# Approximation classes for real number optimization problems

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**Abstract.** A fundamental research area in relation with analyzing the complexity of optimization problems are approximation algorithms. For combinatorial optimization a vast theory of approximation algorithms has been developed, see [1]. Many natural optimization problems involve real numbers and thus an uncountable search space of feasible solutions. A uniform complexity theory for real number decision problems was introduced by Blum, Shub, and Smale [4]. However, approximation algorithms were not yet formally studied in their model.
In this paper we develop a structural theory of optimization problems and approximation algorithms for the BSS model similar to the above mentioned one for combinatorial optimization. We introduce a class $NPO_{\mathbb{R}}$ of real optimization problems closely related to $NP_{\mathbb{R}}$. The class $NPO_{\mathbb{R}}$ has four natural subclasses. For each of those we introduce and study real approximation classes $APX_{\mathbb{R}}$ and $PTAS_{\mathbb{R}}$ together with reducibility and completeness notions. As main results we establish the existence of natural complete problems for all these classes.

## 1 Introduction

Many important problems in mathematics and computer science are optimization problems. In the framework of complexity theory a lot of such problems are obtained as optimization versions of $NP$-complete decision problems. Typical examples are the TSP problem, the MAX-SAT problem or the Knapsack problem in its optimization form. Such problems constitute the class $NPO$ of combinatorial optimization problems having an exponential size search space. An important field of complexity theory is dealing with approximation properties for such $NP$-hard optimization problems [1].

Another tradition of approximation algorithms comes from continuous mathematics. A typical example is Newton's method for approximating a zero of a function. Another one is the approximation of the number of solutions of a polynomial system, being extremely important for numerical solution algorithms, see [8] as an example for such investigations.

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A structural complexity theory for real number problems along the lines of the classical study of decision problems was developed in [4], where the BSS model was introduced. It is a hypercomputer in that the possibility to use real number machine constants implies the computability of any function $f : \mathbb{N} \to \mathbb{N}$ in the model. In [4] the authors study a real version $NP_{\mathbb{R}}$ of $NP$ together with an analogue $P_{\mathbb{R}} = NP_{\mathbb{R}}?$ of the famous $P$ versus $NP$ question. A typical $NP_{\mathbb{R}}$-complete problem is QPS which asks for deciding whether a system of quadratic real polynomials has a common real zero. Real number decision problems have been extensively studied in recent years, see [11] and [7] for getting a good impression. Given the immense importance of the theory of approximation algorithms within the Turing model it is tempting to develop a similar framework for real number optimization problems as well. A typical example of such a problem is to approximate the maximal number of commonly solvable real polynomials in a system. This problem has strong relations to a potential real version of the PCP theorem as shown in [9]. Another such problem is to minimize the norm of a root of a given polynomial; good bounds on this minimum (i.e. good approximations) are of huge importance for algorithms dealing with problems in semi-algebraic geometry [2, 10].

In the present paper we start the structural investigation of approximation algorithms for certain classes of real number optimization problems. The idea is to take the classical notions of approximation algorithms, polynomial approximation schemes and AP-reducibility as starting point and transfer them in a suitable way to the real number model. However, in doing so one has to take care of some intrinsic differences. One of the most fundamental such difference is that our optimization problems constituting the real class $NPO_{\mathbb{R}}$ allow for uncountable sets of feasible solutions. This is a quite natural aspect of real number problems, yet it leads to some subtle differences when defining the main concepts below. As example, approximation algorithms will not have to compute a feasible solution with an approximation ratio but only guarantee its existence. This requirement is intrinsically related to the impossibility of computing in general polynomial roots exactly by a BSS algorithm. Another difference will be that $NPO_{\mathbb{R}}$ naturally gives rise to study four subclasses, each representing a different kind of optimization problems. We introduce as well real versions $APX_{\mathbb{R}}, PTAS_{\mathbb{R}}$ of the corresponding discrete classes; they similarly have four natural subclasses.

Our main results are as follows: For each of the four classes that define $NPO_{\mathbb{R}}, APX_{\mathbb{R}}$, and $PTAS_{\mathbb{R}}$ we show the existence of complete problems under a real version of AP-reducibility (to be defined). This holds both for maximization and minimization problems. In addition, some further completeness results for natural optimization problems will be given.

Our hope is that the present approach will serve as starting point for an analysis of approximation algorithms for real number optimization problems as fruitful as the corresponding approach for combinatorial optimization problems.

## 2 Definitions

In the BSS model over $\mathbb{R}$ real numbers are considered as entities. The basic arithmetic operations $+, -, *, /$ can be performed at unit cost and there is a test operation "is $x_i \geq 0$?" reflecting the underlying ordering of the reals. Decision problems are subsets $L \subseteq \mathbb{R}^\infty := \bigcup_{n=1}^{\infty} \mathbb{R}^n$. The (algebraic) size of a point $x \in \mathbb{R}^n$, denoted $|x|$, is $n$. Having fixed these notions it is easy to define real analogues $P_\mathbb{R}$ and $NP_\mathbb{R}$ of the classes $P$ and $NP$ as well as the notion of $NP_\mathbb{R}$-completeness. $DNP_\mathbb{R}$ (Digital $NP_\mathbb{R}$) is the subset of $NP_\mathbb{R}$ where we are restricted to guesses encoded over $\mathbb{Z}_2$. For more details on the BSS model we refer to [3].

### 2.1 The class $NPO_\mathbb{R}$ and its subclasses

In classical complexity the most relevant class of optimization problems is $NPO$ which consist of combinatorial optimization problems with exponential size search space. We shall define a real number counterpart $NPO_\mathbb{R}$. There is more than just one immediate way of doing so. Whereas the problem instances for such a class are encoded over $\mathbb{R}$, problems naturally differ as to where feasible solutions are located and what quantity is measured by the objective function. Feasible solutions could be encoded over both $\mathbb{R}$ or $\mathbb{Z}_2$ depending on the search space of the problem. Similarly, the measure of a feasible solution could either be a non-negative real number or a non-negative integer depending on the problem. It turns out that all four resulting combinations are meaningful.

In the definition below we use $R_s$ to denote the ring over which feasible solutions are encoded. We consider $R_s \in \{\mathbb{R}, \mathbb{Z}_2\}$. By $R_m$ we denote the image set of a measure function; here $R_m \in \{\mathbb{R}^{\geq 0}, \mathbb{Z}_2^\infty\}$ are suitable choices.

We are now ready to define real analogues of $NPO$:

**Definition 1.** *a) A triple $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x \in \mathcal{I}}, m)$ belongs to the class $NPO_{\mathbb{R},max}^{R_s,R_m}$ if it is a maximization problem over $\mathbb{R}$ that consists of the following:*

*i) A set $\mathcal{I} \subseteq \mathbb{R}^\infty$ as instances of the problem;*
*ii) for every $x \in \mathcal{I}$ a set $Sol(x) \subseteq R_s^\infty$ of feasible solutions. For every $x \in \mathcal{I}$ and every $y \in Sol(x)$ we require $|y| \leq p(|x|)$ for a fixed polynomial $p$;*
*iii) a function $m : \{(x, y) | x \in \mathcal{I}, y \in Sol(x)\} \to R_m$. The value $m(x, y)$ is called the* measure *of the feasible solution $y$.*

*In addition, the following has to hold:*

*iv) For a given $x \in \mathbb{R}^\infty$ it is decidable in polynomial time in $|x|$ in the BSS model if $x \in \mathcal{I}$;*
*v) for all $x \in \mathcal{I}$ and for any arbitrary $y \in R_s^\infty$ of size at most $p(|x|)$ it is decidable in polynomial time in $|x|$ in the BSS model if $y \in Sol(x)$;*
*vi) for all $x \in \mathcal{I}$ and for all $y \in Sol(x)$ the measure function $m$ is computable in polynomial time in $|x|$ in the BSS model.*

*We are looking for the optimal solution measure* $m^*(x) = \sup_{y \in Sol(x)} m(x,y)$.

*b) The classes* $NPO^{R_s,R_m}_{\mathbb{R},min}$ *of minimization problems is defined similarly.*

*c)* $NPO^{R_s,R_m}_{\mathbb{R}}$ *is the union of classes* $NPO^{R_s,R_m}_{\mathbb{R},max}$ *and* $NPO^{R_s,R_m}_{\mathbb{R},min}$*. Similarly,* $NPO_{\mathbb{R}}$ *denotes the union of all optimization problems defined above.*

*Remark 1.* a) In case the measure function takes values in $\mathbb{R}^{\geq 0}$ we just output a single real number. If the measure values are non-negative integers we output the number in binary representation.

b) Since a single real number has algebraic size 1 deciding if an arbitrary real number is integral cannot be done in polynomial time in its algebraic size. Thus, if some components of an instance are required to be integral or rational numbers we need to encode these in binary in order to satisfy condition a,iv) above.

b) $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$ is the only class introduced in this paper for which problems can have instances with a non-empty set of feasible solutions and $m^*(x)$ undefined. If $R_s = \mathbb{Z}_2$ this follows from the finite search space; and if $R_m = \mathbb{Z}_2^\infty$ it is a consequence of part a,vi) of the definition.

*Example 1.* We illustrate the four variants of optimization problems by giving examples of problems in each class. $NPO^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}_{\mathbb{R}}$ contains problems like Binpacking with real weights on the items or maximizing the number of real polynomials in a given system that share a zero in $\{0,1\}^n$. The TSP problem with real weights on the edges and the Knapsack optimization problem with real numbers as weights belong to $NPO^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}_{\mathbb{R}}$. Typical problems in $NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R}}$ are the counterpart of the above problem for polynomial systems when solutions in $\mathbb{R}^n$ are searched. Similarly, the problem of finding the maximal solvable subsystem of a given system of linear inequalities is in this class. Finally, minimizing the function value of a multivariate real polynomial, minimizing the maximum norm of a root of such a polynomial, or the linear and quadratic programming problems with real data are members in $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R}}$. We easily obtain the following relations between these classes:

$$NPO^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}_{\mathbb{R}} \subset NPO^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}_{\mathbb{R}} \cap NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R}} \subset NPO^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}_{\mathbb{R}} \cup NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R}} \subset NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R}}$$

**Definition 2.** *An optimization problem* $\mathcal{P}$ *in class* $NPO_{\mathbb{R}}$ *is* polynomially bounded *if there exists a polynomial* $q$ *such that for all* $x$ *the value* $m(x,y)$ *is bounded by* $q(|x|)$ *for all feasible solutions* $y$*.*

### 2.2 Approximation classes $\boldsymbol{APX}_{\mathbb{R}}$, $\boldsymbol{PTAS}_{\mathbb{R}}$ and $\boldsymbol{FPTAS}_{\mathbb{R}}$

In order to study approximation properties of these types of optimization problems we define approximation classes similar to those well known in the classical setting. The definition, however, needs some care.

**Definition 3.** *Given an instance $x$ and a feasible solution $y \in Sol(x)$ with measure $v$, the* ratio *of this feasible solution is defined as $R(x,v) = max(\frac{v}{m^*(x)}, \frac{m^*(x)}{v})$. The ratio is undefined if either $v = 0$, $m^*(x) = 0$ or $m^*(x)$ is undefined. This definition applies to both maximization and minimization problems.*

**Definition 4.** *An optimization problem $\mathcal{P}$ in class $NPO^{R_s,R_m}_{\mathbb{R},max}$ is in class $APX^{R_s,R_m}_{\mathbb{R},max}$ if there exists a real number $r > 1$ and polynomial time BSS machines $M_1$, $M_2$ and $M_3$ such that:*

*i) For any input $x \in \mathcal{I}$ machine $M_1$ decides if $Sol(x) = \emptyset$;*
*ii) for any input $x \in \mathcal{I}$, $Sol(x) \neq \emptyset$ machine $M_2$ decides if $m^*(x) = 0$ or $m^*(x)$ is undefined;*
*iii) for any input $x \in \mathcal{I}$, $Sol(x) \neq \emptyset$, $m^*(x)$ is defined and $m^*(x) \neq 0$ machine $M_3$ computes a value $v \in R_m$ such that there exists a $y \in Sol(x)$ with $m(x,y) \geq v$ and $R(x,v) \leq r$.*

*Similarly for optimization problems in class $NPO^{R_s,R_m}_{\mathbb{R},min}$.*

*Remark 2.* Most important we do not require $M_3$ to compute a feasible solution $y$. The output of the computation is only a bound for the measure of a feasible solution which *must* exist and this bound must be within a constant factor of the measure of the optimal solution. The reason for this is that though one might guarantee such a $y$ to exist it is impossible to compute it by a BSS algorithm (f.e. if $y$ has to satisfy an equation $y^2 - s = 0$ for some $\sqrt{s}$ not belonging to the algebraic closure of the machine constants).

$APX_{\mathbb{R}}$ is then defined as the union of all the subclasses. In a similar way we can define approximation classes *log-*$APX_{\mathbb{R}}$, *poly-*$APX_{\mathbb{R}}$ and *exp-*$APX_{\mathbb{R}}$. For these classes we require the *ratio* of the feasible solution $y$ to be logarithmic, polynomial and exponential in $|x|$, respectively. Note that membership in *exp-*$APX_{\mathbb{R}}$ requires that we in polynomial time can decide if the set of feasible solutions is empty.

Finally, the class $PTAS_{\mathbb{R}}$ of problems allowing a polynomial time approximation scheme is defined by changing Definition 4 in the obvious way: The ratio $r$ now is given as part of the input to $M_1$, $M_2$ and $M_3$. For approximation class $FPTAS_{\mathbb{R}}$ we additionally require the running time of $M_1$, $M_2$ and $M_3$ to be polynomial in $|x|$ and $\frac{1}{r-1}$.

*Example 2.* The following is an example of a problem in $APX^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$. Let $k \in \mathbb{N}$ be fixed and $> 1$. The problem MAX QUADRATIC POLYNOMIAL SYSTEM($k$), for short MAX QPS($k$), is defined as follows: Given polynomials $p_1, \ldots, p_m$ with real coefficients in $n$ variables, each of degree at most 2, each depending on at most 3 of the variables $x_1, \ldots, x_n$ and each variable occurs in at most $k$ of the polynomials, find the maximal number of polynomials $p_i$, $1 \leq i \leq m$, that have a common root in $\mathbb{R}^n$. For $k > 2$ MAX QPS($k$) is an $NP_{\mathbb{R}}$-hard maximization problem due to [4]. It is not hard to see that MAX QPS($k$) is in class $APX^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$. First decide for each $p_i$ whether it has a root; this is polynomial time decidable

since only three variables occur in each single $p_i$. Then, for the solvable ones fixing the occurring variables will at most influence $3(k-1)$ other polynomials. Thus an approximation ratio $r := 3(k-1)+1$ can be guaranteed. □

### 2.3 Approximation preserving reductions

In order to show the existence of complete problems for the above classes of optimization problems we define approximation preserving reductions.

**Definition 5.** *Let $\mathcal{P}_1$ and $\mathcal{P}_2$ be maximization problems in class $NPO_{\mathbb{R}}$. $\mathcal{P}_1$ is AP-reducible to $\mathcal{P}_2$, denoted $\mathcal{P}_1 \leq_{AP} \mathcal{P}_2$, if functions $f$ and $g$ and a real constant $\alpha \geq 1$ exists such that:*

*i) For any $x \in \mathcal{I}_1$ and any real number $r > 1$ it is $f(x,r) \in \mathcal{I}_2$;*
*ii) for any $x \in \mathcal{I}_1$ and any real number $r > 1$ if $Sol_1(x) \neq \emptyset$, then $Sol_2(f(x,r)) \neq \emptyset$;*
*iii) $f(x,r)$ is polynomial time computable in $|x|$;*
*iv) for any $x \in \mathcal{I}_1$ and any fixed real number $r > 1$ if there exists a $y \in Sol_2(f(x,r))$ with measure at least $v$ such that $R_2(f(x,r),v) \leq r$, then there exists a $z \in Sol_1(x)$ with measure at least $w$ such that $R_1(x,w) \leq 1+\alpha\cdot(r-1)$, and $w = g(x,r,v)$;*
*v) $g(x,r,v)$ is polynomial time computable in $|x|$.*

A similar definition can be made for minimization problems.

*Remark 3.* a) In the classical setting we also require a function which computes a feasible solution to the instance of $\mathcal{P}_1$ given a feasible solution to the instance of $\mathcal{P}_2$ we reduced it to. As mentioned previously we do not require the actual solution to be computed; the fact that it exists is sufficient.

b) Most AP-reductions in this paper will use $\alpha = 1$, do not depend on $r$, and satisfy $g(x,r,v) = v$. We will explicitly state when this is not the case.

**Definition 6.** *Given a class $\mathcal{C}$ of optimization problems, $\mathcal{P}$ is* $\mathcal{C}$-hard *with respect to AP-reducibility if for all $\mathcal{P}' \in \mathcal{C}$, $\mathcal{P}' \leq_{AP} \mathcal{P}$. If furthermore $\mathcal{P} \in \mathcal{C}$, then $\mathcal{P}$ is* $\mathcal{C}$-complete *with respect to AP-reducibility.*

### 2.4 Relation of $NPO_{\mathbb{R}}$ to decision problems

Let us shortly justify our definition of $NPO_{\mathbb{R}}$ and its subclasses from another point of view, namely the relation of each class to a corresponding class of decision problems.

In the classical setting $PO$ is the subset of $NPO$ for which an optimal solution and its measure can be computed in polynomial time. It is $PO = NPO$ iff $P = NP$. Just as we have defined four subclasses of $NPO_{\mathbb{R}}$ we can define four subclasses of $PO_{\mathbb{R}}$.

In order for an optimization problem in $NPO_{\mathbb{R}}^{R_s,R_m}$ to be in the subclass $PO_{\mathbb{R}}^{R_s,R_m}$ we require that we in polynomial time can decide for any given instance

if the set of feasible solutions is empty. Then for each of the four classes $PO_{\mathbb{R}}^{R_s,R_m}$ introduced we add the following additional requirements necessary for a problem in order to belong to the class:

For $PO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}$ and $PO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}$ an optimal solution and for $PO_{\mathbb{R}}^{\mathbb{R},\mathbb{Z}_2^\infty}$ the optimal measure value have to be computable in polynomial time. Finally, for $PO_{\mathbb{R}}^{\mathbb{R},\mathbb{R}^{\geq 0}}$ the following questions have to be decidable in polynomial time: given $x$, is $m^*(x)$ is defined; given $v \in \mathbb{R}^{\geq 0}$ is there a solution with measure $v$; given $v_1 < v_2 \in \mathbb{R}^{\geq 0}$, is there is a solution with measure in $]v_1, v_2[$.

**Theorem 1.** *The following relations hold between the classes of optimization problems and the corresponding classes of decision problems:*

*a)* $P_{\mathbb{R}} = NP_{\mathbb{R}} \iff PO_{\mathbb{R}}^{\mathbb{R},\mathbb{R}^{\geq 0}} = NPO_{\mathbb{R}}^{\mathbb{R},\mathbb{R}^{\geq 0}} \iff PO_{\mathbb{R}}^{\mathbb{R},\mathbb{Z}_2^\infty} = NPO_{\mathbb{R}}^{\mathbb{R},\mathbb{Z}_2^\infty}$

*b)* $P_{\mathbb{R}} = DNP_{\mathbb{R}} \iff PO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}} = NPO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}} \iff PO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{Z}_2^\infty} = NPO_{\mathbb{R}}^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}$

## 3 Completeness results for maximization

We turn to the main results of this paper. In this section we show the existence of complete problems for each of the classes of maximization problems $NPO_{\mathbb{R},max}^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}$, $NPO_{\mathbb{R},max}^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}$, $NPO_{\mathbb{R},max}^{\mathbb{R},\mathbb{Z}_2^\infty}$ and $NPO_{\mathbb{R},max}^{\mathbb{R},\mathbb{R}^{\geq 0}}$ as well as the corresponding subclasses of $APX_{\mathbb{R}}$. Minimization will be treated in the next section.

### 3.1 $NPO_{\mathbb{R}}$-completeness

For each of the four classes completeness of a particular problem will be shown. Let us start with

*Example 3.* The MAX WEIGHTED 4FEAS problem is defined as:
INPUT: $n \in \mathbb{N}$, a degree 4 polynomial with real coefficients in $n$ variables. Each variable has a non-negative integer weight;
FEASIBLE SOLUTION: $x \in \mathbb{R}^n$ such that the polynomial evaluates to zero;
MEASURE: Sum of weights of all variables which have been assigned a value different from zero.

The problem is easily shown to belong to class $NPO_{\mathbb{R},max}^{\mathbb{R},\mathbb{Z}_2^\infty}$. Note that deciding existence of a feasible solution is $NP_{\mathbb{R}}$-complete so this problem does not belong to class *exp-*$APX_{\mathbb{R}}$ if $P_{\mathbb{R}} \neq NP_{\mathbb{R}}$.

**Theorem 2.** *MAX WEIGHTED 4FEAS is complete for* $NPO_{\mathbb{R},max}^{\mathbb{R},\mathbb{Z}_2^\infty}$ *under AP-reducibility.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be a problem in class $NPO_{\mathbb{R},max}^{\mathbb{R},\mathbb{Z}_2^\infty}$. We design an AP-reduction from $\mathcal{P}$ to MAX WEIGHTED 4FEAS.

Let $x \in \mathcal{I}$ be fixed. Consider the following decision problem: On input $(y, v)$ decide if $y \in Sol(x)$, if $v$ is the binary encoding of an integer, and if $m(x, y) = v$.

Since $\mathcal{P} \in NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$ we can easily construct a BSS machine $M_x$ which in polynomial time in $|x|$ decides this problem. On the other hand, deciding *if* an input $(y, v)$ exists such that $M_x$ accepts is a decision problem in class $NP_\mathbb{R}$.

An AP-reduction $(f, g)$ is obtained as follows: On input $x \in \mathcal{I}$ $f$ constructs $M_x$. It then uses the reduction described in [4] to generate a degree 4 polynomial $p$ which has a real root iff there exists $(y, v)$ which fulfills the above conditions. In $p$ we identify the polynomially many variables which correspond to those registers that store the binary representation of $v$ at the beginning of the computation performed by $M_x$ on input $(y, v)$. Each of these variables is assigned a weight $2^i$ corresponding to which part of the binary representation of $v$ it represents. All other variables in $p$ are assigned the weight 0. Adding weights to the variables in $p$ produces a MAX WEIGHTED 4FEAS instance $q = f(x)$. By choosing $\alpha = 1$ and $g(x, r, v) = v$ we obtain an AP-reduction since:

- If $Sol(x) \neq \emptyset$, then there exists a $(y, v)$ such that $M_x$ on input $(y, v)$ will accept. This implies $q$ will have a real root so $Sol(q) \neq \emptyset$. Moreover, $f$ is polynomial time computable.
- For any feasible solution $z'$ to $q$ we can identify the variables which correspond to the registers at the beginning of the computation performed by $M_x$ on input $(y, v)$. This $y$ is in $Sol(x)$ and has measure $v$. The assignment of weights to the variables in $q$ implies that $m(x, y)$ equals the measure of the feasible solution $z'$ of $q$. This especially holds for optimal feasible solutions, so we have $m^*(x) = m^*(q)$. If there exists a feasible solution $s'$ for $q$ with measure $w$ such that $R(q, w) \leq r$, then there exists a $y \in Sol(x)$ with measure $w$ such that $R(x, w) \leq r$. □

*Example 4.* The MAX VARIABLE SUM problem is defined as:
INPUT: $n \in \mathbb{N}$, a degree 4 polynomial with real coefficients in $n$ variables;
FEASIBLE SOLUTION: $x \in \mathbb{R}^n$ such that the polynomial evaluates to 0;
MEASURE: Sum $\sum_{i=1}^{n} x_i$ of all components of $x$.

**Theorem 3.** *MAX VARIABLE SUM is complete for* $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$ *under AP-reducibility.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be a problem in class $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$. We will show $\mathcal{P} \leq_{AP}$ MAX VARIABLE SUM. Let $x \in \mathcal{I}$ be a fixed instance for $\mathcal{P}$. We once more consider machine $M_x$ which on input $(y, v)$ decides if $y \in Sol(x)$ and if $m(x, y) = v$. Again the corresponding decision problem is in $NP_\mathbb{R}$. It can be reduced in polynomial time to a 4FEAS instance, i.e. the question whether a polynomial $p$ of degree 4 has a real zero. Moreover one particular component of each root $z^*$, say $z_1^*$, represents the value $v$ of the guess $(y, v)$ accepted by $M_x$. Let $\hat{n}$ be the number of variables that $p$ depends on. We introduce $\hat{n} - 1$ new variables denoted by $u_2, \ldots, u_{\hat{n}}$. Construct a MAX VARIABLE SUM instance as follows:

$$q(z, u) = p(z) + (z_2 + u_2)^2 + (z_3 + u_3)^2 + \ldots + (z_{\hat{n}} + u_{\hat{n}})^2$$

Clearly $q(z,u)$ has a real root iff $p(z)$ does. In order for $q(z,u)$ to evaluate to 0 we need $z_i = -u_i$ for $2 \leq i \leq \hat{n}$. Thus, for all feasible solutions to this MAX VARIABLE SUM instance the measure is equal to the value assigned to $z_1$. □

For the two remaining classes the following problems are complete:

*Example 5.* The MAX SUBCIRCUIT VALUE problem is defined as:
INPUT: $n \in \mathbb{N}$, an algebraic circuit (see [3]) with $n$ input gates and two output gates; the latter are addressed as first and second output;
FEASIBLE SOLUTION: $x \in \{0,1\}^n$ such that the first output gate evaluates to 1;
MEASURE: Output from the second output gate.

Note that deciding whether the set of feasible solutions is empty is complete for the complexity class $DNP_{\mathbb{R}}$ [6]. The search space is finite, but of exponential size.

*Example 6.* The MAX WEIGHTED BINARY CIRCUIT is defined as:
INPUT: $n \in \mathbb{N}$, an algebraic circuit with $n$ input gates, a non-negative integer weight for each of the input gates;
FEASIBLE SOLUTION: $x \in \{0,1\}^n$ such that the circuit evaluates to 1;
MEASURE: Sum of weights of all input gates which have been assigned value 1.

**Theorem 4.** *MAX SUBCIRCUIT VALUE is complete for* $NPO^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$ *and MAX WEIGHTED BINARY CIRCUIT is complete in* $NPO^{\mathbb{Z}_2,\mathbb{Z}_2^\infty}_{\mathbb{R},max}$ *under AP-reductions.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be a problem in $NPO^{\mathbb{Z}_2,\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$. We will show $\mathcal{P} \leq_{AP}$ MAX SUBCIRCUIT VALUE. Let $x \in \mathcal{I}$ be a fixed instance for $\mathcal{P}$. Once again we obtain a decision problem: On input $y \in \{0,1\}^\infty$ decide if $y \in Sol(x)$. This decision problem is in $DNP_{\mathbb{R}}$. So it reduces to the Binary Circuit Satisfiability problem in polynomial time. The output gate of this circuit is the first output gate in the instance we produce. Now add another circuit which as input gates has copies of the input gates in the above circuit. This circuit computes the function $m(x,y)$, where $x$ is fixed and $y$ comes from the input gates. The output gate of this circuit is the second output gate in the instance we produce.

For MAX WEIGHTED BINARY CIRCUIT apply ideas from Theorem 2 and Theorem 4. □

Many important problems are polynomially bounded, i.e. the numerical size of all solutions is polynomially bounded in the input size. For the related class $NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$ the following problem turns out to be important:

*Example 7.* The MAX ZERO VARIABLES problem is defined as:
INPUT: $n \in \mathbb{N}$, a degree 4 polynomial with real coefficients in $n$ variables;
FEASIBLE SOLUTION: $x \in \mathbb{R}^n$ such that the polynomial evaluates to 0;
MEASURE: Number of variables that are assigned the value 0.

The problem is in $NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$; since the optimal solution has measure at most $n$ the problem is polynomially bounded. Note that the corresponding problem for linear inequalities is related to finding non-degenerated bases in Linear Programming.

**Theorem 5.** *MAX ZERO VARIABLES is complete for the set of polynomially bounded problems in class $NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$ with respect to AP-reducibility.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be a polynomially bounded problem in class $NPO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$. Let $x \in \mathcal{I}$ be a fixed instance for this problem. Construct a BSS machine $M_x$ which on input $(y, v) \in \mathbb{R}^\infty \times \mathbb{Z}_2^\infty$ decides if $y \in Sol(x)$ and if $m(x, y) = v$. As before, this decision problem reduces to the 4FEAS decision problem. The proof involves two steps: The first is to modify the reduction in [4] such that the degree 4 polynomial we produce does not have real roots involving 0-components. The second step is to construct an optimization problem by introducing new variables which are allowed to take the value 0 and are related to the value $v$ given to $M_x$ as part of its input.

We assume the reader is familiar with the reduction in [4]. As part of that reduction there occur variables $X_{t,r}$ representing the entry of a register $r$ at step $t$ of the computation of $M_x$. In the original reduction a variable $X_{t,r}$ can attain the value 0. We change the reduction in such a way that no solution of the resulting polynomial system has a 0-component. Towards this aim introduce additional variables $S_{t,r}$ which encode the value 0 in the following way: If in the computation performed by $M_x$ on input $(y, v)$ register $r$ at time-step $t$ holds a value $z \neq 0$, then we will enforce $S_{t,r} = 1$ and $X_{t,r} = z$. If register $r$ at time-step $t$ holds the value 0, then we will enforce $S_{t,r} = -1$ and $X_{t,r} = -1$.

Using a semi-algebraic description we both can simulate the computation of $M_x$ and enforce avoidance of the value 0 in intermediate computations. The parts of the constructed formula that simulate algebraic operations are a bit more involved than in the usual proof. We have 8 different combinations of whether the 3 registers involved in the algebraic operation hold values different from or equal to 0. Nevertheless the total size of the produced formula is still polynomial in $|x|$. Each strict inequality can as usual be replaced by an equality by introducing one additional variable $V$. Some of these variables will end up in equalities which might not be fulfilled in an accepting computation of $M_x$. In such a case these variables can be assigned any value, including 0. This is resolved by introducing one more variable $W$, for each strict inequality and in addition requiring the equality $V \cdot W = 1$ to be satisfied.

For the second step of the proof we introduce new variables in order to obtain the desired optimization problem: Assume $v$ as part of the input to $M_x$ to be encoded in unary notation. For each register position $i$ involved in this unary encoding we introduce a variable $M_i$ and require $S_{1,i} + M_i = 1$. Note that $S_{1,i}$ represents the entry of register $i$ at the start $t = 1$ of the computation; this entry according to our setting is given the corresponding component of (the unary notation of) $v$. Thus, for any real root of the constructed formula the number of variables among the $M_i$'s that are assigned the value 0 equals the

value $v$ of the guess $(y, v)$ accepted by $M_x$. Moreover, since no other variables have been assigned the value 0 in a real root of the formula, this implies the total number of variables assigned the value 0 equals the value $v$. □

### 3.2 Completeness in $APX_{\mathbb{R}}$ and $PTAS_{\mathbb{R}}$

In this section we outline how both $APX_{\mathbb{R}}$-complete and $PTAS_{\mathbb{R}}$-complete problems will be constructed. This is done by exploiting a certain property of the previously introduced problems. This property leads in a canonical way to complete problems for both classes (where for the latter the reducibility notion has to be strengthened appropriately).

**Definition 7.** *We call* $\mathcal{P}$ measure preserving *with respect to* $NPO^{R_s,R_m}_{\mathbb{R},max}$ *under AP-reductions iff the following conditions are satisfied. First,* $\mathcal{P}$ *is* $NPO^{R_s,R_m}_{\mathbb{R},max}$*-complete under AP-reductions. Second, for every problem* $\mathcal{Q} \in NPO^{R_s,R_m}_{\mathbb{R},max}$ *there is a reduction algorithm which preserves the measure. The latter means that for every instance* $x_1$ *of* $\mathcal{Q}$*, if* $x$ *is the* $\mathcal{P}$ *instance obtained when we apply the AP-reduction from* $\mathcal{Q}$ *to* $\mathcal{P}$ *on input* $x_1$*, then* $x_1$ *has a feasible solution with measure* $t$ *iff* $x$ *has a feasible solution with measure* $t$*.*

We show $APX^{R_s,R_m}_{\mathbb{R},max}$-completeness by using bounds on the measure for instances of a measure preserving problem. Since the proof is very similar to what is done in [5] for the classical setting we only include parts of it.

**Definition 8.** *Let* $\mathcal{P}$ *be in class* $NPO^{R_s,R_m}_{\mathbb{R},max}$*. We define a maximization problem* $\mathcal{P}'$ *as follows:*
INPUT: *A triple* $X := (x, k, b)$ *with* $x$ *an instance of* $\mathcal{P}$*,* $k \geq 2$ *integral,* $b > 0$ *a real;*
FEASIBLE SOLUTION: $y \in \mathbb{R}^{\infty}$ *of size at most* $p(|x|)$ *for a polynomial* $p$*,* $p$ *only depends on* $\mathcal{P}$*;*
MEASURE: *Let* $A(X) = (k-2) \cdot b$*. The measure is* $m_{\mathcal{P}'}(X, y) = A(X) + b$ *if either* $y \notin Sol(x)$ *or* $y \in Sol(x)$ *and* $m_{\mathcal{P}}(x, y) < b$*; and it is* $m_{\mathcal{P}'}(X, y) = A(X) + min(k \cdot b, m_{\mathcal{P}}(x, y))$ *if* $y \in Sol(x)$ *and* $m_{\mathcal{P}}(x, y) \geq b$*.*

The following two lemmas are crucial.

**Lemma 1.** *For all* $\mathcal{P}$ *in* $NPO^{R_s,R_m}_{\mathbb{R},max}$ *we have* $\mathcal{P}' \in APX^{R_s,R_m}_{\mathbb{R},max}$*.*

**Lemma 2.** *If* $\mathcal{P}$ *is measure preserving w.r.t.* $NPO^{R_s,R_m}_{\mathbb{R},max}$ *and AP-reductions* $\mathcal{P}'$ *is* $APX^{R_s,R_m}_{\mathbb{R},max}$*-hard with respect to AP-reductions.*

*Proof.* (of Lemma 2) Let $\mathcal{Q}$ be a problem in class $APX^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},max}$. Then there exists a real constant $k > 1$ such that for all instances of $\mathcal{Q}$ we can in polynomial time can compute a $k$-approximation. W.l.o.g. we assume $k \in \mathbb{N}$. Since $k$ is fixed once $\mathcal{Q}$ is fixed we can have $\alpha$ and $g$ in our AP-reduction from $\mathcal{Q}$ to $\mathcal{P}'$ depend on $k$. The AP-reduction works as follows:

Let $x_1$ be an instance of $\mathcal{Q}$. Compute a real number $b$ such that there exists a feasible solution for $x_1$ with measure at least $b$ and $R_{\mathcal{Q}}(x_1, b) \leq k$. By using $x_1$ as input to the AP-reduction from $\mathcal{Q}$ to $\mathcal{P}$ we obtain a $\mathcal{P}$ instance $x$ with the property that $x_1$ has a feasible solution with measure $t$ iff $x$ has a feasible solution with measure $t$. We construct a $\mathcal{P}'$ instance $X := (x, k, b)$.

Consider the following two cases:

$k = 2$: We define $\alpha = 1$ and $g(w, r, v) = v$. We see that $m^*_{\mathcal{P}'}(X) = m^*_{\mathcal{P}}(x)$. This is not true for $\mathcal{P}'$ instances in general but it is true in this case since $b \leq m^*_{\mathcal{P}}(x) \leq k \cdot b$ (these inequalities are true because of the measure preserving property of $\mathcal{P}$ and because $x$ was constructed from a $\mathcal{Q}$ instance $x_1$ for which we had a $k$-approximation).

Assume $y$ is a feasible solution to $X$ with $R_{\mathcal{P}'}(X, m_{\mathcal{P}'}(X, y)) \leq r$: If $y$ is *not* a feasible solution to $x$, or $y$ is a feasible solution to $x$ with $m_{\mathcal{P}}(x, y) < b$, then $m_{\mathcal{P}'}(X, y) = b$. But there exists a feasible solution $u$ to $x$ with $m_{\mathcal{P}}(x, u) \geq b$ and this feasible solution $u$ will have $R_{\mathcal{P}}(x, m_{\mathcal{P}}(x, u)) \leq r$. If $y$ *is* a feasible solution to $x$ with $m_{\mathcal{P}}(x, y) \geq b$ then we also have the $R_{\mathcal{P}}(x, m_{\mathcal{P}}(x, y)) \leq r$.

The measure preserving property of $\mathcal{P}$ then implies there exists a feasible solution to $x_1$ with measure at least $m_{\mathcal{P}'}(X, y)$ and $R_{\mathcal{Q}}(x_1, m_{\mathcal{P}'}(X, y)) \leq r$.

$k > 2$: We define $\alpha = (k - 1)$ and $g(w, r, v) = v - A(X)$. We have $m^*_{\mathcal{P}'}(X) = A(X) + m^*_{\mathcal{P}}(x)$ by the same argument as before. Assume there exists a feasible solution $y$ to the $\mathcal{P}'$ instance $X$ such that $R_{\mathcal{P}'}(X, m_{\mathcal{P}'}(X, y)) \leq r$: If $y$ is *not* a feasible solution to $x$, or $y$ is a feasible solution to $x$ with $m_{\mathcal{P}}(x, y) < b$, then $m_{\mathcal{P}'}(X, y) = A(X) + b$. Since there exists a feasible solution $u$ to $x$ with measure at least $b$ we obtain

$$\frac{m^*_{\mathcal{P}'}(X)}{m_{\mathcal{P}'}(X, y)} \leq r \Leftrightarrow \frac{m^*_{\mathcal{P}}(x)}{b} \leq (r - 1) \cdot \alpha + 1$$

If $y$ *is* a feasible solution to $x$ with $m_{\mathcal{P}}(x, y) \geq b$ it follows similarly:

$$\frac{m^*_{\mathcal{P}'}(X)}{m_{\mathcal{P}'}(X, y)} \leq r \Rightarrow \frac{m^*_{\mathcal{P}}(x)}{m_{\mathcal{P}}(x, y)} \leq (r - 1) \cdot (k - 2) + r = (r - 1) \cdot \alpha + 1$$

□

**Theorem 6.** *All the four subclasses of $APX_{\mathbb{R},max}$ has complete problems under AP-reductions.*

*Proof.* According to the the previous two lemmas it is sufficient to establish the existence of measure preserving problems in each of the four related subclasses of $NPO_{\mathbb{R},max}$. The completeness proofs for the classes in $NPO_{\mathbb{R},max}$ in the previous section show that all the problems considered together with the given reductions are measure preserving. □

$PTAS_{\mathbb{R}}$-completeness can be shown similarly. First, the notion of F-reductions needs to be transferred to the real number setting in order to establish a reduction that preserves membership in $FPTAS_{\mathbb{R}}$. The reductions in Section 3.1 are all F-reductions as well. Then a canonical problem similar to the one of Definition 8 is used to prove

**Theorem 7.** *The above subclasses of $PTAS_{\mathbb{R}}$ do have complete problems under F-reductions.*

## 4 Completeness results for minimization

So far we have been dealing with maximization problems. Nevertheless, the result from the previous section can easily be used to show

**Theorem 8.** *All four subclasses of $NPO_{\mathbb{R},min}$ have complete problems under AP-reductions.*

*Proof.* For the problems studied in Examples 3, 4, 5, 6 and 7, respectively, the corresponding minimization versions are complete for their respective minimization classes. The above proofs can be adapted almost word by word. □

In this section we focus on completeness results for some further natural minimization problems. The results so far have all been for problems for which already a potential membership in class *exp-APX*$_{\mathbb{R}}$ implies $P_{\mathbb{R}} = NP_{\mathbb{R}}$. The next result deals with a problem for which feasible solutions easily are found. Consider the minimization problem MIN DIFFERENT FUNCTION VALUES: Let $n$, $m \in \mathbb{N}$, $m$ degree polynomials of degree 4 with real coefficients in $n$ variables be given. We ask for the minimal number of different function values obtained when evaluating the $m$ polynomials in a point $x \in \mathbb{R}^n$. The related maximization problem is easily shown to be in class $PO^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},max}$.

**Theorem 9.** *MIN DIFFERENT FUNCTION VALUES is complete for the class of polynomially bounded problems in poly-$APX^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},min}$ with respect to AP-reducibility.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be polynomially bounded in class *poly-*$APX^{\mathbb{R},\mathbb{Z}_2^\infty}_{\mathbb{R},min}$. Let $x \in \mathcal{I}$ be a fixed instance for this problem and let $p$ be a bound for the measure of all feasible solutions. For each $1 \le i \le p$ we consider a polynomial time BSS machine $M_{x,i}$ which on input $y$ decides if $y \in Sol(x)$ and $m(x,y) \le i$. For each $M_{x,i}$, $1 \le i \le p$, we apply the reduction in [4] to obtain a 4FEAS instance which depends on variables $y$ and additional variables $Z_i$. We obtain $p$ polynomials $f_1, \ldots, f_p$ with the following properties:

$$\begin{aligned} &\exists Z_1 \ \textit{such that} \ f_1(y,Z_1) = 0 \Longleftrightarrow y \in Sol(x) \ \textit{and} \ m(x,y) \le 1 \\ &\qquad\qquad \vdots \\ &\exists Z_p \ \textit{such that} \ f_p(y,Z_p) = 0 \Longleftrightarrow y \in Sol(x) \ \textit{and} \ m(x,y) \le p \end{aligned}$$

By the construction of these $p$ polynomials we know their function values always are non-negative. The next step is to obtain the desired instance of MIN DIFFERENT FUNCTION VALUES. The output $x'$ of our AP-reduction is given as $p$ further polynomials $g_1, \ldots, g_p$ obtained as follows:

$$\begin{aligned} g_p(y,Z_p) = f_p(y,Z_p) \ , \ g_{p-1}(y,Z_{p-1},Z_p) &= f_{p-1}(y,Z_{p-1}) + f_p(y,Z_p) \\ &\vdots \\ g_1(y,Z_1,\ldots,Z_p) &= f_1(y,Z_1) + \ldots + f_p(y,Z_p) \end{aligned}$$

For all assignments of values to the variables of this set of polynomials the following inequalities will hold: $g_p \leq g_{p-1} \leq \ldots \leq g_1$. If the optimal solution to the instance $x$ we reduced from has measure $j$, then we have for all assignments of values to the variables the following strict inequalities: $g_j < g_{j-1} < \ldots < g_1$. This implies that the optimal solution for $x'$ has measure greater than or equal to the measure of the optimal solution for $x$, since there will be at least $j$ different function values in $x'$. On the other hand, we can also find assignments to the variables of $g_p, \ldots, g_j$ such that all $g_i$'s attain the same function value 0. So $x$ and $x'$ have the same optimal measure and the requirements for an AP-reduction are satisfied. □

Our final result deals with minimizing the norm of a root of a given polynomial. This and related questions are important in studying algorithms in semi-algebraic geometry. There, in many situations a first important task is to find good upper bounds on the norm of representatives in each connected component of a semi-algebraic set. For more on this see [10].

*Example 8.* The MIN $L^1$-NORM problem is defined as:
INPUT: $n \in \mathbb{N}$, a degree 4 polynomial with real coefficients in $n$ variables;
FEASIBLE SOLUTION: $x \in (\mathbb{R}\backslash\{0\})^n$ such that the polynomial evaluates to 0;
MEASURE: The $L^1$-norm $\sum_{i=1}^{n} |x_i|$ of $x$.

**Theorem 10.** *MIN $L^1$-NORM is complete for $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},min}$ under AP-reducibility.*

*Proof.* Let $\mathcal{P} := (\mathcal{I}, \{Sol(x)\}_{x\in\mathcal{I}}, m)$ be a problem in class $NPO^{\mathbb{R},\mathbb{R}^{\geq 0}}_{\mathbb{R},min}$ and let $x \in \mathcal{I}$ be a fixed instance of this problem. Construct a BSS machine $M_x$ which on input $(y, v) \in \mathbb{R}^\infty \times \mathbb{R}$ decides if $y \in Sol(x)$ and $m(x, y) = v$. As before, this reduces to a degree 4 polynomial $q_x$ such that $q_x$ has a real root iff an accepting guess for $M_x$ exists. Moreover assume $z_n$ to be the variable which for any root of $q_x$ gets the measure $v$ of the encoded feasible solution.

A straightforward idea now is to consider homogeneous polynomials. Let $\tilde{p}(z_0, z_1, \ldots, z_n)$ be the homogenization of the above $q_x$ with $deg(\tilde{p}) = 4$ and $z_0$ as the homogenization variable. If $z^* \in \mathbb{R}^n$ is a root of $q_x$ then for all $\lambda \neq 0$ we have $\tilde{p}(\lambda, \lambda \cdot z^*) = \lambda^4 \cdot \tilde{p}(1, z^*) = 0$ and thus the infimum of the $L^1$-norm for roots of $\tilde{p}$ with $\lambda \neq 0$ is 0. We finally have to increase the infimum to $v$ to obtain the desired result. Define $p(z_0, \ldots, z_n, z_{n+1}) = \tilde{p}(z_0, \ldots, z_n) + (z_n - z_0 \cdot z_{n+1})^2$. Now for any root $(z_0^*, \ldots, z_{n+1}^*)$ of $p$ with $z_0^* \neq 0$ we have $z_{n+1}^* = \frac{z_n^*}{z_0^*}$ and $q_x(\frac{z_1^*}{z_0^*}, \ldots, \frac{z_n^*}{z_0^*}) = 0$. Thus $v = z_{n+1}^*$ is always a lower bound for the $L^1$-norm of solution points obtained by varying $z_0$ above. It follows that the infimum of $z_{n+1}^*$ (and thus the infimum of the $L^1$-norm) for all roots of $p$ always equals the infimum of the optimal solution measure for $x$. □

Note that above the only variable we require to be different from 0 is the homogenization variable $z_0$. It would be interesting to see whether we can free ourselves from this restriction. The result as well holds for any other (computable) norm.

## 5 Conclusions

In the present paper we started the development of a structural analysis of approximation algorithms for real number optimization problems based on the BSS model of computation. Our goal was to set up a framework analogue to the study of combinatorial optimization problems in Turing complexity theory. We did so by introducing classes of real number optimization problems $NPO_{\mathbb{R}}$, $APX_{\mathbb{R}}$ and $PTAS_{\mathbb{R}}$ together with a completeness notion for these classes. It turned out that each of the above classes gives rise to consider four natural subclasses. For each of those we established the existence of complete problems. Given the many optimization problems being of immense interest it might be promising to further study the present approach in order to get a clearer view of the approximation properties of $NP_{\mathbb{R}}$-hard problems just as it is done since many years in combinatorial optimization.

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