lemma 10.3. □

Suppose that $\mathcal{C}$ does not satisfy Assumption 10.1 (v), which implies $\boldsymbol{D}^k = \boldsymbol{D}^*$ $(\exists k)$. Then (66) does not always have a solution because $\operatorname{rank} \boldsymbol{D}^* = N^a - 1$, which implies the necessity of Assumption 10.1 (v) so as to satisfy Lemma 10.4.

From Corollary 4.2 and Lemma 10.4, the existence of an equilibrium configuration $(\boldsymbol{u}^*, \boldsymbol{y}^*)$, $(\boldsymbol{q}^*, \boldsymbol{v}^*)$ of $\mathcal{C}$ has been shown for any external loads. The uniqueness of the solution will be discussed in the following section.

11 Uniqueness of the solution.

Consider (Π^{C}) for the structure $\mathcal{C}$ satisfying Assumption 10.1. Because of the strict convexity of $\phi_i^C(q_i)$ (Corollary 3.6 (ii)) and $\|\boldsymbol{v}_i\|$, $w_i^C(\boldsymbol{v}_i)$ defined by (28) is also a strictly convex function of $\boldsymbol{v}_i$. Hence $\Pi^C(\boldsymbol{v})$ is a strictly convex function of $\boldsymbol{v}_i$ $(i = 1, \cdots, N^m)$.

Since (Π^{C}) consists of the linear constraint and the strictly convex objective function, the optimal solution $\boldsymbol{v}^{\mathrm{II}}$ is unique if it exists. From Lemma 4.1, q_i^* is also unique. It follows from Corollary 3.5 and (23) that y_i^* obtained as $y_i^* = \psi_i^{-1}(q_i^*)$ is also unique. However, the inversion of relation (18) is not unique for $y_i^* = 0$, then $\varepsilon_i^{\mathrm{II}}$ is not always unique, which implies $\boldsymbol{u}^{\mathrm{II}}$ is sometimes not unique in the case where some cable members are not in tension.

It follows from the definition as listed in Table 1 that $\|\boldsymbol{h}_i^{\mathrm{II}}\| = \varepsilon_i^{\mathrm{II}} + l_i^0$. Suppose the ith member is in tensile state; i.e., $\boldsymbol{v}_i^{\mathrm{II}} \neq \boldsymbol{0}$. From (31), we obtain

$$\boldsymbol{h}_i^{\mathrm{II}} = -(\varepsilon_i^{\mathrm{II}} + l_i^0)\frac{\boldsymbol{v}_i^{\mathrm{II}}}{\|\boldsymbol{v}_i^{\mathrm{II}}\|}. \tag{67}$$

Since $\varepsilon_i^{\mathrm{II}}$ is uniquely determined from (18) for $y_i^* > 0$, (67) guarantees the uniqueness of $\boldsymbol{h}_i^{\mathrm{II}}$. Hence, $\boldsymbol{v}_i^{\mathrm{II}} \neq \boldsymbol{0}$ is a sufficient condition for the uniqueness of the configuration of the member at Γ^{II}.

We shall investigate a sufficient condition for the uniqueness of the nodal displacements $\boldsymbol{u}^{\mathrm{II}}$. The uniqueness of infinitesimal increment of displacements was investigated by Panagiotopoulos [18]. However, to the authors' knowledge, no study has been published concerning the uniqueness of equilibrium configuration allowing large deformation. Consider $\overline{\mathcal{C}}$ obtained by removing all the members satisfying $\boldsymbol{v}_i^{\mathrm{II}} = \boldsymbol{0}$ from $\mathcal{C}$. Because all the remaining members are in tension at Γ^{II}, the equilibrium configuration of $\overline{\mathcal{C}}$ is unique. The unconstrained cable network $\overline{\mathcal{C}}^*$ can be determined for $\overline{\mathcal{C}}$ by using the Definition 10.2. Note that $\mathcal{G}(\overline{\mathcal{C}}^*)$ is a subgraph of $\mathcal{G}(\mathcal{C}^*)$. Suppose that $\mathcal{G}(\overline{\mathcal{C}}^*)$ has a subgraph which is a spanning tree of $\mathcal{G}(\mathcal{C}^*)$. Then, $\overline{\mathcal{C}}$ has all the nodes of $\mathcal{C}$, which implies that the equilibrium configuration $\boldsymbol{u}^{\mathrm{II}}$ of $\mathcal{C}$ is unique and the equilibrium state is stable. In particular, $\boldsymbol{u}^{\mathrm{II}}$ is unique if all the members of $\mathcal{C}$ are in tension.

Note that the total potential energy of $\mathcal{C}$ can be rewritten only in terms of $\boldsymbol{u}$, which is a continuous function. Then, the tangent stiffness matrix is obtained as the Hessian of the total potential energy. Therefore, if $\boldsymbol{u}^{\mathrm{II}}$ is a unique minimizer, the tangent stiffness matrix at $\boldsymbol{u}^{\mathrm{II}}$ is guaranteed to be positive semidefinite. This implies that our result contains that of Volokh and Vilnay [39] which stated that the tangent stiffness matrix is positive definite if all the members are in tension.

12 Conclusions.

The minimum principle of complementary energy has been established for cable networks in geometrically nonlinear elasticity theory with piecewise linear constitutive relation.

The minimum principle of total potential energy has been first formulated under the assumption of large rotation and small strain, and its convex formulation (P) has been presented. Then the dual of problem (P^{C}) is derived, whose optimizer corresponds to the set of internal force vectors and axial forces at the equilibrium state. The principle of minimum complementary energy has been obtained from (P^{C}) involving only the internal force vectors as variables. It should be emphasized that the presented principle of minimum complementary energy is formulated purely in terms of generalized stress components. The alternative reciprocal formulations of potential and complementary energy theorem that are to be used for formulating the force method have also been presented, and the relationship of these four principles has been discussed.

The strong duality between (P) and (P^{C}) has been shown, which guarantees that the objective function of the derived dual principle is the actual complementary energy function. The compatibility of the directions between stress and strain components are guaranteed by the optimality conditions. It is known that, in general, the complementary energy function expressed only stress components cannot be uniquely determined, and minimum principle cannot be established even if the structure is stable. On the contrary, it has been shown in this paper that cable networks have the unique complementary energy function and the minimum principle of complementary energy

can be established irrespective to the stability of the equilibrium state. Therefore the presented principles may be actually useful in practical application such as the force method.

The existence and uniqueness of the solution to the minimum complementary energy have been investigated. Based on the graph theory, we have presented the sufficient condition for existence of the solution to the problem of minimum complementary energy allowing large deformation. This condition is related to the topology and the support condition of cable networks, but is independent of the geometry of structures and the property of external loads. From the strict convexity of the proposed complementary energy function, the sufficient condition for the uniqueness of equilibrium configuration of a cable network has also been presented, which is also independent of its geometry and magnitudes of axial forces.

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A Proof of Lemma 3.9.

We start with proving 'if' part of Lemma 3.9. It follows from (9), (18) and Corollary 3.6 (iii) that $\phi_i(y_i^*) = w_i(\varepsilon_i^{\mathrm{II}})$, which implies

$$\Phi(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}}) = \Pi(\boldsymbol{u}^{\mathrm{II}}). \tag{68}$$

For $\hat{\boldsymbol{u}} \in \Re^{Nd}$, define $\hat{\boldsymbol{\varepsilon}} = (\hat{\varepsilon}_i) \in \Re^{Nm}$ and $\hat{\boldsymbol{y}}^* = (\hat{y}_i^*) \in \Re^{Nm}$ as

$$\hat{\varepsilon}_i = \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \hat{\boldsymbol{u}}) - \hat{\boldsymbol{x}}_i^0\| - l_i^0, \quad \hat{y}_i^* = \begin{cases} \hat{\varepsilon}_i & (\hat{\varepsilon}_i \geq 0), \\ 0 & (-l_i^0 \leq \hat{\varepsilon}_i < 0). \end{cases} \tag{69}$$

Note that $(\hat{\boldsymbol{\varepsilon}}, \hat{\boldsymbol{u}})$ is a feasible solution of (Π). In the same manner as (68), we obtain

$$\Phi(\hat{\boldsymbol{y}}^*, \hat{\boldsymbol{u}}) = \Pi(\hat{\boldsymbol{u}}). \tag{70}$$

Because $(\hat{\boldsymbol{y}}^*, \hat{\boldsymbol{u}})$ is a feasible solution of (P),

$$\Phi(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}}) \leq \Phi(\hat{\boldsymbol{y}}^*, \hat{\boldsymbol{u}}) \tag{71}$$

is satisfied. By substituting (68) and (70) into (71), we see that $\Pi(\boldsymbol{u}^{\mathrm{II}}) \leq \Pi(\hat{\boldsymbol{u}})$ holds for any $\hat{\boldsymbol{u}} \in \Re^{N^d}$, which implies that $(\boldsymbol{\varepsilon}^{\mathrm{II}}, \boldsymbol{u}^{\mathrm{II}})$ is an optimal solution of (Π).

For 'only if' part, we have $\Pi(\boldsymbol{u}^{\mathrm{II}}) \leq \Pi(\hat{\boldsymbol{u}})$ because $\boldsymbol{u}^{\mathrm{II}}$ is an optimal solution of (Π), and

$$\Phi(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}}) \leq \Phi(\hat{\boldsymbol{y}}^*, \hat{\boldsymbol{u}}) \tag{72}$$

is derived by using (68) and (70) again. From Corollary 3.6, we obtain

$$\min\{\phi_i(y_i) |\ y_i \geq \hat{\varepsilon}_i\} = \begin{cases} \phi_i(\hat{\varepsilon}_i) & (\hat{\varepsilon}_i \geq 0), \\ \phi_i(0) = 0 & (-l_i^0 \leq \hat{\varepsilon}_i < 0). \end{cases}$$

Hence, $\phi_i(\hat{y}_i^*) \leq \phi_i(\hat{y}_i)$ is satisfied for any $\hat{y}_i \geq \hat{\varepsilon}_i$, which implies that

$$\Phi(\hat{\boldsymbol{y}}^*, \hat{\boldsymbol{u}}) \leq \Phi(\hat{\boldsymbol{y}}, \hat{\boldsymbol{u}}) \tag{73}$$

holds for any feasible solution $(\hat{\boldsymbol{y}}, \hat{\boldsymbol{u}})$ of (P). It follows from (72) and (73) that $\Phi(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}}) \leq \Phi(\hat{\boldsymbol{y}}, \hat{\boldsymbol{u}})$, which implies that $(\boldsymbol{y}^*, \boldsymbol{u}^{\mathrm{II}})$ is an optimal solution of (P).

B Derivation of (25).

By substituting (23) and (24) to (19), we obtain

$$L^D(\boldsymbol{q}, \boldsymbol{v}) = \sum_{i=1}^{N^m} \Big(\phi_i - \frac{d\phi_i}{dy_i} y_i \Big) - \sum_{i=1}^{N^m} (l_i^0 q_i + (\boldsymbol{B}_i \boldsymbol{x}^0)^\top \boldsymbol{v}_i) + \sum_{i=1}^{N^m} \bar{\boldsymbol{u}}^\top \boldsymbol{v}_i. \tag{74}$$

From (7) and (8),

$$\phi_i(y_i) + \phi_i^C(q_i) = \int_0^{y_i} \psi_i(y_i) dy_i + \int_0^{q_i} \psi_i^{-1}(q_i) dq_i. \tag{75}$$

It follows from Corollary 3.5, the second term in the right-hand side of (75) can be converted into the integral by the variable $y_i = \psi_i^{-1}(q_i)$; i.e.,

$$\begin{aligned} \phi_i(y_i) + \phi_i^C(q_i) &= \int_0^{y_i} \Big(\psi_i(y_i) + y_i \frac{d\psi_i(y_i)}{dy_i} \Big) dy_i \\ &= \int_0^{y_i} \frac{d}{dy_i} (y_i \psi_i(y_i)) dy_i \\ &= y_i \psi_i(y_i) \\ &= \frac{d\phi_i}{dy_i} y_i. \end{aligned} \tag{76}$$

By using (76), (74) is reduced to (25).

C Standard forms of $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$.

It will be shown that $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ are reduced to $\mathcal{P}_{\rm SOCP}$ and $\mathcal{D}_{\rm SOCP}$ which make a primal and dual pair of SOCP. For simplicity, however, we restrict ourselves to the case of Example 3.7; i.e.,

$$S_i = \Big\{(t_i, y_i)\Big|\ t_i \geq \frac{k_i}{2} y_i^2\Big\}. \tag{77}$$

By writing (77) in a form of the second-order cone, $\overline{\mathcal{P}}$ is reduced to

$$\left.\begin{array}{lll} \overline{\mathcal{P}}': & \text{Minimize} & \overline{\Phi} = \displaystyle\sum_{i=1}^{N^m} t_i - \bar{\boldsymbol{f}}^\top \boldsymbol{u} \\ & \text{subject to} & y_i + l_i^0 \geq \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i\|, \quad (i = 1, \cdots, N^m), \\ & & \dfrac{t_i}{2k_i} + 1 \geq \Big[\Big(\dfrac{t_i}{2k_i} - 1\Big)^2 + y_i^2\Big]^{1/2}, \quad (i = 1, \cdots, N^m). \end{array}\right\} \tag{78}$$

Let $\boldsymbol{\theta}^i = (\theta_j^i) \in \Re^{N^m}$ denote the vector such that

$$\theta_j^i = \begin{cases} 1 & (j = i), \\ 0 & (\text{otherwise}). \end{cases}$$

Define $\mathbf{0} = (0, \cdots, 0) \in \Re^{N^m}$ and $\mathbf{1} = (1, \cdots, 1) \in \Re^{N^m}$. Let

$$\boldsymbol{w} = (\boldsymbol{t}, \boldsymbol{y}, \boldsymbol{u}) \in \Re^{2N^m + N^d}, \tag{79}$$

$$\boldsymbol{b} = (\mathbf{1}, \mathbf{0}, -\bar{\boldsymbol{f}}) \in \Re^{2N^m + N^d}, \tag{80}$$

then we can see that $\overline{\Phi} = \boldsymbol{b}^\top \boldsymbol{w}$.

To embed the constraints $y_i + l_i^0 \geq \|\boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i\|$ of $\overline{\mathcal{P}}'$ into those of $\mathcal{P}_{\rm SOCP}$, define

$$\boldsymbol{A}_i = \begin{bmatrix} \mathbf{0}^\top & \boldsymbol{\theta}^{i\top} & \mathbf{0}^\top \\ \mathbf{O} & \mathbf{O} & \boldsymbol{B}_i \end{bmatrix} \in \Re^{4 \times (2N^m + N^d)}, \quad \boldsymbol{c}_i = (-l_i^0, -\boldsymbol{B}_i \boldsymbol{x}^0 + \bar{\boldsymbol{x}}_i) \in \Re^4, \quad (i = 1, \cdots, N^m). \tag{81}$$

It follows from the equality constraints in $\mathcal{P}_{\rm SOCP}$ that $\boldsymbol{x}_i = \boldsymbol{A}_i \boldsymbol{w} - \boldsymbol{c}_i = (y_i + l_i^0, \boldsymbol{B}_i(\boldsymbol{x}^0 + \boldsymbol{u}) - \bar{\boldsymbol{x}}_i)$. Then, we can see that $x_{i0} \geq \|\boldsymbol{x}_{i1}\|$ is equivalent to the first N^m inequality constraints in $\overline{\mathcal{P}}'$, which validates that the definitions of $\boldsymbol{A}_i$ and $\boldsymbol{c}_i$ are correct.

For the remaining inequality constraints of $\overline{\mathcal{P}}'$, define

$$\boldsymbol{A}_{N^m+i} = \begin{bmatrix} \dfrac{1}{2k_i}\boldsymbol{\theta}^{i\top} & \mathbf{0}^\top & \mathbf{0}^\top \\ \dfrac{1}{2k_i}\boldsymbol{\theta}^{i\top} & \mathbf{0}^\top & \mathbf{0}^\top \\ \mathbf{0}^\top & \boldsymbol{\theta}^{i\top} & \mathbf{0}^\top \end{bmatrix} \in \Re^{3 \times (2N^m + N^d)}, \quad \boldsymbol{c}_{N^m+i} = (-1, 1, 0) \in \Re^3, \quad (i = 1, \cdots, N^m). \tag{82}$$

From (79), we obtain

$$\boldsymbol{x}_{N^m+i} = \boldsymbol{A}_{N^m+i}\boldsymbol{w} - \boldsymbol{c}_{N^m+i} = \left\{\begin{array}{c} \dfrac{t_i}{2k_i} + 1 \\ \dfrac{t_i}{2k_i} - 1 \\ y_i \end{array}\right\}, \quad (i = 1, \cdots, N^m),$$

and it can be seen that $x_{(N^m+i)0} \geq \|\boldsymbol{x}_{(N^m+i)1}\|$ is equivalent to the last inequality constraint in $\overline{\mathcal{P}}'$. Thus, $\overline{\mathcal{P}}$ can be embedded into the standard form of SOCP.

For $\mathcal{D}_{\rm SOCP}$, define the dual variable vector $\boldsymbol{z} \in \Re^{7N^m}$ as

$$\boldsymbol{z}_i = (q_i, \boldsymbol{v}_i) \in \Re^4, \quad (i = 1, \cdots, N^m), \tag{83}$$

$$\boldsymbol{z}_i = (z_{i0}, \boldsymbol{z}_{i1}) = (z_{i0}, z_{i1}, z_{i2}) \in \Re^3, \quad (i = N^m + 1, \cdots, 2N^m). \tag{84}$$

From (81) and (83), we obtain

$$\boldsymbol{A}_i^\top \boldsymbol{z}_i = \left\{ \begin{array}{c} \boldsymbol{0} \\ q_i \boldsymbol{\theta}^i \\ \boldsymbol{B}_i^\top \boldsymbol{v}_i \end{array} \right\} \in \Re^{3 \times 2N^m + N^d}, \quad \boldsymbol{c}_i^\top \boldsymbol{z}_i = -l_i^0 q_i - (\boldsymbol{B}_i \boldsymbol{x}^0)^\top \boldsymbol{v}_i + \bar{\boldsymbol{x}}_i^\top \boldsymbol{v}_i. \tag{85}$$

By using (82) and (84), we obtain

$$\begin{aligned} \boldsymbol{A}_{N^m+i}^\top \boldsymbol{z}_{N^m+i} &= \left\{ \begin{array}{c} \dfrac{z_{(N^m+i)0} + z_{(N^m+i)1}}{2k_i} \boldsymbol{\theta}^i \\ z_{(N^m+i)2} \boldsymbol{\theta}^i \\ \boldsymbol{0} \end{array} \right\} \quad (i = 1, \cdots, N^m), \\ \boldsymbol{c}_i^\top \boldsymbol{z}_i &= -z_{i0} + z_{i1}, \quad (i = N^m + 1, \cdots, 2N^m). \end{aligned} \tag{86}$$

It follows from (80), (85)and (86) that the system of equations $\sum_{i=1}^{2N^m} \boldsymbol{A}_i^\top \boldsymbol{z}_i = \boldsymbol{b}$ is reduced to

$$\sum_{i=1}^{N^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}, \tag{87}$$

$$\frac{z_{(N^m+i)0} + z_{(N^m+i)1}}{2k_i} = 1, \quad (i = 1, \cdots, N^m), \tag{88}$$

$$q_i + z_{(N^m+i)2} = 0, \quad (i = 1, \cdots, N^m). \tag{89}$$

From (88), $z_{(N^m+i)0}$ and $z_{(N^m+i)1}$ are rewritten by introducing new variable t_i^C as

$$z_{(N^m+i)0} = \frac{t_i^C}{2} + k_i, \quad z_{(N^m+i)1} = -\frac{t_i^C}{2} + k_i. \tag{90}$$

By using (90), we obtain $\boldsymbol{c}_{N^m+i}^\top \boldsymbol{z}_{N^m+i} = -t_i^C$. Then, the dual problem of $\overline{\mathcal{P}}'$ in the standard form is obtained as $\mathcal{D}_{\rm SOCP}$; i.e.,

$$\left. \begin{array}{lll} \overline{\mathcal{D}}': & \text{Maximize} & -\displaystyle\sum_{i=1}^{N^m} t_i^C - \sum_{i=1}^{N^m} (l_i^0 q_i + (\boldsymbol{B}_i \boldsymbol{x}^0)^\top \boldsymbol{v}_i) + \sum_{i=1}^{N^m} \bar{\boldsymbol{x}}_i^\top \boldsymbol{v}_i \\ & \text{subject to} & \displaystyle\sum_{i=1}^{N^m} \boldsymbol{B}_i^\top \boldsymbol{v}_i + \bar{\boldsymbol{f}} = \boldsymbol{0}, \quad q_i + z_{i2} = 0, \\ & & q_i \geq \|\boldsymbol{v}_i\|, \quad z_{i0} \geq \|\boldsymbol{z}_{i1}\|, \quad (i = 1, \cdots, N^m). \end{array} \right\} \tag{91}$$

Note again that $\overline{\mathcal{P}}'$ and $\overline{\mathcal{D}}'$ make a pair of primal and dual SOCP problems. By using (89) and (90), the second-order cone $z_{i0} \geq \|\boldsymbol{z}_{i1}\|$ is reduced to

$$t_i^C \geq \frac{1}{2k_i} q_i^2,$$

which validates that $\overline{\mathcal{D}}'$ is equivalent to $\overline{\mathcal{D}}$ with

$$S_i^C = \left\{ (t_i^C, q_i) \middle| \; t_i^C \geq \frac{1}{2k_i} q_i^2 \right\}.$$

Therefore, $\overline{\mathcal{P}}$ and $\overline{\mathcal{D}}$ make a pair of primal and dual SOCP problems.

Note that the derivation of $\overline{\mathcal{D}}$ shown above is restricted to the case of a linear elastic material. It seems to be possible to obtain ($\mathrm{P^C}$) by converting (P) to $\overline{\mathcal{P}}'$, deriving the dual $\overline{\mathcal{D}}'$, and reducing $\overline{\mathcal{D}}'$ to ($\mathrm{P^C}$). In this context, the most of discussions in Section 5 may not be necessary because the duality between $\overline{\mathcal{P}}'$ and $\overline{\mathcal{D}}'$ is immediately guaranteed by Theorem 2.1. However, the transformation of (90) is not obvious, or may be impossible, for general cases such that $\psi_i(y_i)$ is a piecewise linear function. Therefore, it may be also complicated to reduce $\overline{\mathcal{D}}'$ into ($\mathrm{P^C}$). On the contrary, the Lagrangian in (19) was defined quite naturally based on the Lagrangian duality theory and the self-duality of second-order cone, and ($\mathrm{P^C}$) was obtained only by very simple operations.