Nonlinear Analysis: Real World Applications 13 (2012) 2006–2016

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

An algebraic approach to proving the global stability of a class of epidemic models

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ARTICLE INFO

Article history:
Received 27 November 2011
Accepted 27 December 2011

Keywords:
Global stability
Lyapunov function
The LaSalle's Invariance Principle

ABSTRACT

The global stability of an autonomous differential equation system is an important issue for ecological, epidemiological and virus dynamical models. By means of the direct Lyapunov method and the LaSalle's Invariance Principle, an algebraic approach to proving the global stability is presented in this paper. This approach gives a logic and possibly programming method on how to choose coefficients a_i based on the classic Lyapunov function of the form $\sum_{i=1}^{n} a_i(x_i - x_i^* - x_i^* \ln x_i/x_i^*)$ such that the derivative of the Lyapunov function is negative definite or semidefinite. As an application, the global stability of an SVS-SEIR epidemic model with vaccination and the latent stage is examined. The generality of the approach is also shown by discussing certain cases.

1. Introduction

For a multidimensional autonomous differential equation system, it is an important issue to investigate the global behaviors of the ecological, epidemiological, and virus dynamic models. So far the most successful method to such a problem is the direct Lyapunov method and the LaSalle's Invariance Principle. These two methods need to construct a suitable Lyapunov function so that its derivative along solutions of the system is negative definite or semidefinite.

For ecological, epidemiological, and virus dynamical models, the set

$$R_+^n = \{x = (x_1, x_2, \ldots, x_n) : x_i \geq 0, i = 1, 2, \ldots, n\}$$

is often feasible and positively invariant. The remarkable Lyapunov function of the form

$$L = \sum_{i=1}^{n} a_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*} \right), \tag{1}$$

where $a_i > 0$ $(i = 1, 2, \ldots, n)$, is positive definite in int R_+^n and greatly used in ecological models [1–3]. Thereafter, Lyapunov functions of this type were also applied with a great success to a variety of models in mathematical epidemiology [4–15] and virus dynamics [16–19]. And the extended form of function (1) were recently used to epidemic and virus dynamical models with nonlinear incidence [20–23].

Although the Lyapunov function of form (1) provides us with an effective method to investigate the global stability of many differential systems, there are some difficulties to apply the stability theorem or principle to some epidemic models

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1468-1218/$ – see front matter © 2012 Published by Elsevier Ltd
doi:10.1016/j.nonrwa.2011.12.022

with higher dimensions. The first thing is how to choose a set of suitable coefficients a_i in (1) and how to rearrange or analyze the derivative dL/dt such that the derivative dL/dt is negative definite or semidefinite. Second, when dL/dt is negative semidefinite, finding the set in which the derivative equals zero is also a mathematical technique for applying the LaSalle's Invariance Principle.

For some simple epidemic models, it is relatively easy to get the reasonable constants a_i $(i = 1, 2, \ldots, n)$ in L such that the derivative dL/dt along solutions of the system is negative definite or semidefinite via observation and experience [8,9,11,14,15]. For multi-group models Guo et al. [12,13] presented a graph-theoretic approach to choosing a_i and determining the negative definiteness or semidefiniteness of the derivative. In references mentioned above, the available Lyapunov functions are unique, so are the associated forms of the derivative. However, for some epidemic models, our recent investigations [24–26] have shown that the suitable Lyapunov functions of form (1) are not unique in terms of different choice of parameters a_i $(i = 1, 2, \ldots, n)$ and consequently the different derivatives. Therefore, how to choose a set of suitable numbers a_i $(i = 1, 2, \ldots, n)$ and hence determine whether or not the derivative is negative definite or semidefinite is a key issue for proving the global stability.

For high dimensional differential systems, little is known about the general roles of choice of the parameters a_i or method of proving that the derivative is negative definite or semidefinite given the Lyapunov function of form (1), and falls within the scope of this study. Our purpose is to develop and generalize an algebraic approach to initially rearranging the derivation and then choosing a suitable set of parameters such that the derivative is negative definite or semidefinite by using the relation between the arithmetic and the associated geometric means. The method is illustrated by proving the global stability of an epidemic model with vaccination and the latent stage.

This paper is organized as follows. In the next section, we begin with an example to show our ideas. The example comes from the model formulated by Guo and Li in [5]. In Section 3, we introduce an algebraic approach to proving the global stability by applying the Lyapunov function of form (1), and the detailed steps are presented. An example of applying the approach is given in Section 4, where the example considers an epidemic model with vaccination and the latent stage. Finally, the application of the approach is discussed, and the generality of the approach is shown by considering certain cases.

2. The global stability of a staged-progression model

In [5], Guo and Li considered a 6-stage SP epidemic model with arbitrary amelioration. The global stability of the endemic equilibrium was investigated by utilizing a Lyapunov function. However, their method was quite complicated on determining the coefficients of the Lyapunov function and its negative semidefiniteness, which may not be easily generalized. Here our purpose is to take a simple 3-stage SP model as an example to illustrate our ideas, which is different from Guo and Li's in choosing coefficients of the Lyapunov function.

In the following, we investigate a 3-stage SP epidemic model

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - d_0 S - S(\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3), \\ \dfrac{dI_1}{dt} = S(\beta_1 I_1 + \beta_2 I_2 + \beta_3 I_3) - (d_1 + \delta_{21}) I_1 + \delta_{12} I_2 + \delta_{13} I_3, \\ \dfrac{dI_2}{dt} = \delta_{21} I_1 - (d_2 + \delta_{12} + \delta_{32}) I_2 + \delta_{23} I_3, \\ \dfrac{dI_3}{dt} = \delta_{32} I_2 - (d_3 + \delta_{13} + \delta_{23} + \delta_{43}) I_3, \end{cases} \tag{2}$$

and $dT/dt = \delta_{43} I_3 - d_T T$. System (2), including three stages, is simple version of Guo and Li's model [5] in which the six stages are considered. Variables S, I_i $(i = 1, 2, 3)$ and all the parameters have the same meanings as those in [5]. It can be verified easily that the endemic equilibrium of our model is feasible if the basic reproduction number (we omit it here) is greater than unity, which is similar to that in [5]. In the following we will prove that the endemic equilibrium of system (2) is globally stable in the feasible region by using our distinct approach.

Assume that system (2) has an endemic equilibrium, $P^*(S^*, I_1^*, I_2^*, I_3^*)$ (where $S^* > 0, I_i^* > 0, i = 1, 2, 3$); we define a Lyapunov function

$$L(S, I_1, I_2, I_3) = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + \sum_{i=1}^{3} a_i \left(I_i - I_i^* - I_i^* \ln \frac{I_i}{I_i^*}\right), \tag{3}$$

where $a_i > 0$ $(i = 1, 2, 3)$ are left unspecified. For simplicity, denote

$$y_0 = \frac{S}{S^*}, \qquad y_1 = \frac{I_1}{I_1^*}, \qquad y_2 = \frac{I_2}{I_2^*}, \qquad y_3 = \frac{I_3}{I_3^*};$$

then the derivative of function $L(S, I_1, I_2, I_3)$ along solutions of system (2) is given by

$$\left.\frac{dL}{dt}\right|_{(2)} = C + (a_1 - 1) S^* (\beta_1 I_1^* y_1 + \beta_2 I_2^* y_2 + \beta_3 I_3^* y_3) y_0$$

$$
\begin{aligned}
&- (d_0S^* + a_1\beta_1S^*I_1^*)y_0 - \left[a_1(d_1+\delta_{21}) - \beta_1S^* - a_2\delta_{21}\right]I_1^*y_1\\
&- \left[a_2(d_2+\delta_{12}+\delta_{32}) - \beta_2S^* - a_1\delta_{12} - a_3\delta_{32}\right]I_2^*y_2\\
&- \left[a_3(d_3+\delta_{13}+\delta_{23}+\delta_{43}) - \beta_3S^* - a_1\delta_{13} - a_2\delta_{23}\right]I_3^*y_3\\
&- \Lambda\frac{1}{y_0} - a_1\beta_2I_2^*S^*\frac{y_0y_2}{y_1} - a_1\beta_3I_3^*S^*\frac{y_0y_3}{y_1} - a_1\delta_{12}I_2^*\frac{y_2}{y_1}\\
&- a_1\delta_{13}I_3^*\frac{y_3}{y_1} - a_2\delta_{21}I_1^*\frac{y_1}{y_2} - a_2\delta_{23}I_3^*\frac{y_3}{y_2} - a_3\delta_{32}I_2^*\frac{y_2}{y_3}\\
\triangleq\ & G(y_0,y_1,y_2,y_3),
\end{aligned}
$$

where $C = \Lambda + d_0S^* + a_1(d_1+\delta_{21})I_1^* + a_2(d_2+\delta_{12}+\delta_{32})I_2^* + a_3(d_3+\delta_{13}+\delta_{23}+\delta_{43})I_3^*$.

To choose the suitable constants $a_i > 0$ $(i = 1, 2, 3)$ so that function $G(y_0, y_1, y_2, y_3)$ is negative definite or semidefinite, we plan to rewrite function $G(y_0, y_1, y_2, y_3)$ with constants $a_i > 0$ $(i = 1, 2, 3)$ as the following form

$$
-\sum_{k=1}^{K} b_k\left(h_{k,1} + h_{k,2} + \cdots + h_{k,n_k} - n_k\right), \tag{4}
$$

where $b_k \ge 0$ $(k = 1, 2, \ldots, K)$, $h_{k,i}$ is an expression only including multiplication and division of y_j $(j = 0, 1, 2, 3)$ and $\Pi_{i=1}^{n_k}h_{k,i} = 1$. According to the property that the arithmetic mean is not less than the associated geometric mean, $h_{k,1} + h_{k,2} + \cdots + h_{k,n_k} - n_k \ge 0$ (the equality holds if and only if $h_{k,1} = h_{k,2} = \cdots = h_{k,n_k} = 1$), we could easily prove that function given in (4) is negative semidefinite, and so is function $G(y_0, y_1, y_2, y_3)$ or the derivative dL/dt.

Note that, for the nonconstant terms of $G(y_0, y_1, y_2, y_3)$, the groups satisfying $\Pi_{i=1}^{n_k}h_{k,i} = 1$ totally have the following six cases:

$$
\begin{aligned}
&\left\{y_0, \frac{1}{y_0}\right\}, \quad \left\{\frac{y_2}{y_1}, \frac{y_1}{y_2}\right\}, \quad \left\{\frac{y_3}{y_2}, \frac{y_2}{y_3}\right\},\\
&\left\{\frac{1}{y_0}, \frac{y_0y_2}{y_1}, \frac{y_1}{y_2}\right\}, \quad \left\{\frac{y_3}{y_1}, \frac{y_1}{y_2}, \frac{y_2}{y_3}\right\}, \quad \left\{\frac{1}{y_0}, \frac{y_0y_3}{y_1}, \frac{y_2}{y_3}, \frac{y_1}{y_2}\right\},
\end{aligned}
$$

then, corresponding to expression (4), we define the function

$$
\begin{aligned}
H(y_0,y_1,y_2,y_3) = &-b_1\left(y_0 + \frac{1}{y_0} - 2\right) - b_2\left(\frac{y_2}{y_1} + \frac{y_1}{y_2} - 2\right) - b_3\left(\frac{y_3}{y_2} + \frac{y_2}{y_3} - 2\right)\\
&- b_4\left(\frac{1}{y_0} + \frac{y_0y_2}{y_1} + \frac{y_1}{y_2} - 3\right) - b_5\left(\frac{y_3}{y_1} + \frac{y_1}{y_2} + \frac{y_2}{y_3} - 3\right)\\
&- b_6\left(\frac{1}{y_0} + \frac{y_0y_3}{y_1} + \frac{y_2}{y_3} + \frac{y_1}{y_2} - 4\right),
\end{aligned} \tag{5}
$$

where $b_i \ge 0$ $(i = 1, 2, \ldots, 6)$ are left unspecified.

In the following we let $G(y_0, y_1, y_2, y_3) = H(y_0, y_1, y_2, y_3)$ to determine the coefficients a_i $(i = 1, 2, 3)$ and b_j $(j = 1, 2, \ldots, 6)$. Since the terms y_0y_1, y_0y_2, y_0y_3, y_1, y_2 and y_3, of function G do not appear in function H, their coefficients should be equal to zero, which gives

$$
\begin{cases}
a_1 - 1 = 0,\\
a_1(d_1+\delta_{21})I_1^* = \beta_1I_1^*S^* + a_2\delta_{21}I_1^*,\\
a_2(d_2+\delta_{12}+\delta_{32})I_2^* = \beta_2I_2^*S^* + a_1\delta_{12}I_2^* + a_3\delta_{32}I_2^*,\\
a_3(d_3+\delta_{13}+\delta_{23}+\delta_{43})I_3^* = \beta_3I_3^*S^* + a_1\delta_{13}I_3^* + a_2\delta_{23}I_3^*.
\end{cases} \tag{6}
$$

Notice that S^*, I_1^*, I_2^* and I_3^* satisfy that the functions at the right hand side of system (2) equal to zero, and it follows from Eqs. (6) that a_1, a_2 and a_3 can be uniquely determined as

$$
\begin{cases}
a_1 = 1,\\
a_2 = \dfrac{S^*(\beta_2I_2^* + \beta_3I_3^*) + \delta_{12}I_2^* + \delta_{13}I_3^*}{\delta_{21}I_3^*},\\
a_3 = \dfrac{1}{\delta_{32}I_2^*}\left\{\dfrac{\delta_{23}I_3^*}{\delta_{21}I_1^*}\left[S^*(\beta_2I_2^* + \beta_3I_3^*) + \delta_{12}I_2^* + \delta_{13}I_3^*\right] + \left(\beta_3S^* + \delta_{13}\right)I_3^*\right\}.
\end{cases} \tag{7}
$$

Consequently, the Lyapunov function (3) is specified. Function G is in turn given as

$$\begin{aligned} G(y_0, y_1, y_2, y_3) &= C - (d_0 S^* + a_1\beta_1 S^* I_1^*) y_0 - \Lambda \frac{1}{y_0} - a_1\beta_2 I_2^* S^* \frac{y_0 y_2}{y_1} \\ &\quad - a_1\beta_3 I_3^* S^* \frac{y_0 y_3}{y_1} - a_1\delta_{12} I_2^* \frac{y_2}{y_1} - a_1\delta_{13} I_3^* \frac{y_3}{y_1} - a_2\delta_{21} I_1^* \frac{y_1}{y_2} - a_2\delta_{23} I_3^* \frac{y_3}{y_2} - a_3\delta_{32} I_2^* \frac{y_2}{y_3} \\ &\triangleq \bar{G}(y_0, y_1, y_2, y_3). \end{aligned}$$

Further, letting $\bar{G}(y_0, y_1, y_2, y_3) = H(y_0, y_1, y_2, y_3)$ and comparing the coefficients of the like terms between them yields

$$\begin{aligned} &b_1 = d_0 S^* + a_1\beta_1 I_1^* S^*, \qquad b_2 = a_1\delta_{12} I_2^*, \qquad b_3 = a_2\delta_{23} I_3^*, \\ &b_4 = a_1\beta_2 I_2^* S^*, \qquad b_5 = a_1\delta_{13} I_3^*, \qquad b_6 = a_1\beta_3 I_3^* S^*. \end{aligned} \tag{8}$$

So function $H(y_0, y_1, y_2, y_3)$ is also uniquely determined, the derivative of the Lyapunov function is given by

$$\begin{aligned} \left.\frac{dL}{dt}\right|_{(2)} &= -b_1\left(y_0 + \frac{1}{y_0} - 2\right) - b_2\left(\frac{y_2}{y_1} + \frac{y_1}{y_2} - 2\right) - b_3\left(\frac{y_3}{y_2} + \frac{y_2}{y_3} - 2\right) \\ &\quad - b_4\left(\frac{1}{y_0} + \frac{y_0 y_2}{y_1} + \frac{y_1}{y_2} - 3\right) - b_5\left(\frac{y_3}{y_1} + \frac{y_1}{y_2} + \frac{y_2}{y_3} - 3\right) - b_6\left(\frac{1}{y_0} + \frac{y_0 y_3}{y_1} + \frac{y_2}{y_3} + \frac{y_1}{y_2} - 4\right), \end{aligned}$$

where $b_i > 0$ $(i = 1, 2, \ldots, 6)$ are determined by (8). According to the relation between the arithmetic and the associated geometric means, we have $dL/dt|_{(2)} \le 0$ and the equality holds if and only if $y_0 = 1$ and $y_1 = y_2 = y_3$, that is, $S = S^*$ and $I_1/I_1^* = I_2/I_2^* = I_3/I_3^*$. It can be easily verified that the largest invariant set of system (2) on the set $\{(S, I_1, I_2, I_3) \in R_+^4 : S = S^*, I_1/I_1^* = I_2/I_2^* = I_3/I_3^*\}$ is the singleton $\{P^*\}$. Therefore, by the LaSalle's Invariance Principle [27], it follows that the endemic equilibrium P^* of (2) is globally stable in the feasible region when it exists.

3. Formulation of the algebraic approach

On the basis of the above process of proving the global stability of the endemic equilibrium of system (2), in this section we shall generalize it and propose an algebraic approach to proving the global stability of autonomous differential systems. To this end, consider the autonomous system of ordinary differential equations

$$\frac{dx}{dt} = f(x) \tag{9}$$

where $x = (x_1, x_2, \ldots, x_n)^T$, $f(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$, and function $f_i(x)$ $(i = 1, 2, \ldots, n)$ is continuous in R_+^n and satisfies the local Lipschitz condition with respect to variable x. Assume that system (9) has a unique equilibrium $P^*(x_1^*, x_2^*, \ldots, x_n^*)$, where $x_i^* > 0$ $(i = 1, 2, \ldots, n)$, and that the set R_+^n is positively invariant to system (9). By the direct Lyapunov method or the LaSalle's Invariance Principle, we present an algebraic approach to proving the global stability of equilibrium P^* of system (9) in int R_+^n in the following. The detailed steps are listed as follows.

Step 1. Define a Lyapunov function

$$L(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} a_i \left(x_i - x_i^* - x_i^* \ln \frac{x_i}{x_i^*}\right),$$

where $a_1 = 1$ and $a_i > 0$ $(i = 2, 3, \ldots, n)$ are unspecified. Denote the derivative of function $L(x_1, x_2, \ldots, x_n)$ along solutions of (9) by $\bar{G}(x_1, x_2, \ldots, x_n)$, that is, $dL/dt|_{(9)} = \bar{G}(x_1, x_2, \ldots, x_n)$. By making the transformation of variables, $y_i = x_i/x_i^*$ $(i = 1, 2, \ldots, n)$, we have

$$\left.\frac{dL}{dt}\right|_{(9)} = G(y), \quad y = (y_1, y_2, \ldots, y_n)^T. \tag{10}$$

Step 2. Define a function of variables y by the following process.

First, construct a function set Γ by letting all the coefficients of functions in a set Γ' be one, where the set Γ' consists of all the nonconstant terms of function $G(y)$.

Second, select some functions from the set Γ and make a group such that the product of these functions within a group is unity. Note that any function in Γ may be chosen to group for multiple times. Without loss of generality, we assume that there are at most K groups formed from Γ, and that there are n_k terms including $h_{k,1}, h_{k,2}, \ldots, h_{k,n_k}$ in the k-th group with $\Pi_{j=1}^{n_k} h_{k,j} = 1$ $(k = 1, 2, \ldots, K)$.

Finally, define function

$$H(y) = -\sum_{k=1}^{K} b_k \left(h_{k,1} + h_{k,2} + \cdots + h_{k,n_k} - n_k\right), \tag{11}$$

where $b_k \geq 0$ $(k = 1, 2, \ldots, K)$ are left unspecified. By the relationship between the arithmetic and the associated geometric means, $h_{k,1} + h_{k,2} + \cdots + h_{k,n_k} - n_k \geq 0$, so $H(y) \leq 0$. In particular, $H(y) = 0$ if and only if the arithmetic and the associated geometric means are equal to each other, that is, $h_{k,1} = h_{k,2} = \cdots = h_{k,n_k}$ for $k = 1, 2, \ldots, K$.

Step 3. Choose suitable parameters $a_i > 0$ $(i = 2, \ldots, n)$ and $b_k \geq 0$ $(k = 1, \ldots, K)$ such that $G(y) = H(y)$. By equating the coefficients of like terms of equation $G(y) = H(y)$ we get a system of algebraic equations from which parameters $a_i > 0$ $(i = 2, \ldots, n)$ and $b_k \geq 0$ $(k = 1, \ldots, K)$ may be solved. Note that, for a_i $(i = 2, \ldots, n)$ and b_k $(k = 1, \ldots, K)$ obtained above, we have $dL/dt|_{(9)} = G(y) = H(y) \leq 0$.

Step 4. Find the set

$$\Omega = \left\{ (x_1, x_2, \ldots, x_n) \in \operatorname{int} R^n_+ : \left. \frac{dL}{dt} \right|_{(9)} = 0 \right\}. \tag{12}$$

It follows from the transformation $y_i = x_i/x_i^*$, (10), (11) and $H(y) = G(y)$ that the set gives

$$\begin{aligned} \Omega &= \left\{ (y_1, y_2, \ldots, y_n) \in \operatorname{int} R^n_+ : H(y_1, \ldots, y_n) = 0 \right\} \\ &= \left\{ (y_1, y_2, \ldots, y_n) \in \operatorname{int} R^n_+ : h_{k,1} = h_{k,2} = \cdots = h_{k,n_k}, \text{ for any } k \right\}. \end{aligned} \tag{13}$$

If Ω is a singleton $\{P^*\}$, it follows by the direct Lyapunov method that equilibrium P^* is globally stable in $\operatorname{int} R^n_+$. If Ω is not a singleton, we further need to prove that there is no positive semiorbit of system (9) on the set Ω. If so, it follows by the LaSalle's Invariance Principle that equilibrium P^* is globally stable in $\operatorname{int} R^n_+$.

In order to make our ideas clearly and apply the above approach conveniently, we make some explanations and give remark in the following.

Remark. Given the Lyapunov function of the form defined in Step 1, it is important to prove that the derivative is negative definite or semidefinite, that is $G(y) \leq 0$. Step 2 says that we try our best to rearrange function $G(y)$ such that it has the form of function $H(y)$, that is, $G(y) = H(y)$, which is obviously negative definite or semidefinite with respect to $y_i = 1$ $(i = 1, 2, \ldots, n)$. Furthermore, the realization of Step 3 is a key to applying this approach. If we can find suitable parameter values $a_i > 0$ $(i = 2, 3, 4, \ldots, n)$ and $b_k \geq 0$ $(k = 1, 2, \ldots, K)$ such that $G(y) = H(y)$, then the approach is usable; otherwise, the approach is unfeasible.

4. Application of the algebraic approach to an SVS-SEIR model

In this section, we apply the approach presented in the previous section to consider an epidemic model with vaccination and the latent stage

$$\begin{cases} \dfrac{dS}{dt} = q_1\mu A - (\mu + p)S - \beta SI + \varepsilon V, \\ \dfrac{dV}{dt} = q_2\mu A + pS - (\mu + \varepsilon)V, \\ \dfrac{dE}{dt} = \beta SI - (\mu + \gamma)E, \\ \dfrac{dI}{dt} = \gamma E - (\mu + \alpha + \delta)I, \end{cases} \tag{14}$$

and $dR/dt = \delta I - \mu R$. Here, the total population is divided into the following epidemiological compartments: S, susceptible; V, vaccinated; E, latent; I, infectious; R, recovered. $S = S(t)$, $V = V(t)$, $E = E(t)$, $I = I(t)$ and $R = R(t)$ denote the numbers of individuals in each compartment at time t, respectively. μ is the per capita natural death rate; μA is the birth rate; q_2 $(0 < q_2 < 1)$ is the vaccinated fraction of the newborn, $q_1 = 1 - q_2$ is the unvaccinated fraction of the newborn; p is the per capita vaccination rate for the susceptible individuals; β is the transmission coefficient of the infection; ε is the per capita rate of losing the immunity from the vaccination; γ is the per capita rate of transfer from the latent compartment to the infectious one; α is the per capita disease-induced death rate; δ is the per capita rate of recovery from the disease.

It is easy to know that the set

$$D = \left\{ (S, V, E, I) \in R^4_+ : S + V + E + I \leq A \right\}$$

is positively invariant to system (14). By applying the method of the next generation matrix in [28], we get the basic reproduction number of system (14), which gives

$$R_0 = \frac{\beta\gamma A(\varepsilon + q_1\mu)}{(\mu + \gamma)(\mu + \alpha + \delta)(p + \mu + \varepsilon)}.$$

And direct calculation shows that, when $R_0 \leq 1$, system (14) only has the disease-free equilibrium $P_0(S_0, V_0, 0, 0)$; when $R_0 > 1$, in addition to P_0, system (14) also has a unique endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$, where

$$S_0 = \frac{A(\varepsilon + q_1\mu)}{p + \mu + \varepsilon}, \qquad V_0 = \frac{A(p + q_2\mu)}{p + \mu + \varepsilon},$$
$$S^* = \frac{(\mu + \alpha + \delta)(\mu + \gamma)}{\beta\gamma}, \qquad V^* = \frac{q_2\mu A + pS^*}{\mu + \varepsilon},$$
$$E^* = \frac{\mu + \gamma}{\gamma} I^*, \quad I^* = \frac{\mu\gamma A(\varepsilon + q_1\mu)}{(\mu + \gamma)(\mu + \varepsilon)(\mu + \alpha + \delta)} \left(1 - \frac{1}{R_0}\right).$$

With respect to the global stability of equilibria of system (14), the following results hold.

Theorem 1. *For system (14), the disease-free equilibrium P_0 is globally stable on the set D if $R_0 \leq 1$; the endemic equilibrium P^* is globally stable in the set D if $R_0 > 1$.*

Proof. First, we investigate the global stability of the disease-free equilibrium P_0. For the disease-free equilibrium $P_0(S_0, V_0, 0, 0)$, S_0 and V_0 satisfy the following equations

$$\begin{cases} (\mu + p)S - \varepsilon V = q_1\mu A, \\ (\mu + \varepsilon)V - pS = q_2\mu A, \end{cases}$$

then system (14) can be rewritten as the following form

$$\begin{cases} \dfrac{dS}{dt} = -\left[(\mu + p) + \beta I\right](S - S_0) - \beta S_0 I + \varepsilon(V - V_0), \\ \dfrac{dV}{dt} = p(S - S_0) - (\mu + \varepsilon)(V - V_0), \\ \dfrac{dE}{dt} = \beta(S - S_0)I + \beta S_0 I - (\mu + \gamma)E, \\ \dfrac{dI}{dt} = \gamma E - (\mu + \alpha + \delta)I. \end{cases} \tag{15}$$

Define a Lyapunov function

$$L_1(S, V, E, I) = \frac{1}{2}(S - S_0)^2 + \frac{m}{2}(V - V_0)^2 + S_0\left(E + \frac{\mu + \gamma}{\gamma} I\right),$$

where $m = \left[2\mu(\mu + p + \varepsilon) + p\varepsilon\right]/p^2$, then the derivative of function L_1 with respect to t along solutions of system (15) is given by

$$\begin{aligned} \left.\frac{dL_1}{dt}\right|_{(15)} &= -(\mu + p)(S - S_0)^2 + (mp + \varepsilon)(S - S_0)(V - V_0) \\ &\quad - m(\mu + \varepsilon)(V - V_0)^2 + S_0\left[\beta S_0 - \frac{(\mu + \alpha + \delta)(\mu + \gamma)}{\gamma}\right] I - \beta I(S - S_0)^2 \\ &\leq -(\mu + p)(S - S_0)^2 + (mp + \varepsilon)(S - S_0)(V - V_0) \\ &\quad - m(\mu + \varepsilon)(V - V_0)^2 + \frac{(\mu + \gamma)(\mu + \alpha + \delta)S_0}{\gamma}(R_0 - 1)\, I. \end{aligned}$$

Since

$$(mp + \varepsilon)^2 - 4m(\mu + p)(\mu + \varepsilon) = -\frac{4\mu(\mu + p)(\mu + \varepsilon)(\mu + p + \varepsilon)}{p^2} < 0,$$

function $-(\mu + p)(S - S_0)^2 + (mp + \varepsilon)(S - S_0)(V - V_0) - m(\mu + \varepsilon)(V - V_0)^2$ is negative definite with respect to $S = S_0$ and $V = V_0$. So $(dL_1/dt)|_{(15)} \leq 0$ for $R_0 \leq 1$.

When $R_0 < 1$, $(dL_1/dt)|_{(15)} = 0$ if and only if $S = S_0, V = V_0$ and $I = 0$; when $R_0 = 1$, $(dL_1/dt)|_{(15)} = 0$ if and only if $S = S_0$ and $V = V_0$. Whether $R_0 < 1$ or $R_0 = 1$, it is easy to verify that the largest invariant set of system (15) on the set $\left\{(S, V, E, I) \in D : (dL_1/dt)|_{(15)} = 0\right\}$ is the singleton $\{P_0\}$. Therefore, by the LaSalle's Invariance Principle [27], the disease-free equilibrium P_0 is globally stable on the set D when $R_0 \leq 1$.

Note that it is relatively easy to determine that $dL_1/dt|_{(15)}$ is negative semidefinite for $R_0 \leq 1$, our proposed method is then not necessary to apply to prove the global stability of the disease-free equilibrium. But, for the global stability of the endemic state P^* in the set D we shall show in the following how our approach presented in the previous section is effective. For this purpose we follow steps presented in the previous section.

Step 1. Define a Lyapunov function

$$L_2(S,V,E,I) = \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + a_2\left(V - V^* - V^* \ln \frac{V}{V^*}\right) + a_3\left(E - E^* - E^* \ln \frac{E}{E^*}\right) + a_4\left(I - I^* - I^* \ln \frac{I}{I^*}\right), \tag{16}$$

where a_i ($i = 2, 3, 4$) are positive, and left unspecified, then the derivative of function L_2 along solutions of system (14) reads

$$\begin{aligned}\left.\frac{dL_2}{dt}\right|_{(14)} &= C - [(\mu + p) - a_2 p]S - (1 - a_3)\beta SI - [a_2(\mu + \varepsilon) - \varepsilon]V - [a_4(\mu + \alpha + \delta) - \beta S^*]I \\ &\quad - [a_3(\mu + \gamma) - a_4\gamma]E - q_1\mu A\frac{S^*}{S} - \varepsilon\frac{S^*V}{S} - a_2q_2\mu A\frac{V^*}{V} - a_2p\frac{V^*S}{V} - a_3\beta\frac{E^*SI}{E} - a_4\gamma\frac{I^*E}{I} \\ &\triangleq \bar{G}(S,V,E,I),\end{aligned}$$

where $C = q_1\mu A + a_2q_2\mu A + (\mu + p)S^* + a_2(\mu + \varepsilon)V^* + a_3(\mu + \gamma)E^* + a_4(\mu + \alpha + \delta)I^*$.

Let

$$x = \frac{S}{S^*}, \qquad y = \frac{V}{V^*}, \qquad z = \frac{E}{E^*}, \qquad u = \frac{I}{I^*};$$

then $\bar{G}(S,V,E,I)$ becomes

$$\begin{aligned}G(x,y,z,u) &= C - [(\mu + p) - a_2p]S^*x - (1 - a_3)\beta S^*I^*xu - [a_2(\mu + \varepsilon) - \varepsilon]V^*y - [a_4(\mu + \alpha + \delta) - \beta S^*]I^*u \\ &\quad - [a_3(\mu + \gamma) - a_4\gamma]E^*z - q_1\mu A\frac{1}{x} - \varepsilon V^*\frac{y}{x} - a_2q_2\mu A\frac{1}{y} - a_2pS^*\frac{x}{y} - a_3\beta S^*I^*\frac{xu}{z} - a_4\gamma E^*\frac{z}{u}.\end{aligned}$$

Step 2. Construct the function set Γ

$$\Gamma = \left\{x, y, z, u, xu, \frac{1}{x}, \frac{1}{y}, \frac{x}{y}, \frac{y}{x}, \frac{z}{u}, \frac{xu}{z}\right\}.$$

There are at most seven groups associated with Γ such that the product of all functions within each group is unity. The seven groups are, respectively,

$$\left\{x, \frac{1}{x}\right\}; \quad \left\{y, \frac{1}{y}\right\}; \quad \left\{\frac{y}{x}, \frac{x}{y}\right\}; \quad \left\{x, \frac{1}{y}, \frac{y}{x}\right\};$$
$$\left\{\frac{1}{x}, y, \frac{x}{y}\right\}; \quad \left\{\frac{1}{x}, \frac{xu}{z}, \frac{z}{u}\right\}; \quad \left\{\frac{1}{y}, \frac{y}{x}, \frac{xu}{z}, \frac{z}{u}\right\}.$$

Further, according to the above groups, we define function

$$\begin{aligned}H(x,y,z,u) &= b_1\left(2 - x - \frac{1}{x}\right) + b_2\left(2 - y - \frac{1}{y}\right) + b_3\left(2 - \frac{y}{x} - \frac{x}{y}\right) + b_4\left(3 - x - \frac{1}{y} - \frac{y}{x}\right) \\ &\quad + b_5\left(3 - \frac{1}{x} - \frac{xu}{z} - \frac{z}{u}\right) + b_6\left(3 - \frac{1}{x} - y - \frac{x}{y}\right) + b_7\left(4 - \frac{1}{y} - \frac{y}{x} - \frac{xu}{z} - \frac{z}{u}\right)\end{aligned} \tag{17}$$

with the coefficients b_k ($k = 1, \ldots, 7$) left unspecified.

Step 3. We would like to determine suitable parameters $a_i > 0$ ($i = 2, 3, 4$) and $b_k \geq 0$ ($k = 1, 2, \ldots, 7$) such that $G(x,y,z,u) = H(x,y,z,u)$. Equating the coefficients of the like terms at its two sides yields the following equations

$$\begin{cases} 1 - a_3 = 0, \\ a_4(\mu + \alpha + \delta) - \beta S^* = 0, \\ a_3(\mu + \gamma) - a_4\gamma = 0, \\ [(\mu + p) - a_2p]\,S^* = b_1 + b_4, \\ [a_2(\mu + \varepsilon) - \varepsilon]\,V^* = b_2 + b_6, \\ q_1\mu A = b_1 + b_5 + b_6, \\ \varepsilon V^* = b_3 + b_4 + b_7, \\ a_2q_2\mu A = b_2 + b_4 + b_7, \\ a_2pS^* = b_3 + b_6, \\ a_3\beta S^*I^* = b_5 + b_7, \\ a_4\gamma E^* = b_5 + b_7, \\ C = 2(b_1 + b_2 + b_3) + 3(b_4 + b_5 + b_6) + 4b_7. \end{cases} \tag{18}$$

Since $S^* = (\mu+\alpha+\delta)(\mu+\gamma)/(\beta\gamma)$, it follows from the first three equations of (18) that

$$a_3 = 1, \qquad a_4 = \frac{\mu+\gamma}{\gamma}. \tag{19}$$

Note that, for the endemic equilibrium $P^*(S^*, V^*, E^*, I^*)$, S^*, V^*, E^* and I^* satisfy the following equations

$$\begin{cases} q_1\mu A - (\mu+p)S^* - \beta S^* I^* + \varepsilon V^* = 0, \\ q_2\mu A + pS^* - (\mu+\varepsilon)V^* = 0, \\ \beta S^* I^* - (\mu+\gamma)E^* = 0, \\ \gamma E^* - (\mu+\alpha+\delta)I^* = 0, \end{cases} \tag{20}$$

then substituting (19) into other equations of (18) yields the following equivalent form of (18)

$$\begin{cases} b_1 + b_4 = [(\mu+p) - a_2 p]\, S^*, \\ b_2 + b_6 = [a_2(\mu+\varepsilon) - \varepsilon]\, V^*, \\ b_1 + b_5 + b_6 = q_1\mu A, \\ b_3 + b_4 + b_7 = \varepsilon V^*, \\ b_2 + b_4 + b_7 = a_2 q_2 \mu A, \\ b_3 + b_6 = a_2 p S^*, \\ b_5 + b_7 = (\mu+\gamma)E^*. \end{cases} \tag{21}$$

Further, (21) is equivalent to the following equations

$$\begin{cases} b_1 = q_1\mu A - \beta S^* I^* - b_6 + b_7, \\ b_2 = [a_2(\mu+\varepsilon) - \varepsilon]\, V^* - b_6, \\ b_3 = a_2 p S^* - b_6, \\ b_4 = \varepsilon V^* - a_2 p S^* + b_6 - b_7, \\ b_5 = \beta S^* I^* - b_7. \end{cases} \tag{22}$$

Since $b_i \geq 0$ $(i = 1, 2, 4, 6)$, it follows from the first two equations of (21) that a_2 should satisfy the following inequalities

$$\frac{\varepsilon}{\mu+\varepsilon} \leq a_2 \leq \frac{\mu+p}{p}. \tag{23}$$

In order to assure $b_i \geq 0$ $(i = 1, \ldots, 5)$, it follows from (22) that b_6 and b_7 must satisfy the following inequalities

$$\begin{cases} 0 \leq b_6 \leq \min\left\{[a_2(\mu+\varepsilon) - \varepsilon]\, V^*, a_2 p S^*\right\}, \\ 0 \leq b_7 \leq (\mu+\gamma)E^*, \\ a_2 p S^* - \varepsilon V^* \leq b_6 - b_7 \leq q_1\mu A - \beta S^* I^*. \end{cases} \tag{24}$$

By using (20), we have

$$(q_1\mu A - \beta S^* I^*) - (a_2 p S^* - \varepsilon V^*) = [(\mu+p) - a_2 p]\, S^*,$$

then (23) assures

$$\min\left\{[a_2(\mu+\varepsilon) - \varepsilon]\, V^*, a_2 p S^*\right\} \geq 0,$$

and

$$a_2 p S^* - \varepsilon V^* \leq q_1\mu A - \beta S^* I^*.$$

So, when a_2 satisfies inequalities (23), it is easy to know geometrically that there must be nonnegative constants b_6 and b_7 satisfying (24). Especially, for $\varepsilon/(\mu+\varepsilon) < a_2 < (\mu+p)/p$, there must be positive numbers b_6 and b_7 such that $b_i > 0$ $(i = 1, 2, \ldots, 5)$ in (22). This implies that, for $\varepsilon/(\mu+\varepsilon) < a_2 < (\mu+p)/p$, positive numbers b_i $(i = 1, 2, \ldots, 7)$ do exist such that $G(x, y, z, u) = H(x, y, z, u)$.

Step 4. When $b_i > 0$ $(i = 1, 2, \ldots, 7)$, it is easy to see that $H(x, y, z, u) = 0$ if and only if $x = y = 1$ and $z = u$. Thus, according to the above steps, we have the Lyapunov function for system (14)

$$\begin{aligned} L_2(S, V, E, I) = {} & \left(S - S^* - S^* \ln \frac{S}{S^*}\right) + a_2\left(V - V^* - V^* \ln \frac{V}{V^*}\right) \\ & + \left(E - E^* - E^* \ln \frac{E}{E^*}\right) + \frac{\mu+\gamma}{\gamma}\left(I - I^* - I^* \ln \frac{I}{I^*}\right), \end{aligned}$$

where $\varepsilon/(\mu+\varepsilon) < a_2 < (\mu+p)/p$, and $dL_2/dt|_{(14)} \leq 0$ and the equality holds true if and only if $S = S^*$, $V = V^*$ and $E/E^* = I/I^*$. It can be verified easily that the largest invariant set of system (14) on the set $\{(S, V, E, I) \in D : S = S^*, V = V^*, E/E^* = I/I^*\}$ is the singleton $\{P^*\}$. Therefore, it follows from the LaSalle's Invariance Principle [27] that the endemic equilibrium P^* of (14) is globally stable in the set D when $R_0 > 1$. This completes the proof. $\square$

Remark. Since the different values of a_2 corresponds to the different Lyapunov functions, the investigation here shows that the suitable Lyapunov function is nonunique for the model in terms of different parameters for the given structure of Lyapunov function (1). We note that choosing the different values of a_2 definitely leads to the different derivatives, and consequently the procedures of proving that the derivative is negative semidefinite are highly dissimilar. We demonstrate this by considering the following three choices of a_2:

$$\text{Case 1: } a_2 = \frac{\varepsilon}{\mu+\varepsilon}; \qquad \text{Case 2: } a_2 = \frac{\mu+p}{p}; \qquad \text{Case 3: } a_2 = 1.$$

Case 1. Since $b_2 \geq 0$ and $b_6 \geq 0$, then, from the second equation of (21) we have $b_2 = b_6 = 0$. Further, it follows from (20) and (21) that

$$\begin{cases} b_1 = q_1\mu A - \beta S^* I^* + b_7, \\ b_3 = \dfrac{\varepsilon p S^*}{\mu+\varepsilon}, \\ b_4 = \dfrac{\varepsilon q_2 \mu A}{\mu+\varepsilon} - b_7, \\ b_5 = \beta S^* I^* - b_7. \end{cases} \tag{25}$$

In order to assure $b_i \geq 0$ $(i = 1, 3, 4, 5)$, from (25) we know that b_7 must satisfy the inequalities

$$\max\left\{0, \beta S^* I^* - q_1\mu A\right\} \leq b_7 \leq \min\left\{\frac{\varepsilon q_2\mu A}{\mu+\varepsilon}, \beta S^* I^*\right\}. \tag{26}$$

Using (20) yields

$$\frac{\varepsilon q_2\mu A}{\mu+\varepsilon} - \left(\beta S^* I^* - q_1\mu A\right) = \frac{\mu(\mu+p+\varepsilon)}{\mu+\varepsilon} S^* > 0;$$

then inequalities (26) must have nonnegative solution, and so is (25). That is, when $a_2 = \varepsilon/(\mu+\varepsilon)$, (21) must have nonnegative solution.

Case 2. Similarly, it follows from (20) and (21) that $b_1 = b_4 = 0$ and

$$\begin{cases} b_2 = \dfrac{(\mu+p)q_2\mu A}{p} - b_7, \\ b_3 = \varepsilon V^* - b_7, \\ b_5 = \beta S^* I^* - b_7, \\ b_6 = (\mu+p)S^* - \varepsilon V^* + b_7. \end{cases} \tag{27}$$

In order to assure $b_i \geq 0$ $(i = 2, 3, 5, 6)$, b_7 must satisfy the inequalities

$$\max\left\{0, \varepsilon V^* - (\mu+p)S^*\right\} \leq b_7 \leq \max\left\{\frac{(\mu+p)q_2\mu A}{p}, \beta S^* I^*, \varepsilon V^*\right\}. \tag{28}$$

Using (20) yields

$$\beta S^* I^* - \left[\varepsilon V^* - (\mu+p)S^*\right] = q_1\mu A > 0$$

and

$$\frac{(\mu+p)q_2\mu A}{p} - \left[\varepsilon V^* - (\mu+p)S^*\right] = \mu\left(\frac{q_2\mu A}{p} + S^* + I^*\right) > 0,$$

then inequalities (28) must have nonnegative solution. It implies that (21) must also have nonnegative solution when $a_2 = (\mu+p)/p$.

Case 3. System (21) becomes

$$\begin{cases} b_1 = q_1\mu A - \beta S^* I^* - b_6 + b_7, \\ b_2 = \mu V^* - b_6, \\ b_3 = pS^* - b_6, \\ b_4 = \varepsilon V^* - pS^* + b_6 - b_7, \\ b_5 = \beta S^* I^* - b_7. \end{cases} \tag{29}$$

In order to assure $b_i \geq 0$ $(i = 1, 2, \ldots, 5)$, it follows from (29) that b_6 and b_7 must satisfy the following inequalities

$$\begin{cases} 0 \leq b_6 \leq \min\left\{\mu V^*, pS^*\right\}, \\ 0 \leq b_7 \leq (\mu+\gamma)E^*, \\ pS^* - \varepsilon V^* \leq b_6 - b_7 \leq q_1\mu A - \beta S^* I^*. \end{cases} \tag{30}$$

Obviously, the existence of nonnegative solution of (30) is similar to that of (24).

The above three cases show that choosing the different Lyapunov functions may lead to various difficulties in rearranging the derivative $dL/dt|_{(14)}$ into the form of function H in (11) to prove $dL/dt|_{(14)} \leq 0$. For the first two cases, we only need to determine the value of b_7, then the associated values of b_i ($i = 1, 2, \ldots, 6$) can be obtained. However, for Case 3, in order to obtain the suitable values of b_i ($i = 1, 2, \ldots, 5$), two other values (i.e., b_6 and b_7) need to be chosen. Thus, adopting the last case is relatively complicated to prove $dL/dt|_{(14)} = H \leq 0$.

5. Discussion

In this paper, we presented an algebraic approach to choosing suitable parameters a_i in the Lyapunov function of form (1) such that the derivative of the Lyapunov function along the given system is negative definite or semidefinite. In particular, the proposed approach actually give a logic and possibly programming method on how to choose coefficients a_i based on the classic Lyapunov function of the form $\sum_{i=1}^{n} a_i(x_i - x_i^* - x_i^* \ln x_i/x_i^*)$ such that the negative definite or semidefinite derivative of the Lyapunov function is derived. The key ideas are to rearrange the derivative dL/dt to be a negative definite or semidefinite function which involves the arithmetic and the associated geometric means of the variables, as the form of (11), and then determine the unspecified parameters. Further, the approach provides a relatively easy way to derive the largest invariant set of the system on the set $\{x \in R_+^n : dL/dt = 0\}$, and hence the equilibrium is globally stable based on the LaSalle's Invariance Principle.

In Section 4, we proved the global stability of the endemic equilibrium of model (14) by choosing $a_3 = 1$, $a_4 = (\mu+\gamma)/\gamma$ and a_2 satisfying $\varepsilon/(\mu + \varepsilon) < a_2 < (\mu + p)/p$ for the Lyapunov function (16). Correspondingly, the associated coefficients $b_k (k = 1, 2, \ldots, 7)$ of function H appeared in (17) were chosen to be a positive solution of the associated Eqs. (22). This implies that, for system (14), the suitable Lyapunov function is nonunique, and so is the associated form of function H for each suitable Lyapunov function. Fortunately, this approach can provide all possibly suitable Lyapunov functions of form (1) and all the available forms of function H corresponding to each Lyapunov function are given. In particular, it follows from Remark in Section 4 that the more appropriate Lyapunov function may make proving $dL/dt \leq 0$ easier.

We illustrate our approach by proving the global stability of the endemic equilibrium of an SVS-SEIR epidemic model with vaccination and the latent stage. This approach is also suitable for proving the global stability of the endemic equilibrium of models in [5–16], as shown in Section 2. Note that when this approach is applied to those models, the solution of the equations obtained in Step 3 is uniquely solved. In contrast with the method used in the previous literature, our approach can not only be more generous and more concise, but also find all possible Lyapunov functions of form (1).

The discussion above has theoretically shown the generality of the algebraic approach. In addition, when the derivative dL/dt is complicated, that is, the function set Γ defined in Step 2 in Section 3 includes more elements, it may not be easy to select all the groups from the set Γ. In order to avoid losing some groups, a set of Matlab program may be applied. Again, Matlab code may also be used to determine the existence of required solutions of the equations established in Step 3, since the coefficients of b_i in the equations are 1 or 0. Hence this algebraic approach possibly provides a programming method on choosing suitable coefficients a_i for the classic Lyapunov function of form (1) and improves the efficacy of calculations.

Acknowledgments

We are very grateful to the anonymous referees for their valuable comments and helpful suggestions, which led to a great improvement of the original manuscript. This work was supported by the National Natural Science Foundation of China (11071256, 11071283, 11171268), the National Megaproject of Science Research of China (2012ZX10001001), the Fundamental Research Funds for the Central Universities (08143042), the Program for New Century Excellent Talents in University (NCET-08-0430), and the Natural Science Foundation of Shanxi Province (2009011005-3).

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