A NEW PROOF OF COMPLETENESS OF S4 WITH RESPECT TO THE REAL LINE

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Abstract

It was proved in McKinsey and Tarski [7] that every finite well-connected closure algebra is embedded into the closure algebra of the power set of the real line **R**. Pucket [10] extended this result to all finite connected closure algebras by showing that there exists an open map from **R** to any finite connected topological space. We simplify his proof considerably by using the correspondence between finite topological spaces and finite quasi-ordered sets. As a consequence, we obtain that the propositional modal system **S4** of Lewis is complete with respect to Boolean combinations of countable unions of convex subsets of **R**, which is strengthening of McKinsey and Tarski's original result. We also obtain that the propositional modal system **Grz** of Grzegorczyk is complete with respect to Boolean combinations of open subsets of **R**. Finally, we show that McKinsey and Tarski's result can not be extended to countable connected closure algebras by proving that no countable Alexandroff space containing an infinite ascending chain is an open image of **R**.

1 Introduction

In [7] and [8] McKinsey and Tarski introduced closure algebras as algebraic models of the propositional modal system **S4** of Lewis, and proved that **S4** is complete with respect to this semantics. They showed that the variety of closure algebras is generated by its finite well-connected members, thus obtaining the finite model property of **S4**, which, together with the finite axiomatizabilty of **S4**, implies its decidability. They also showed that every closure algebra is represented as an algebra of subsets of a topological space, thus giving an alternative adequate topological semantics for **S4**. They proved that every finite well-connected closure algebra is embedded into the closure algebra of the power set of the real line **R**, hence obtaining that the variety of closure algebras is generated by the power set algebra of **R**. In logical terms, this means that **S4** is complete with respect to **R**. Whether or not every finite connected closure algebra is embedded into the power set algebra of **R** was stated as an open problem in [7] and was subsequently answered in the affirmative by Pucket in [10]. He proved that every finite connected topological space is an open image of **R**.

In recent years the interest in topological semantics of modal logic has been renewed. In particular, Shehtman [11] extended the McKinsey and Tarski result to a bimodal language capable of expressing connectedness, Mints [9] proposed a new proof of completeness of **S4** with respect to the Cantor space $\mathcal{C}$, while Aiello et al [2] supplied a new proof of completeness of **S4** with respect to **R** using a new technique of topo-bisimulation first developed in Aiello and van Benthem [1]. In this paper we present yet another proof of completeness of **S4** with respect to **R** exploring the isomorphism between the categories of finite topological spaces with open maps and finite quasi-ordered sets with p-morphisms, respectively. This provides a considerable simplification of Pucket's construction. In additon, it leads to completeness of **S4** with respect to Boolean combinations of countable unions of convex subsets of **R**, as well as with respect to Borel sets over open subsets of **R**. These results strengthen the original result by McKinsey and Tarski. Another consequence of our theorem is completeness of the modal system **Grz** of Grzegorczyk with respect to Boolean combinations of open subsets of **R**.

The paper is organized as follows. Section 2 consists of preliminaries. In it we recall the one-to-one correspondence between Alexandroff spaces and quasi-ordered sets and between Alexandroff T_0-spaces and partially ordered sets. We present the order-theoretical equivalents of finite connected and well-connected spaces, and introduce the tree like and the quasi-tree like topological spaces. We show that every finite well-connected T_0-space is an open image of a finite tree like topological space, and that every finite well-connected topological space is an open image of a finite quasi-tree like topological space. Most of these results are well-known and are scattered throughout the literature. For references we use the standard textbooks in general topology by Engelking [3], and by Kelley [5], as well as the papers by Kirk [6] and by Aiello et al [2]. We also introduce the tree sum of finitely many finite trees and the quasi-tree sum of finitely many finite quasi-trees. We prove that every finite connected T_0-space is an open image of the tree sum of finitely many finite trees, and that every finite connected space is an open image of the quasi-tree sum of finitely many finite quasi-trees. These results appear to be new, but see Shehtman [11] for somewhat similar constructions. In Section 3 we prove that a finite T_0-space is an open image of $\mathbf{R}$ iff it is connected. In Section 4 we extend this result to all finite topological spaces by showing that a finite topological space is an open image of $\mathbf{R}$ iff it is connected. In Section 5, as a consequence of our construction, we show that in order to refute a non-theorem of **S4** it is sufficient to consider only Boolean combinations of countable unions of convex subsets of $\mathbf{R}$, and that in order to refute a non-theorem of **Grz** it is sufficient to consider only Boolean combinations of open subsets of $\mathbf{R}$. Hence, every finite connected closure algebra is embedded into the closure algebra generated by the countable unions of convex subsets of $\mathbf{R}$, and every finite connected Grzegorczyk algebra is embedded into the Grzegorczyk algebra generated by the open subsets of $\mathbf{R}$. In logical terms, this means that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbf{R}$, and that **Grz** is complete with respect to Boolean combinations of open subsets of $\mathbf{R}$. Finally, in Section 6 we show that McKinsey and Tarski's result can not be extended to the countable case by proving that no countable Alexandroff space containing an infinite ascending chain is an open image of $\mathbf{R}$.

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2 Preliminaries

Denote by **Top** the category of topological spaces and continuous maps. Also let **Qos** denote the category of quasi-ordered sets and order preserving maps. A subset Y of a quasi-ordered set $(X, \leq)$ is said to be an *up-set* if $x \in Y$ and $x \leq y$ imply $y \in Y$. With every quasi-ordered set $(X, \leq)$ we can associate the topology $\tau_{\leq}$ on X whose opens are exactly the up-sets of $(X, \leq)$. Moreover, a map $f : (X_1, \leq_1) \to (X_2, \leq_2)$ is order preserving iff $f : (X_1, \tau_{\leq_1}) \to (X_2, \tau_{\leq_2})$ is continuous. Hence, we have a functor $F : \mathbf{Qos} \to \mathbf{Top}$.

Call a topological space (X, τ) an *Alexandroff space* if the intersection of any family of opens of (X, τ) is again an open of (X, τ). Denote by **Alex** the category of Alexandroff spaces and continuous maps. It is obvious that **Alex** is a full subcategory of **Top**. Moreover, one can easily check that F is one-to-one, and that the F-image of **Qos** is **Alex**.

To construct a functor $G : \mathbf{Top} \to \mathbf{Qos}$, recall that the *specialization order* on a topological space (X, τ) is defined by putting

$$x \leq_\tau y \text{ iff } x \in \overline{\{y\}}.$$

It is routine to check that $\leq_\tau$ is reflexive and transitive. Hence, $(X, \leq_\tau) \in \mathbf{Qos}$. Moreover, if $f : (X_1, \tau_1) \to (X_2, \tau_2)$ is continuous, then $f : (X_1, \leq_{\tau_1}) \to (X_2, \leq_{\tau_2})$ is order preserving. Therefore, $G : \mathbf{Top} \to \mathbf{Qos}$ is well-defined.

An easy calculation shows that G is a right adjoint to F, and when restricted to **Alex** these functors are inverse isomorphisms between the categories **Qos** and **Alex**. In addition, the identity map $id_X : FG(X, \tau) \to (X, \tau)$ is continuous and one can show that **Alex** is a coreflective subcategory of **Top**.

Since every finite topological space is an Alexandroff space, we obtain as an easy consequence that the category $\mathbf{Qos}_f$ of finite quasi-ordered sets and order preserving maps is isomorphic to the category $\mathbf{Top}_f$ of finite topological spaces and continuous maps.

Also recall that a map $f : (X_1, \leq_1) \to (X_2, \leq_2)$ is called a *p-morphism* if it is order preserving and

$$f(x) \leq_2 y \text{ implies } (\exists z \in X_1)(x \leq_1 z \ \& \ f(z) = y),$$

for any $x \in X_1$ and $y \in X_2$. We call a map $f : (X_1, \tau_1) \to (X_2, \tau_2)$ *open* if it is continuous and the f-image of any open set in (X_1, τ_1) is open in (X_2, τ_2). Now we have that $f : (X_1, \leq_1) \to (X_2, \leq_2)$ is a p-morphism iff $f : (X_1, \tau_{\leq_1}) \to (X_2, \tau_{\leq_2})$ is open. Hence, the category $\mathbf{Qos}^P$ of quasi-ordered sets and p-morphisms is isomorphic to the category $\mathbf{Alex}^O$ of Alexandroff spaces and open maps. As a particular case we obtain that the category $\mathbf{Qos}_f^P$ of finite quasi-ordered sets and p-morphisms is isomorphic to the category $\mathbf{Top}_f^O$ of finite topological spaces and open maps.

Let us also mention that $(X, \leq)$ is a partial order iff $(X, \tau_{\leq})$ is a T_0-space. So, we obtain that the category **Pos** of partially ordered sets and order preserving maps is isomorphic to the category $\mathbf{Alex}_{T_0}$ of Alexandroff T_0-spaces and continuous maps, and that the category $\mathbf{Pos}_f$ of finite partially ordered sets and order preserving maps is isomorphic to the category $(\mathbf{Top}_{T_0})_f$ of finite T_0-spaces and continuous maps.

Finally, if we restrict ourselves to open maps, we get that the category $\mathbf{Pos}^P$ of partially ordered sets and p-morphisms is isomorphic to the category $\mathbf{Alex}^O_{T_0}$ of Alexandroff T_0-spaces and open maps, and that the category $\mathbf{Pos}^P_f$ of finite partially ordered sets and p-morphisms is isomorphic to the category $(\mathbf{Top}^O_{T_0})_f$ of finite T_0-spaces and open maps.

In the light of this correspondence let us give order analogues of the topological notions of connectedness and well-connectedness. Recall that a subset of X is said to be *clopen* if it is both closed and open. A topological space (X, τ) is called *connected* if there are no clopens in (X, τ) other than $\emptyset$ and X.

For a quasi-ordered set $(X, \leq)$ and $x \in X$ let $\uparrow x = \{y \in X : x \leq y\}$ and $\downarrow x = \{y \in X : y \leq x\}$. Also for $Y \subseteq X$ let $\uparrow Y = \bigcup_{x \in Y} \uparrow x$ and $\downarrow Y = \bigcup_{x \in Y} \downarrow x$. It is obvious that Y is an up-set iff $Y = \uparrow Y$. We call Y a *down-set* if $Y = \downarrow Y$.

We say that there exists a $\leq$*-path* between two points x, y of a quasi-ordered set $(X, \leq)$ if there exists a sequence $w_1, \ldots, w_n$ of points of X such that $w_1 = x$, $w_n = y$, and either $w_i \leq w_{i+1}$ or $w_{i+1} \leq w_i$ for any $1 \leq i \leq n-1$. $(X, \leq)$ is said to be a *connected component* if there is a $\leq$-path between any two points of X.

Lemma 1 *A finite topological space (X, τ) is connected iff $(X, \leq_\tau)$ is a connected component.*

Proof: Recall that open sets of (X, τ) correspond to up-sets of $(X, \leq_\tau)$, and closed sets of (X, τ) correspond to down-sets of $(X, \leq_\tau)$. Hence, Y is a clopen in (X, τ) iff it is both up- and down-set in $(X, \leq_\tau)$. Now X is the only non-empty clopen of (X, τ) iff we can get X by applying $\uparrow$ and $\downarrow$ finitely many times to any $x \in X$. Hence, (X, τ) is connected iff $(X, \leq_\tau)$ is a connected component. **q.e.d.**

A topological space (X, τ) is called *well-connected* if there exists a least non-empty closed set in (X, τ). Since the complement of a clopen Y is also a clopen disjoint from Y, it follows that every well-connected space is connected. The converse however is not true.

For an element x of a quasi-ordered set $(X, \leq)$ let $C(x) = \{y \in X : x \leq y \;\&\; y \leq x\}$. $C \subseteq X$ is called a *cluster* if there is $x \in X$ such that $C = C(x)$. A quasi-ordered set $(X, \leq)$ is said to be *rooted* if there exists $r \in X$ such that $r \leq x$ for any $x \in X$. We call r a *root* of X. Note that r is not unique. Indeed, every element of $C(r)$ will also serve as a root of X. Obviously every rooted quasi-ordered set is a connected component, but not vice versa.

Lemma 2 *A finite topological space* (X, τ) *is well-connected iff* $(X, \leq_\tau)$ *is rooted.*

Proof: If $(X, \leq_\tau)$ is rooted with a root r, then $C(r)$ is a least non-empty closed subset of (X, τ). Hence, (X, τ) is well-connected. Conversely, suppose $(X, \leq_\tau)$ is not rooted. Then there exist $x, y \in X$ such that $\downarrow x \cap \downarrow y = \emptyset$. Hence, there is no least non-empty closed subset of (X, τ), and (X, τ) is not well-connected. **q.e.d.**

Suppose $(X, \leq)$ is a finite partially ordered set. A subset Y of X is said to be a *chain* if either $x \leq y$ or $y \leq x$ for any $x, y \in Y$. The *depth* of $x \in X$ is the number of elements of a maximal chain with the root x. The *depth* of $(X, \leq)$ is the supremum of the depths of all $x \in X$. Call y an *immediate successor* of x if $x < y$ and there is no z such that $x < z < y$. Call n the *branching* of $x \in X$ if n is the number of immediate successors of x. Call n the *branching* of $(X, \leq)$ if n is the supremum of branchings of all $x \in X$.

A finite partially ordered set $(X, \leq)$ is said to be a *tree* if $\downarrow x$ is a chain for any $x \in X$. A tree $(X, \leq)$ is said to be an *n-tree* if the branching of every element of X is n.

Lemma 3 *Kirk [6]*

(1) Every finite rooted partially ordered set is a p-morphic image of a finite tree.

(2) Every tree of branching n and depth m is a p-morphic image of the n-tree of depth m.

Proof (Sketch):

(1) Suppose $(X, \leq)$ is a finite partially ordered set with the root r. Let $T = \{(x_1, \ldots, x_n) : x_i \in X,\; r = x_1 < x_2 < \ldots < x_n\}$. Put $(x_1, \ldots, x_n) \leq_T$

$(y_1, \dots, y_m)$ if $(x_1, \dots, x_n)$ is the initial segment of $(y_1, \dots, y_m)$. Define $f : T \to X$ by putting $f(x_1, \dots, x_n) = x_n$. Then it is easy to check that $(T, \leq_T)$ is a finite tree, and that f is a p-morphism from $(T, \leq_T)$ onto $(X, \leq)$.

(2) Suppose $(T, \leq)$ is a tree of branching n and depth m. Let $(T_n^m, \leq_n^m)$ be the n-tree of depth m. To construct $g : T_n^m \to T$ start from the bottom and send the root $r_{T_n^m}$ of T_n^m to the root r_T of T. Then go one level up. If $x_1, \dots, x_k$, $k \leq n$, are the immediate successors of r_T and $y_1, \dots, y_n$ are the immediate successors of $r_{T_n^m}$, then send y_i to x_i for $i < k$ and send y_i to x_k for $k \leq i \leq n$. In order g to be a p-morphism the successors of y_i need to be send to the corresponding successors of y_j for $k \leq i \neq j \leq n$. After this go one more level up, and do the same. Eventually, after going through all m levels, we will get g which is a p-morphism from $(T_n^m, \leq_{T_n^m})$ onto $(T, \leq)$. **q.e.d.**

Corollary 4 *For every finite rooted partially ordered set $(X, \leq)$ there exists n such that $(X, \leq)$ is a p-morphic image of a finite n-tree.* **q.e.d.**

Suppose a finite quasi-ordered set $(X, \leq)$ is given. Define an equivalence relation $\sim$ on X by putting $x \sim y$ iff x, y belong to the same cluster. Denote the quotient of X under $\sim$ by $(X/\sim, \leq_\sim)$. Obviously $(X/\sim, \leq_\sim)$ is a partial order, which we call the *skeleton* of $(X, \leq)$.

Call $(X, \leq)$ a *quasi-tree* if $(X/\sim, \leq_\sim)$ is a tree. Call $(X, \leq)$ a *quasi-n-tree* if $(X/\sim, \leq_\sim)$ is an n-tree. Call $(X, \leq)$ a *quasi-(q, n)-tree* if $(X, \leq)$ is a quasi-n-tree and every cluster of $(X, \leq)$ consists of q elements.

The following lemma is an easy generalization of Corollary 4 to quasi-ordered sets.

Lemma 5 *For every finite rooted quasi-ordered set $(X, \leq)$ there exist q, n such that $(X, \leq)$ is a p-morphic image of a finite quasi-(q, n)-tree.*

Proof: Let q be the supremum of the cardinalities of $C(x)$ for all $x \in X$. Then replacing every cluster of X by a q-element cluster, we get a new quasi-ordered set $(Y, \leq)$, which is regular in the sense that every cluster of Y contains exactly q elements. Obviously there is a p-morphism from Y to X: suppose a cluster of X consists of m elements $x_1, \dots, x_m$, and the corresponding cluster of Y consists of q elements $y_1, \dots, y_q$. Send y_i to x_i for $i < m$, and send y_i to x_m for $m \leq i \leq q$.

Note that $(X/\sim, \leq_\sim)$ is isomorphic to $(Y/\sim, \leq_\sim)$. Now from the previous corollary we know that there exists n such that $(Y/\sim, \leq_\sim)$ is a p-morphic image of an n-tree $(T_n, \leq_n)$. Let this p-morphism be f. Denote by $(T_{q,n}, \leq_{q,n})$ the quasi-tree obtained from $(T_n, \leq_n)$ by replacing every node t of T_n by a q-element cluster $[t] = \{t_1, \dots, t_q\}$. Obviously $(T_{q,n}, \leq_{q,n})$ is a finite quasi-(q,n)-tree. Suppose $[y] = \{y_1, \dots, y_q\}$ is an element of $Y/\sim$ and t is an element of T_n. Define $h : T_{q,n} \to Y$ by putting $h(t_i) = y_i$ if $f(t) = y$, $t_i \in [t]$ and $y_i \in [y]$ for $1 \leq i \leq q$. Since $h[t] = f(t)$ and f is an onto p-morphism, so is h. So, $(Y, \leq)$ is a p-morphic image of $(T_{q,n}, \leq_{q,n})$, and since $(X, \leq)$ is a p-morphic image of $(Y, \leq)$, $(X, \leq)$ is a p-morphic image of $(T_{q,n}, \leq_{q,n})$ too. **q.e.d.**

The topological spaces corresponding to trees and quasi-trees will be called *tree like* and *quasi-tree like*, respectively. The following is the topological version of Corollary 4 and Lemma 5.

Corollary 6 *(1) For every finite well-connected T_0-space (X, τ) there exists n such that (X, τ) is an open image of a finite n-tree like topological space.*

(2) For every finite well-connected topological space (X, τ) there exist q, n such that (X, τ) is an open image of a finite quasi-(q,n)-tree like topological space. **q.e.d.**

To extend this result to finite connected topological spaces we will use the following construction. Suppose $T_1, \dots, T_n$ are finite trees (of branching ≥ 2). Let t_i^l and t_i^r denote two distinct maximal nodes of T_i.[1] Consider the disjoint union $\bigsqcup_{i=1}^n T_i$, and identify t_i^l with t_{i-1}^r and t_i^r with t_{i+1}^l, respectively. Call this construction the *tree sum* of $T_1, \dots, T_n$ and denote it by $\bigoplus_{i=1}^n T_i$.

We can generalize this construction to quasi-trees. Suppose $T_1, \dots, T_n$ are finite *q-regular* quasi-trees (of branching ≥ 2), meaning that every cluster of each T_i consists of q elements. Let C_i^l and C_i^r denote two distinct maximal clusters of T_i.[2] Consider the disjoint union $\bigsqcup_{i=1}^n T_i$, and identify C_i^l with C_{i-1}^r and C_i^r with C_{i+1}^l, respectively. Call this construction the *regular quasi-tree sum* of $T_1, \dots, T_n$ and denote it by $\bigoplus_{i=1}^n T_i$.

[1]Recall that an element x of a partially ordered set $(X, \leq)$ is said to be *maximal* if $x \leq y$ implies $x = y$ for any $y \in X$. An element $x \in X$ is said to be *minimal* if $y \leq x$ implies $y = x$ for any $y \in X$.

[2]Recall that a cluster C is called *maximal* if the elements of C can see only the elements of C. A cluster C is called *minimal* if the elements of the other clusters can not see the elements of C.

Lemma 7 *(1) For every finite partially ordered connected component* $(X, \leq)$ *there exist trees* $T_1, \ldots, T_n$ *such that* $(X, \leq)$ *is a p-morphic image of* $\bigoplus_{i=1}^{n} T_i$.

(2) For every finite connected component $(X, \leq)$ *there exist q-regular quasi-trees* $T_1, \ldots, T_n$ *such that* $(X, \leq)$ *is a p-morphic image of* $\bigoplus_{i=1}^{n} T_i$.

Proof: (1) is a particular case of (2).

(2) Suppose $(X, \leq)$ is a finite connected component. Let $C_1, \ldots, C_n$ be all minimal clusters of $(X, \leq)$. Consider $(\uparrow C_1, \leq_1), \ldots, (\uparrow C_n, \leq_n)$, where $\leq_i$ is the restriction of $\leq$ to $\uparrow C_i$. Obviously each $(\uparrow C_i, \leq_i)$ is a finite rooted quasi-ordered set and $\bigcup_{i=1}^{n} C_i = X$. As follows from Lemma 5, for each $(\uparrow C_i, \leq_i)$ there exist q_i, m_i such that $(\uparrow C_i, \leq_i)$ is a p-morphic image of a finite quasi-(q_i, m_i)-tree. Let $q = sup\{q_i\}_{i=1}^{n}$, and consider quasi-(q, m_i)-trees $T_1, \ldots, T_n$. Obviously every $(\uparrow C_i, \leq_i)$ is a p-morphic image of T_i. Denote these p-morphisms by f_i. Since every T_i is q-regular, we can form $\bigoplus_{i=1}^{n} T_i$. Assume without loss of generality that f_i agrees with f_{i-1} on C_i^l and C_{i-1}^r, and that f_i agrees with f_{i+1} on C_i^r and C_{i+1}^l, which are identified in $\bigoplus_{i=1}^{n} T_i$. Now define $f : \bigoplus_{i=1}^{n} T_i \to X$ by putting $f(t) = f_i(t)$, if $t \in T_i$. It is routine to check that f is well-defined and that it is an onto p-morphism. **q.e.d.**

The topological version of this lemma is expressed as follows.

Corollary 8 *(1) For every finite connected* T_0*-space* (X, τ) *there exist tree like topological spaces* $T_1, \ldots, T_n$ *such that* (X, τ) *is an open image of* $\bigoplus_{i=1}^{n} T_i$.

(2) For every finite connected topological space (X, τ) *there exist q-regular quasi-tree like topological spaces* $T_1, \ldots, T_n$ *such that* (X, τ) *is an open image of* $\bigoplus_{i=1}^{n} T_i$. **q.e.d.**

3 Finite T_0 open images of R

Now we are in a position to characterize finite T_0 open images of **R**. Our strategy is the following. First we show that every finite n-tree like topological space is an open image of **R**. Then we prove that actually the tree sum of finitely many finite tree like topological spaces is also an open image of **R**. It will imply that every finite connected T_0-space is an open image of **R**. Since **R** is connected and open (even continuous) onto maps preserve connectedness, it will follow that a finite T_0-space is an open image of **R** iff it is connected.

We start by showing that the n-tree T of depth 2 shown in Fig.1 below is an open image of any bounded interval $I \subseteq \mathbf{R}$.

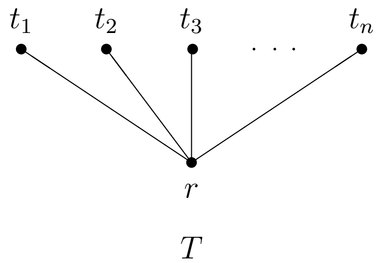


Fig.1

Suppose $a, b \in \mathbf{R}$, $a < b$, $I = (a, b)$, $I = [a, b)$, $I = (a, b]$, or $I = [a, b]$. Recall that the Cantor set $\mathcal{C}$ is constructed inside I by taking out open intervals from I infinitely many times. More precisely, in step 1 of the construction, the open interval

$$I_1^1 = (a + \frac{b-a}{3}, a + \frac{2(b-a)}{3})$$

is taken out. Denote the remaining closed intervals by J_1^1 and J_2^1, respectively.

In step 2, the open intervals

$$I_1^2 = (a + \frac{b-a}{3^2}, a + \frac{2(b-a)}{3^2}) \text{ and } I_2^2 = (a + \frac{7(b-a)}{3^2}, a + \frac{8(b-a)}{3^2})$$

are taken out. Denote the remaining closed intervals by J_1^2, J_2^2, J_3^2 and J_4^2, respectively.

In general, in step m, the open intervals $I_1^m, \ldots, I_{2^{m-1}}^m$ are taken out, and the closed intervals $J_1^m, \ldots, J_{2^m}^m$ remain. We will use the construction of $\mathcal{C}$ to obtain T as an open image of I.

Lemma 9 *T is an open image of I.*

Proof: Define $f_I^T : I \to T$ by putting

$$f_I^T(x) = \begin{cases} t_k & \text{if } x \in \bigcup_{m \equiv k (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \\ r & \text{otherwise} \end{cases}$$

Obviously, f_I^T is a well-defined onto map. Moreover,

$$(f_I^T)^{-1}(t_k) = \bigcup_{m \equiv k (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I_p^m \quad \text{and} \quad (f_I^T)^{-1}(r) = \mathcal{C}.$$

Let us show that f_I^T is open. Since the singletons $\{t_k\}$ $(1 \leq k \leq n)$ form a subbasis of T, continuity of f_I^T is obvious. Suppose U is an open interval in I. If $U \cap \mathcal{C} = \emptyset$, then $f_I^T(U) \subseteq \{t_1, \ldots, t_n\}$ and hence is open. Suppose $U \cap \mathcal{C} \neq \emptyset$. Then there exists $c \in U \cap \mathcal{C}$. Since $c \in \mathcal{C}$, $f_I^T(c) = r$. From $c \in U$ it follows that there is $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. Pick m so that $\frac{b-a}{3^m} < \varepsilon$. Since $c \in \mathcal{C}$, there is $k \in \{1, \ldots, 2^m\}$ such that $c \in J_m^k$. Moreover, since the length of J_m^k is equal to $\frac{b-a}{3^m}$, we have $J_m^k \subseteq U$. Therefore, U contains the points removed from J_m^k in the subsequent iterations of the construction of $\mathcal{C}$. Thus, $f_I^T(U) \supseteq \{t_1, \ldots, t_n\}$ and $f_I^T(U) = T$. Hence, $f_I^T(U)$ is open for any open interval U of I. It follows that f_I^T is an onto open map. **q.e.d.**

Theorem 10 *Every finite n-tree is an open image of I.*

Proof: We define a map $f_I : I \to T$ for an arbitrary finite n-tree T by induction on the depth of T. If the depth of T is 2, then $f_I(x) = f_I^T(x)$ for any $x \in I$. As follows from the previous lemma, f_I is onto and open. Suppose the depth of T is $d+1$, $d \geq 2$. Let $t_1, \ldots, t_m$ be the elements of T of depth d, and let T_d be the subtree of T of all elements of T of depth $\leq d$. Note that $\uparrow t_k$ is isomorphic to the n-tree of depth 2 for any $k \in \{1, \ldots, m\}$, and that T_d is the n-tree of depth d. So, by the induction hypothesis there exists an onto open map $f_I^d : I \to T_d$. We use f_I^d to define a map $f_I : I \to T$ as follows. For each $k \in \{1, \ldots, m\}$ and $x \in (f_I^d)^{-1}(t_k)$, define I_x to be the connected component of $(f_I^d)^{-1}(t_k)$ containing x. Now set

$$f_I(x) = \begin{cases} f_I^d(x) & \text{if } f_I^d(x) \notin \{t_1, \ldots, t_m\} \\ f_{I_x}^{\uparrow t_k}(x) & \text{if } f_I^d(x) = t_k \end{cases}$$

It is clear that f_I is a well-defined onto map. To show that f_I is continuous observe that for $t \in T - T_d$ there is t_k such that $t_k < t$. Hence we have

$$f_I^{-1}(t) = \bigcup \{(f_{I'}^{\uparrow t_k})^{-1}(t) : I' \text{ is a connected component of } (f_I^d)^{-1}(t_k)\},$$

and that for $t \in T_d$ we have

$$f_I^{-1}(\uparrow_T t) = (f_I^d)^{-1}(\uparrow_{T_d} t).$$

Now since $\{t\}$ for $t \in T - T_d$ and $\uparrow_T t$ for $t \in T_d$ form a subbasis of T, f_I is continuous.

To show that f_I is open, let $U = (c, d)$ be an open interval in I. If $U \subseteq I'$ where I' is a connected component of $(f_I^d)^{-1}(t_k)$ for some k, then $f_I(U) = f_{I'}^{\uparrow t_k}(U)$, which is open by the previous lemma. Assume $U \not\subseteq I'$ for any k and I'. Then $f_I(U) = \uparrow f_I^d(U)$. Indeed, if $t \in T - \uparrow \{t_1, \ldots, t_m\}$, then $f_I^{-1}(t) = (f_I^d)^{-1}(t)$, and thus $t \in f_I(U)$ iff $t \in f_I^d(U)$. Suppose $t \in \uparrow t_k$ for some k. Then if $t \in f_I(U)$, there is $x \in U$ with $f_I(x) = t$. Hence, by the definition of f_I, there exists a connected component I' of $(f_I^d)^{-1}(t_k)$ with $x \in I'$ and $f_I(x) = f_{I'}^{\uparrow t_k}(x)$. Therefore, $x \in U \cap (f_I^d)^{-1}(t_k)$, which implies $t_k \in f_I^d(U)$. Hence $t \in \uparrow t_k \subseteq \uparrow f_I^d(U)$. Conversely, if $t \in \uparrow f_I^d(U)$, then there exist $k \in \{1, \ldots, m\}$ and $x \in U$ with $f_I^d(x) = t_k \leq t$. Hence $x \in (f_I^d)^{-1}(t_k)$, and there is a connected component $I' = (p, q)$ of $(f_I^d)^{-1}(t_k)$ containing x. Now since $U \cap I' \neq \emptyset$ and $U \not\subseteq I'$ by assumption, we have $U \cap I'$ is either (p, d) or (c, q). But $f_I(U) \supseteq f_I(U \cap I') = f_{I'}^{\uparrow t_k}(U \cap I') = \uparrow t_k$ since both (p, d) and (c, q) must intersect the Cantor set constructed on I' and $f_{I'}^{\uparrow t_k}$ is open. Thus $t \in \uparrow t_k \subseteq f_I(U)$. Therefore, $f_I(U) = \uparrow f_I^d(U)$, which is open. Hence f_I is an onto open map, and T is an open image of I. **q.e.d.**

Corollary 11 *Every finite tree is an open image of I.*

Proof: This directly follows from Lemma 3, Theorem 10 and the fact that the composition of open maps is open as well. **q.e.d.**

Theorem 12 *The tree sum of finitely many finite trees is an open image of* $\mathbf{R}$.

Proof: Suppose $T_1, \ldots, T_n$ are finite trees. Consider $\bigoplus_{k=1}^n T_k$. For $2 \leq k \leq n-1$ let t_k^l and t_k^r denote the maximal nodes of T_k which got identified with the corresponding nodes t_{k-1}^r of T_{k-1} and t_{k+1}^l of T_{k+1}, respectively. Also let $I_1 = (0, 1], I_k = [2k-2, 2k-1]$, for $k \in \{2, \ldots, n-1\}$, and $I_n = [2n-2, 2n-1)$. From the previous corollary it follows that for each I_k there exists an onto open map $f_{I_k} : I_k \to T_k$. Define $f : (0, 2n-1) \to \bigoplus_{k=1}^n T_k$ by putting

$$f(x) = \begin{cases} f_{I_k}(x) & \text{if } x \in I_k \\ t_k^r & \text{if } x \in (2k-1, 2k) \\ f_{I_{k+1}}(x) & \text{if } x \in I_{k+1} \end{cases}$$

Here $k \in \{1, \ldots, n-1\}$. It is obvious that f is a well-defined onto map. For $t \in T_k$, observe that if $t_k^l, t_k^r \notin \uparrow t$, then

$$f^{-1}(\uparrow t) = f_{I_k}^{-1}(\uparrow t);$$

if $t_k^l \in \uparrow t$ and $t_k^r \notin \uparrow t$, then

$$f^{-1}(\uparrow t) = f_{I_k}^{-1}(\uparrow t) \cup f_{I_{k-1}}^{-1}(t_{k-1}^r) \cup (2k-3, 2k-2);$$

if $t_k^l \notin \uparrow t$ and $t_k^r \in \uparrow t$, then

$$f^{-1}(\uparrow t) = f_{I_k}^{-1}(\uparrow t) \cup f_{I_{k+1}}^{-1}(t_{k+1}^l) \cup (2k-1, 2k);$$

and finally if $t_k^l, t_k^r \in \uparrow t$, then

$$f^{-1}(\uparrow t) = f_{I_k}^{-1}(\uparrow t) \cup f_{I_{k-1}}^{-1}(t_{k-1}^r) \cup f_{I_{k+1}}^{-1}(t_{k+1}^l) \cup (2k-3, 2k-2) \cup (2k-1, 2k).$$

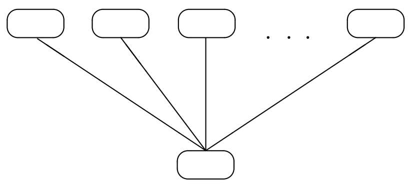
Hence, f is continuous. To show f is open, notice that for each $k \in \{1, \ldots, n\}$ any open (i.e. up-set) of T_k is open (i.e. up-set) in $\bigoplus_{k=1}^n T_k$. Also observe that for an open interval $U \subseteq (0, 2n-1)$, if $U \subseteq I_k$, then $f(U) = f_{I_k}(U)$, and if $U \subseteq (2k-1, 2k)$, then $f(U) = \{t_k^r\}$. In either case $f(U)$ is open in $\bigoplus_{k=1}^n T_k$. Now every open interval $U \subseteq (0, 2n-1)$ is the union $U = U_1 \cup \ldots \cup U_{2n}$, where $U_{2k} = U \cap (2k-1, 2k)$ for $k = 1, \ldots, n-1$, and $U_{2k+1} = U \cap I_k$ for $k = 0, \ldots, n-1$. Thus, $f(U) = f(U_1) \cup \ldots \cup f(U_{2n-1})$, which is a union of open sets in $\bigoplus_{k=1}^n T_k$. Hence f is an onto open map, and $\bigoplus_{k=1}^n T_k$ is an open image of $(0, 2n-1)$. Since $(0, 2n-1)$ is homeomorphic to $\mathbf{R}$, $\bigoplus_{k=1}^n T_k$ is an open image of $\mathbf{R}$. q.e.d.

Corollary 13 *A finite T_0-space is an open image of* $\mathbf{R}$ *iff it is connected.*

Proof: It follows from Corollary 8 and Theorem 12 that every finite connected T_0-space is an open image of $\mathbf{R}$. Conversely, since $\mathbf{R}$ is connected and open (even continuous) images of connected spaces are connected, finite T_0 images of $\mathbf{R}$ are connected. q.e.d.

4 Finite open images of R

In this section we will generalize our results of Section 3. Most importantly, we prove that a finite topological space is an open image of $\mathbf{R}$ iff it is connected. Our strategy is similar to Section 3. But this time we will work with quasi-trees rather than trees. We start by showing that the quasi-(q,n)-tree T of depth 2 shown in Fig.2 below is an open image of I.



T

Fig.2

Recall that a subset A of a topological space X is called *dense* if $\overline{A} = X$. Dually, A is called *boundary* if $Int(A) = \emptyset$. Here $\overline{()}$ and $Int()$ denote the closure and the interior operators on X, respectively.

Lemma 14 *If X has a countable basis and every countable subset of X is boundary, then for any natural number n there exist dense and boundary disjoint subsets $A_1, \ldots, A_n$ of X such that $X = \bigcup_{i=1}^{n} A_i$.*

Proof: Suppose $\{B_i\}_{i=1}^{\infty}$ is a countable basis of X. From each B_i pick out a point x_i^1 so that $x_i^1 \neq x_j^1$ if $i \neq j$, and let $A_1 = \{x_i^1\}_{i=1}^{\infty}$. Now from each $B_i - \{x_i^1\}$ pick out a point x_i^2 so that $x_i^2 \neq x_j^2$ if $i \neq j$, and let $A_2 = \{x_i^2\}_{i=1}^{\infty}$. Do the same construction $(n-1)$-times, and put $A_n = X - \bigcup_{i=1}^{n-1} A_i$. Note that since every countable subset of X is boundary, every B_i is uncountable. So, we can perform our construction. It is clear then that all A_i are disjoint and $X = \bigcup_{i=1}^{n} A_i$. Further, every A_i contains at least one point from every open base set. Hence, every A_i is dense. Furthermore, no open base set is a subset of any A_i. Therefore, every A_i is boundary. **q.e.d.**

Lemma 15 *T is an open image of I.*

Proof: Since the Cantor set $\mathcal{C}$ as well as every I_p^m $(1 \leq p \leq 2^{m-1}$, $m \in \omega)$ satisfy the conditions of the previous lemma, each of them can be divided into q-many dense and boundary disjoint subsets. For $\mathcal{C}$ let them be $\mathcal{C}_1, \ldots, \mathcal{C}_q$ and for I_p^m let them be $(I_p^m)^1, \ldots, (I_p^m)^q$. Denote the least cluster of T by r

and its elements by $r_1, \ldots, r_q$. Also for $1 \leq i \leq n$ denote the i-th maximal cluster of T by t^i and its elements by $t^i_1, \ldots, t^i_q$. Define $f^T_I : I \to T$ by putting

$$f^T_I(x) = \begin{cases} t^i_k & \text{if } x \in \bigcup_{m \equiv i (mod\ n)} \bigcup_{p=1}^{2^{m-1}} (I^m_p)^k \\ r_k & \text{if } x \in \mathcal{C}_k \end{cases}$$

Here $k = 1, \ldots, q$. Similarly to Lemma 9, we have

$$(f^T_I)^{-1}(t^i) = \bigcup_{m \equiv i (mod\ n)} \bigcup_{p=1}^{2^{m-1}} I^m_p \quad \text{and} \quad (f^T_I)^{-1}(r) = \mathcal{C}.$$

Hence, f^T_I is continuous. Suppose U is an open interval in I. If $U \cap \mathcal{C} = \emptyset$, then $f^T_I(U) \subseteq \bigcup_{i=1}^n t^i$. Moreover, since $(I^m_p)^1, \ldots, (I^m_p)^q$ partition I^m_p into q-many dense (and boundary) disjoint subsets, $U \cap I^m_p \neq \emptyset$ implies $U \cap (I^m_p)^k \neq \emptyset$ for any $k \in \{1, \ldots, q\}$. Hence, if $f^T_I(U)$ contains an element of a cluster, it contains the whole cluster. Thus, $f^T_I(U)$ is open. Suppose $U \cap \mathcal{C} \neq \emptyset$. Then there exists $c \in U \cap \mathcal{C}$. Since $c \in \mathcal{C}$, $f^T_I(c) \in r$. Again since $\mathcal{C}_1, \ldots, \mathcal{C}_q$ partition $\mathcal{C}$ into q-many dense (and boundary) disjoint subsets, $r \subseteq f^T_I(U)$. Now the same argument as in the proof of Lemma 9 guarantees that every point greater than a point in r also belongs to $f^T_I(U)$. Hence $f^T_I(U) = T$, implying that f^T_I is an open map. **q.e.d.**

Theorem 16 *Every finite quasi-(q, n)-tree is an open image of I.*

Proof: This follows along the same lines as the proof of Theorem 10 but is based on Lemma 15 instead of Lemma 9. **q.e.d.**

Corollary 17 *Every finite quasi-tree is an open image of I.*

Proof: This directly follows from Lemma 5, Theorem 16, and the fact that the composition of open maps is open as well. **q.e.d.**

Theorem 18 *The regular tree sum of finitely many finite q-regular quasi-trees is an open image of* $\mathbf{R}$.

Proof: This follows along the same lines as the proof of Theorem 12 but is based on Theorem 16 instead of Theorem 10. In addition, according to Lemma 14, for $k = 1, \ldots, n-1$ we divide each interval $(2k-1, 2k)$ into q-many dense and boundary disjoint subsets $J_1^k, \ldots, J_q^k$ and define $f : (0, 2n-1) \to \bigoplus_{k=1}^n T_k$ by putting

$$f(x) = \begin{cases} f_{I_k}(x) & \text{if } x \in I_k \\ (t_k^r)_i & \text{if } x \in J_i^k \\ f_{I_{k+1}}(x) & \text{if } x \in I_{k+1} \end{cases}$$

As a result, we obtain that $\bigoplus_{k=1}^n T_k$ is an open image of $(0, 2n-1)$, and hence is an open image of **R**. **q.e.d.**

Corollary 19 *A finite topological space is an open image of* **R** *iff it is connected.*

Proof: It follows from Corollary 8 and Theorem 18 that every finite connected topological space is an open image of **R**. Conversely, since **R** is connected and open (even continuous) images of connected spaces are connected, finite images of **R** are connected. **q.e.d.**

5 Completeness of S4 and Grz

In this section we observe that the results of Section 3 lead to completeness of **Grz** with respect to the closure algebra generated by the open sets of **R**, and that the results of Section 4 lead to completeness of **S4** with respect to the closure algebra generated by the countable unions of convex subsets of **R**. This will also imply that **S4** is complete with respect to the closure algebra of Borel sets over open subsets of **R**. The last two observations provide a strengthening of the result by McKinsey and Tarski [7], [8].

Recall that McKinsey and Tarski proved every finite well-connected closure algebra is embedded into the closure algebra of the power set of **R**. Pucket [10] extended their result to all finite connected closure algebras. We are in a position now to show that every finite connected Grzegorczyk algebra is embedded into the closure algebra generated by open subsets of **R**, and that every finite connected closure algebra is embedded into the closure algebra generated by countable unions of convex subsets of **R**.

Denote by $Op(\mathbf{R})$ the set of all open subsets of **R**. Let $B(Op(\mathbf{R}))$ denote the Boolean algebra generated by $Op(\mathbf{R})$. Recall that a subset X of **R** is said to be *convex* if $x, y \in X$ implies every $z \in [x, y]$ also belongs to X. Denote by $C(\mathbf{R})$ the set of all convex subsets of $\mathbf{R}$, and by $C_\infty(\mathbf{R})$ the set of all countable unions of convex subsets of $\mathbf{R}$. Obviously every open interval of $\mathbf{R}$ belongs to $C(\mathbf{R})$. Now since every open subset of $\mathbf{R}$ is a countable union of open intervals of $\mathbf{R}$, every open subset of $\mathbf{R}$ belongs to $C_\infty(\mathbf{R})$. Moreover, since every singleton subset of $\mathbf{R}$ belongs to $C(\mathbf{R})$, every countable subset of $\mathbf{R}$ also belongs to $C_\infty(\mathbf{R})$. However, not every subset of $\mathbf{R}$ belongs to $C_\infty(\mathbf{R})$. An example is the Cantor set $\mathcal{C}$. Since $\mathcal{C}$ is the complement of an element of $C_\infty(\mathbf{R})$, $C_\infty(\mathbf{R})$ does not form a Boolean algebra. Let $B(C_\infty(\mathbf{R}))$ denote the Boolean algebra generated by $C_\infty(\mathbf{R})$. Since $Op(\mathbf{R}) \subseteq C_\infty(\mathbf{R})$, it is obvious that $B(Op(\mathbf{R})) \subseteq B(C_\infty(\mathbf{R}))$. Moreover, this inclusion is proper since the set $\mathbf{Q}$ of all rational numbers belongs to $B(C_\infty(\mathbf{R}))$ but does not belong to $B(Op(\mathbf{R}))$.

Denote by **Borel** the Boolean algebra of Borel sets over open subsets of $\mathbf{R}$. Obviously $B(C_\infty(\mathbf{R})) \subseteq$ **Borel**. This inclusion is also proper since $B(C_\infty(\mathbf{R}))$ is contained within a finite level of the Borel hierarchy over $\mathbf{R}$. Finally, let $P(\mathbf{R})$ denote the power set algebra of $\mathbf{R}$. Then **Borel** is a proper subalgebra of $P(\mathbf{R})$ since every element of **Borel** is measurable, while there exist non-measurable subsets of $\mathbf{R}$.

Hence we obtain the four Boolean algebras over $\mathbf{R}$ forming a proper chain: $B(Op(\mathbf{R})) \subset B(C_\infty(\mathbf{R})) \subset$ **Borel** $\subset P(\mathbf{R})$. Since all closed sets of $\mathbf{R}$ belong to $B(Op(\mathbf{R}))$, we have that each of these four algebras forms a closure algebra with respect to the closure operator $\overline{()}$ on $\mathbf{R}$. Now we will show that **Grz** is complete with respect to $(B(Op(\mathbf{R})), \overline{()})$, and that **S4** is complete with respect to any of the other three closure algebras.

Lemma 20 *(1) Every finite connected Grzegorczyk algebra is embedded into the closure algebra* $(B(Op(\mathbf{R})), \overline{()})$.

(2) Every finite connected closure algebra is embedded into the closure algebra $(B(C_\infty(\mathbf{R})), \overline{()})$.

Proof: (1) It is well-known (see e.g. Esakia [4]) that finite Grzegorczyk algebras are the power set algebras of finite partially ordered sets, and hence the power set algebras of finite T_0-spaces. Now as follows from Corollary 13, every finite connected T_0-space X is an open image of $\mathbf{R}$. Moreover, as follows from Theorems 10 and 12, the inverse image of a subset of X is a countable union of intervals of $\mathbf{R}$ and the Cantor sets constructed on intervals of $\mathbf{R}$, which belong to $B(Op(\mathbf{R})$. Hence a finite connected Grzegorczyk algebra is embedded into $(B(Op(\mathbf{R})), \overline{()})$.

(2) It is well-known (see e.g. McKinsey and Tarski [7]) that finite closure algebras are the power set algebras of finite topological spaces. Now as follows from Corollary 19, every finite connected space X is an open image of $\mathbf{R}$. Moreover, as follows from Theorems 16 and 18, the inverse image of a subset of X is a countable union of dense and boundary subsets of intervals of $\mathbf{R}$ and the Cantor sets constructed on intervals of $\mathbf{R}$. The proof of Lemma 14 guarantees that those subsets of $\mathbf{R}$ belong to $B(C_\infty(\mathbf{R})$. Hence a finite connected closure algebra is embedded into $(B(C_\infty(\mathbf{R})), \overline{()})$. q.e.d.

Theorem 21 *(1) The variety of Grzegorczyk algebras is generated by the Grzegorczyk algebra* $(B(Op(\mathbf{R})), \overline{()})$.

(2) The variety of closure algebras is generated by any of the following three closure algebras $(B(C_\infty(\mathbf{R})), \overline{()})$, $(\mathbf{Borel}, \overline{()})$, *and* $(P(\mathbf{R}), \overline{()})$.

Proof: (1) It is well-known (see e.g. Esakia [4]) that the variety of Grzegorczyk algebras is generated by finite (well-)connected Grzegorczyk algebras. Moreover, $(B(Op(\mathbf{R})), \overline{()})$ is a Grzegorczyk algebra since $Op(\mathbf{R})$ is a Heyting algebra and the Boolean algebra generated by a Heyting algebra always forms a Grzegorczyk algebra ([4]). Now apply Lemma 20 (1).

(2) It is well-known (see e.g. McKinsey and Tarski [7]) that the variety of closure algebras is generated by finite (well-)connected closure algebras. Applying Lemma 20 (2) we obtain that the variety of closure algebras is generated by $(B(C_\infty(\mathbf{R})), \overline{()})$. Now since $(B(C_\infty(\mathbf{R})), \overline{()})$ is a subalgebra of both $(\mathbf{Borel}, \overline{()})$ and $(P(\mathbf{R}), \overline{()})$, the result follows. q.e.d.

In logical terms, Theorem 21 tells us that **Grz** is complete with respect to Boolean combinations of open subsets of $\mathbf{R}$, and that **S4** is complete with respect to Boolean combinations of countable unions of convex subsets of $\mathbf{R}$, as well as with respect to Borel sets over open subsets of $\mathbf{R}$.

6 Countable connected spaces which are not open images of R

In this final section we show that our results of Sections 3 and 4 can not be generalized to the countable case. For this we need to recall the Baire category theorem:

Theorem 22 *(Baire) A complete metric space X is not the union of countably many closed boundary subsets of X.* q.e.d.

Now we are in a position to prove the following:

Theorem 23 *If $(X, \leq)$ is a countable quasi-ordered set containing an infinite ascending chain, then $(X, \tau_\leq)$ is not an open image of* $\mathbf{R}$.

Proof: Suppose $(X, \leq)$ is a countable quasi-ordered set, $x_1 < x_2 < x_3 < \ldots$ is an infinite ascending chain in X, and there is an onto open map $f : \mathbf{R} \to X$. Let $Y = \downarrow \{x_1, x_2, \ldots\}$. Obviously Y is a closed subset of $(X, \tau_\leq)$. Hence $f^{-1}(Y)$ is a closed subset of $\mathbf{R}$. Since $\mathbf{R}$ is a complete metric space and $f^{-1}(Y)$ is a closed subset of $\mathbf{R}$, $f^{-1}(Y)$ with the subspace topology is a complete metric space. Moreover,

$$Y = \bigcup_{i=1}^{\infty} \downarrow x_i \quad \text{and hence} \quad f^{-1}(Y) = \bigcup_{i=1}^{\infty} f^{-1}(\downarrow x_i).$$

Obviously, $\downarrow x_i$ is closed in Y. Hence, $f^{-1}(\downarrow x_i)$ is closed in $f^{-1}(Y)$. Moreover, if $f^{-1}(\downarrow x_i)$ is not a boundary subset of $f^{-1}(Y)$, then there is an open interval I of $\mathbf{R}$ such that $I \cap f^{-1}(Y) \subseteq f^{-1}(\downarrow x_i)$. But then $f(I) \cap Y \subseteq \downarrow x_i$, and $f(I) \cap Y$ is not open in Y. Hence, $f(I)$ is not open in X, which contradicts openness of f. Hence, $f^{-1}(\downarrow x_i)$ is a closed and boundary subset of $f^{-1}(Y)$, and since $f^{-1}(Y)$ is a complete metric space, by Baire's theorem, it can not be the union of the sets $f^{-1}(\downarrow x_i)$, $i \geq 1$. This is a contradiction and thus no such open map exists. q.e.d.

Obviously, the simplest Alexandroff space containing an infinite ascending chain is $(\mathbf{N}, \tau_\leq)$, where $(\mathbf{N}, \leq)$ denotes the set of natural numbers with its standard order. Observe that $(\mathbf{N}, \tau_\leq)$ is a well-connected T_0-space. Hence, there exist even countable well-connected Alexandroff T_0-spaces which are not open images of $\mathbf{R}$.

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