MAGIC CUBES

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Since antiquity mathematicians (and not only them) have taken interest in constructing magic squares. Probably the first magic square ever created is the one shown in Fig.1. Its origin is shrouded in the mystical legends of ancient China. It became to be known as *Luo Shu (Luo river writing)*. There was no clear connection between this configuration and mathematical study until the time of *Yang Hui*, even though it was described in the sixth century. Another very famous magic square (Fig. 2) is in the paiting *Melancholy* ([3], p. 6) made by the famous renaissance artist *Albrecht Dürer* in 1514 (the year is formed in the middle of the lowest row).

4	9	2
3	5	7
8	1	6

FIGURE 1

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

FIGURE 2

A *magic square* of order n is a square matrix (square table) of order n containing natural numbers $1, 2, 3, \ldots, n^2$ such that the sum of the numbers along any row, column, or main diagonal is a fixed constant. It is easy to see that this constant in a square of order n is $\frac{n(n^2+1)}{2}$.

In [3] and elsewhere we can find constructions of magic squares of order n for all natural numbers $n \neq 2$.

A generalization of magic squares are magic cubes. In Fig. 3 a magic cube of order 4 is depicted.

Definition. A *magic cube* of order n is a 3-dimensional matrix

$$\mathbf{Q}_n = [\mathbf{q}(i,j,k);\ 1 \leq i,j,k \leq n]$$

containing natural numbers $1, 2, 3, \ldots, n^3$ such that

$$\sum_{x=1}^{n} \mathbf{q}(x,j,k) = \sum_{x=1}^{n} \mathbf{q}(i,x,k) = \sum_{x=1}^{n} \mathbf{q}(i,j,x) = \frac{n(n^3+1)}{2} \qquad \text{for all} \quad i,j,k = 1,\ldots,n$$

Every element of a magic cube is uniquely determined by a triple of numbers (i,j,k) called coordinates of the element.

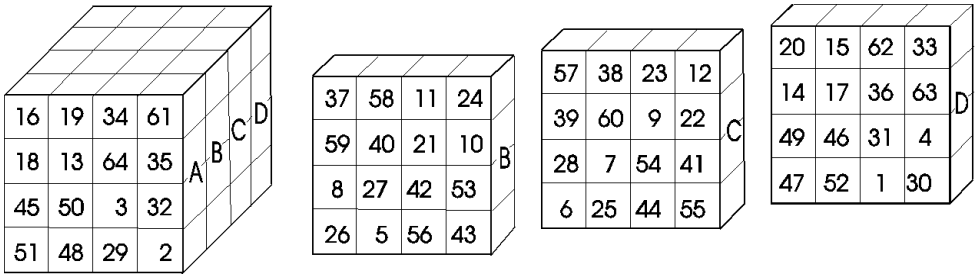

FIGURE 3

A magic square of order 1 is a magic cube of order 1. Because a magic square of order 2 does not exist a magic cube of order 2 does not exist either.

In this paper we prove the following theorem:

Theorem. *For every natural number $n \neq 2$ there exists a magic cube of order n.*

Before we prove this theorem we will consider Latin squares.

A *Latin square* $\mathbf{R}_n = [\mathbf{r}(i,j);\ 1 \leq i,j \leq n]$ of order n is a square matrix of order n such that every row and column is a permutation of the set of natural numbers $\{1,2,\ldots,n\}$. Two Latin squares $\mathbf{R}_n = [\mathbf{r}(i,j)]$ a $\mathbf{S}_n = [\mathbf{s}(i,j)]$ of order n are called *orthogonal*, if an n^2 ordered pairs $[\mathbf{r}(i,j), \mathbf{s}(i,j)]$, where $i,j \in \{1,2,\ldots,n\}$, are pairwise different.

Two orthogonal Latin squares of order 5 are depicted in Fig.4.

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

FIGURE 4

If you look carefully on this pair of orthogonal Latin squares you can easily find the rule for constructing any pair of orthogonal squares for every odd natural number n. This fact was already known to *L.Euler* at the beginning of 18th century. But it was not until 1960 when *R.C.Bose, S.S.Shrikhande a E.T.Parker* proved that two orthogonal Latin squares of order n exist if and only if $n \neq 2,6$. We use this statement to prove our theorem.

Proof of the Theorem. Let $\mathbf{R}_n = [\mathbf{r}(i,j); 1 \leq i,j \leq n]$ and $\mathbf{S}_n = [\mathbf{s}(i,j); 1 \leq i,j \leq n]$ be two orthogonal Latin squares of order n and $\mathbf{M}_n = [\mathbf{m}(i,j); 1 \leq i,j \leq n]$ be a magic square of order n.

Let us define a magic cube $\mathbf{Q}_n = [\mathbf{q}(i,j,k); 1 \leq i,j,k \leq n]$ in the following way:

$$\mathbf{q}(i,j,k) = [\mathbf{s}(i,\mathbf{r}(j,k)) - 1]n^2 + \mathbf{m}(i,\mathbf{s}(j,k)) \qquad \text{for all } 1 \leq i,j,k \leq n$$

We will prove, in four steps, that $\mathbf{Q}_n$ is a magic cube.

1. All elements of squares $\mathbf{R}_n$ a $\mathbf{S}_n$ are from the set $\{1,2,\ldots,n\}$ and all elements of $\mathbf{M}_n$ are from the set $\{1,\ldots,n^2\}$. It follows immediately that for all elements $\mathbf{Q}_n$ we have $1 \le \mathbf{q}(i,j,k) \le n^3$.

2. In this part we prove that no two elements of $\mathbf{Q}_n$ with different indices are identical. Let us suppose $\mathbf{q}(i,j,k) = \mathbf{q}(i',j',k')$. We show that this implies equalities

$$i = i', \quad j = j', \quad k = k'$$

From the definition of the $\mathbf{Q}_n$ it follows

$$[\mathbf{s}(i,\mathbf{r}(j,k)) - 1]n^2 + \mathbf{m}(i,\mathbf{s}(j,k)) = [\mathbf{s}(i',\mathbf{r}(j',k')) - 1]n^2 + \mathbf{m}(i',\mathbf{s}(j',k'))$$

By rearranging this equality we get

$$-[\mathbf{s}(i,\mathbf{r}(j,k)) - \mathbf{s}(i',\mathbf{r}(j',k'))]n^2 = \mathbf{m}(i,\mathbf{s}(j,k)) - \mathbf{m}(i',\mathbf{s}(j',k')) \tag{1}$$

Because all the elements of $\mathbf{M}_n$ are from the interval $< 1, n^2 >$, their difference is not a non-zero multiple of n^2. On the left is a whole multiple of n^2. Equality in (1) arises if $\mathbf{s}(i,\mathbf{r}(j,k)) - \mathbf{s}(i',\mathbf{r}(j',k')) = 0$. Hence

$$\mathbf{s}(i,\mathbf{r}(j,k)) = \mathbf{s}(i',\mathbf{r}(j',k')) \tag{2}$$

$$\mathbf{m}(i,\mathbf{s}(j,k)) = \mathbf{m}(i',\mathbf{s}(j',k')) \tag{3}$$

In a magic square no two elements are identical. If two elements of $\mathbf{M}_n$ are equal, then both their indices are the same, e.g. from (3) we have

$$i = i', \qquad \mathbf{s}(j,k) = \mathbf{s}(j',k')$$

By substitution $i' = i$ in (2) we get $\mathbf{s}(i,\mathbf{r}(j,k)) = \mathbf{s}(i,\mathbf{r}(j',k'))$. Because $\mathbf{S}_n$ is a Latin square, from the equality of the first indices the equality of the second indices follows, the following is true: $\mathbf{r}(j,k) = \mathbf{r}(j',k')$.

From the assumption that $\mathbf{S}_n, \mathbf{R}_n$ are orthogonal Latin squares and from the relation

$$[\mathbf{s}(j,k), \mathbf{r}(j,k)] = [\mathbf{s}(j',k'), \mathbf{r}(j',k')]$$

we get $j = j', \quad k = k'$.

3. Next we prove that the sum of numbers in every row is the same.

$$\sum_{x=1}^{n} \mathbf{q}(x,j,k) = \sum_{x=1}^{n} [\mathbf{s}(x,\mathbf{r}(j,k)) - 1]n^2 + \sum_{x=1}^{n} \mathbf{m}(x,\mathbf{s}(j,k)) =$$

$$= [\tfrac{n(n+1)}{2} - n]n^2 + n.\tfrac{n^2+1}{2} = n\tfrac{n^3+1}{2}$$

Similarly we get

$$\sum_{x=1}^{n} \mathbf{q}(i,x,k) = \sum_{x=1}^{n} \mathbf{q}(i,j,x) = n\tfrac{n^3+1}{2}$$

4. The above construction of magic squares is based on the use of two orthogonal Latin squares and therefore is not valid for $n = 6$. The magic cube $\mathbf{Q}_6$ can be found in [4].

Remark 1. If we found out how to construct a pair of orthogonal Latin squares for odd n, then, from the following, we can easily make a computer program which constructs magic cubes for every odd n. You can substitute a magic square $\mathbf{M}_n$ by the square $\mathbf{M}_n^* = [\mathbf{m}^*(i,j)]$, where $\mathbf{m}^*(i,j) = [\mathbf{r}(i,j)-1]n^2 + \mathbf{s}(i,j)$ for all $1 \le i,j \le n$. In this square the sums on the diagonals need not be the same, therefore we do not use it.

Remark 2. The construction of the magic cube of order 4, shown in Fig.3, was based on the magic square shown in Fig.2 and from two orthogonal Latin squares shown in Fig.5.

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

FIGURE 5

Remark 3. By a generalization of the presented method magic hypercubes of order n ($n \neq 2, 6$) in the p-dimensional Euclidean space ($p \ge 4$) can be constructed. (See [5].)

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