AN APPLICATION OF RADO'S THEOREM TO DISJOINT MATCHINGS IN BIPARTITE GRAPHS

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For the terminology and notation we follow Bondy and Murty [1] in graph theory and Welsh [3] in matroid theory. All sets considered are finite.

Let G be a bipartite graph with bipartition (X, Y) and let V and E denote, respectively, the vertex and edge sets of G. A matching M in G is called a *matching of X into Y* if every vertex in X is M-saturated. If S is any subset of V, we shall write $\delta(S)$ to denote the set of edges incident with S; we simplify $\delta(\{y\})$ to $\delta(y)$. Lebensold [2] proved the following result.

THEOREM. *Given a positive integer k, G has k (edge-) disjoint matchings of X into Y if and only if*

$$\sum_{y \in Y} \min \{k, |\delta(S) \cap \delta(y)|\} \geqslant k\,|S| \quad \textit{for all} \quad S \subseteq X. \tag{1}$$

The necessity part of this theorem is easily verified. The purpose of this note is to give a simple proof of the sufficiency part using the well-known theorem of Rado (see [3; page 98]).

Define a subset I of E to be *independent* if

$$|I \cap \delta(y)| \leqslant k \quad \text{for all} \quad y \in Y.$$

If $\mathscr{I}$ is the collection of all such independent subsets of E, then $Q = (E, \mathscr{I})$ is a matroid. Let ρ denote the rank function of this matroid. It can be seen that the left-hand side of (1) is simply $\rho(\delta(S))$.

Let $X_1, X_2, \ldots, X_k$ be k disjoint sets, each of cardinality $|X|$. Put

$$\mathbf{X} = X_1 \cup X_2 \cup \ldots \cup X_k$$

and for each i, $1 \leqslant i \leqslant k$, choose a bijection f_i of X_i into X. Now obtain a bipartite graph H with bipartition $(\mathbf{X}, E)$ by joining an $x \in X_i$ to an $e \in E$ if $f_i(x)$ is incident with e in G. Note that corresponding to each edge e of G, there are k edges in H with e as their common end vertex.

According to Rado's theorem, H has a matching M of $\mathbf{X}$ into E such that the M-saturated subset of E is independent in Q if and only if

$$\rho(N_H(\mathbf{S})) \geqslant |\mathbf{S}| \quad \text{for all} \quad \mathbf{S} \subseteq \mathbf{X} \tag{2}$$

where $N_H(\mathbf{S})$ denotes the neighbour set of $\mathbf{S}$ in H.

We proceed to show that (1) implies (2). Let $\mathbf{S}$ be any subset of $\mathbf{X}$. Write

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$f_i(\mathbf{S} \cap X_i) = S_i$ for $1 \leqslant i \leqslant k$. We have

$$\begin{aligned} k\rho(N_H(\mathbf{S})) &= k\rho\left(\bigcup_{i=1}^{k} \delta(S_i)\right) \\ &\geqslant \sum_{i=1}^{k} \rho(\delta(S_i)) \\ &\geqslant \sum_{i=1}^{k} k\,|S_i| \qquad \text{(by (1))} \\ &= k\,|\mathbf{S}|. \end{aligned}$$

This yields (2).

Therefore, if (1) holds, by Rado's theorem there exists a matching M of $\mathbf{X}$ into E in H such that the M-saturated subset of E is independent in Q. Let G' be the subgraph of G induced by the subset of E which is M-saturated in H. The fact that M is a matching in H ensures that the degree in G' of each vertex in X is exactly k, and the fact that the edge set of G' is independent in Q ensures that the degree in G' of each vertex in Y is at most k.

Let R denote a k-regular bipartite graph which contains G' as an induced subgraph. By the marriage theorem (see [1; page 73]) R contains k disjoint perfect matchings. The restrictions of these matchings to $E(G')$ constitute k disjoint matchings of X into Y in G.

References

1. J. A. Bondy and U. S. R. Murty, *Graph theory with Applications* (MacMillan Press, London, 1976).
2. K. Lebensold, " Disjoint Matchings of Graphs ", *J. Combinatorial theory* (*B*), 22 (1977), 207–210.
3. D. J. A. Welsh, *Matroid Theory* (Academic Press, London, 1976).

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