Fibring Non-Truth-Functional Logics: Completeness Preservation*

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Abstract

Fibring has been shown to be useful for combining logics endowed with truth-functional semantics. One wonders if fibring can be extended in order to cope with logics endowed with non-truth-functional semantics as, for example, paraconsistent logics. The first main contribution of the paper is a positive answer to this question. Furthermore, it is shown that this extended notion of fibring preserves completeness under certain reasonable conditions. This completeness transfer result, the second main contribution of the paper, generalizes the one established by Zanardo *et al.* and is obtained using a new technique exploiting the properties of the meta-logic where the (possibly non-truth-functional) valuations are defined. The modal paraconsistent logic of da Costa and Carnielli is obtained by fibring and its completeness is so established.

1 Introduction

In recent years, the problem of combining logics has gained the attention of many researchers in mathematical logic. Besides leading to very interesting applications whenever it is necessary to work with different logics at the same time, combinations of logics are also of great interest on purely theoretical grounds [3].

The practical impact of the problem is clear, at least from the point of view of those working in knowledge representation (within artificial intelligence) and in formal specification and verification (within software engineering). Indeed, in these fields, the need for working with several formalisms at the same time is the rule rather than the exception. For instance, in a knowledge representation problem it may be necessary to work with both temporal and deontic aspects.

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And in a software specification problem it may be necessary to work with both equational and temporal specifications.

¿From a theoretical point of view, one might be tempted, for instance, to look at predicate temporal logic as resulting from the combination of first order logic and propositional temporal logic. But the approach will be of significance only if general preservation results are available about the mechanism used for combining the logics. For example, if it had been established that completeness is preserved by the combination mechanism $\circ$ and it is known that logic $\mathcal{L}$ is given by $\mathcal{L}' \circ \mathcal{L}''$, then the completeness of the combination $\mathcal{L}$ would follow from the completeness of $\mathcal{L}'$ and $\mathcal{L}''$. No wonder that so much theoretical effort has been dedicated to establishing preservation results and/or finding preservation counterexamples within the community of logicians working in the problem of combining logics.

Among the different techniques for combining logics, *fibring* [11, 12, 21, 22, 23] deserves close attention. But what is fibring? The answer can be given in a few paragraphs for the special case of logics with a propositional base, that is, with propositional variables and connectives of arbitrary arity.

The language of the fibring is obtained by the free use of the language constructors (atomic symbols and connectives) from the given logics. So, for example, when fibring a temporal logic and a deontic logic, mixed formulae, like $((G\alpha) \supset (O(F\beta)))$, appear in the resulting logic. Naturally, in many cases one wants to share some of the symbols. The previous example would involve the constrained form of fibring imposed by sharing a common propositional part.

At the deductive system level, provided that the two given logics are endowed with deductive systems of the same type, the deductive system of the fibring will be obtained by the free use of the inference rules from both. This approach will be of interest only if the two given deductive systems are schematic in the sense that their inference rules are open for application to formulae with foreign symbols. For instance, when one represents *Modus Ponens* by the rule MP, $\{(\xi_1 \supset \xi_2), \xi_1\} \vdash \xi_2$, in some Hilbert system, one may implicitly assume that the instantiation of the schema variables ξ_1, ξ_2 by any formulae, possibly with symbols from both logics, is allowed when applying MP in the fibring.

Although fibring at the syntactic level is quite simple as described above, the semantics of fibring is much more complicated and it is advisable to consider only the special case where both logics have semantics with similar models. Following [21, 23], a convenient, but quite general, model for a wide class of logics with propositional base is provided by a triple $\langle U, \mathcal{B}, \nu \rangle$ where U is a set (of points, worlds, states, whatever), $\mathcal{B} \subseteq \wp U$, and $\nu(c) : \mathcal{B}^n \to \mathcal{B}$ for each language constructor c of arity $n \geq 0$. We look at the pair $\langle \mathcal{B}, \nu \rangle$ as an algebra of truth-values. In this sense, such models are said to be *truth-functional*.

Given two logics $\mathcal{L}', \mathcal{L}''$ with models of this type, what is the semantics of their fibring? As first shown in [21], it is a class of models of the same type, such that at each point $u \in U$ it is possible to extract a model from $\mathcal{L}'$ and one from $\mathcal{L}''$. Clearly, if symbols are shared, the two extracted models should agree on them. In order to visualize the semantics of fibring, consider the fibring of a propositional linear temporal logic with a propositional linear space logic. Each model of the fibring will be a cloud U of points such that at each point

one knows the time line and the space line crossing there. For instance, at the point $\langle$*Berlin, 10h15m 25 March 2000*$\rangle$ one knows the time line (of past, present and future) of Berlin and the space line (the universe taken as a line for the sake of the example) at that time. However, so far, as sketched above, those working on the semantics of fibring have assumed that the logics at hand are truth-functional and, therefore, have not considered non-truth-functional logics like paraconsistent logics.

Paraconsistent logics were introduced in [6] and since then have been the object of continued attention, because of their theoretical and practical significance. In particular, the paraconsistent systems $\mathcal{C}_n$ of [6] are subsystems of propositional classical logic in which the principle of *Pseudo Scotus* $\gamma, \neg\gamma \vdash \delta$ does not hold. Some systems of paraconsistent modal logic were investigated in [9, 10, 8, 20]. One might wonder if we should try to obtain such a mixed logic as the fibring of the underlying modal and paraconsistent logics.

The first main contribution of this paper is a positive answer to this question. To this end, previous work on the semantics of fibring is extended to a more general notion of logic system encompassing the case of non-truth-functional valuations. Furthermore, it is shown that this extended notion of fibring preserves completeness under reasonable conditions. This completeness transfer result, the second main contribution of this paper, generalizes the one established in [23] and is obtained using a new technique exploiting the properties of the meta-logic where the (possibly non-truth-functional) valuations are defined.

In Section 2, the notion of *interpretation system presentation* as a specification of the intended valuations within a suitable meta-logic of equational nature is introduced. The *interpretation structures* appear as models (algebras) of the specification. We adopt CEQ (conditional equational logic [13, 18]) as the meta-logic. Section 3 defines the notions of *unconstrained* and *constrained fibring* of interpretation system presentations. The main example, fibring the paraconsistent system $\mathcal{C}_1$ and the modal system KD, is discussed in Section 4 at the semantic level. Section 5 contains a brief account of the appropriate proof-theoretic notions and returns to the main example at the deduction level. Section 6 establishes the completeness preservation theorem and applies it for proving the completeness of the modal paraconsistent logic in [8]. Section 7 discusses applications of self-fibring, namely in the context of the $\mathcal{C}_n$ hierarchy of paraconsistent systems. Section 8 concludes with an assessment of what was achieved and what lays ahead.

2 Specifying valuation semantics

Since the object logics under investigation are propositional-based, the following notion of signature suffices for our purposes:

Definition 2.1 An *object signature* is a family $C = \{C_k\}_{k\in\mathbb{N}}$ where each C_k is a set (of *connectives* of arity k).

In particular, the set of propositional symbols is included in C_0.

We assume given once and for all the set $\Xi = \{\xi_1, \xi_2, \dots\}$ of *propositional schema variables*, to be used in *inference rules*, such that $\Xi \cap C_0 = \emptyset$.

We denote by $L(C, \Xi)$ the set of *schema formulae* inductively built from C and Ξ. For example, $\xi_1 \supset (p \vee \neg \xi_2)$ is a schema formula, if $p \in C_0$, $\neg \in C_1$ and $\supset, \vee \in C_2$.

Our next step will be to define an equational signature induced by a given object signature C as a meta-linguistic device which permits to talk about the semantics of logics based on C. For this purpose, it is convenient to consider two new sorts, ϕ (for formulae) and τ (for truth-values). As usual, ϵ denotes the empty string over $\mathbf{S} = \{\phi, \tau\}$ and, if $w \in \mathbf{S}^*$ and $s \in \mathbf{S}$, then $O_{w\,s}$ denotes the set of operations with domain w and codomain s.

Definition 2.2 Given an object signature C, the *induced meta-signature* is the 2-sorted equational signature $\Sigma(C, \Xi) = \langle \mathbf{S}, O \rangle$ where $\mathbf{S} = \{\phi, \tau\}$ and:

- $O_{\epsilon\,\phi} = C_0 \cup \Xi$;
- $O_{\phi^k\,\phi} = C_k$ for $k > 0$;
- $O_{\phi\,\tau} = \{v\}$;
- $O_{\epsilon\,\tau} = \{\top, \bot\}$;
- $O_{\tau\,\tau} = \{-\}$;
- $O_{\tau\tau\,\tau} = \{\sqcap, \sqcup, \Rightarrow\}$;
- $O_{\omega\,s} = \emptyset$ in the other cases.

We shall use $\Sigma(C)$ to denote the subsignature $\Sigma(C, \emptyset)$, that is, where $O_{\epsilon\,\phi} = C_0$.

In this context, ϕ and τ are the sorts of schema formulae and truth-values, respectively. The symbols $\top$, $\bot$, $-$, $\sqcap$, $\sqcup$ and $\Rightarrow$ are used as generators of truth-values. The symbol v will be interpreted as a valuation map.

We consider the following sets of variables for $\Sigma(C)$ and $\Sigma(C, \Xi)$: $X_\tau = \{x_1, x_2, \dots\}$ and $X_\phi = \{y_1, y_2, \dots\}$. Recall that a term t is called a ground term if it does not contain variables, and that a substitution θ is said to be ground if it replaces every variable by a ground term.

We want to write valuation specifications (within the adopted meta-logic CEQ) over $\Sigma(C)$ and X. Recall that a CEQ-specification is composed of conditional equations of the general form:

$$(\text{equation}_1\ \&\ \dots\ \&\ \text{equation}_n\ \rightarrow\ \text{equation})$$

with $n \geq 0$. Conditional equations are universally quantified, although, for the sake of simplicity, we omit the quantifier, contrarily to the notation used in [18]. For example, $(\ \rightarrow v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$ is a conditional equation of sort τ, supposing that $\wedge \in C_2$. It is clear, from this and also the forthcoming examples, that we only need to consider specifications containing exclusively conditional equations (or *meta-axioms*) of sort τ. Such specifications are called τ-specifications in the sequel.

An important remark is that, in the context of the meta-signature $\Sigma(C, \Xi)$, it might seem that we have two different ways to represent arbitrary formulae: by means of propositional schema variables (i.e., ξ_1, ξ_2, etc.) and by means of variables of sort ϕ (i.e., y_1, y_2, etc.). The former shall indeed represent arbitrary formulae but only in the context of Hilbert Calculi (to be defined in Section 5). The latter represent arbitrary formulae in the meta-language of CEQ. In this meta-language, propositional schema variables appear as constants.

Definition 2.3 An *interpretation system presentation* (isp) is a pair $\mathcal{S} = \langle C, S \rangle$ where C is an object signature and S is a τ-specification over $\Sigma(C)$.

Definition 2.4 Given an isp $\mathcal{S}$, the class $\mathrm{Int}(\mathcal{S})$ of *interpretation structures* presented by $\mathcal{S}$ is the class of all Heyting algebras over $\Sigma(C, \Xi)$ satisfying the specification S.

We denote by $S^\bullet$ the specification composed of the meta-axioms in S plus τ-equations over $\Sigma(C)$ specifying the class of all Heyting algebras. Note that $\mathrm{Int}(\mathcal{S})$, that is, the class of all algebras over $\Sigma(C, \Xi)$ satisfying $S^\bullet$, is always non-empty. Indeed, the trivial algebra with singleton carrier sets for all sorts satisfies any set of conditional equations.

For the sake of economy of presentation, we introduce the following abbreviations: $x_1 \leq x_2$ for $\sqcap(x_1, x_2) = x_1$, and $\Leftrightarrow(x_1, x_2)$ for $\sqcap(\Rightarrow(x_1, x_2), \Rightarrow(x_2, x_1))$. The relation symbol $\leq$ denotes a partial order on truth-values. Furthermore, the partial order is a bounded lattice with meet $\sqcap$, join $\sqcup$, top $\top$ and bottom $\bot$ (cf. [2]). As expected, given an algebra $\mathcal{A}$, $a_1 \leq_{\mathcal{A}} a_2$ and $\Leftrightarrow_{\mathcal{A}}(a_1, a_2)$ are abbreviations of $\sqcap_{\mathcal{A}}(a_1, a_2) = a_1$ and $\sqcap_{\mathcal{A}}(\Rightarrow_{\mathcal{A}}(a_1, a_2), \Rightarrow_{\mathcal{A}}(a_2, a_1))$, respectively. It is also well known that the Heyting algebra axioms further entail the following result:

Proposition 2.5 Let $\mathcal{S}$ be an isp, t_1 and t_2 terms of sort τ and $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$. Then, for every assignment ρ over $\mathcal{A}$:

$$[\![t_1]\!]^\rho_{\mathcal{A}} \leq_{\mathcal{A}} [\![t_2]\!]^\rho_{\mathcal{A}} \quad \text{iff} \quad \Rightarrow_{\mathcal{A}} ([\![t_1]\!]^\rho_{\mathcal{A}}, [\![t_2]\!]^\rho_{\mathcal{A}}) = \top_{\mathcal{A}}$$

and

$$[\![t_1]\!]^\rho_{\mathcal{A}} = [\![t_2]\!]^\rho_{\mathcal{A}} \quad \text{iff} \quad \Leftrightarrow_{\mathcal{A}} ([\![t_1]\!]^\rho_{\mathcal{A}}, [\![t_2]\!]^\rho_{\mathcal{A}}) = \top_{\mathcal{A}}.$$

As explained, our framework is intended to study properties of fibring of non-truth-functional logics in general. We now illustrate the notion of isp with two examples that will be used throughout the rest of the paper.

Example 2.6 *Paraconsistent system* $\mathcal{C}_1$ [6]:

- Object signature - C:
  - $C_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;
  - $C_1 = \{\neg\}$;
  - $C_2 = \{\wedge, \vee, \supset\}$.

- Meta-axioms - S:
  - Truth-values axioms – further axioms in order to obtain a specification of the class of all Boolean algebras, e.g., adding the equation:
    - $(\ \rightarrow -(-(x_1)) = x_1)$.
  - Valuation axioms:
    - $(\ \rightarrow v(\mathbf{t}) = \top)$;
    - $(\ \rightarrow v(\mathbf{f}) = \bot)$;
    - $(\ \rightarrow v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$;
    - $(\ \rightarrow v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2)))$;
    - $(\ \rightarrow v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2)))$;
    - $(v(y_1) = \bot \rightarrow v(\neg\, y_1) = \top)$;
    - $(v(\neg\, \neg\, y_1) = \top \rightarrow v(y_1) = \top)$;
    - $(v(y_2^\circ) = \top\ \&\ v(y_1 \supset y_2) = \top\ \&\ v(y_1 \supset \neg\, y_2) = \top \rightarrow v(y_1) = \bot)$;
    - $(v(y_1^\circ) = \top\ \&\ v(y_2^\circ) = \top \rightarrow v((y_1 \wedge y_2)^\circ) = \top)$;
    - $(v(y_1^\circ) = \top\ \&\ v(y_2^\circ) = \top \rightarrow v((y_1 \vee y_2)^\circ) = \top)$;
    - $(v(y_1^\circ) = \top\ \&\ v(y_2^\circ) = \top \rightarrow v((y_1 \supset y_2)^\circ) = \top)$.

  As usual in the $\mathcal{C}_n$ systems, γ° is an abbreviation of $\neg(\gamma \wedge \neg\, \gamma)$.

  The reader should be warned that we are using Boolean algebras here as a metamathematical environment sufficient to carry out the computations of truth-values for the formulae in $\mathcal{C}_1$. Specifically we are not introducing any unary operator in the Boolean algebras corresponding to paraconsistent negation, but we are computing the values of formulae of the form $\neg\, \gamma$ by means of conditional equations in the algebras. In other words, $\neg$ does not correspond to the Boolean algebra complement $-$. Therefore we are not attaching any algebraic meaning to $\mathcal{C}_1$.[1] $\triangle$

In the context of an isp $\mathcal{S}$, a *truth-functional derived connective* of arity k is a λ-term $\lambda y_1 \ldots y_k \,.\, \delta$ where the set of schema variables occurring in the schema formula δ is $\{y_1, \ldots, y_k\}$ and such that some equation of the form

$$v(\delta) = t\, \theta_x^{v(y)}$$

is derived from $S^\bullet$ (within the meta-logic CEQ), where t is some τ-term in which only the variables $x_1, ..., x_k$ may occur, and $\theta_x^{v(y)}$ is the substitution such that $\theta_x^{v(y)}(x_n) = v(y_n)$ for every $n \geq 1$.

In $\mathcal{C}_1$, classical negation $\sim\ := \lambda y_1 \,.\, \neg\, y_1 \wedge y_1^\circ$ (take t as $-(x_1)$) and equivalence $\equiv\ := \lambda y_1 y_2 \,.\, (y_1 \supset y_2) \wedge (y_2 \supset y_1)$ (take t as $\Leftrightarrow(x_1, x_2)$) are both truth-functional derived connectives. And, of course, so are the primitive conjunction $\lambda y_1 y_2 \,.\, y_1 \wedge y_2$, disjunction $\lambda y_1 y_2 \,.\, y_1 \vee y_2$, and implication $\lambda y_1 y_2 \,.\, y_1 \supset y_2$. But, paraconsistent negation $\lambda y_1 \,.\, \neg\, y_1$ is not truth-functional.

[1] The question of algebraizing paraconsistent logic is a separate issue and we refer the interested reader to [19, 16].

In the following isp, all derived connectives are truth-functional, but the modality $\lambda y_1 \,.\, L\, y_1$ would require in $\Sigma(C)$ the extra generator $\Box$ in $O_{\tau\,\tau}$ satisfying:

- $(\ \rightarrow \Box(\top) = \top)$;
- $(\ \rightarrow \Box(\sqcap(x_1, x_2)) = \sqcap(\Box(x_1), \Box(x_2)))$;
- $(\ \rightarrow \sqcap(\Box(x_1), -(\Box(-(x_1))) = \Box(x_1))$;
- $(\ \rightarrow v(L\, y_1) = \Box(v(y_1)))$.

Example 2.7 *Modal system KD* [14, 15]:

- Object signature - C:
  - $C_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;
  - $C_1 = \{\neg, L\}$;
  - $C_2 = \{\wedge, \vee, \supset\}$.
- Meta-axioms - S:
  - Truth-values axioms:
    - Further axioms in order to obtain a specification of the class of all Boolean algebras as in the previous example.
  - Valuation axioms:
    - $(\ \rightarrow v(\mathbf{t}) = \top)$;
    - $(\ \rightarrow v(\mathbf{f}) = \bot)$;
    - $(\ \rightarrow v(\neg\, y_1) = -(v(y_1)))$;
    - $(\ \rightarrow v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2)))$;
    - $(\ \rightarrow v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2)))$;
    - $(\ \rightarrow v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2)))$;
    - $(\ \rightarrow v(L\, \mathbf{t}) = \top)$;
    - $(\ \rightarrow v(L(y_1 \wedge y_2)) = \sqcap(v(L\, y_1), v(L\, y_2)))$;
    - $(\ \rightarrow \sqcap(v(L\, y_1), v(\neg\, L\, \neg\, y_1)) = v(L\, y_1))$;
    - $(v(y_1) = v(y_2) \rightarrow v(L\, y_1) = v(L\, y_2))$.

  Note that the last four axioms are related to the axioms on $\Box$. Their inclusion allows us to specify the intended modal algebras while avoiding the use of the $\Box$ that is not in the signature $\Sigma(C)$. $\triangle$

Returning to the first example (paraconsistent system $\mathcal{C}_1$), it is straightforward to verify that every paraconsistent valuation introduced in [7] has a counterpart, up to isomorphism because of the built-in freedom in the truth-values, in $\mathrm{Int}(\mathcal{S})$. Furthermore, the additional interpretation structures do not change the semantic entailment (as defined below). Note that it is easy to extend this example in order to set up the isp's for the whole hierarchy $\mathcal{C}_n$ by specifying the paraconsistent n-valuations introduced in [17].

With respect to the second example (modal system KD), it is also straightforward to verify that every Kripke model has a counterpart in $\mathrm{Int}(\mathcal{S})$: consider the algebra of truth-values given by the power set of the set of worlds. Furthermore, every general model in [23] also has a counterpart in $\mathrm{Int}(\mathcal{S})$: take $\langle \mathcal{B}, \nu \rangle$ as the algebra of the truth-values. Again, the extra interpretation structures do not change the semantic entailment.

We are now ready to define the (global and local) semantic entailments. To this end, we need to refer to the denotation $[\![t]\!]^{\rho}_{\mathcal{A}}$ of a meta-term t given an assignment ρ over an algebra $\mathcal{A}$. As expected, an assignment maps each variable to an element in the carrier set of the sort of the variable. In the case of a ground term t, as usual, we just write $[\![t]\!]_{\mathcal{A}}$ for its denotation in $\mathcal{A}$.

Definition 2.8 Given an isp $\mathcal{S}$, a set Γ of schema formulae and a schema formula δ, we say that:

- $\Gamma \models^{\mathcal{S}}_{g} \delta$ (Γ *globally entails* δ) iff, for every $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$, $v_{\mathcal{A}}([\![\delta]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$ whenever $v_{\mathcal{A}}([\![\gamma]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$ for every $\gamma \in \Gamma$;
- $\Gamma \models^{\mathcal{S}}_{l} \delta$ (Γ *locally entails* δ) iff, for every $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$ and $b \in \mathcal{A}_{\phi}$, $v_{\mathcal{A}}(b) \leq_{\mathcal{A}} v_{\mathcal{A}}([\![\delta]\!]_{\mathcal{A}})$ whenever $v_{\mathcal{A}}(b) \leq_{\mathcal{A}} v_{\mathcal{A}}([\![\gamma]\!]_{\mathcal{A}})$ for every $\gamma \in \Gamma$.

Observe that $\Gamma \models^{\mathcal{S}}_{l} \delta$ implies that $\Gamma \models^{\mathcal{S}}_{g} \delta$, provided that $v_{\mathcal{A}}(b) = \top_{\mathcal{A}}$ for some $b \in \mathcal{A}_{\phi}$. On the other hand, if either $\Gamma = \emptyset$ or $\mathcal{A}_{\tau} = \{\bot_{\mathcal{A}}, \top_{\mathcal{A}}\}$ with $\bot_{\mathcal{A}} \leq_{\mathcal{A}} \top_{\mathcal{A}}$, then $\Gamma \models^{\mathcal{S}}_{l} \delta$ iff $\Gamma \models^{\mathcal{S}}_{g} \delta$.

Before turning our attention to the problem of fibring isp's, we first define the categories **Sig** and **Isp**. The objects of the category **Sig** are object signatures. A morphism $h : C \to C'$ in **Sig** is of the form $h = \{h_k : C_k \to C'_k\}_{k \in \mathbb{N}}$, each h_k being a map. The objects of the category **Isp** are isp's. The appropriate notion of morphism in the category **Isp** is as follows: each $h : \langle C, S \rangle \to \langle C', S' \rangle$ is a morphism $h : C \to C'$ in **Sig** such that for each $s \in S$, $\hat{h}(s)$ is in $S'^{\bullet}$. Here, $\hat{h}$ is the unique map from the meta-language over $\Sigma(C)$ to the meta-language over $\Sigma(C')$ that extends the map h. Note that such a morphism imposes the condition that for every $\mathcal{A}' \in \mathrm{Int}(\mathcal{S}')$ its reduct to $\Sigma(C)$, via h and the identity on the generators, is in $\mathrm{Int}(\mathcal{S})$.

In what follows, we shall also use the forgetful functor N from **Isp** to the category **Sig** of object signatures. This functor maps each $\mathcal{S}$ to the underlying object signature C and each morphism h to the underlying object signature morphism.

3 Fibring non-truth-functional logics

We assume that we are given two isp's $\mathcal{S}'$ and $\mathcal{S}''$. Following the method proposed in [21], we start by considering the notion of *unconstrained fibring* that corresponds to combining the two isp's without sharing any of the symbols of the object signatures C' and C''. That is, if we so combine $\mathcal{C}_1$ and KD we shall obtain in the result of the fibring two different symbols for conjunction,

disjunction, etc. This construction appears as the coproduct of $\mathcal{S}'$ and $\mathcal{S}''$ in the category $\mathbf{Isp}(G)$. Therefore,

$$\mathcal{S}' \oplus \mathcal{S}'' = \langle C, S \rangle$$

where:

- $C = C' \oplus C''$ within **Sig** with the injections i' and i'';
- $S = \hat{i'}(S') \cup \hat{i''}(S'')$.

Proposition 3.1 Given $\mathcal{S}'$ and $\mathcal{S}''$ as above, a $\Sigma(C' \oplus C'', \Xi)$-algebra $\mathcal{A}$ belongs to $\mathrm{Int}(\mathcal{S}' \oplus \mathcal{S}'')$ iff:

- $\mathcal{A}|^{i'}_{\Sigma(C',\Xi)} \in \mathrm{Int}(\mathcal{S}')$;
- $\mathcal{A}|^{i''}_{\Sigma(C'',\Xi)} \in \mathrm{Int}(\mathcal{S}'')$;

where $\mathcal{A}|^{i'}_{\Sigma(C',\Xi)}$ and $\mathcal{A}|^{i''}_{\Sigma(C'',\Xi)}$ are the reducts of $\mathcal{A}$ to the signatures $\Sigma(C', \Xi)$ and $\Sigma(C'', \Xi)$, respectively, via the indicated inclusions and the identity on the generators.

Again following the method in [21], it is easy to introduce the notion of *constrained fibring by sharing connectives and/or propositional symbols* that corresponds to combining the two isp's while sharing some of the symbols of the object signatures C' and C'': the construction appear as a co-Cartesian lifting by the functor N along the signature coequalizer for the envisaged pushout of the signatures. We refrain from dwelling further on this construction since it does not bring any insight to the main issue of this paper (that is, the fibring of logics possibly with non-truth-functional semantics). For an interesting example of constrained fibring see the next section.

4 Application I: modal paraconsistent logic

The idea is to combine $\mathcal{C}_1$ and KD by fibring them while sharing the propositional symbols, conjunction, disjunction, implication, true and false. Let $\mathcal{S}' = \langle C', S' \rangle$ be the isp for $\mathcal{C}_1$ as described in Example 2.6 and $\mathcal{S}'' = \langle C'', S'' \rangle$ the isp for KD as described in Example 2.7.

We work first in the category **Sig**. Consider the propositional-based signature B as follows:

- $B_0 = \{p_n : n \in \mathbb{N}\} \cup \{\mathbf{t}, \mathbf{f}\}$;
- $B_2 = \{\wedge, \vee, \supset\}$;
- $B_k = \emptyset$ for the other values of k.

The propositional-based signature $C' \overset{f'Bf''}{\oplus} C''$ of the envisaged constrained fibring is obtained by the pushout of $C' \xleftarrow{f'} B \xrightarrow{f''} C''$ where f' and f'' are the obvious inclusions.

The unique morphism z from the coproduct $C' \oplus C''$ to $C' \overset{f'Bf''}{\oplus} C''$ is now used to obtain the envisaged isp as briefly described at the end of the previous section. The isp we want,

$$\mathcal{S}' \overset{f'Bf''}{\oplus} \mathcal{S}'',$$

corresponding to the constrained sharing of $\mathcal{S}'$ and $\mathcal{S}''$ while sharing the symbols in B, appears as the co-Cartesian lifting by the functor N along z on the isp $\mathcal{S}' \oplus \mathcal{S}''$. That is,

$$\mathcal{S}' \overset{f'Bf''}{\oplus} \mathcal{S}'' = \langle C' \overset{f'Bf''}{\oplus} C'', \hat{z}(\hat{i'}(S') \cup \hat{i''}(S'')) \rangle.$$

As expected, the interpretation structures presented by this isp are precisely those algebras in the unconstrained fibring that agree on the shared symbols.

Note that we end up having two negations: $z(i'(\neg))$ coming from C' and $z(i''(\neg))$ coming from C''. The former is a paraconsistent negation and the latter is the classical negation inherited from KD. Clearly, the (classical) strong negation $\lambda y_1 \,.\, \neg y_1 \wedge y_1^\circ$ inherited from $\mathcal{C}_1$ collapses into $z(i''(\neg))$.

One particularly interesting application of modal paraconsistent logic is in coping with certain problems of deontic logic having to do with deontic paradoxes and moral dilemmas. In [8] a paraconsistent deontic logic called $\mathcal{C}_1^D$ is introduced by combining the paraconsistent system $\mathcal{C}_1$ with modal KD interpreting the modal operator L as "obligatory". In order to recover $\mathcal{C}_1^D$ at the semantic level in our setting we have to add one additional meta-axiom for valuations:

- $(v(y_1^\circ) = \top \rightarrow v((L\, y_1)^\circ) = \top)$.

Using the terminology introduced in [4], this procedure can be seen as a *splitting* of $\mathcal{C}_1^D$ in the components KD and $\mathcal{C}_1$.

5 Logic systems

This section is devoted to extending the fibring techniques to the syntactical counterpart of isp's. As we have hinted before, we shall use the propositional schema variables in $\Xi = \{\xi_1, \xi_2, \dots\}$ to write *inference rules*.

Definition 5.1 A *Hilbert calculus* over Ξ is a triple $\langle C, P, D \rangle$ in which (1) C is an object signature, (2) P is a subset of $\wp_{fin} L(C, \Xi) \times L(C, \Xi)$, (3) D is a subset of $(\wp_{fin} L(C, \Xi) \setminus \emptyset) \times L(C, \Xi)$, and (4) $D \subseteq P$.

Given any $r = \langle \Gamma, \gamma \rangle$ in P, the (finite) set Γ is the set of premises of r and γ is the conclusion; we will often write $r = \langle Prem(r), Conc(r) \rangle$. If $Prem(r) = \emptyset$, then r is said to be an *axiom schema*; otherwise, it is said to be a *proof rule schema*. Each r in D is said to be a *derivation rule schema*.

Example 5.2 In the modal Hilbert calculus KD we have:

- $P = \{\langle \emptyset, \xi_1 \supset (\xi_2 \supset \xi_1)\rangle,$
  $\langle \emptyset, (\xi_1 \supset (\xi_2 \supset \xi_3)) \supset ((\xi_1 \supset \xi_2) \supset (\xi_1 \supset \xi_3))\rangle,$
  $\langle \emptyset, (\neg \xi_1 \supset \neg \xi_2) \supset (\xi_2 \supset \xi_1)\rangle,$
  $\langle \emptyset, L(\xi_1 \supset \xi_2) \supset (L\,\xi_1 \supset L\,\xi_2)\rangle,$
  $\langle \emptyset, L\,\xi_1 \supset \neg\, L \neg \xi_1\rangle,$
  $\langle \emptyset, (\xi_1 \vee \xi_2) \equiv (\neg \xi_1 \supset \xi_2)\rangle,$
  $\langle \emptyset, (\xi_1 \wedge \xi_2) \equiv \neg(\neg \xi_1 \vee \neg \xi_2)\rangle,$
  $\langle \emptyset, \mathbf{t} \equiv (\xi_1 \supset \xi_1)\rangle,$
  $\langle \emptyset, \mathbf{f} \equiv (\xi_1 \wedge \neg \xi_1)\rangle,$
  $\langle \{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle,$
  $\langle \{\xi_1\}, L\,\xi_1\rangle\};$
- $D = \{\langle \{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle\}.$

Example 5.3 The Hilbert calculus $\mathcal{C}_1$ is easily defined adapting the well known axiomatics presented in [7].

Definition 5.4 A schema formula $\delta \in L(C, \Xi)$ is *provable* from the set of schema formulae $\Gamma \subseteq L(C, \Xi)$ in the Hilbert calculus $\langle C, P, D\rangle$, denoted by $\Gamma \vdash_p^{PD} \delta$, iff there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C, \Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either

(1) $\gamma_i \in \Gamma$, or

(2) there exist a rule $r \in P$ and a schema variable substitution $\sigma : \Xi \to L(C, \Xi)$ such that $Conc(r)\sigma = \gamma_i$ and $Prem(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

Definition 5.5 A schema formula $\delta \in L(C, \Xi)$ is *derivable* from the set of schema formulae $\Gamma \subseteq L(C, \Xi)$ in the Hilbert calculus $\langle C, P, D\rangle$, denoted by $\Gamma \vdash_d^{PD} \delta$, iff there is a sequence $\gamma_1, \ldots, \gamma_m \in L(C, \Xi)^+$ such that $\gamma_m = \delta$ and, for $i = 1$ to m, either

(1) $\gamma_i \in \Gamma$, or

(2) γ_i is provable from the empty set of formulae, or

(3) there exist a rule $r \in D$ and a schema variable substitution σ such that $Conc(r)\sigma = \gamma_i$ and $Prem(r)\sigma \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$.

Clearly, if $\Gamma \vdash_d^{PD} \delta$ then also $\Gamma \vdash_p^{PD} \delta$. Furthermore, if $\vdash_p^{PD} \delta$ then $\vdash_d^{PD} \delta$. The following *structurality* properties are also immediate: for every schema variable substitution σ, if $\Gamma \vdash_p^{PD} \delta$ then $\Gamma\sigma \vdash_p^{PD} \delta\sigma$, and if $\Gamma \vdash_d^{PD} \delta$ then $\Gamma\sigma \vdash_d^{PD} \delta\sigma$.

Definition 5.6 The *unconstrained fibring* of the Hilbert calculi $\langle C', P', D'\rangle$ and $\langle C'', P'', D''\rangle$ is the Hilbert calculus

$$\langle C', P', D'\rangle \oplus \langle C'', P'', D''\rangle \;=\; \langle C' \oplus C'', \hat{i}'(P') \cup \hat{i}''(P''), \hat{i}'(D') \cup \hat{i}''(D'')\rangle.$$

Definition 5.7 The *constrained fibring* of the Hilbert calculi $\langle C', P', D' \rangle$ and $\langle C'', P'', D'' \rangle$ sharing C according to the injective morphisms $f' : C \to C'$ and $f'' : C \to C''$ is the Hilbert calculus $\langle C', P', D' \rangle \overset{f'Cf''}{\oplus} \langle C'', P'', D'' \rangle$ defined as follows:

$$\langle C' \overset{f'Cf''}{\oplus} C'', \hat{z}(\hat{i}'(P')) \cup \hat{z}(\hat{i}''(P'')), \hat{z}(\hat{i}'(D')) \cup \hat{z}(\hat{i}''(D'')) \rangle.$$

By adopting the notion of Hilbert calculus morphism proposed in [21], both forms of fibring above again appear as universal constructions (coproduct and co-Cartesian lifting). It follows that there is a morphism from each given Hilbert calculus to the fibring, e.g., h' from $\langle C', P', D' \rangle$ to $\langle C, P, D \rangle$. Therefore:

- if $\Gamma \vdash_p^{P'D'} \delta$ then $\hat{h}'(\Gamma) \vdash_p^{PD} \hat{h}'(\delta)$;
- if $\Gamma \vdash_d^{P'D'} \delta$ then $\hat{h}'(\Gamma) \vdash_d^{PD} \hat{h}'(\delta)$.

The fibring of Hilbert calculi for $\mathcal{C}_1$ and KD, sharing the propositional symbols, conjunction, disjunction, implication, true and false, is the Hilbert calculus where we have all the proof and derivation rules for both $\mathcal{C}_1$ and KD. In order to get the deontic paraconsistent system $\mathcal{C}_1^D$ of [8], at the proof-theoretic level, we need to introduce the following proof rule:

- $\langle \emptyset, \xi_1^\circ \supset (L\, \xi_1)^\circ \rangle$.

Definition 5.8 A *logic system* is a tuple $\mathcal{L} = \langle C, S, P, D \rangle$ where the pair $\langle C, S \rangle$ constitutes an isp and the triple $\langle C, P, D \rangle$ constitutes a Hilbert calculus.

As expected, the (unconstrained and constrained) fibring of logic systems is obtained by the corresponding fibring of the underlying isp's and Hilbert calculi.

Example 5.9 The logic systems for $\mathcal{C}_1$ and KD will be denoted by $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$, respectively, and their fibring while sharing B will be denoted by $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$.

Definition 5.10 Given a logic system $\mathcal{L} = \langle C, S, P, D \rangle$, we say that the deduction system $\langle C, P, D \rangle$ is *sound w.r.t. the isp* $\langle C, S \rangle$, or simply that $\mathcal{L}$ is *sound*, iff for every set Γ of schema formulae and every schema formula δ:

- $\Gamma \vdash_p^{PD} \delta$ implies $\Gamma \models_g^{\mathcal{S}} \delta$;
- $\Gamma \vdash_d^{PD} \delta$ implies $\Gamma \models_l^{\mathcal{S}} \delta$.

We say that $\langle C, P, D \rangle$ is *adequate w.r.t.* $\langle C, S \rangle$, or simply that $\mathcal{L}$ is *adequate*, iff for every set Γ of schema formulae and every schema formula δ:

- $\Gamma \vdash_p^{PD} \delta$ is implied by $\Gamma \models_g^{\mathcal{S}} \delta$;
- $\Gamma \vdash_d^{PD} \delta$ is implied by $\Gamma \models_l^{\mathcal{S}} \delta$.

Furthermore, we say that $\langle C, P, D \rangle$ is *complete w.r.t.* $\langle C, S \rangle$, or simply that $\mathcal{L}$ is *complete*, iff it is both sound and adequate.

Example 5.11 The logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$ are complete.

6 Preservation results

The main goal is to establish the preservation of completeness by fibring. To this end, it is convenient to take advantage of the completeness of CEQ as proved for instance in [13, 18] by encoding the deduction system of the meta-logic CEQ in the object Hilbert calculus.

In order to deal with local reasoning at the meta-level, we shall take advantage of the following two schema variable substitutions:

- σ^{+1} such that $\sigma^{+1}(\xi_i) = \xi_{i+1}$ for every $i \geq 1$;
- σ^{-1} such that $\sigma^{-1}(\xi_1) = \xi_1$ and $\sigma^{-1}(\xi_i) = \xi_{i-1}$ for every $i \geq 2$.

Note that if γ is a schema formula then $\gamma\sigma^{+1}$ is a variant of γ where ξ_1 does not occur. Furthermore, easily, $\gamma\sigma^{+1}\sigma^{-1} = \gamma$.

6.1 Encoding

First we analyse what can be obtained proof-theoretically within CEQ. Given an isp $\mathcal{S} = \langle C, S\rangle$, we adopt the following abbreviations, where $\Gamma \cup \{\delta\} \subseteq L(C, \Xi)$:

- $\Gamma \vdash_g^{\mathcal{S}} \delta$ for $S^\bullet \cup \{(\to v(\gamma) = \top) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\delta) = \top$;
- $\Gamma \vdash_l^{\mathcal{S}} \delta$ for $S^\bullet \cup \{(\to v(\xi_1) \leq v(\gamma\sigma^{+1})) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\xi_1) \leq v(\delta\sigma^{+1})$.

Theorem 6.1 Given an isp $\mathcal{S} = \langle C, S\rangle$ and $\Gamma \cup \{\delta\} \subseteq L(C, \Xi)$, we have:

- $\Gamma \models_g^{\mathcal{S}} \delta$ iff $\Gamma \vdash_g^{\mathcal{S}} \delta$;
- $\Gamma \models_l^{\mathcal{S}} \delta$ iff $\Gamma \vdash_l^{\mathcal{S}} \delta$.

Proof: This is an immediate consequence of the completeness of the meta-logic CEQ. In the local case it is essential to note that, since schema variables cannot occur in $S^\bullet$, we can freely change the denotation of schema variables given by an algebra $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$ (namely according to σ^{+1} or σ^{-1}) and still obtain an algebra in $\mathrm{Int}(\mathcal{S})$. The fact that ξ_1 cannot occur in schema formulae instantiated by σ^{+1} does the rest. QED

For the envisaged encoding, it is necessary to assume that the logic system at hand is sufficiently expressive:

Definition 6.2 A logic system $\mathcal{L} = \langle C, S, P, D\rangle$ is said to be *rich* iff:

1. $\mathbf{t}, \mathbf{f} \in C_0$ and $\wedge, \vee, \supset \in C_2$;
2. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\mathbf{t}) = \top$;
3. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\mathbf{f}) = \bot$;
4. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \wedge y_2) = \sqcap(v(y_1), v(y_2))$;

5. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \vee y_2) = \sqcup(v(y_1), v(y_2))$;

6. $S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(y_1 \supset y_2) = \Rightarrow(v(y_1), v(y_2))$;

7. $\langle\{\xi_1, \xi_1 \supset \xi_2\}, \xi_2\rangle \in D$.

Example 6.3 Both logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$, as well as many other common logics, are rich.

Within a rich logic system it is possible to translate from the meta-logic level to the object logic level. A ground term of sort τ over $\Sigma(C, \Xi)$ is mapped to a formula in $L(C, \Xi)$ according to the following rules:

- $v(\gamma)^*$ is γ
- $\top^*$ is $\mathbf{t}$;
- $\bot^*$ is $\mathbf{f}$;
- $-(t)^*$ is $t^* \supset \mathbf{f}$;
- $\sqcap(t_1, t_2)^*$ is $t_1^* \wedge t_2^*$;
- $\sqcup(t_1, t_2)^*$ is $t_1^* \vee t_2^*$;
- $\Rightarrow(t_1, t_2)^*$ is $t_1^* \supset t_2^*$.

Moreover, a ground τ-equation $(t_1 = t_2)$ is translated to $(t_1 = t_2)^*$ given by $t_1^* \equiv t_2^*$. Finally, if E is a set of ground τ-equations, then E^* will denote the set $\{\mathrm{eq}^* : \mathrm{eq} \in E\}$.

Lemma 6.4 Let $\mathcal{L}$ be a rich logic system and t a ground τ-term over $\Sigma(C, \Xi)$. Then:

$$S^\bullet \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(t^*) = t.$$

Proof: Immediate by definition, taking into account the requirements of richness and the completeness of CEQ. QED

Lemma 6.5 Let $\mathcal{L}$ be a rich logic system, t_1 and t_2 ground τ-terms over $\Sigma(C, \Xi)$ and $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$. Then:

$$[\![t_1]\!]_\mathcal{A} \leq_\mathcal{A} [\![t_2]\!]_\mathcal{A} \quad \text{iff} \quad v_\mathcal{A}([\![t_1^* \supset t_2^*]\!]_\mathcal{A}) = \top_\mathcal{A}$$

and

$$[\![t_1]\!]_\mathcal{A} = [\![t_2]\!]_\mathcal{A} \quad \text{iff} \quad v_\mathcal{A}([\![t_1^* \equiv t_2^*]\!]_\mathcal{A}) = \top_\mathcal{A}.$$

Proof: Direct corollary of Proposition 2.5 using the previous Lemma and taking into account the completeness of CEQ. QED

In a rich logic system, under certain conditions (cf. Definition 6.6 below), one can encode the relevant part of the meta-reasoning into the object calculus.

Definition 6.6 A rich logic system $\mathcal{L}$ is said to be *equationally appropriate* iff for every conditional equation $(\mathrm{eq}_1 \;\&\; \ldots \;\&\; \mathrm{eq}_n \;\rightarrow\; \mathrm{eq})$ in $S^\bullet$ and every ground substitution θ:

$$\{(\mathrm{eq}_1\theta)^*, \ldots, (\mathrm{eq}_n\theta)^*\} \vdash_p^{PD} (\mathrm{eq}\,\theta)^*.$$

Finally, we obtain the main results of this Section relating adequacy to equational appropriateness. Such results are important because it is much easier to analyze the preservation by fibring of equational appropriateness than of adequacy directly.

Theorem 6.7 Every rich and adequate logic system is equationally appropriate.

Proof: Assume that $\mathcal{L}$ is a rich and adequate logic system and $\mathcal{A} \in \mathrm{Int}(\mathcal{S})$, and let $(t_1 = s_1 \;\&\; \ldots \;\&\; t_n = s_n \;\rightarrow\; t = s)$ be a conditional equation in $S^\bullet$ and θ a ground substitution.

If it is the case that $v_{\mathcal{A}}([\![(t_i\theta)^* \equiv (s_i\theta)^*]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$ for $i = 1, \ldots, n$, then, according to the previous Lemma, this means precisely that $[\![t_i\theta]\!]_{\mathcal{A}} = [\![s_i\theta]\!]_{\mathcal{A}}$ for $i = 1, \ldots, n$. Consider the assignment $\rho = [\![\_]\!]_{\mathcal{A}} \circ \theta$. It is straightforward to verify that $[\![r\theta]\!]_{\mathcal{A}} = [\![r]\!]^{\rho}_{\mathcal{A}}$, for every τ-term r over $\Sigma(C, \Xi)$. So, since by definition of $\mathrm{Int}(\mathcal{S})$ we know that $\mathcal{A}$ is a model of the conditional equation, it immediately follows that also $[\![t\theta]\!]_{\mathcal{A}} = [\![s\theta]\!]_{\mathcal{A}}$, or equivalently, $v_{\mathcal{A}}([\![(t\theta)^* \equiv (s\theta)^*]\!]_{\mathcal{A}}) = \top_{\mathcal{A}}$.

Therefore, we have $\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \models_g^{\mathcal{S}} (t\theta)^* \equiv (s\theta)^*$. Now, equational appropriateness follows easily since, from adequacy, we must have $\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \vdash_p^{PD} (t\theta)^* \equiv (s\theta)^*$. QED

Before proving the converse of this theorem, we need to establish some technical lemmas.

Lemma 6.8 Let $\mathcal{L}$ be an equationally appropriate logic system and $\Gamma \cup \{\delta\}$ be a set of schema formulae where ξ_1 does not occur. If $\{\xi_1 \supset \gamma : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta$ then $\Gamma \vdash_d^{PD} \delta$.

Proof: First of all we note that, since $S^\bullet$ must contain a specification of the class of all Heyting algebras, equational appropriateness implies that every intuitionistic theorem written with $\mathbf{t}, \mathbf{f}, \wedge, \vee, \supset$ must be provable in the Hilbert calculus. In the sequel, we shall use this fact without further notice.

Let us assume that $\{\xi_1 \supset \gamma : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta$. Immediately, by the finite character of derivations, $\{\xi_1 \supset \gamma_1, \ldots, \xi_1 \supset \gamma_n\} \vdash_p^{PD} \xi_1 \supset \delta$, where each $\gamma_i \in \Gamma$. Let now γ be the schema formula $\gamma_1 \wedge \ldots \wedge \gamma_n$ and take the schema variable substitution σ such that $\sigma(\xi_1) = \gamma$ and $\sigma(\xi_i) = \xi_i$ for every $i \geq 2$. Since ξ_1 does not occur in Γ or δ, the structurality of proofs easily implies that we must also have $\{\gamma \supset \gamma_1, \ldots, \gamma \supset \gamma_n\} \vdash_p^{PD} \gamma \supset \delta$. But it is clear by easy intuitionistic reasoning that $\vdash_p^{PD} \gamma \supset \gamma_i$ for $i = 1, \ldots, n$. So, it follows that $\vdash_p^{PD} \gamma \supset \delta$, and since by further intuitionistic reasoning we have $\vdash_p^{PD} \gamma_1 \supset (\ldots \supset (\gamma_n \supset \gamma) \ldots)$, the derivation rule of *Modus Ponens* immediately implies that $\Gamma \vdash_d^{PD} \delta$. QED

Lemma 6.9 Let $\mathcal{L}$ be an equationally appropriate logic system, E a set of ground τ-equations over $\Sigma(C,\Xi)$ and θ a ground substitution. If

$$S^\bullet \cup \{(\rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_1 = t_2$$

then either t_1, t_2 are the same term of sort ϕ, or t_1, t_2 are of sort τ and

$$E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*.$$

Proof: The rules of CEQ [18] are the usual Reflexivity, Symmetry, Transitivity, Congruence and a form of *Modus Ponens* that allows us to obtain $\mathrm{eq}\,\theta$ from $\mathrm{eq}_1\theta, \ldots, \mathrm{eq}_n\theta$ for any given conditional equation $(\mathrm{eq}_1\ \&\ \ldots\ \&\ \mathrm{eq}_n \rightarrow \mathrm{eq})$.

Given a ground substitution θ, we show by induction on the length n of a proof of $S^\bullet \cup \{(\rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_1 = t_2$ that $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*$.
Base: $n = 1$.
(i) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained by CEQ *Modus Ponens* from $(\rightarrow s_1 = s_2) \in S^\bullet$.
Obviously t_1 and t_2 are τ-terms. Immediately, by equational appropriateness, we have $\vdash^{PD}_p (s_1\theta'\theta)^* \equiv (s_2\theta'\theta)^*$ and therefore $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*$ by the monotonicity of provability.
(ii) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained by CEQ *Modus Ponens* from $(\rightarrow s_1 = s_2)$ with $s_1 = s_2 \in E$.
Obviously t_1 and t_2 are closed τ-terms, $s_1\theta'$ is s_1 and $s_2\theta'$ is s_2. Thus, $(t_1\theta)^* \equiv (t_2\theta)^* \in E^*$, i.e., $t_1^* \equiv t_2^* \in E^*$ and $E^* \vdash^{PD}_p t_1^* \equiv t_2^*$ by the extensiveness of provability.
(iii) t_1 and t_2 are the same term, of either sort ϕ or τ, and $t_1 = t_2$ is obtained by Reflexivity.
If the sort is ϕ we are done. Otherwise, obviously, $(t_1\theta)^*$ and $(t_2\theta)^*$ are the same formula and trivial intuitionistic reasoning allows us to conclude that $\vdash^{PD}_p \xi_1 \equiv \xi_1$. Therefore, $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*$ by the structurality and monotonicity of provability.
Step: $n > 1$.
(i) $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_2 = t_1$ by Symmetry.
If the terms have sort ϕ, by induction hypothesis, they coincide. Otherwise, also by induction hypothesis, we know that $E^* \vdash^{PD}_p (t_2\theta)^* \equiv (t_1\theta)^*$. Elementary intuitionistic reasoning allows us to conclude that $\vdash^{PD}_p (\xi_1 \equiv \xi_2) \supset (\xi_2 \equiv \xi_1)$, and therefore also $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*$.
(ii) $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_1 = t_3$, $t_3 = t_2$ by Transitivity.
If the terms have sort ϕ by induction hypothesis they coincide. Otherwise, also by induction hypothesis, we know that $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_3\theta)^*$, $(t_3\theta)^* \equiv (t_2\theta)^*$. Simple intuitionistic reasoning allows us to conclude that $\{\xi_1 \equiv \xi_2, \xi_2 \equiv \xi_3\} \vdash^{PD}_p \xi_1 \equiv \xi_3$, and therefore also $E^* \vdash^{PD}_p (t_1\theta)^* \equiv (t_2\theta)^*$.
(iii) t_1 is $f(t_{11}, \ldots, t_{1k})$, t_2 is $f(t_{21}, \ldots, t_{2k})$ and $t_1 = t_2$ is obtained from $S^\bullet \cup \{(\rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} t_{11} = t_{21}, \ldots, t_{1k} = t_{2k}$ by Congruence.
It t_1 and t_2 have sort ϕ then $f \in C_k$ and therefore all the terms t_{ij} are also

of sort ϕ. By induction hypothesis, then, t_{1j} and t_{2j} must be identical and t_1 coincides with t_2. Otherwise, f can either be v or a generator among $-, \sqcap, \sqcup, \Rightarrow$. In the first case $k = 1$ and by induction hypothesis t_{11} coincides with t_{21} since they must have sort ϕ. Therefore, t_1 and t_2 also coincide and we repeat step (iii) of the Base to obtain $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$. Finally, if f is a generator then all the terms t_{ij} are of sort τ. Thus, by induction hypothesis, $E^* \vdash_p^{PD} (t_{11}\theta)^* \equiv (t_{21}\theta)^*, \ldots, (t_{1k}\theta)^* \equiv (t_{2k}\theta)^*$. Using again intuitionistic reasoning we have that $\{\xi_1 \equiv \xi_2, \xi_3 \equiv \xi_4\} \vdash_p^{PD} (\xi_1 \wedge \xi_3) \equiv (\xi_2 \wedge \xi_4)$, $(\xi_1 \vee \xi_3) \equiv (\xi_2 \vee \xi_4)$, $(\xi_1 \supset \xi_3) \equiv (\xi_2 \supset \xi_4)$. Given that $-$ is translated using $\mathbf{f}$ and $\supset$ this is enough to guarantee that $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$.
(iv) t_1 is $s_1\theta'$, t_2 is $s_2\theta'$ and $t_1 = t_2$ is obtained using the conditional equation $(s_{11} = s_{21}\ \&\ \ldots\ \&\ s_{1k} = s_{2k} \rightarrow s_1 = s_2) \in S^\bullet$ from $S^\bullet \cup \{(\ \rightarrow \mathrm{eq}) : \mathrm{eq} \in E\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} s_{11}\theta' = s_{21}\theta', \ldots, s_{1k}\theta' = s_{2k}\theta'$ by CEQ *Modus Ponens*.
Obviously, all the terms are τ-terms and the induction hypothesis implies that $E^* \vdash_p^{PD} (s_{11}\theta'\theta)^* \equiv (s_{21}\theta'\theta)^*, \ldots, (s_{1k}\theta'\theta)^* \equiv (s_{2k}\theta'\theta)^*$. Moreover, by equational appropriateness, we have $\{(s_{11}\theta'\theta)^* \equiv (s_{21}\theta'\theta)^*, \ldots, (s_{1k}\theta'\theta)^* \equiv (s_{2k}\theta'\theta)^*\} \vdash_p^{PD} (s_1\theta'\theta)^* \equiv (s_2\theta'\theta)^*$. Therefore, $E^* \vdash_p^{PD} (t_1\theta)^* \equiv (t_2\theta)^*$. QED

Theorem 6.10 Every equationally appropriate logic system is adequate.

Proof: Let $\mathcal{L}$ be an equationally appropriate logic system and $\Gamma \cup \{\delta\} \subseteq L(C, \Xi)$.

If $\Gamma \models_g^{\mathcal{S}} \delta$ then also $\Gamma \vdash_g^{\mathcal{S}} \delta$, i.e., $S^\bullet \cup \{(\ \rightarrow v(\gamma) = \top) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\delta) = \top$. Using the previous Lemma, we have that $\{\gamma \equiv \mathbf{t}\} \vdash_p^{PD} \delta \equiv \mathbf{t}$. Trivial intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} \xi_1 \equiv (\xi_1 \equiv \mathbf{t})$, and therefore it follows that $\Gamma \vdash_p^{PD} \delta$.

If $\Gamma \models_l^{\mathcal{S}} \delta$ then also $\Gamma \vdash_l^{\mathcal{S}} \delta$, i.e., $S^\bullet \cup \{(\ \rightarrow v(\xi_1) \leq v(\gamma\sigma^{+1})) : \gamma \in \Gamma\} \vdash^{\mathrm{CEQ}}_{\Sigma(C,\Xi)} v(\xi_1) \leq v(\delta\sigma^{+1})$. Using the previous Lemma, we have that $\{(\xi_1 \wedge \gamma\sigma^{+1}) \equiv \xi_1 : \gamma \in \Gamma\} \vdash_p^{PD} (\xi_1 \wedge \delta\sigma^{+1}) \equiv \xi_1$. Trivial intuitionistic reasoning allows us to conclude that $\vdash_p^{PD} (\xi_1 \supset \xi_2) \equiv ((\xi_1 \wedge \xi_2) \equiv \xi_1)$, and therefore it follows that $\{\xi_1 \supset \gamma\sigma^{+1} : \gamma \in \Gamma\} \vdash_p^{PD} \xi_1 \supset \delta\sigma^{+1}$. Thus, we already know that $\Gamma\sigma^{+1} \vdash_d^{PD} \delta\sigma^{+1}$ and by structurality, using σ^{-1}, we have $\Gamma \vdash_d^{PD} \delta$. QED

The equivalence between adequacy and equational appropriateness for rich systems will be used below for showing that adequacy is preserved by fibring rich systems, but this equivalence may also be useful for establishing the adequacy of logics endowed with a semantics presented by conditional equations. Indeed, it is a much easier task to verify equational appropriateness than to establish adequacy directly.

6.2 Preservation of completeness by fibring

We consider in turn the preservation of soundness and of adequacy by fibring.

Theorem 6.11 Soundness is preserved by fibring.

Proof: Let $\mathcal{L}$ be the fibring of two sound logic systems $\mathcal{L}'$ and $\mathcal{L}''$. It is enough to prove the following: $\Gamma \vdash_p^{PD} \delta$ implies that $\Gamma \vdash_g^{\mathcal{S}} \delta$, and $\Gamma \vdash_d^{PD} \delta$ implies that $\Gamma \vdash_l^{\mathcal{S}} \delta$, by Theorem 6.1. Moreover, it is enough to prove that $Prem(r) \vdash_g^{\mathcal{S}} Conc(r)$ for every $r \in P$, and $Prem(r) \vdash_l^{\mathcal{S}} Conc(r)$ for every $r \in D$. Let $r \in P$. Assume, without loss of generality, that $r = \hat{h}'(r')$. Then, by definition of proof, $Prem(r)' \vdash_p^{P'D'} Conc(r)'$ and, so, $Prem(r)' \vdash_g^{\mathcal{S}'} Conc(r)'$, by the soundness of $\mathcal{L}'$. This means that

$$S'^{\bullet} \cup \{(\rightarrow v(\gamma') = \top) : \gamma' \in Prem(r)'\} \vdash^{CEQ}_{\Sigma(C',\Xi)} v(Conc(r)') = \top$$

and then, by the uniformness of CEQ under change of notation by $\hat{h}'$ (cf. [18]), we obtain

$$\hat{h}'(S')^{\bullet} \cup \{(\rightarrow v(\hat{h}'(\gamma')) = \top) : \gamma' \in Prem(r)'\} \vdash^{CEQ}_{\Sigma(C,\Xi)} v(\hat{h}'(Conc(r)')) = \top.$$

This immediately implies $\hat{h}'(Prem(r)') \vdash_g^{\mathcal{S}} \hat{h}'(Conc(r)')$, that is, $Prem(r) \vdash_g^{\mathcal{S}} Conc(r)$. The proof for derivations is similar. QED

We, finally, consider the problem of preservation of adequacy by fibring, taking advantage of the technical machinery presented before on the encoding of the meta-logic in the object Hilbert calculus.

Lemma 6.12 Richness is preserved by fibring provided that conjunction, disjunction, implication, true and false are shared.

Proof: It is trivial that the signature and valuation requirements are preserved since we are sharing conjunction, disjunction, implication, true and false. Moreover, it is clear that *Modus Ponens* is still a derivation rule in the fibring. QED

Lemma 6.13 Equational appropriateness is preserved by fibring provided that conjunction, disjunction, implication, true and false are shared.

Proof: Let $\mathcal{L}'$ and $\mathcal{L}''$ be equationally appropriate logic systems, and $\mathcal{L}$ their fibring by sharing conjunction, disjunction, implication, true and false. From the previous Lemma we already know that $\mathcal{L}$ is rich.

Now, let ceq be $(t_1 = s_1 \ \& \ \ldots \ \& \ t_n = s_n \rightarrow t = s) \in S^{\bullet}$, and θ a ground substitution. Clearly, by definition of fibring, ceq must be the translation of a conditional equation in some of the components. Let us assume, without loss of generality, that ceq comes from $\mathcal{L}'$, i.e., ceq is $\hat{h}'(\text{ceq}')$, where ceq$'$ is the conditional equation $(t'_1 = s'_1 \ \& \ \ldots \ \& \ t'_n = s'_n \rightarrow t' = s') \in S'^{\bullet}$. Since we know that $\mathcal{L}'$ is equationally appropriate, it follows that

$$\{(t'_1\theta')^* \equiv (s'_1\theta')^*, \ldots, (t'_n\theta')^* \equiv (s'_n\theta')^*\} \vdash_p^{P'D'} (t'\theta')^* \equiv (s'\theta')^*,$$

where θ' is the following ground substitution:

- for every $i \geq 1$, $\theta'(x_i) = v(\xi_{2i-1})$ and $\theta'(y_i) = \xi_{2i}$.

By definition of fibring of Hilbert calculi, then, it must be also the case that

$$\{\hat{h}'((t'_1\theta')^* \equiv (s'_1\theta')^*), \ldots, \hat{h}'((t'_n\theta')^* \equiv (s'_n\theta')^*)\} \vdash_p^{PD} \hat{h}'((t'\theta')^* \equiv (s'\theta')^*).$$

Consider now the substitution σ on schema variables defined by:

- $\sigma(\xi_{2i-1}) = \theta(x_i)^*$;
- $\sigma(\xi_{2i}) = \theta(y_i)$.

Using σ and the structurality of the Hilbert calculus, now, we must also have

$$\{\hat{h}'((t'_1\theta')^* \equiv (s'_1\theta')^*), \ldots, \hat{h}'((t'_n\theta')^* \equiv (s'_n\theta')^*)\}\sigma \vdash_p^{PD} \hat{h}'((t'\theta')^* \equiv (s'\theta')^*)\sigma.$$

But, in fact, a straightforward inductive proof allows us to conclude that $\hat{h}'((u'\theta')^*)\sigma = (\hat{h}'(u')\theta)^*$ for every term u' of sort τ over $\Sigma(C', \Xi)$, and therefore

$$\{(t_1\theta)^* \equiv (s_1\theta)^*, \ldots, (t_n\theta)^* \equiv (s_n\theta)^*\} \vdash_p^{PD} (t\theta)^* \equiv (s\theta)^*.$$

Thus, $\mathcal{L}$ is equationally appropriate. QED

Theorem 6.14 Given two rich and complete logic systems, their fibring while sharing conjunction, disjunction, implication, true and false is also complete.

Proof: The preservation of adequacy is an immediate consequence of the two previous lemmas and the equivalence between equational appropriateness and adequacy for rich systems. QED

Example 6.15 By fibring while sharing conjunction, disjunction, implication, true and false the logic systems $\mathcal{L}_{\mathcal{C}_1}$ and $\mathcal{L}_{\mathrm{KD}}$ we obtain a new modal paraconsistent logic system $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$ that is complete. Observe that if we add to $\mathcal{L}_{\mathcal{C}_1 \oplus \mathrm{KD}}$:

- $(v(y_1^\circ) = \top \rightarrow v((L\, y_1)^\circ) = \top)$ as a meta-valuation axiom;
- $\langle \emptyset, \xi_1^\circ \supset (L\, \xi_1)^\circ \rangle$ as an axiom in the Hilbert calculus;

we obtain a complete logic system that is equivalent to the system $\mathcal{C}_1^D$ of [8] both at the proof-theoretic and the semantic levels.

7 Application II: self-fibring of truth-functional and non-truth-functional logics

It is not difficult to see that, if $\mathcal{S}$ is a truth-functional isp (that is, all the connectives are truth-functional derived connectives) then the self-fibring $\mathcal{S} \oplus \mathcal{S}$ of $\mathcal{S}$ with itself, without sharing of connectives (just sharing the propositional symbols in C_0), produces a copy of $\mathcal{S}$ where each connective appears duplicate. In fact, if $c \in C_k$ $(k > 0)$ and c' is its duplicate then $v_{\mathcal{A}}([\![c(t_1, ..., t_k)]\!]^\rho_{\mathcal{A}})$ is equal to $v_{\mathcal{A}}([\![c(t_1, ..., t_k)]\!]^\rho_{\mathcal{A}})$ for every interpretation $\mathcal{A}$, assignment ρ over $\mathcal{A}$ and terms $t_1, .., t_k$ of sort ϕ. Additionally, if $c \in C_0$ is a constant symbol, such

as $\mathbf{t}$ or $\mathbf{f}$, then $v_{\mathcal{A}}([\![c]\!]_{\mathcal{A}})$ is also equal to $v_{\mathcal{A}}([\![c']\!]_{\mathcal{A}})$. Of course, this property can be extended to every rich and complete logic system $\mathcal{L}$ while sharing C_0, conjunction, disjunction and implication in the self-fibring $\mathcal{L} \oplus \mathcal{L}$. In such cases the formulae $c(t_1, ..., t_k)$ and $c'(t_1, ..., t_k)$ will be equivalent in $\mathcal{L} \oplus \mathcal{L}$, even if c and c' are not explicitly shared.

On the other hand, if $\mathcal{S}$ is an isp with some non-truth-functional connective c, then $v_{\mathcal{A}}([\![c(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ and $v_{\mathcal{A}}([\![c'(t_1, ..., t_k)]\!]^{\rho}_{\mathcal{A}})$ do not necessarily coincide in the models of $\mathcal{S} \oplus \mathcal{S}$ and, consequently, the formulae $c(t_1, ..., t_k)$ and $c'(t_1, ..., t_k)$ are not necessarily equivalent in $\mathcal{L} \oplus \mathcal{L}$, unless c and c' are explicitly shared.

As a concrete instance, consider the isp $\mathcal{S}$ of example 2.6, representing the semantics of the paraconsistent calculus $\mathcal{C}_1$ (cf. [6]). If we perform the fibring $\mathcal{S} \oplus \mathcal{S}$ of $\mathcal{S}$ with itself, while sharing the symbols in C_0, then we obtain two families of connectives: $\{\wedge, \vee, \supset, \neg\}$ and $\{\wedge', \vee', \supset', \neg'\}$. A model for $\mathcal{S} \oplus \mathcal{S}$ gives a valuation map $v_{\mathcal{A}}$ such that

$$v_{\mathcal{A}}([\![t_1 * t_2]\!]^{\rho}_{\mathcal{A}}) = v_{\mathcal{A}}([\![t_1 *' t_2]\!]^{\rho}_{\mathcal{A}}) \qquad \text{for } * \in \{\wedge, \vee, \supset\},$$

because those connectives are truth-functional. On the other hand, $v_{\mathcal{A}}([\![\neg t]\!]^{\rho}_{\mathcal{A}})$ does not coincide necessarily with $v_{\mathcal{A}}([\![\neg' t]\!]^{\rho}_{\mathcal{A}})$. For example, let v_1 and v_2 be two $\mathcal{C}_1$-bivaluations such that $v_1(p_1) = v_1(\neg p_1) = 1$ and $v_2(p_1) = 1$, $v_2(\neg p_1) = 0$. Moreover, assume that $v_1(p) = v_2(p)$ for every other propositional symbol $p \in C_0$. We can thus define an interpretation $\mathcal{A}$ of $\mathcal{S} \oplus \mathcal{S}$ as follows:

- $\mathcal{A}_\tau = \{0, 1\}$ (with its usual Boolean algebra structure);
- $v_{\mathcal{A}}$ restricted to the fragment with $\neg$ coincides with v_1;
- $v_{\mathcal{A}}$ restricted to the fragment with $\neg'$ coincides with v_2;
- $v_{\mathcal{A}}$ extended to the mixed language $C \oplus C'$ is obtained from v_1 and v_2 by using the same techniques used in the proof of Proposition 6.1 in [4].

The interpretation $\mathcal{A}$ satisfies: $v_{\mathcal{A}}([\![\neg p_1]\!]_{\mathcal{A}}) \neq v_{\mathcal{A}}([\![\neg' p_1]\!]_{\mathcal{A}})$, showing that $\neg$ and $\neg'$ do not collapse. Considering the fibring at the logic system level, we obtain, by the completeness-preservation theorem 6.15, that $\neg t_1$ and $\neg' t_1$ are not equivalent formulae (unless they are both theorems).

The example above shows that the fibring of $\mathcal{C}_n, n \geq 1$ with itself produces, for every n, two disjoint copies of $\mathcal{C}_n$ (as the same argument can be applied to the whole hierarchy of paraconsistent calculi $\mathcal{C}_n$). We exclude $\mathcal{C}_0$(which is just the Classical Propositional Calculus) since in this case the self -fibring will collapse with $\mathcal{C}_0$, because this system is truth-functional. On the other hand, the fibring of $\mathcal{C}_n$ with $\mathcal{C}_m$, with $m > n$, produces a new paraconsistent system with two paraconsistent negations, $\neg_n$ and $\neg_m$, whose axioms correspond to adding the axioms of $\mathcal{C}_n$ and of $\mathcal{C}_m$, and whose interpretations are given by maps which are simultaneously $\mathcal{C}_n$ and $\mathcal{C}_m$ valuations. It is an open question whether or not there exists a formula (in the language of $\mathcal{C}_m$) which encodes in $\mathcal{C}_m$ the paraconsistent negation $\neg_n$, for $m > n$ (this, in fact, happens with the classical negation, which is representable in every calculi $\mathcal{C}_n$). If this question has a positive answer, then the fibring of $\mathcal{C}_n$ and $\mathcal{C}_m$ (without sharing the negation)

will be equivalent to $\mathcal{C}_m$, for $m > n$. Of course, the fibring of $\mathcal{C}_n$ and $\mathcal{C}_m$ (while sharing the negation) will be equivalent to $\mathcal{C}_n$. If we generalize this argument, including in the object signature C an unary symbol $\neg_n$ for every $n \geq 0$, then the infinite fibring of the whole hierarchy $\mathcal{C}_n$, without sharing the negations $\neg_n$, will produce a new paraconsistent system with infinitely many paraconsistent negations. If negations are shared in the fibring, the result coincides with the Classical Propositional Calculus $\mathcal{C}_0$. It is worth remarking that, althought we concentrate most of the time on paraconsistent calculi because these are excellent examples of interesting non-truth-functional logic systems, it is clear that our treatment is fully general.

8 Concluding remarks

The first main contribution of this paper is a general semantics for fibring propositional-based logics encompassing systems with non-truth-functional valuations. This goal was achieved by recognizing that such valuations are specified in some appropriate meta-logic, in this case conditional equational logic. Although restricted to systems with a finitary propositional base, the proposed semantics deals with a wide variety of logics from paraconsistent to modal, many-valued and intuitionistic systems. Within this setting, fibring appears as a universal construction within the underlying category, generalizing previous results for truth-functional systems [21].

The second main contribution of this paper is the completeness preservation theorem that generalizes the result established in [23] and is obtained using a new technique exploiting the properties of conditional equational logic where the (possibly non-truth-functional) valuations are defined.

As an example of application of our techniques, the use of fibring for combining paraconsistent logics with other logics is illustrated by recovering the modal paraconsistent logic $\mathcal{C}_1^D$ of da Costa and Carnielli [8] as a fibring. We believe that fibring is a very natural way of establishing new combined systems involving non-truth-functional logics. This approach is more widely applicable as it appears to be: a large family of logics (including many-valued and intuitionistic) admits bivalued non-truth-functional semantics (cf. [1]). Whenever that happens, the completeness preservation theorem can then be used for establishing the completeness of the result as long as the given logics are complete and fulfill the requirements of richness.

Other lines of research are obvious, towards relaxing the assumptions of this paper. For instance, we may want to work with more general object logics (e.g., predicate logics), or with a more general meta-logic (e.g., disjunctive conditional equational logic), or with an even more general universe of truth-values (e.g., involving less or extra generators), or with weaker richness requirements and still obtain completeness preservation by fibring.

Still other lines of research are related to more general forms of fibring, namely heterogeneous forms of fibring where we want to combine two (or more) logics that are defined in quite different forms (either at the deductive system level or at the semantic level). To summarize, our approach offers a framework

formalizing the minimal metamathematical requirements that are sufficient to express a large variety of logic systems; such representations of logic systems constitute a category and they can be combined by means of fibring (that is, through universal constructions in the category). We have also proved that the completeness of such logic systems is preserved by fibring under certain reasonable assumptions on the logic systems, guaranteeing the important property of non-destructiveness of our fibring constructions.

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