Combining Logics
::: A Tutorial :::

Carlos Caleiro

Center for Logic and Computation
Department of Mathematics
IST, Technical University of Lisbon, Portugal

1st World School on Universal Logic
Montreux, Switzerland – March 2005

Overview

- **Lecture #1**
  Why so, and how to?

  Combined logics, mechanisms for combining logics, transference results, the idea of fibring, logical consequence, an abstract framework for fibring, logical systems, free and constrained fibring, examples

- **Lecture #2**
  How good does it get?

  Fibred deduction and fibred semantics, soundness and completeness, local and global reasoning, more examples, soundness preservation (and why it is simple), completeness preservation (and why it is hard)

- **Lecture #3**
  Are we done?

  The collapsing problem, examples, cryptofibring and cryptofibred semantics, solving the collapsing problem, extensionality, soundness and completeness yet again

Combining Logics - Lecture #1
::: Why so, and how to? :::

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Plan

- **Motivation**
  Combined logics, combination mechanisms and transference results
- **An overview of fibring**
  The idea, the modal case
- **Basic notions**
  Logical consequence, signatures and structurality
- **An abstract framework for fibring**
  Categories and colimits, logical systems
- **Free and constrained fibring**
  Hilbert calculi and logical matrices

Combined logics

- Many examples
  First-order modal logics
  Many-dimensional modal logics
    Temporal, dynamic, spatial, epistemic, doxastic
    Multi-agent, distributed
  Arithmetic, equational, probabilistic
  First-order, or even higher-order reasoning
  Features like partiality, fuzziness
- Many application areas
  Artificial intelligence
    Knowledge representation
  Software engineering
    Formal specification and verification
    Algorithms and protocols
  Linguistics

Mechanisms for combining logics

- **Modularity, transference results**

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Mechanisms for combining logics

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Taming complexity, divide and conquer

Define and study $\mathcal{L}_1$ and $\mathcal{L}_2$

and then build $\mathcal{L}_1 \bullet \mathcal{L}_2$

Mechanisms for combining logics

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Taming complexity, divide and conquer

Define and study $\mathcal{L}_1$ and $\mathcal{L}_2$

and then build $\mathcal{L}_1 \bullet \mathcal{L}_2$

Transference results

When does a property of both $\mathcal{L}_1$ and $\mathcal{L}_2$

transfer to $\mathcal{L}_1 \bullet \mathcal{L}_2$?

Mechanisms for combining logics

- ## Trios

  P. Blackburn, M. de Rijke. Zooming in, zooming out, Journal of Logic Language and Information, 1997

Mechanisms for combining logics

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- ## Synchronization

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- ## Temporalization

  M. Finger, D. Gabbay. Adding a temporal dimension to a logic system. Journal of Logic, Language and Information, 1992

  F. Wolter, M. Zakharyaschev. Temporalizing description logics. Frontiers of Combining Systems 2, 2000

Mechanisms for combining logics

- **Parameterization**

  C. Caleiro, C. Sernadas, A. Sernadas. Parameterisation of logics. LNCS 1589, 1999

Mechanisms for combining logics

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  C. Caleiro, C. Sernadas, A. Sernadas. Parameterisation of logics. LNCS 1589, 1999

- **Institutions and parchments**

  T. Mossakowski. Using limits of parchments to systematically construct institutions of partial algebras. LNCS 1130, 1996

  T. Mossakowski, A. Tarlecki, W. Pawłowski. Combining and representing logical systems using model-theoretic parchments. LNCS 1376, 1998

Mechanisms for combining logics

- ## Product and fusion of modal logics

  S. Thomason. Independent Propositional Logics. Studia Logica, 1980

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- ## Fibring

  D. Gabbay. Fibred semantics and the weaving of logics (1). Journal of Symbolic Logic, 1996

  D. Gabbay. Fibring logics. Oxford UP, 1999

  A. Sernadas, C. Sernadas, C. Caleiro. Fibring of logics as a categorial construction. Journal of Logic and Computation, 1999

Overview of fibring

Given logics $\mathcal{L}_1$ and $\mathcal{L}_2$ their fibring $\mathcal{L}_1 * \mathcal{L}_2$ is the least logic over the mixed language that extends both $\mathcal{L}_1$ and $\mathcal{L}_2$

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$\Longrightarrow$ **Addition of schema axiomatizations**

Well understood from the beginning!

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$\Longrightarrow$ **Semantics?**

The fibring function ...

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$\Longrightarrow$ **Addition of schema axiomatizations**

Well understood from the beginning

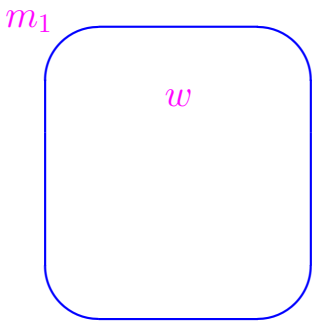
$\Longrightarrow$ **Semantics?**

The fibring function ...

$\mathcal{L}_1$ and $\mathcal{L}_2$ are propositional monomodal normal logics with operators $\Box_1$ and $\Box_2$, respectively, and $\varphi_1 \in \mathcal{L}_1$

How to evaluate the formula $\Box_1(\neg(\Box_2\,\varphi_1))$?

The fibring function F



$\Box_1(\neg(\Box_2\,\varphi_1))$ at w of m_1

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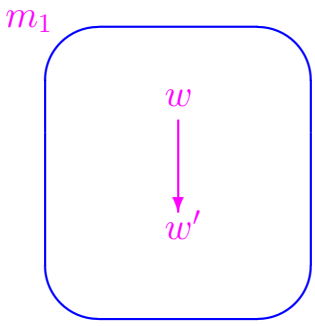

$\Box_1(\neg(\Box_2\,\varphi_1))$ at w of m_1
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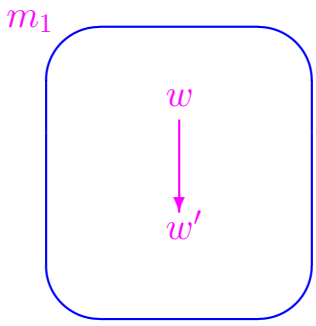

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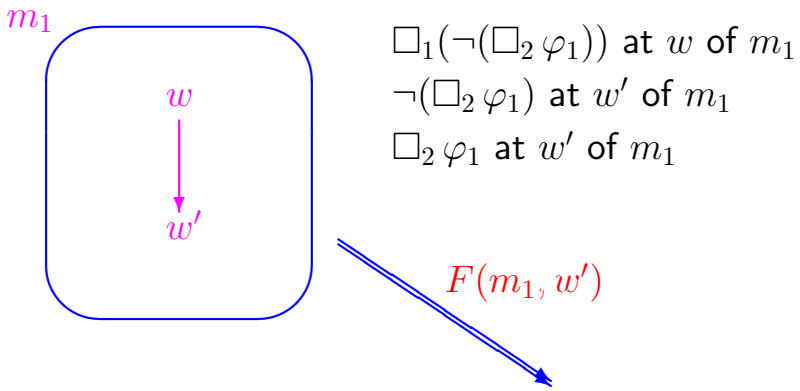

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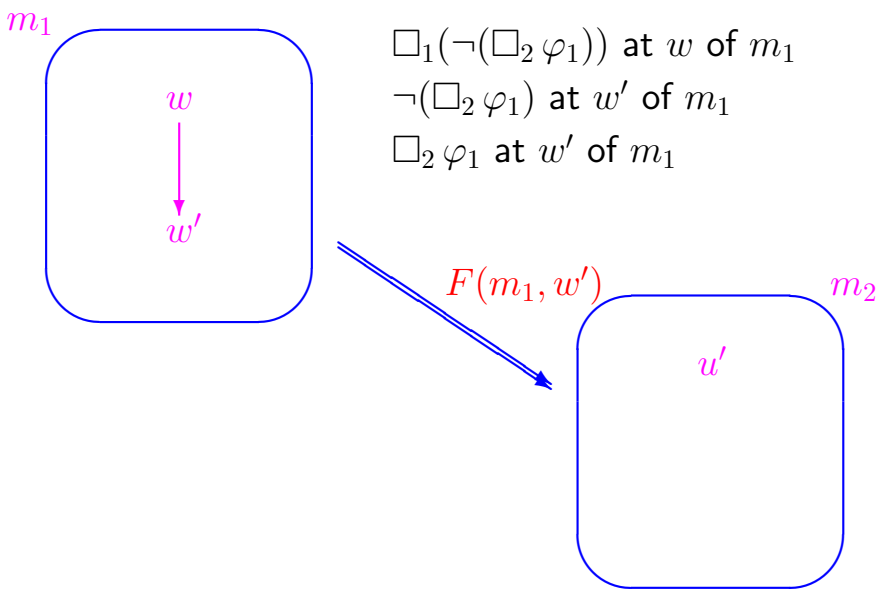

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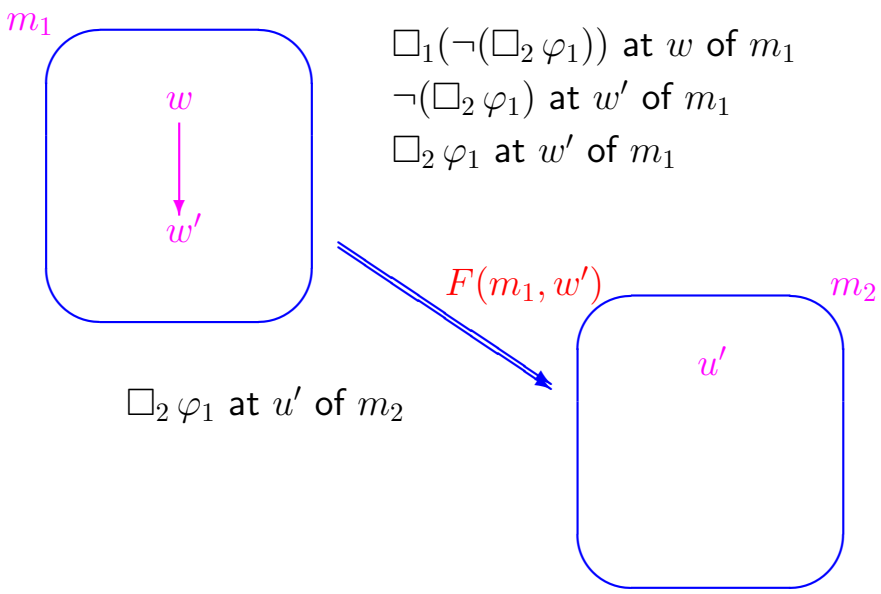

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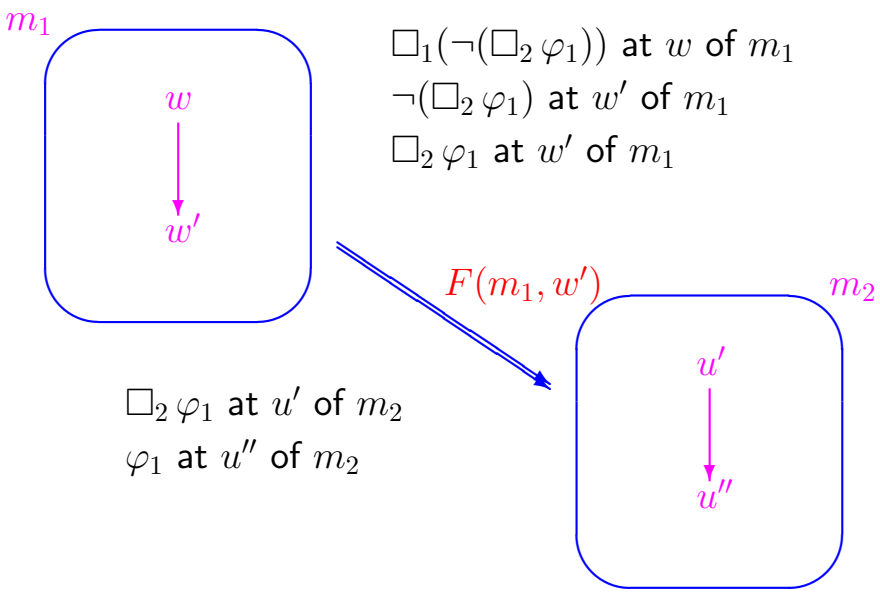

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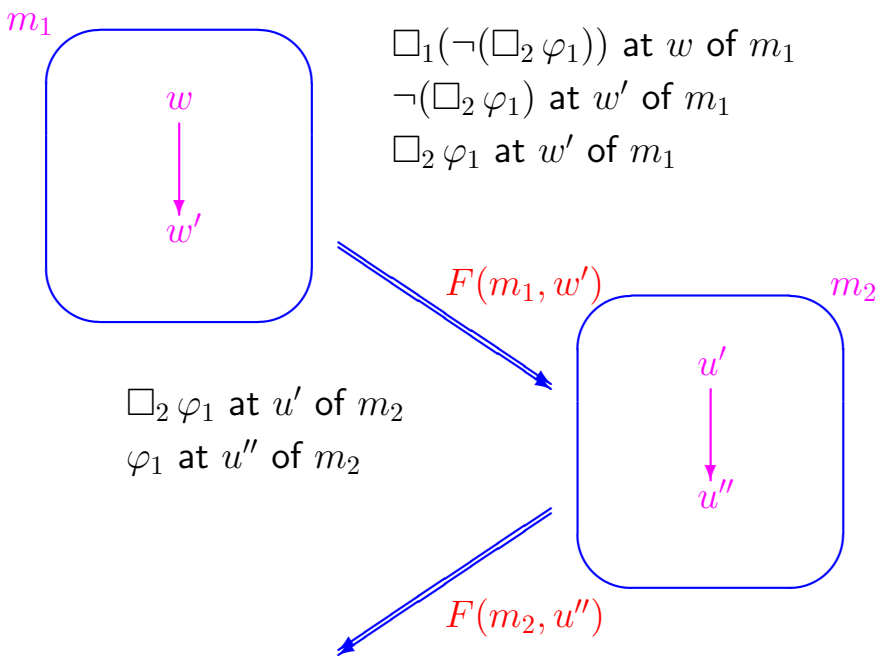

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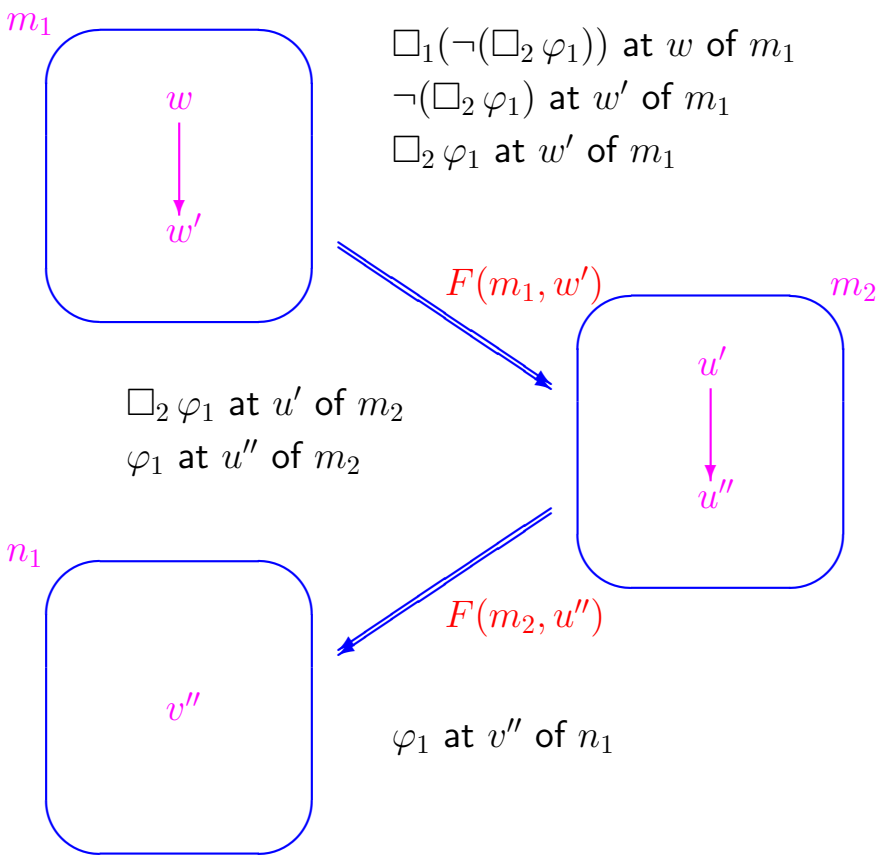

Fusion

$\mathcal{L}_1$ and $\mathcal{L}_2$ are propositional monomodal normal logics with operators $\Box_1$ and $\Box_2$, respectively, and $\varphi_1 \in \mathcal{L}_1$

A fused model (or fibred model) is simply a Kripke structure $M = \langle W, R_1, R_2, V \rangle$ such that $M_1 = \langle W, R_1, V \rangle$ is a model of $\mathcal{L}_1$ and $M_2 = \langle W, R_2, V \rangle$ is a model of $\mathcal{L}_2$

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The associated fibring function F_M is such that $F_M(M_1, w) = (M_2, w)$ and $F_M(M_2, w) = (M_1, w)$

Overview of fibring

Logic?

Mixed language?

One logic extends the other?

Least extension?

Overview of fibring

Logic?

Consequence system

$\langle L, \vdash \rangle$ **where** L **is a set and** $\vdash\colon \wp(L) \to \wp(L)$ **is a closure operator**

- $\Phi \subseteq \Phi^{\vdash}$
- $\Phi \subseteq \Psi$ **implies** $\Phi^{\vdash} \subseteq \Psi^{\vdash}$
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Morphism of consequence systems

$g : \langle L_1, \vdash_1 \rangle \to \langle L_2, \vdash_2 \rangle$ **is a function** $g : L_1 \to L_2$ **such that**

- **if** $\Phi \vdash_1 \psi$ **then** $g[\Phi] \vdash_2 g(\psi)$

Overview of fibring

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Mixed language?

Overview of fibring

Structural logical consequence

Signature

$\Sigma = \langle S, O \rangle$ **an algebraic many-sorted signature with distinguished sort** $\phi \in S$

Schema variables

Ξ **a denumerable set** (ξ **schema variable**)

Syntax

$L_\Sigma = |\mathcal{W}_\Sigma|_\phi$ (φ, ψ **formulas**) (Φ, Ψ **sets**)
$L_\Sigma(\Xi) = |\mathcal{W}_\Sigma(\Xi)|_\phi$ (γ, δ **schema formulas**) (Γ, Δ **sets**)

Overview of fibring

Structural logical consequence

Structural consequence system

$\langle \Sigma, \vdash \rangle$ **where** Σ **is a signature and** $\langle L_\Sigma(\Xi), \vdash \rangle$ **a consequence system such that**

- **if** $\Gamma \vdash \delta$ **then** $\sigma[\Gamma] \vdash \sigma(\delta)$**, substitution** $\sigma : \Xi \to \mathcal{W}_\Sigma(\Xi)$

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Morphism of structural consequence systems

$g : \langle \Sigma_1, \vdash_1 \rangle \to \langle \Sigma_2, \vdash_2 \rangle$ **is a signature morphism** $g : \Sigma_1 \to \Sigma_2$ **preserving** ϕ **such that** $g : \langle L_{\Sigma_1}(\Xi), \vdash_1 \rangle \to \langle L_{\Sigma_2}(\Xi), \vdash_2 \rangle$ **is a morphism of consequence systems**

Overview of fibring

Least extension?

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- **objects: structural consequence systems**
- **arrows: extension morphisms**
- **category Csy**

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J. Adámek, H. Herrlich, G. Strecker. Abstract and Concrete Categories. John Wiley, 1990

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Categorial methodology

Least extension = Colimit

Categorial methodology

Least extension = Colimit

"Given a category of widgets, the operation of putting a system of widgets together to form a super widget corresponds to taking a colimit of the diagram of widgets that shows how to interconnect them"

J. Goguen, 1970s

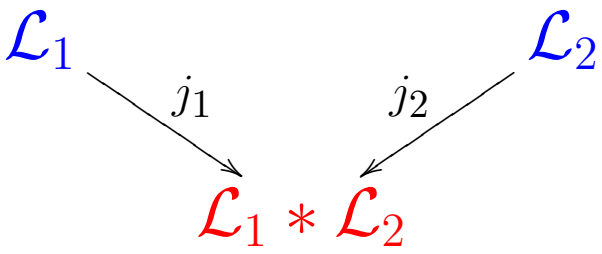
Colimit

Coproduct, or sum

$\mathcal{L}_1$ $\mathcal{L}_2$

Colimit

Coproduct, or sum



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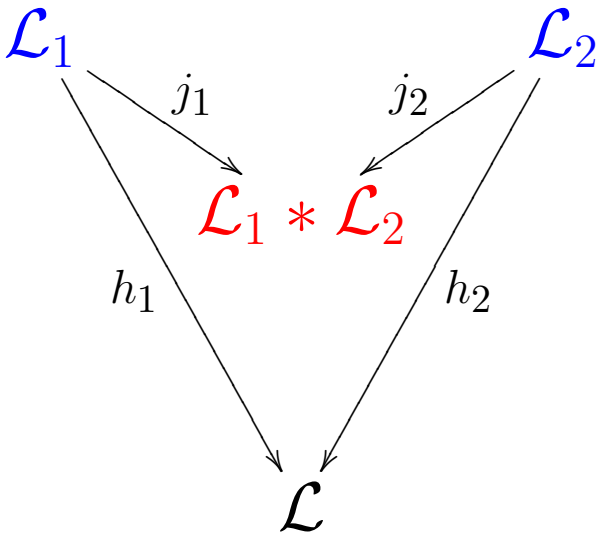

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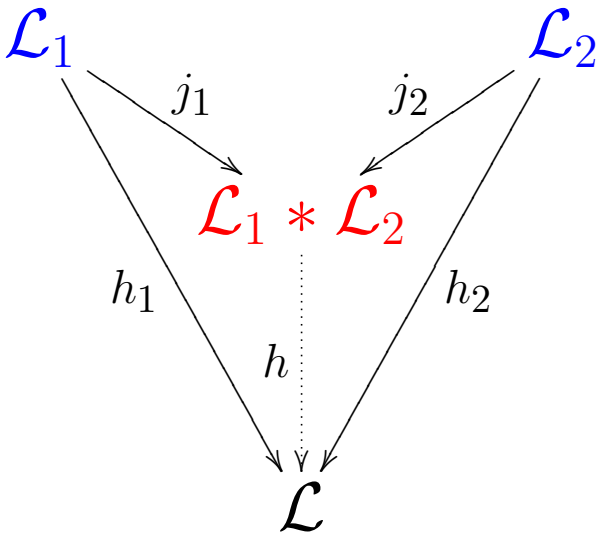
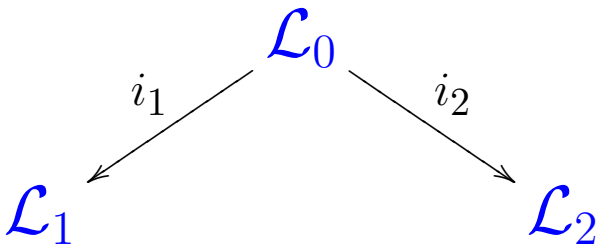


Colimit

Pushout



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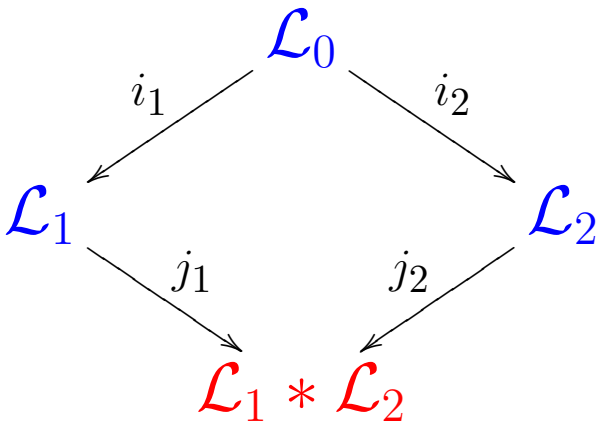

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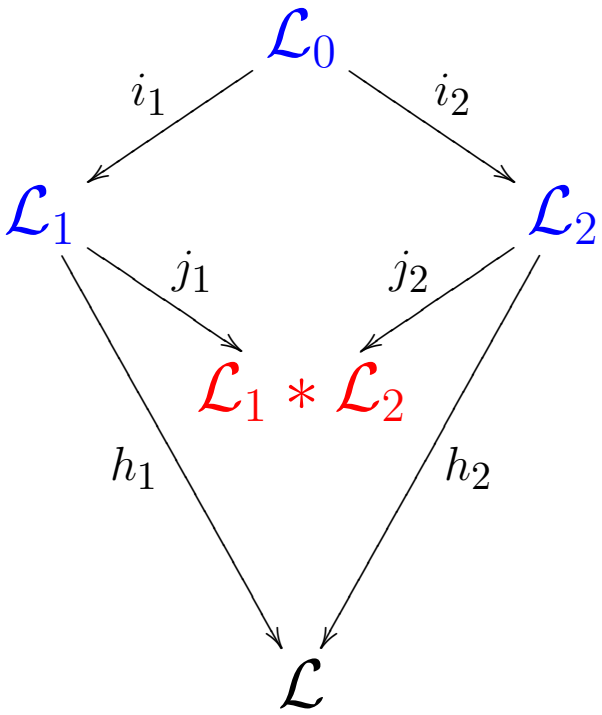

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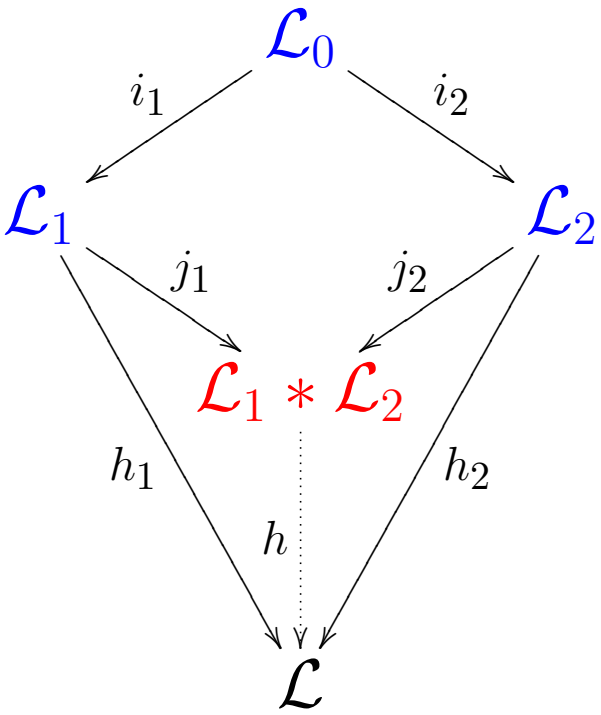

Abstract fibring

In abstract, fibring is a colimit in Csy

Given $\langle \Sigma_1, \vdash_1 \rangle$ and $\langle \Sigma_2, \vdash_2 \rangle$,
the common subsignature $\Sigma_0 = \Sigma_1 \cap \Sigma_2$ is shared

- **Unconstrained fibring (when $\Sigma_0 = \langle \{\phi\}, \emptyset \rangle$) is a sum**
- **Constrained fibring (when $\Sigma_0 \neq \langle \{\phi\}, \emptyset \rangle$) is a pushout**

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How does the sharing affect the colimit in Csy?

$$Sg : \mathbf{Csy} \longrightarrow \mathbf{Sig}$$

Abstract fibring

Sg : **Csy** $\longrightarrow$ **Sig is a topological functor**

Abstract fibring

Sg : **Csy** $\longrightarrow$ **Sig is a topological functor**

Adjoint and Coadjoint

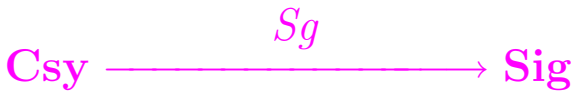
Fibration and Cofibration

Csy is fibre complete

Csy has all colimits because Sig does

Sg **preserves limits and colimits**

Abstract fibring



$\langle \Sigma_1, \vdash_1 \rangle$ $\langle \Sigma_2, \vdash_2 \rangle$

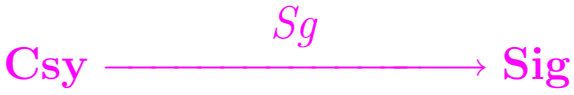
Abstract fibring

$$\mathbf{Csy} \xrightarrow{\quad Sg \quad} \mathbf{Sig}$$

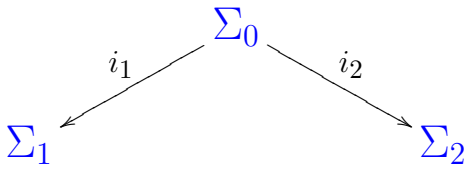
$\Sigma_1 \cap \Sigma_2$

$\langle \Sigma_1, \vdash_1 \rangle$ $\langle \Sigma_2, \vdash_2 \rangle$ Σ_1 Σ_2

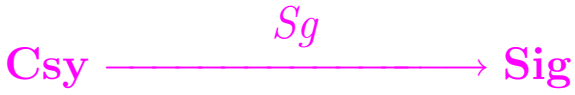
Abstract fibring

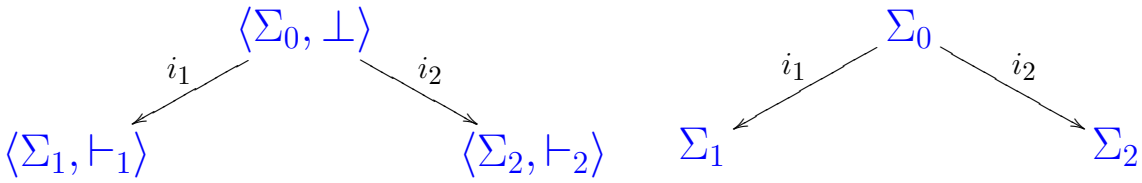


$\langle \Sigma_1, \vdash_1 \rangle$ $\langle \Sigma_2, \vdash_2 \rangle$



Abstract fibring

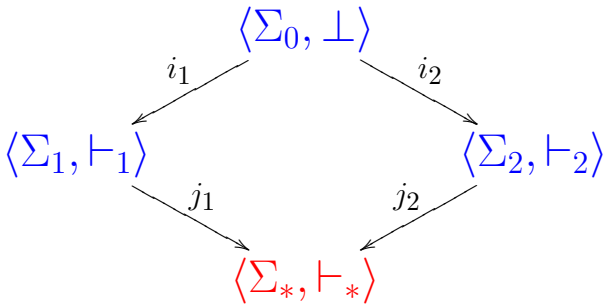


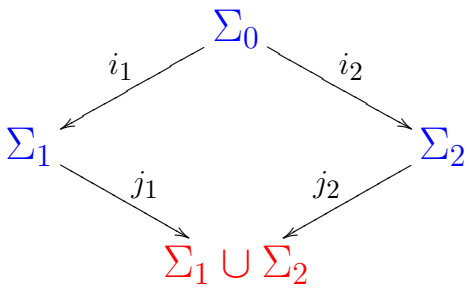


$$\Gamma^{\perp} = \Gamma$$

Abstract fibring

$$\mathbf{Csy} \xrightarrow{\quad Sg \quad} \mathbf{Sig}$$





$$\Gamma^{\perp} = \Gamma$$

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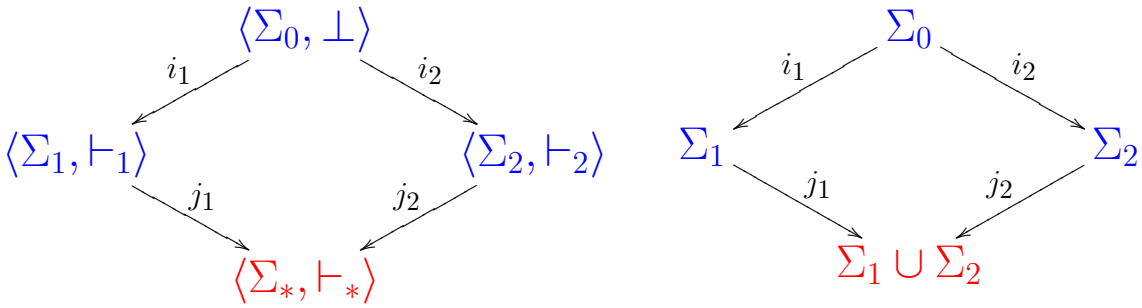

$$\Gamma^{\perp} = \Gamma$$

$$\Sigma_* = \Sigma_1 \cup \Sigma_2$$

$\vdash_*$ **is the closure generated by**

all instances with substitutions $\sigma : \Xi \to L_{\Sigma_*}(\Xi)$

of schematic inferences $\Gamma \vdash_1 \delta$ **or** $\Gamma \vdash_2 \delta$

Abstract fibring

Definition

$$\langle \Sigma_1, \vdash_1 \rangle * \langle \Sigma_2, \vdash_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \vdash_* \rangle$$

$\Gamma^{\vdash_*}$ is the least set that contains Γ and

if $\Delta \vdash_1 \delta$ and $\sigma[\Delta] \subseteq \Gamma^{\vdash_*}$ then $\sigma(\delta) \in \Gamma^{\vdash_*}$ and

if $\Delta \vdash_2 \delta$ and $\sigma[\Delta] \subseteq \Gamma^{\vdash_*}$ then $\sigma(\delta) \in \Gamma^{\vdash_*}$ and

for every mixed language substitution $\sigma : \Xi \to L_{\Sigma_1 \cup \Sigma_2}(\Xi)$

Characterization

$\langle \Sigma_1 \cup \Sigma_2, \vdash_* \rangle$ is a pushout in Sig of

$i_1 : \langle \Sigma_1 \cap \Sigma_2, \bot \rangle \to \langle \Sigma_1, \vdash_1 \rangle$ and $i_2 : \langle \Sigma_1 \cap \Sigma_2, \bot \rangle \to \langle \Sigma_2, \vdash_2 \rangle$

where i_1 and i_2 are the obvious inclusions of signatures

Abstract fibring

Nice ...

Abstract fibring

Nice ...

But what about concrete fibring?

Abstract fibring

Nice ...

But what about concrete fibring?

Logical systems are usually presented by means of

Deductive calculi

Hilbert style, tableaux, sequents

Semantic systems

Interpretation structures, logical matrices, ordered algebras

Abstract fibring

Nice ...

But what about concrete fibring?

Logical systems are usually presented by means of

Deductive calculi

Hilbert style, tableaux, sequents

Semantic systems

Interpretation structures, logical matrices, ordered algebras

Can we fibre them directly?

Concrete fibring

Logical systems must form a category Lsy

and come endowed with a functor $Cn : \mathbf{Lsy} \to \mathbf{Csy}$

Concrete fibring

Logical systems must form a category Lsy

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$$\mathbf{Lsy} \xrightarrow{\quad Cn \quad} \mathbf{Csy} \xrightarrow{\quad Sg \quad} \mathbf{Sig}$$

$\mathcal{L}_1$ $\mathcal{L}_2$

Concrete fibring

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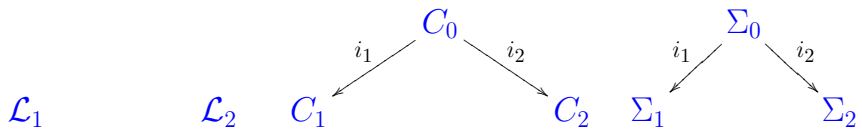

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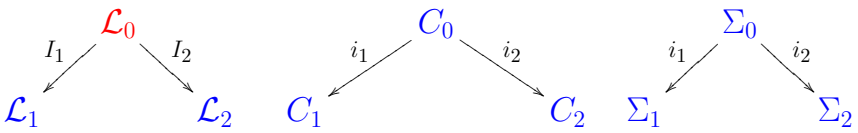

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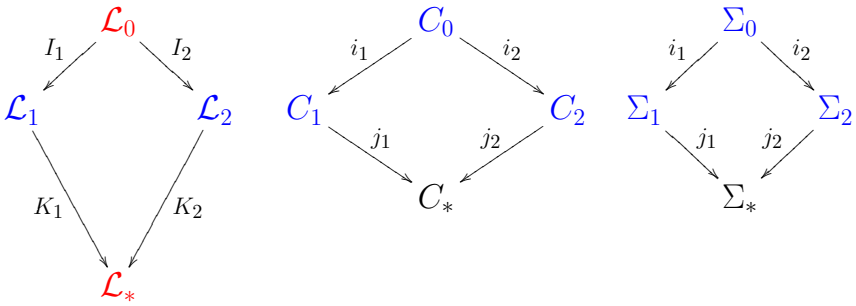

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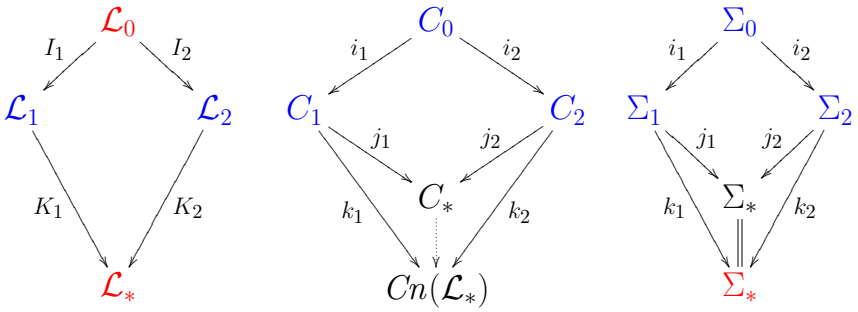


Concrete fibring

A category of logical systems with $Cn : \mathbf{Lsy} \to \mathbf{Csy}$ is suitable for fibring

for every Σ_0 there exists $\mathcal{L}_0$ s.t. $Cn(\mathcal{L}_0) = \langle \Sigma_0, \bot \rangle$

and

given $\mathcal{L}$ with $Cn(\mathcal{L}) = \langle \Sigma, \vdash \rangle$, and $h : \Sigma_0 \to \Sigma$

there exists a unique $g : \mathcal{L}_0 \to \mathcal{L}$ s.t. $Cn(g) = h : \langle \Sigma_0, \bot \rangle \to \langle \Sigma, \vdash \rangle$

Lsy has colimits

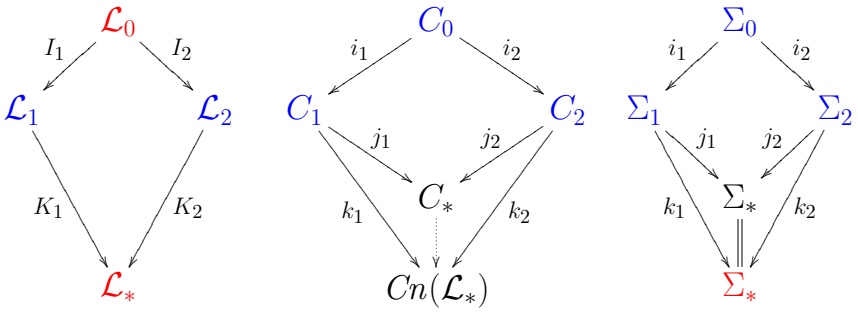
$Sg \circ Cn$ preserves colimits

Concrete fibring

Logical systems must form a category Lsy

and come endowed with a functor $Cn : \mathbf{Lsy} \to \mathbf{Csy}$

$$\mathbf{Lsy} \xrightarrow{\quad Cn \quad} \mathbf{Csy} \xrightarrow{\quad Sg \quad} \mathbf{Sig}$$

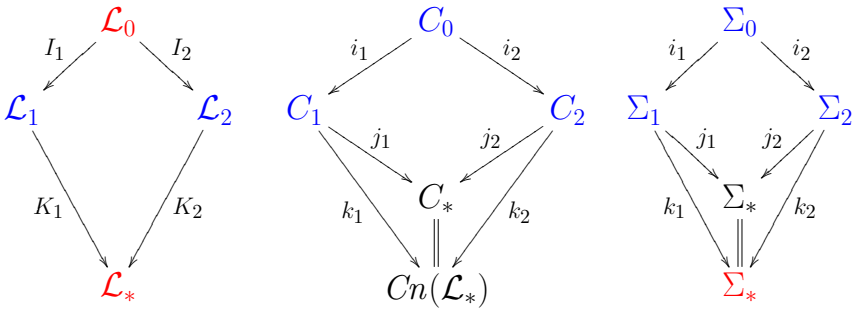


Concrete fibring

Logical systems must form a category Lsy

and come endowed with a functor Cn : Lsy $\to$ Csy

$$\mathbf{Lsy} \xrightarrow{\quad Cn \quad} \mathbf{Csy} \xrightarrow{\quad Sg \quad} \mathbf{Sig}$$



Concrete fibring

A suitable category of logical systems with $Cn : \mathbf{Lsy} \to \mathbf{Csy}$ is exact for fibring

Cn preserves colimits

Examples ...

Logical systems

Hilbert calculi

Hilbert system

$\langle \Sigma, R \rangle$ **with** Σ **a signature and** $R \subseteq \wp_{fin}(L_\Sigma(\Xi)) \times L_\Sigma(\Xi)$ **induces** $\vdash\colon \wp(L_\Sigma(\Xi)) \to \wp(L_\Sigma(\Xi))$

Logical systems

Hilbert calculi

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$$\Gamma \vdash \gamma$$

there exists a sequence $\gamma_1, \gamma_2, \ldots, \gamma_n$ **st** $\gamma_n = \gamma$ **and**

for each γ_i **either**

$\gamma_i \in \Gamma$ **or**

for some $\langle \Delta, \delta \rangle \in R$ **and substitution** σ

$\sigma[\Delta] \subseteq \{\gamma_1, \ldots, \gamma_{i-1}\}$ **and substitution** $\sigma(\delta) = \gamma_i$

Logical systems

Hilbert calculi

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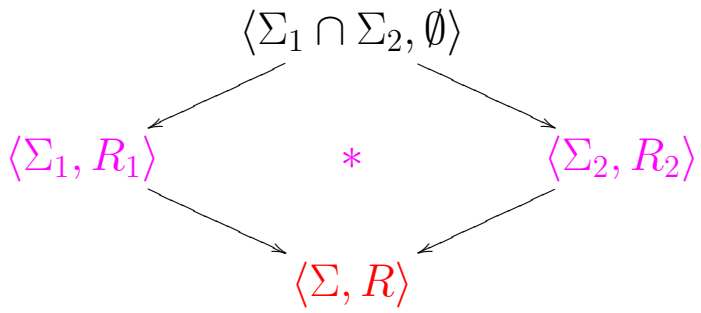
Morphism

$g : \langle \Sigma_1, R_1 \rangle \to \langle \Sigma_2, R_2 \rangle$ **is a signature morphism** $g : \Sigma_1 \to \Sigma_2$ **such that** $g(\delta) \in g[\Gamma]^{\vdash_2}$ **for each** $\langle \Gamma, \delta \rangle \in R_1$

Category Hsy
Suitable for fibring. Exact?

Logical systems

Hilbert calculi



$$\Sigma = \Sigma_1 \cup \Sigma_2$$

$$R = R_1 \cup R_2$$

Logical systems

Matrix semantics

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ **interpretation systems based on logical matrices**

- Σ **signature**
- $\mathcal{M}$ **class (of models)**
- $\mathbb{A}$ **such that** $\mathbb{A}(m) = \langle \mathcal{A}_m, T_m \rangle$ **for** $m \in \mathcal{M}$ **where** $\mathcal{A}_m$ **is a** Σ**-algebra,** $|\mathcal{A}_m|_\phi$ **the set of truth-values and** $T_m \subseteq |\mathcal{A}_m|_\phi$ **is the set of designated values**

$$\Gamma \models \delta$$

$[\![\delta]\!]_{m,\alpha} \in T_m$ **whenever** $[\![\Gamma]\!]_{m,\alpha} \subseteq T_m$ **for each** $m \in \mathcal{M}$ **and** $\alpha : \Xi \to \mathcal{A}_m$

Logical systems

Matrix semantics

Interpretation system

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ **based on logical matrices (**Σ **unsorted)**

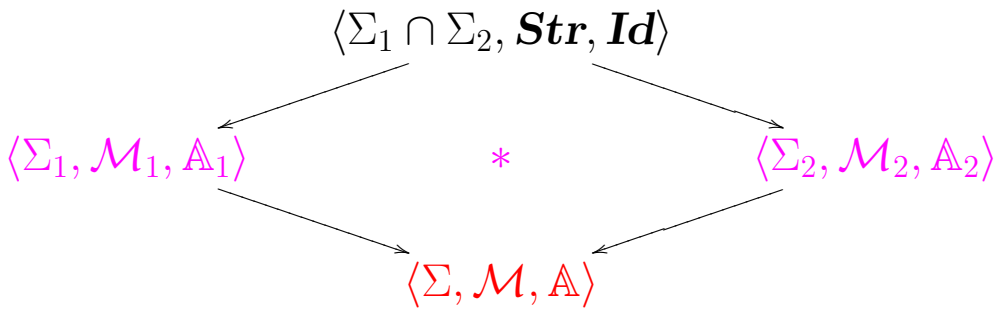
Morphism of interpretation systems

$\langle g, \mu \rangle : \langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \to \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle$ **where** $g : \Sigma_1 \to \Sigma_2$ **is a signature morphism and** $\mu : \mathcal{M}_2 \to \mathcal{M}_1$ **is a map such that** $\mathbb{A}_1(\mu(m)) = \langle \mathcal{A}_m|_g, T_m \rangle = \mathbb{A}_2(m)|_g$ **for each** $m \in \mathcal{M}_2$

Category Int
Suitable for fibring. Exact?

Logical systems

Matrix semantics



$$\Sigma = \Sigma_1 \cup \Sigma_2$$

$$\mathcal{M} = \{(m_1, m_2) : |\mathcal{A}_{m_1}| = |\mathcal{A}_{m_2}|,\ T_{m_1} = T_{m_2},\ \underline{@}_{m_1} = \underline{@}_{m_2} \textbf{ if } @ \in \Sigma_1 \cap \Sigma_2\}$$

$$\mathbb{A}(m_1, m_2) = \langle \mathcal{A}, T \rangle \textbf{ where}$$

$\mathcal{A}$ **is the unique algebra with** $|\mathcal{A}| = |\mathcal{A}_{m_1}| = |\mathcal{A}_{m_2}|$ **that extends** $\mathcal{A}_m$

$$T = T_{m_1} = T_{m_2}$$

References

- **Center for Logic and Computation**

  At `clc.math.ist.utl.pt`, namely Project FibLog

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C. Caleiro, P. Mateus, J. Ramos, A. Sernadas. Combining logics: Parchments revisited. LNCS 2267, 2001

C. Caleiro, W. Carnielli, J. Rasga, C. Sernadas. Fibring of logics as a universal construction. To appear in Handbook of Philosophical Logic

- ## First-order

A. Sernadas, C. Sernadas, C. Caleiro, T. Mossakowski. Categorial fibring of logics with terms and binding operators. In Frontiers of combining systems 2, 2000

A. Sernadas, C. Sernadas, and A. Zanardo. Fibring modal first-order logics: Completeness preservation. Logic Journal of the IGPL, 2002

C. Caleiro, P. Gouveia, J. Ramos. Completeness results for fibred parchments: Beyond the propositional base. LNCS 2755, 2003

What's next?

- **Fibred semantics and deduction**
- **Transference results**
- **Soundness and completeness preservation**

FfT and Exercises

- Check that fusions of modal logics match the idea behind the fibring function
- Define Sig and compute colimits
- Properties of Csy, Hsy, Int, and functors
- Colimits in Csy, Hsy, Int
- Formulate your favourite logics at this level
- Study some category theory, (co)limits, (co)adjoints
- Suitability and exactness for Hsy and Int
- Propose other categories of logical systems and check for suitability and exactness for fibring

Combining Logics - Lecture #2
::: How good does it get? :::

Carlos Caleiro

Center for Logic and Computation

Department of Mathematics

IST, Technical University of Lisbon, Portugal

1st World School on Universal Logic
Montreux, Switzerland – March 2005

Plan

- **Fibring**
  Fibred deduction and fibred semantics, soundness and completeness
- **Local and global reasoning**
  An upgrade
- **Soudness**
  Why is it easy? Unconditional preservation
- **Completeness**
  Why is it hard? Fullness, congruence, implication and conjunction, sufficient conditions for preservation

Logical systems

Hilbert calculi

Hilbert system

$\langle \Sigma, R \rangle$ with Σ a signature and $R \subseteq \wp_{fin}(L_\Sigma(\Xi)) \times L_\Sigma(\Xi)$
induces $\vdash$: $\wp(L_\Sigma(\Xi)) \to \wp(L_\Sigma(\Xi))$

$$\Gamma \vdash \gamma$$

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Category Hsy
Suitable for fibring. Exact?

Logical systems

Hilbert calculi

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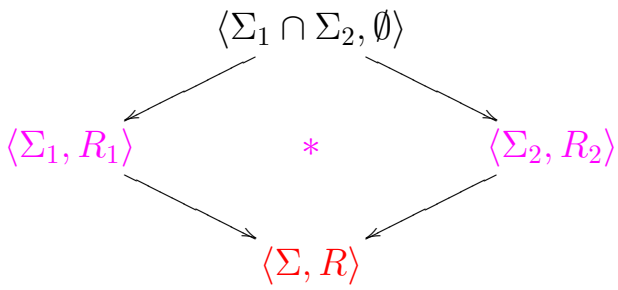
Category Hsy
Suitable and exact for fibring!

Logical systems

Hilbert calculi



$$\Sigma = \Sigma_1 \cup \Sigma_2$$

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Logical systems

Matrix semantics

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ interpretation systems based on logical matrices

- Σ signature
- $\mathcal{M}$ class (of models)
- $\mathbb{A}$ such that $\mathbb{A}(m) = \langle \mathcal{A}_m, T_m \rangle$ for $m \in \mathcal{M}$ where $\mathcal{A}_m$ is a Σ-algebra, $|\mathcal{A}_m|_\phi$ the set of truth-values and $T_m \subseteq |\mathcal{A}_m|_\phi$ is the set of designated values

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Logical systems

Matrix semantics

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$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ based on logical matrices (Σ unsorted)

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Category Int
Suitable for fibring. Exact?

Logical systems

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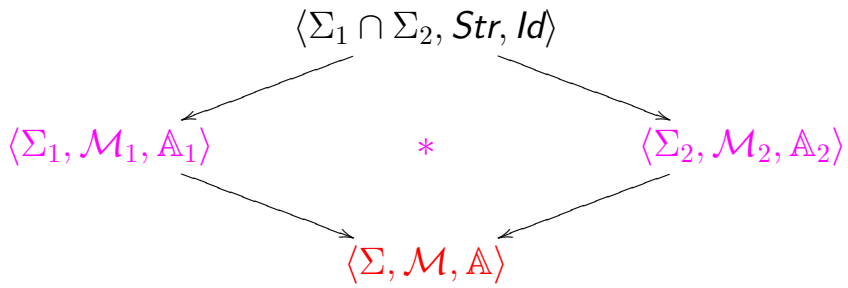
Category Int
Suitable but not exact for fibring!

Logical systems

Matrix semantics



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$$T = T_{m_1} = T_{m_2}$$

Fibring

Soundness and completeness

Given

$\mathit{hsy} = \langle \Sigma, R \rangle$ a deduction system inducing $\vdash$

$\mathit{int} = \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ an interpretation system inducing $\models$

Soundness of $\langle$hsy, int$\rangle$

if $\Gamma \vdash \delta$ then $\Gamma \models \delta$

Completeness of $\langle$hsy, int$\rangle$

if $\Gamma \models \delta$ then $\Gamma \vdash \delta$

Fibring

Soundness and completeness

Given

$\mathit{hsy} = \langle \Sigma, R \rangle$ a deduction system inducing $\vdash$

$\mathit{int} = \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ an interpretation system inducing $\models$

Soundness of $\langle$hsy, int$\rangle$

if $\Gamma \vdash \delta$ then $\Gamma \models \delta$

Completeness of $\langle$hsy, int$\rangle$

if $\Gamma \models \delta$ then $\Gamma \vdash \delta$

Finite completeness of $\langle$hsy, int$\rangle$

if Γ is finite and $\Gamma \models \delta$ then $\Gamma \vdash \delta$

Weak completeness of $\langle$hsy, int$\rangle$

if $\models \delta$ then $\vdash \delta$

Fibring

Preservation results

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
then is $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ sound?

Completeness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are complete,
is $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ complete?

Other properties?

Fibring

Preservation results

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
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Very different answers!

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is $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ complete?

Very different answers!

Why?

Global and local reasoning

An upgrade

Substitution in Equational Logic: $\dfrac{t_1 = t_2}{t_1\sigma = t_2\sigma}$

Generalization in First–Order Logic: $\dfrac{\gamma}{\forall x\, \gamma}$

Necessitation in Modal Logic: $\dfrac{\gamma}{\Box\gamma}$

versus

Modus Ponens: $\dfrac{\gamma \quad \gamma \Rightarrow \delta}{\delta}$

Global and local reasoning

The modal case

Kripke model $M = \langle W, R, V \rangle$

- **local satisfaction**
  given $w \in W$**,** $M, w \Vdash \gamma$
- **global satisfaction**
  $M \Vdash \gamma$ **if** $M, w \Vdash \gamma$ **for every** $w \in W$

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Consequence

- **local entailment**
  $\Gamma \models_l \delta$ **if** $M, w \Vdash \Gamma$ **implies** $M, w \Vdash \delta$
- **global entailment**
  $\Gamma \models_g \delta$ **if** $M \Vdash \Gamma$ **implies** $M \Vdash \delta$

Global and local reasoning

The modal case

Necessitation in Modal Logic: $\dfrac{\gamma}{\Box\gamma}$

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Necessitation holds globally, but not locally!

Modus Ponens holds globally and locally!

Global and local reasoning

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Modus Ponens holds globally and locally!

In general

$\models_l \subseteq \models_g$ **ie, if** $\Gamma \models_l \delta$ **then** $\Gamma \models_g \delta$

Global and local reasoning

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$\models_l \subseteq \models_g$ **ie, if** $\Gamma \models_l \delta$ **then** $\Gamma \models_g \delta$

but $\models_l \delta$ **iff** $\models_g \delta$

Global and local reasoning

Two layers of consequence

Two-layered structural consequence system

$\langle \Sigma, \vdash_l, \vdash_g \rangle$ where Σ is a signature and $\langle L_\Sigma(\Xi), \vdash_l \rangle$ and $\langle L_\Sigma(\Xi), \vdash_g \rangle$ are structural consequence systems such that

- if $\Gamma \vdash_l \delta$ then $\Gamma \vdash_g \delta$, and $\vdash_g \delta$ implies $\vdash_l \delta$

Global and local reasoning

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One logic extends the other?

Global and local reasoning

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Morphism

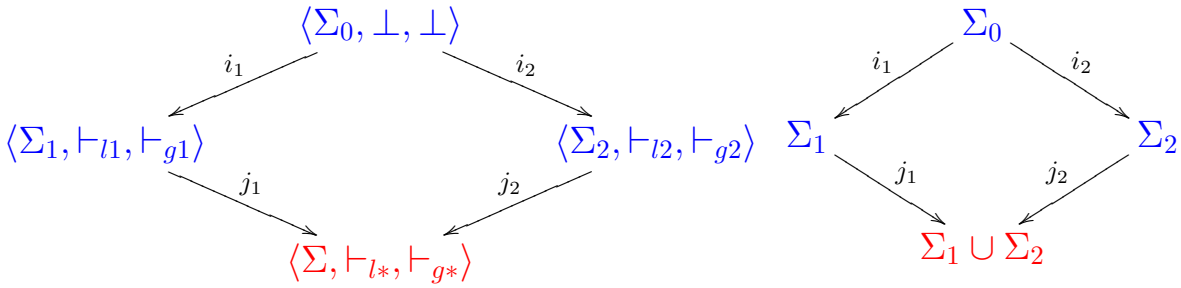
$g : \langle \Sigma_1, \vdash_{l1}, \vdash_{g1} \rangle \longrightarrow \langle \Sigma_2, \vdash_{l2}, \vdash_{g2} \rangle$ is a signature morphism $g : \Sigma_1 \longrightarrow \Sigma_2$ preserving ϕ such that $g : \langle L_{\Sigma_1}(\Xi), \vdash_{l1} \rangle \longrightarrow \langle L_{\Sigma_2}(\Xi), \vdash_{l2} \rangle$ and $g : \langle L_{\Sigma_1}(\Xi), \vdash_{lg} \rangle \longrightarrow \langle L_{\Sigma_2}(\Xi), \vdash_{g2} \rangle$ are morphisms of structural consequence systems

Category 2Csy

Abstract fibring

In abstract, **fibring** is a **colimit** in **2Csy**

$$\mathbf{2Csy} \xrightarrow{\quad 2Sg \quad} \mathbf{Sig}$$



$$\Gamma^{\perp} = \Gamma$$

$$\Sigma_* = \Sigma_1 \cup \Sigma_2$$

$\vdash_{g*}$ is generated by instances of $\vdash_{g1}$ and $\vdash_{g2}$

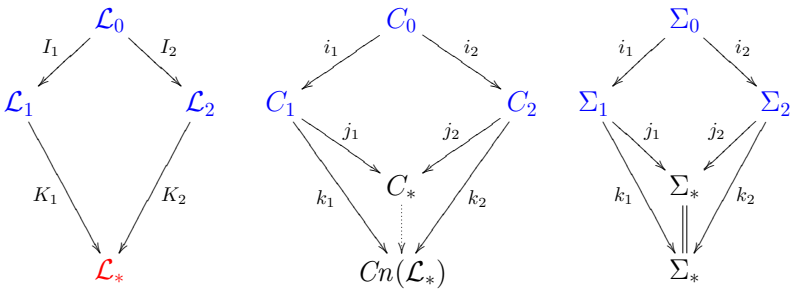
$\vdash_{l*}$ is generated by instances of $\vdash_{l1}$ and $\vdash_{l2}$, plus $\vdash_{g*}$ theorems

Concrete fibring

Two-layered logical systems must form a category $\mathbf{2Lsy}$

and come endowed with a functor $\mathit{2Cn} : \mathbf{2Lsy} \to \mathbf{2Csy}$

$$\mathbf{2Lsy} \xrightarrow{\mathit{2Cn}} \mathbf{2Csy} \xrightarrow{\mathit{2Sg}} \mathbf{Sig}$$



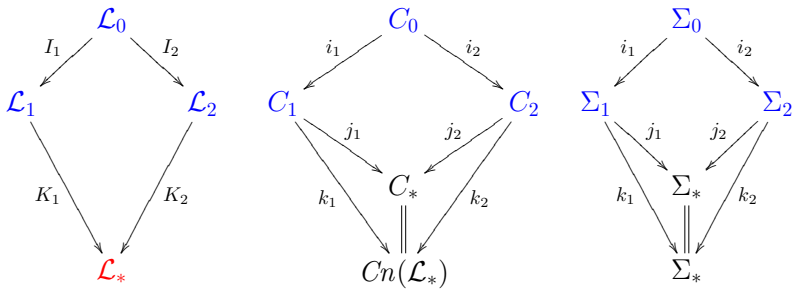
Suitable

Concrete fibring

Two-layered logical systems must form a category $\mathbf{2Lsy}$

and come endowed with a functor $\mathit{2Cn} : \mathbf{2Lsy} \to \mathbf{2Csy}$

$$\mathbf{2Lsy} \xrightarrow{\mathit{2Cn}} \mathbf{2Csy} \xrightarrow{\mathit{2Sg}} \mathbf{Sig}$$



Exact

Global and local reasoning

An upgrade

Everything works as before

Global and local reasoning

An upgrade

Everything works as before

If one is not interested in the distinction

... or one does not see the point

just make $\vdash_l = \vdash_g$ and work with one consequence as before

Global and local reasoning

An upgrade

Everything works as before

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... or one does not see the point

just make $\vdash_l = \vdash_g$ and work with one consequence as before

Examples ...

Two-layered logical systems

Hilbert calculi

Two-layered Hilbert system

$\langle \Sigma, R_l, R_g \rangle$ with Σ a signature, $R_l \subseteq \wp_{fin}(L_\Sigma(\Xi)) \times L_\Sigma(\Xi)$
and $R_g \subseteq (\wp_{fin}(L_\Sigma(\Xi)) \setminus \emptyset) \times L_\Sigma(\Xi)$ induces $\vdash_l, \vdash_g$: $\wp(L_\Sigma(\Xi)) \to \wp(L_\Sigma(\Xi))$

$$\Gamma \vdash_g \gamma$$

inferences built from Γ and instances of rules in $R_l \cup R_g$

$$\Gamma \vdash_l \gamma$$

inferences built from Γ, instances of rules in R_l and $\vdash_g$ theorems

Two-layered logical systems

Hilbert calculi

Two-layered Hilbert system

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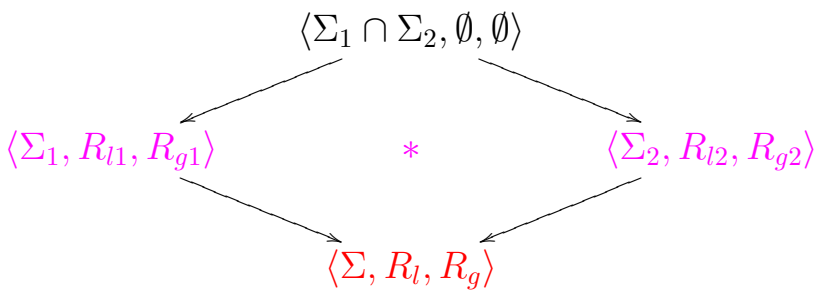
Morphism

$g : \langle \Sigma_1, R_{l1}, R_{g1} \rangle \to \langle \Sigma_2, R_{l2}, R_{g2} \rangle$ is a signature morphism $g : \Sigma_1 \to \Sigma_2$ such that $g(\delta) \in g[\Gamma]^{\vdash_{l2}}$ for each $\langle \Gamma, \delta \rangle \in R_{l1}$ and $g(\delta) \in g[\Gamma]^{\vdash_{g2}}$ for each $\langle \Gamma, \delta \rangle \in R_{g1}$

Category 2Hsy
Suitable and exact for fibring!

Two-layered logical systems

Hilbert calculi



$$\Sigma = \Sigma_1 \cup \Sigma_2$$

$$R_l = R_{l1} \cup R_{l2}$$

$$R_g = R_{g1} \cup R_{g2}$$

Two-layered logical systems

Generalized matrices

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ interpretation systems based on generalized matrices

- Σ signature
- $\mathcal{M}$ class (of models)
- $\mathbb{A}$ such that $\mathbb{A}(m) = \langle \mathcal{A}_m, \mathbf{c}_m \rangle$ for $m \in \mathcal{M}$ where $\mathcal{A}_m$ is a Σ-algebra, $|\mathcal{A}_m|_\phi$ the set of truth-values and $\mathbf{c}_m : \wp(|\mathcal{A}_m|_\phi) \to \wp(|\mathcal{A}_m|_\phi)$ is a closure operator

$$\Gamma \models_g \delta$$

$[\![\delta]\!]_{m,\alpha} \in T_m$ whenever $[\![\Gamma]\!]_{m,\alpha} \subseteq T_m$ for each m and α, letting $T_m = \emptyset^{\mathbf{c}_m}$

$$\Gamma \models_l \delta$$

$[\![\delta]\!]_{m,\alpha} \in ([\![\Gamma]\!]_{m,\alpha})^{\mathbf{c}_m}$ for each m and α

Two-layered logical systems

Generalized matrices

Generalized interpretation system

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ based on generalized logical matrices (Σ unsorted)

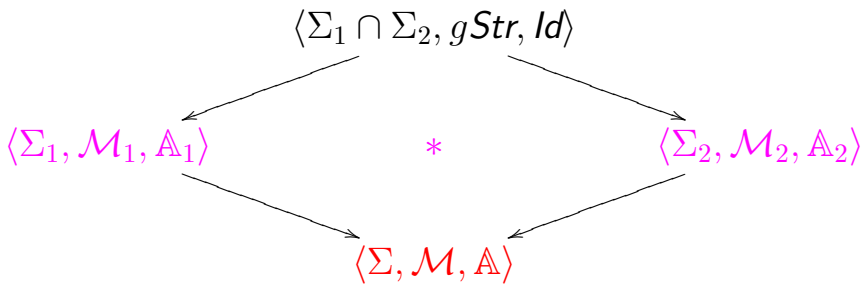
Morphism

$\langle g, \mu \rangle : \langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \to \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle$ where $g : \Sigma_1 \to \Sigma_2$ is a signature morphism and $\mu : \mathcal{M}_2 \to \mathcal{M}_1$ is a map such that $\mathbb{A}_1(\mu(m)) = \langle \mathcal{A}_m|_g, c_m \rangle = \mathbb{A}_2(m)|_g$ for each $m \in \mathcal{M}_2$

Category 2Int
Suitable but not exact for fibring!

Two-layered logical systems

Generalized matrices



$$\Sigma = \Sigma_1 \cup \Sigma_2$$

$$\mathcal{M} = \{(m_1, m_2) : |\mathcal{A}_{m_1}| = |\mathcal{A}_{m_2}|,\ c_{m_1} = c_{m_2},\ \underline{@}_{m_1} = \underline{@}_{m_2} \text{ if } @ \in \Sigma_1 \cap \Sigma_2\}$$

$$\mathbb{A}(m_1, m_2) = \langle \mathcal{A}, c \rangle \text{ where}$$

$\mathcal{A}$ is the unique algebra with $|\mathcal{A}| = |\mathcal{A}_{m_1}| = |\mathcal{A}_{m_2}|$ that extends $\mathcal{A}_{m_1}$ and $\mathcal{A}_{m_2}$

$$c = c_{m_1} = c_{m_2}$$

Two-layered logical systems

Generalized matrices

Particular cases

- **Elementary structure**
  $\langle \mathcal{A}, c \rangle$ **st** $\{a\}^c = \{b\}^c$ **implies** $a = b$

Two-layered logical systems

Generalized matrices

Particular cases

- **Elementary structure**
  $\langle \mathcal{A}, c \rangle$ **st** $\{a\}^c = \{b\}^c$ **implies** $a = b$
- **Pre-order structure**
  $\langle \mathcal{A}, c_{\leq} \rangle$ **for some pre-order** $\langle \mathcal{A}, \leq \rangle$**, where**
  $X^{c_{\leq}} = \{a :$ **if** $y \leq X$ **then** $y \leq a\}$

Two-layered logical systems

Pre-order structures

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ interpretation systems based pre-ordered matrices

- Σ signature
- $\mathcal{M}$ class (of models)
- $\mathbb{A}$ such that $\mathbb{A}(m) = \langle \mathcal{A}_m, \leq_m \rangle$ for $m \in \mathcal{M}$ where $\mathcal{A}_m$ is a Σ-algebra, $|\mathcal{A}_m|_\phi$ the set of truth-values and $\leq_m$ is a pre-order on $|\mathcal{A}_m|_\phi$

$$\Gamma \models_g \delta$$

$[\![\delta]\!]_{m,\alpha} \in T_m$ whenever $[\![\Gamma]\!]_{m,\alpha} \subseteq T_m$ for each m and α, letting T_m be the maximal elements

$$\Gamma \models_l \delta$$

$v \leq_m [\![\delta]\!]_{m,\alpha}$ whenever $v \leq_m [\![\Gamma]\!]_{m,\alpha}$ for each m and α

$\bigwedge_m [\![\Gamma]\!]_{m,\alpha} \leq_m [\![\delta]\!]_{m,\alpha}$ for each m and α

Two-layered logical systems

Generalized matrices

Particular cases

- **Elementary structure**
  $\langle \mathcal{A}, c \rangle$ **st** $\{a\}^c = \{b\}^c$ **implies** $a = b$
- **Pre-order structure**
  $\langle \mathcal{A}, c_{\leq} \rangle$ **for some pre-order** $\langle \mathcal{A}, \leq \rangle$**, where**
  $X^{c_{\leq}} = \{a :$ **if** $y \leq X$ **then** $y \leq a\}$
- **Partial-order structure**
  if the pre-order is a partial order, in which case $c_{\leq}$ **is elementary**

Two-layered logical systems

Generalized matrices

Particular cases

- **Elementary structure**
  $\langle \mathcal{A}, c \rangle$ **st** $\{a\}^c = \{b\}^c$ **implies** $a = b$
- **Pre-order structure**
  $\langle \mathcal{A}, c_{\leq} \rangle$ **for some pre-order** $\langle \mathcal{A}, \leq \rangle$**, where**
  $X^{c_{\leq}} = \{a :$ **if** $y \leq X$ **then** $y \leq a\}$
- **Partial-order structure**
  if the pre-order is a partial order, in which case $c_{\leq}$ **is elementary**
- **Powerset structure**
  $\langle \mathcal{A}, c \rangle$ **st** $A \subseteq \wp(U)$ **and** $U \in A$**, where**
  $X^c = \{a \in A : (\bigcap X) \subseteq a\}$

Fibring

Soundness and completeness

Given

hsy $= \langle \Sigma, R_l, R_g \rangle$ a two-layered deduction system inducing $\vdash_l$ and $\vdash_g$

int $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ a two-layered interpretation system based on generalized matrices inducing $\models_l$ and $\models_g$

Soundness of $\langle$hsy, int$\rangle$

if $\Gamma \vdash_l \delta$ then $\Gamma \models_l \delta$

if $\Gamma \vdash_g \delta$ then $\Gamma \models_g \delta$

Fibring

Soundness and completeness

Given

hsy $= \langle \Sigma, R_l, R_g \rangle$ a two-layered deduction system inducing $\vdash_l$ and $\vdash_g$

int $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ a two-layered interpretation system based on generalized matrices inducing $\models_l$ and $\models_g$

Weak completeness of $\langle$hsy, int$\rangle$

if $\models_l \delta$ then $\vdash_l \delta$, or equivalently, if $\models_g \delta$ then $\vdash_g \delta$

Local completeness/finite completeness of $\langle$hsy, int$\rangle$

if $\Gamma \models_l \delta$ then $\Gamma \vdash_l \delta$

if Γ is finite and $\Gamma \models_l \delta$ then $\Gamma \vdash_l \delta$

Global completeness/finite completeness of $\langle$hsy, int$\rangle$

if $\Gamma \models_g \delta$ then $\Gamma \vdash_g \delta$

if Γ is finite and $\Gamma \models_g \delta$ then $\Gamma \vdash_g \delta$

Fibring

Preservation results

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
then is $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ sound?

Completeness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are complete,
is $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ complete?

Very different answers!

Soundness preservation

Given

hsy $= \langle \Sigma, R_l, R_g \rangle$ inducing $D = \langle \Sigma, \vdash_l, \vdash_g \rangle$

int $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ inducing $S = \langle \Sigma, \models_l, \models_g \rangle$

$\langle$ ***hsy***, ***int*** $\rangle$ **is sound iff** $id_\Sigma : D \to S$ **is a 2Csy-morphism**

Soundness preservation

Given

$\mathit{hsy} = \langle \Sigma, R_l, R_g \rangle$ inducing $D = \langle \Sigma, \vdash_l, \vdash_g \rangle$

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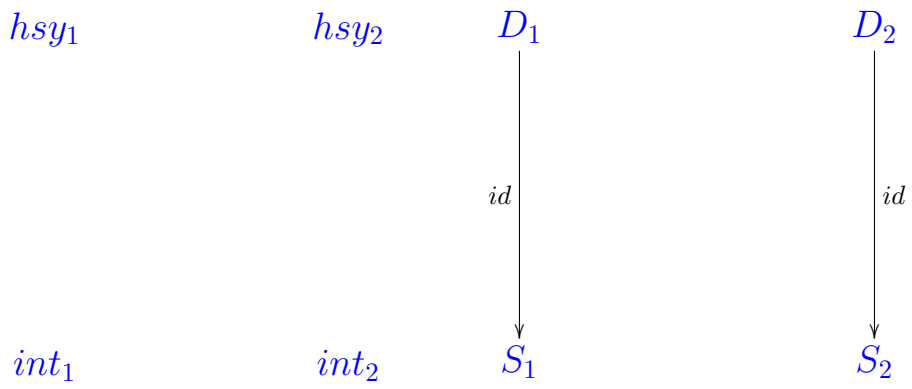

Soundness preservation

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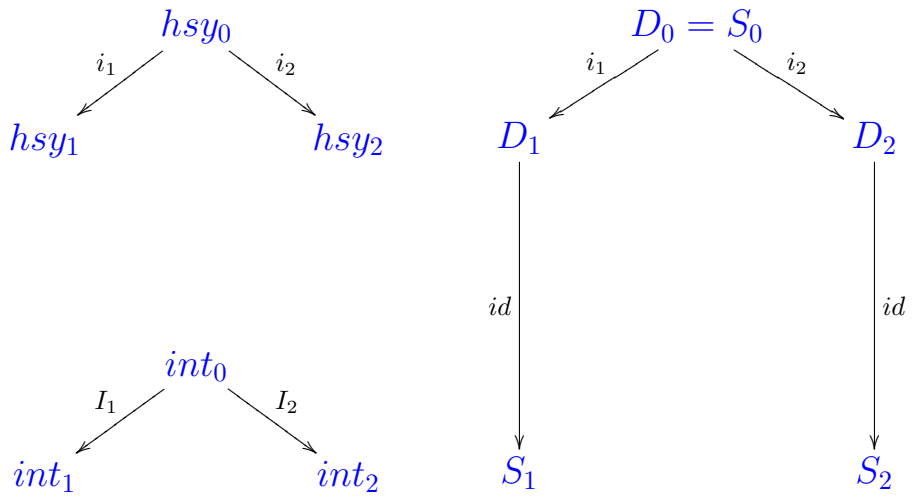

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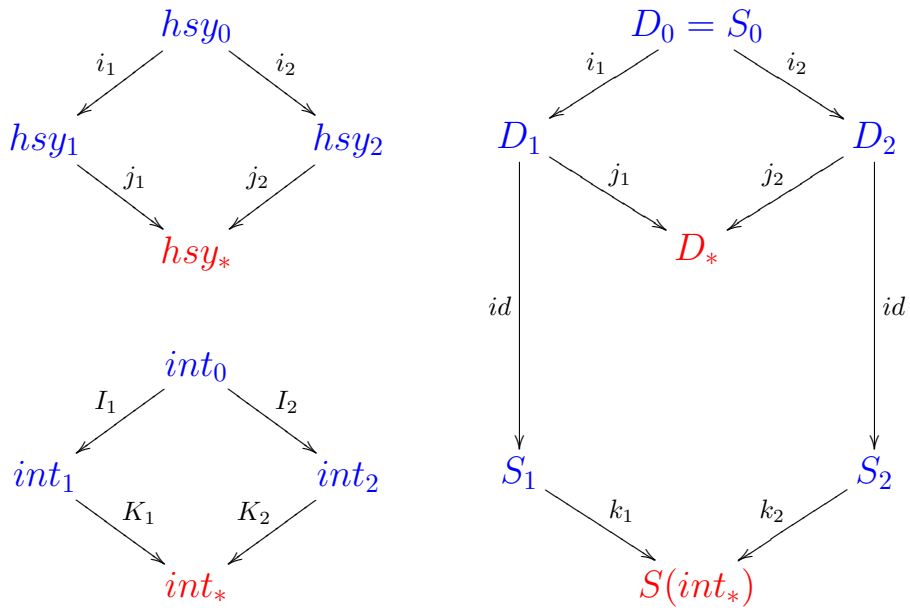

Soundness preservation

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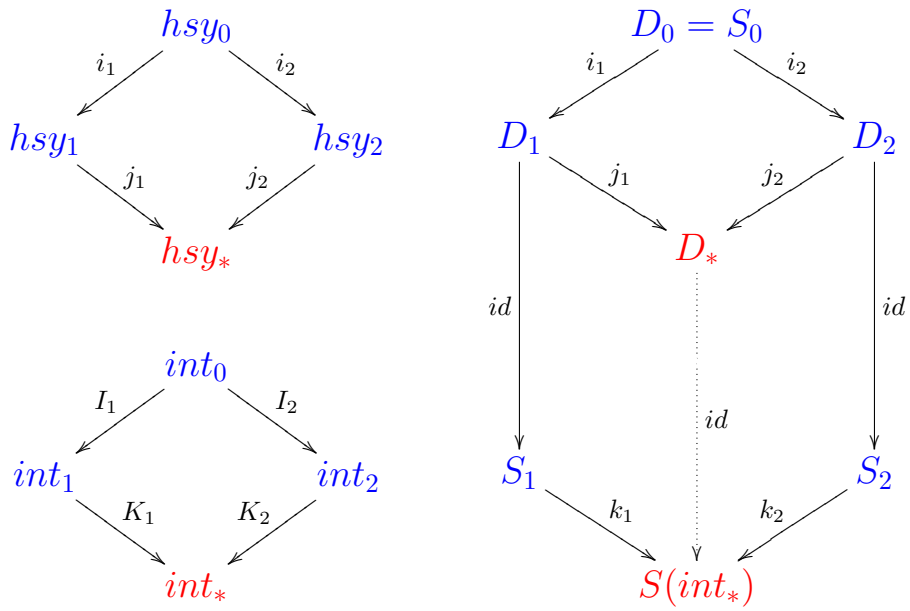

Soundness preservation

Soundness preservation

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Soundness preservation

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
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What about completeness?

Soundness preservation

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
then $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ is sound!

What about completeness?

Not preserved, in general!

No categorial argument will help

Well known examples

Soundness preservation

Soundness preservation

if both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are sound
then $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ is sound!

What about completeness?

Not preserved, in general!

No categorial argument will help

Well known examples

Find sufficient conditions ...

Completeness preservation

Fullness

$\mathcal{I} \subseteq \mathit{Str}$ a class of intended structures

Structure for a local rule

$\langle \mathcal{A}, c \rangle$ is a structure for a local rule $\langle \Gamma, \delta \rangle$ if

$$[\![\delta]\!]_{\mathcal{A},\alpha} \in [\![\Gamma]\!]_{\mathcal{A},\alpha} \text{for each } \alpha : \Xi \to \mathcal{A}$$

Structure for a global rule

$\langle \mathcal{A}, c \rangle$ is a structure for a global rule $\langle \Gamma, \delta \rangle$ if

$$[\![\delta]\!]_{\mathcal{A},\alpha} \in \emptyset^c \text{ whenever } [\![\Gamma]\!]_{\mathcal{A},\alpha} \subseteq \emptyset^c \text{ for each } \alpha : \Xi \to \mathcal{A}$$

Fullness wrt $\mathcal{I}$

$\langle \mathit{hsy}, \mathit{int} \rangle$ is full wrt to $\mathcal{I}$ if

there exists a model of *int* whose structure is $\langle \mathcal{A}, T \rangle \in \mathcal{I}$

whenever $\langle \mathcal{A}, T \rangle$ is a structure for all the local and global rules of *hsy*

Completeness preservation

Fullness

Theorem

Fullness wrt a class of structures is preserved by fibring

Completeness preservation

Fullness

Theorem

Fullness wrt a class of structures is preserved by fibring

ie

If $\langle \mathsf{hsy}_1, \mathsf{int}_1 \rangle$ and $\langle \mathsf{hsy}_2, \mathsf{int}_2 \rangle$ are both full wrt some $\mathcal{I}$

then $\langle \mathsf{hsy}_1 * \mathsf{hsy}_2, \mathsf{int}_1 * \mathsf{int}_2 \rangle$ is also full wrt $\mathcal{I}$

Completeness preservation

Fullness

Theorem

Fullness wrt a class of structures is preserved by fibring

ie

If $\langle \mathsf{hsy}_1, \mathsf{int}_1 \rangle$ and $\langle \mathsf{hsy}_2, \mathsf{int}_2 \rangle$ are both full wrt some $\mathcal{I}$

then $\langle \mathsf{hsy}_1 * \mathsf{hsy}_2, \mathsf{int}_1 * \mathsf{int}_2 \rangle$ is also full wrt $\mathcal{I}$

Possible classes $\mathcal{I}$ of intended structures

all, elementary, pre-order, partial-order, powerset

Overview of fibring

Completeness preservation

Theorem - all structures

If both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are full wrt the class of all structures then $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ is complete

Overview of fibring

Completeness preservation

Theorem - all structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt the class of all structures then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt the class of all structures

Overview of fibring

Completeness preservation

Theorem - all structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt the class of all structures then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt the class of all structures

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt the class of all structures then it is complete

Overview of fibring

Completeness preservation

Theorem - all structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt the class of all structures then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt the class of all structures

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt the class of all structures then it is complete

The flavour is a little bit too syntactic

What about elementary structures?

Overview of fibring

Completeness preservation

Theorem - elementary structures

If both $\langle hsy_1, int_1 \rangle$ and $\langle hsy_2, int_2 \rangle$ are full wrt the class of elementary structures and congruent, and they share an implication connective then $\langle hsy_1 * hsy_2, int_1 * int_2 \rangle$ is complete

Overview of fibring

Completeness preservation

Theorem - elementary structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt the class of elementary structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt elementary structures

Prove that fibring preserves implication, if shared

Prove that with implication, fibring preserves congruence

Conclude $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt elementary structures, congruent and has an implication connective then it is complete

Overview of fibring

Completeness preservation

Theorem - elementary structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt the class of elementary structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt elementary structures

Prove that fibring preserves implication, if shared

Prove that with implication, fibring preserves congruence

Conclude $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt elementary structures, congruent and has an implication connective then it is complete

Can we improve?

Overview of fibring

Completeness preservation

Theorem - partial order structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local weak complete

Overview of fibring

Completeness preservation

Theorem - partial order structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local weak complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt po-structures

As before, conclude $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt po-structures, congruent and has an implication connective then it is global complete and local weak complete

Overview of fibring

Completeness preservation

Theorem - partial order structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local weak complete

Use the fact that fibring preserves fullness to conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt po-structures

As before, conclude $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt po-structures, congruent and has an implication connective then it is global complete and local weak complete

Why weak?

Overview of fibring

Completeness preservation

Theorem - partial order structures, enhanced

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, at least one of them has a conjunction, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local finite complete

Overview of fibring

Completeness preservation

Theorem - partial order structures, enhanced

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, at least one of them has a conjunction, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local finite complete

Conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt po-structures

Also, $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that a conjunction in one of the components yields a conjunction after fibring

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt po-structures, congruent and has a conjunction and an implication connective then it is global complete and local finite complete

Overview of fibring

Completeness preservation

Theorem - partial order structures, enhanced

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt po-structures and congruent, at least one of them has a conjunction, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is global complete and local finite complete

Conclude that $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is full wrt po-structures

Also, $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is congruent and has implication

Show that a conjunction in one of the components yields a conjunction after fibring

Show that if $\langle \boldsymbol{hsy}, \boldsymbol{int} \rangle$ is full wrt po-structures, congruent and has a conjunction and an implication connective then it is global complete and local finite complete

Still perfect if finitary, or if local=global!

Overview of fibring

Completeness preservation

Theorem - powerset structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt powerset structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Overview of fibring

Completeness preservation

Theorem - powerset structures

If both $\langle \boldsymbol{hsy}_1, \boldsymbol{int}_1 \rangle$ and $\langle \boldsymbol{hsy}_2, \boldsymbol{int}_2 \rangle$ are full wrt powerset structures and congruent, and they share an implication connective then $\langle \boldsymbol{hsy}_1 * \boldsymbol{hsy}_2, \boldsymbol{int}_1 * \boldsymbol{int}_2 \rangle$ is complete

Much of the above, plus a Henkin construction

Check

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References

- ## Other transference results, other settings

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What's next?

- **Fibring classical and intuitionistic logics**
- **The collapsing problem**
- **Cryptofibring**
- **Extensionsality**
- **Soundness and completeness revisited**

FfT and Exercises

- **Fibre together some of your favorite logics.**
- **What do the local and global layers mean for first-order logic?**
- **Define two-layered classical logic over Boolean algebras. What do local and global correspond to?**
- **Define two-layered intuitionistic logic over Heyting algebras. What do local and global correspond to?**
- **What about Lukasiewicz's 3-valued logic?**
- **Show that global completeness implies local weak completeness.**
- **What is the result of fibring classical logic with intuitionistic logic?**
- **Prove completeness results for K$*$K.**

Combining Logics - Lecture #3
::: Are we done? :::

Carlos Caleiro

Center for Logic and Computation
Department of Mathematics
IST, Technical University of Lisbon, Portugal

1st World School on Universal Logic
Montreux, Switzerland – March 2005

Plan

- **The collapsing problem**
  Fibring classical and intuitionistic logics
- **Cryptofibring**
  Cryptomorphisms and cryptofibred semantics
- **Solving the collapsing problem**
  Cryptofibring classical and intuitionistic logics, extensionality
- **Soundness and completeness revisited**
  Some problems and a few solutions

The collapsing problem

- **D. Gabbay**

  An overview of fibred semantics and the combination of logics, in F. Baader and K. Schulz (eds), Frontiers of combining systems, Applied Logic Series, Kluwer, 1996, pp. 1–55

- **L. Fariñas del Cerro and A. Herzig**

  Combining classical and intuitionistic logic, in F. Baader and K. Schulz (eds), Frontiers of combining systems, Applied Logic Series, Kluwer, 1996, pp. 93-102

- **C. Sernadas, J. Rasga, W. Carnielli**

  Modulated fibring and the collapsing problem, Journal of Symbolic Logic, 67(4):1541–1569, 2002

The collapsing problem

Classical propositional logic (two-valued matrices)

CPL $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$

- $\Sigma_0 = \{p'_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg'\}$ and $\Sigma_2 = \{\Rightarrow'\}$
- $\mathcal{M} = \{0,1\}^{\Sigma_0}$
- for each $v \in \mathcal{M}$, $\mathbb{A}(v) = \langle \langle \{0,1\}, \cdot_v \rangle, \{1\} \rangle$ where

$$\underline{p'_n}_v = v(p'_n)$$

	$\underline{\neg'}_v$
0	1
1	0

$\underline{\Rightarrow'}_v$	0	1
0	1	1
1	0	1

The collapsing problem

Gödel's 3-valued logic (3-valued matrices)

G3 $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$

- $\Sigma_0 = \{p''_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg''\}$ and $\Sigma_2 = \{\Rightarrow''\}$
- $\mathcal{M} = \{0, \frac{1}{2}, 1\}^{\Sigma_0}$
- for each $v \in \mathcal{M}$, $\mathbb{A}(v) = \langle \langle \{0, \frac{1}{2}, 1\}, \cdot_v \rangle, \{1\} \rangle$ where

$$\underline{p''_n}_v = v(p''_n)$$

	$\underline{\neg''}_v$
0	1
$\frac{1}{2}$	0
1	0

$\underline{\Rightarrow''}_v$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

The collapsing problem

Anything goes ...

G3 $= \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$

- $\Sigma_0 = \{p''_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg''\}$ and $\Sigma_2 = \{\Rightarrow''\}$
- $\mathcal{M} = \{0, \frac{1}{2}, 1\}^{\Sigma_0}$
- for each $v \in \mathcal{M}$, $\mathbb{A}(v) = \langle \langle \{0, \frac{1}{2}, 1\}, \cdot_v \rangle, \{1\} \rangle$ where

$$\underline{p''_n}_v = v(p''_n)$$

	$\underline{\neg''}_v$
0	1
$\frac{1}{2}$	0
1	0

$\underline{\Rightarrow''}_v$	0	$\frac{1}{2}$	1
0	1	1	1
$\frac{1}{2}$	0	1	1
1	0	$\frac{1}{2}$	1

CPL $*$ G3 has no models!!!

The collapsing problem

Gödel's 3-valued logic (2-valued and 3-valued matrices)

G3 $= \langle \Sigma, \mathcal{M}^+, \mathbb{A}^+ \rangle$

- $\Sigma_0 = \{p''_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg''\}$ and $\Sigma_2 = \{\Rightarrow''\}$
- $\mathcal{M}^+ = \{0,1\}^{\Sigma_0} \cup \{0, \frac{1}{2}, 1\}^{\Sigma_0}$
- for each $v : \Sigma_0 \to \{0,1\}$, $\mathbb{A}^+(v) = \langle \langle \{0,1\}, \cdot_v \rangle, \{1\} \rangle$ as before
- for each $v : \Sigma_0 \to \{0, \frac{1}{2}, 1\}$, $\mathbb{A}^+(v) = \langle \langle \{0, \frac{1}{2}, 1\}, \cdot_v \rangle, \{1\} \rangle$ as before

The collapsing problem

Collapses to *CPL*

G3 $= \langle \Sigma, \mathcal{M}^+, \mathbb{A}^+ \rangle$

- $\Sigma_0 = \{p''_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg''\}$ and $\Sigma_2 = \{\Rightarrow''\}$
- $\mathcal{M}^+ = \{0,1\}^{\Sigma_0} \cup \{0, \frac{1}{2}, 1\}^{\Sigma_0}$
- for each $v : \Sigma_0 \to \{0,1\}$, $\mathbb{A}^+(v) = \langle \langle \{0,1\}, \cdot_v \rangle, \{1\} \rangle$ as before
- for each $v : \Sigma_0 \to \{0, \frac{1}{2}, 1\}$, $\mathbb{A}^+(v) = \langle \langle \{0, \frac{1}{2}, 1\}, \cdot_v \rangle, \{1\} \rangle$ as before

CPL $*$ G3 is now a double copy of CPL !!!

The collapsing problem

Collapses to *CPL*

G3 $= \langle \Sigma, \mathcal{M}^+, \mathbb{A}^+ \rangle$

- $\Sigma_0 = \{p''_n : n \in \mathbb{N}\}$, $\Sigma_1 = \{\neg''\}$ and $\Sigma_2 = \{\Rightarrow''\}$
- $\mathcal{M}^+ = \{0,1\}^{\Sigma_0} \cup \{0, \frac{1}{2}, 1\}^{\Sigma_0}$
- for each $v : \Sigma_0 \to \{0,1\}$, $\mathbb{A}^+(v) = \langle \langle \{0,1\}, \cdot_v \rangle, \{1\} \rangle$ as before
- for each $v : \Sigma_0 \to \{0, \frac{1}{2}, 1\}$, $\mathbb{A}^+(v) = \langle \langle \{0, \frac{1}{2}, 1\}, \cdot_v \rangle, \{1\} \rangle$ as before

CPL $*$ G3 is now a double copy of CPL !!!

the same collapse happens with CPL $*$ IPL !!!

CPL - implicative fragment $\Rightarrow$

Deductive system

syntax

$\mathbf{P}$ set of propositional symbols , binary connective $\Rightarrow$

axioms

$$\mathbf{A} \Rightarrow (\mathbf{B} \Rightarrow \mathbf{A})$$

$$(\mathbf{A} \Rightarrow (\mathbf{B} \Rightarrow \mathbf{C})) \Rightarrow ((\mathbf{A} \Rightarrow \mathbf{B}) \Rightarrow (\mathbf{A} \Rightarrow \mathbf{C}))$$

$$((\mathbf{A} \Rightarrow \mathbf{B}) \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}$$

inference rule

$$\frac{\mathbf{A} \qquad \mathbf{A} \Rightarrow \mathbf{B}}{\mathbf{B}}$$

IPL - implicative fragment $\rightarrow$

Deductive system

syntax

$\mathbf{Q}$ set of propositional symbols , binary connective $\rightarrow$

axioms

$$\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A})$$

$$(\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow ((\mathbf{A} \rightarrow \mathbf{B}) \rightarrow (\mathbf{A} \rightarrow \mathbf{C}))$$

inference rule

$$\frac{\mathbf{A} \qquad \mathbf{A} \rightarrow \mathbf{B}}{\mathbf{B}}$$

CPL versus IPL

Deductive consequence

CPL strictly stronger than IPL

$$\vdash_{\mathbf{C}} ((\mathbf{A} \Rightarrow \mathbf{B}) \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}$$
$$\nvdash_{\mathbf{I}} ((\mathbf{A} \to \mathbf{B}) \to \mathbf{A}) \to \mathbf{A}$$

both enjoy the deduction theorem

$$\mathbf{\Gamma}, \mathbf{A} \vdash_{\mathbf{C}} \mathbf{B} \quad \text{iff} \quad \mathbf{\Gamma} \vdash_{\mathbf{C}} \mathbf{A} \Rightarrow \mathbf{B}$$
$$\mathbf{\Gamma}, \mathbf{A} \vdash_{\mathbf{I}} \mathbf{B} \quad \text{iff} \quad \mathbf{\Gamma} \vdash_{\mathbf{C}} \mathbf{A} \to \mathbf{B}$$

CPL∗IPL

Fibred deduction

syntax

$\mathbf{P} \cup \mathbf{Q}$ set of propositional symbols, binary connectives $\Rightarrow$ e $\rightarrow$

axioms

$$\mathbf{A} \Rightarrow (\mathbf{B} \Rightarrow \mathbf{A})$$

$$(\mathbf{A} \Rightarrow (\mathbf{B} \Rightarrow \mathbf{C})) \Rightarrow ((\mathbf{A} \Rightarrow \mathbf{B}) \Rightarrow (\mathbf{A} \Rightarrow \mathbf{C}))$$

$$((\mathbf{A} \Rightarrow \mathbf{B}) \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}$$

$$\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{A})$$

$$(\mathbf{A} \rightarrow (\mathbf{B} \rightarrow \mathbf{C})) \rightarrow ((\mathbf{A} \rightarrow \mathbf{B}) \rightarrow (\mathbf{A} \rightarrow \mathbf{C}))$$

inference rules

$$\frac{\mathbf{A} \qquad \mathbf{A} \Rightarrow \mathbf{B}}{\mathbf{B}} \qquad\qquad \frac{\mathbf{A} \qquad \mathbf{A} \rightarrow \mathbf{B}}{\mathbf{B}}$$

CPL∗IPL

Questions

We may expect $\mathbf{A} \to \mathbf{B} \vdash_{\mathbf{CI}} \mathbf{A} \Rightarrow \mathbf{B}$.

But $\mathbf{A} \Rightarrow \mathbf{B}, \mathbf{A} \vdash_{\mathbf{CI}} \mathbf{B}$ by *modus ponens*,

and we would have $\mathbf{A} \Rightarrow \mathbf{B} \vdash_{\mathbf{CI}} \mathbf{A} \to \mathbf{B}$ by the *deduction theorem* !

Collapse of $\Rightarrow$ and $\to$?

CPL∗IPL

Questions

We may expect $\mathbf{A \to B \vdash_{CI} A \Rightarrow B}$.

But $\mathbf{A \Rightarrow B, A \vdash_{CI} B}$ by *modus ponens*,

and we would have $\mathbf{A \Rightarrow B \vdash_{CI} A \to B}$ by the *deduction theorem* !

Collapse of $\Rightarrow$ and $\to$?

In fact

$$\mathbf{\Gamma, A \vdash_{C} B} \quad \text{iff} \quad \mathbf{\Gamma \vdash_{C} A \Rightarrow B} \quad \text{and} \quad \mathbf{\Gamma, A \vdash_{I} B} \quad \text{iff} \quad \mathbf{\Gamma \vdash_{I} A \to B}$$

do not imply

$$\mathbf{\Gamma, A \vdash_{CI} B} \quad \text{iff} \quad \mathbf{\Gamma \vdash_{CI} A \Rightarrow B} \quad \text{iff} \quad \mathbf{\Gamma \vdash_{CI} A \to B}$$

unless

$$\mathbf{X \Rightarrow A, X \Rightarrow (A \to B) \vdash_{CI} X \Rightarrow B}$$

$$\mathbf{X \to A, X \to (A \Rightarrow B) \vdash_{CI} X \to B}$$

...

CPL∗IPL

No collapse?

It would be enough to show $\not\vdash_{\mathbf{CI}} ((\mathbf{A} \to \mathbf{B}) \to \mathbf{A}) \to \mathbf{A}$!

How? By fibred semantics ...

Semantics

Interpretation systems

Signature

$\Sigma = \{\Sigma_n\}_{n\in\mathbb{N}}$ syntactic constructors of different aritys

Formulas

$L_\Sigma = |\mathcal{F}_\Sigma|$ (A, B formulas) (Γ, Δ sets of formulas)

Interpretation system

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ (based on logical matrices [Tarski, Łukasiewicz])

- Σ signature
- $\mathcal{M}$ class (of models)
- $\mathbb{A}$ such that $\mathbb{A}(m) = \langle \mathcal{A}_m, T_m \rangle \in \mathit{Str}(\Sigma)$ for $m \in \mathcal{M}$, i.e., $\mathcal{A}_m$ is a Σ-algebra, $|\mathcal{A}_m|$ set of truth-values, and $T_m \subseteq |\mathcal{A}_m|$ set of designated values

Semantic consequence

$\Gamma \models A$ iff $[\![A]\!]_m \in T_m$ whenever $[\![\Gamma]\!]_m \subseteq T_m$, for each $m \in \mathcal{M}$

CPL - implicative fragment $\Rightarrow$

Interpretation system

signature

$$\Sigma_0 = P,\ \Sigma_1 = \emptyset,\ \Sigma_2 = \{\Rightarrow\},\ \Sigma_{n+3} = \emptyset$$

models

$m = \langle \bot, \top, v \rangle$ where $v : P \to \{\bot, \top\}$ bivaluation

structures

$\mathbb{A}(m) = \langle \mathcal{A}_m, T_m \rangle$ with $|\mathcal{A}_m| = \{\bot, \top\}$ and $T_m = \{\top\}$

$$p_m = v(p)$$

$\Rightarrow_m$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

IPL - implicative fragment $\rightarrow$

Interpretation system

signature

$$\Sigma_0 = Q,\ \Sigma_1 = \emptyset,\ \Sigma_2 = \{\rightarrow\},\ \Sigma_{n+3} = \emptyset$$

models

Kripke models $m = \langle W, \leq, V\rangle$ where $\langle W, \leq\rangle$ partial order

$$\mathcal{U}_{\leq} = \{X \subseteq W : \text{if } w_1 \in X \text{ and } w_1 \leq w_2 \text{ then } w_2 \in X\}$$

$$V : Q \rightarrow \mathcal{U}_{\leq}$$

structures

$$\mathbb{A}(m) = \langle \mathcal{A}_m, T_m\rangle \text{ with } |\mathcal{A}_m| = \mathcal{U}_{\leq} \text{ and } T_m = \{W\}$$

$$q_m = V(q)$$

$$X \rightarrow_m Y = \{w \in W : \{w' : w \leq w'\} \subseteq (W \setminus X) \cup Y\}$$

Fibred semantics

Morphism of interpretation systems

$$\langle g, \mu \rangle : \langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \to \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

$g : \Sigma' \to \Sigma''$ signature morphism

$\mu : \mathcal{M}'' \to \mathcal{M}'$

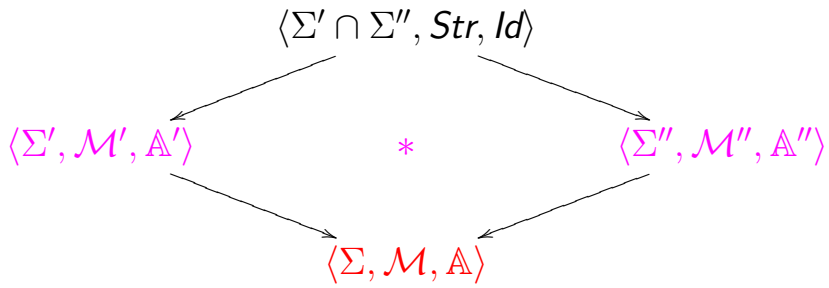
such that

$\mathbb{A}'(\mu(m)) = \langle \mathcal{A}'_{\mu(m)}, T'_{\mu(m)} \rangle = \langle \mathcal{A}''_m|_g, T''_m \rangle = \mathbb{A}''(m)|_g$ for each $m \in \mathcal{M}''$

Category Int

Fibred semantics

Categorial characterization



$$\Sigma = \Sigma' \cup \Sigma''$$

$$\mathcal{M} = \{(m', m'') \in \mathcal{M}' \times \mathcal{M}'' : |\mathcal{A}'_{m'}| = |\mathcal{A}''_{m''}|,\ T'_{m'} = T''_{m''},\ \underline{@}_{m'} = \underline{@}_{m''} \text{ if } @ \in \Sigma' \cap \Sigma''\}$$

$$\mathbb{A}(m', m'') = \langle \mathcal{A}, T \rangle \text{ where}$$

$\mathcal{A}$ is the unique algebra with $|\mathcal{A}| = |\mathcal{A}'_{m'}| = |\mathcal{A}''_{m''}|$ that extends $\mathcal{A}'_{m'}$ and $\mathcal{A}''_{m''}$

$$T = T'_{m'} = T''_{m''}$$

CPL∗IPL - Fibred semantics

signature

$$\Sigma_0 = P \cup Q,\ \Sigma_1 = \emptyset,\ \Sigma_2 = \{\Rightarrow, \rightarrow\},\ \Sigma_{n+3} = \emptyset$$

models

$m = \langle\langle \emptyset, W, v\rangle, \langle W, \leq, V\rangle\rangle$ with $\mathcal{U}_{\leq} = \{\emptyset, W\}$, i.e. W singleton

structures

$$\mathbb{A}(m) = \langle \mathcal{A}_m, T_m\rangle \text{ with } |\mathcal{A}_m| = \{\emptyset, W\} \text{ and } T_m = \{W\}$$

$$p_m = v(p) \qquad q_m = V(q)$$

$\Rightarrow_m$	$\emptyset$	W
$\emptyset$	W	W
W	$\emptyset$	W

$\rightarrow_m$	$\emptyset$	W
$\emptyset$	W	W
W	$\emptyset$	W

Collapse: $\rightarrow_m = \Rightarrow_m$ **!**

CPL∗IPL

Collapse

Fibred semantics collapses $\rightarrow$ in $\Rightarrow$!

How to show that $\not\vdash_{\mathbf{CI}} ((\mathbf{A} \rightarrow \mathbf{B}) \rightarrow \mathbf{A}) \rightarrow \mathbf{A}$?

By cryptofibred semantics ...

The collapsing problem

Solutions

- **Modulated fibring - C. Sernadas, J. Rasga, W. Carnielli**

  Extends fibring by using adjunctions between pre-ordered algebras of truth-values

  Avoids the collapsing problem by using safety-relevant syntactic information

  to control the definition of semantic entailment

  and the instantiation of schematic deduction rules

  Compatible with L. Fariñas del Cerro and A. Herzig's logic C+J

- **Cryptofibring**

  C. Caleiro and J. Ramos. Cryptomorphisms at work, LNCS 3423, pp. 45–60, 2005

  C. Caleiro and J. Ramos. Cryptofibring, in Proceedings of CombLog'04. Extended abstract.

Cryptofibred semantics

Cryptomorphism of interpretation systems

$$\langle g, \mu \rangle : \langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \to \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

$$g : \Sigma' \to \Sigma'' \text{ signature morphism}, \ \mu : \mathcal{M}'' \to \mathcal{M}'$$

such that $\mathbb{A}'(\mu(m)) = \langle \mathcal{A}'_{\mu(m)}, T'_{\mu(m)} \rangle = \langle \mathcal{A}''_m|_g, T''_m \rangle = \mathbb{A}''(m)|_g$ for each $m \in \mathcal{M}''$

Cryptofibred semantics

Cryptomorphism of interpretation systems

$$\langle g, \mu \rangle : \langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \to \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

$g : \Sigma' \to \Sigma''$ signature morphism, $\mu : \mathcal{M}'' \to \mathcal{M}'$

such that $\mathbb{A}'(\mu(m)) = \langle \mathcal{A}'_{\mu(m)}, T'_{\mu(m)} \rangle = \langle \mathcal{A}''_m|_g, T''_m \rangle = \mathbb{A}''(m)|_g$ for each $m \in \mathcal{M}''$

$$\langle g, \mu, h \rangle : \langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \to \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

$g : \Sigma' \to \Sigma''$ signature morphism, $\mu : \mathcal{M}'' \to \mathcal{M}'$, $h = \{h_m\}_{m \in \mathcal{M}''}$

with $h_m : \mathbb{A}'(\mu(m)) \to \mathbb{A}''(m)$ a g-cryptomorphism for each $m \in \mathcal{M}''$

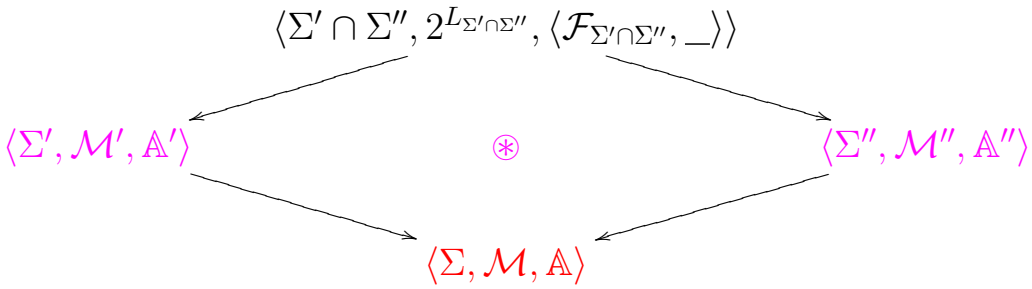
that is

$h_m : \mathcal{A}'_{\mu(m)} \to \mathcal{A}''_m|_g$ homomorphism of Σ'-algebras such that $T'_{\mu(m)} = h_m^{-1}(T'_m)$

Category Crypt

Cryptofibred semantics

Categorial characterization



Colimits in Crypt

Preliminaries

Sig

(many-sorted signatures)

Alg

(flat category of many-sorted algebras)

are both cocomplete

Coproducts in Crypt

Crypt

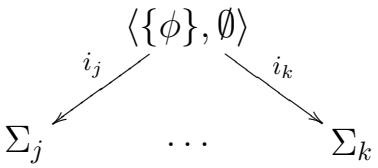
$$\langle \Sigma_j, \mathcal{M}_j, \mathbb{A}_j \rangle \qquad \cdots \qquad \langle \Sigma_k, \mathcal{M}_k, \mathbb{A}_k \rangle$$

Coproducts in Crypt

Crypt

$$\langle \Sigma_j, \mathcal{M}_j, \mathbb{A}_j \rangle \qquad \cdots \qquad \langle \Sigma_k, \mathcal{M}_k, \mathbb{A}_k \rangle$$

Sig



Coproducts in Crypt

Crypt

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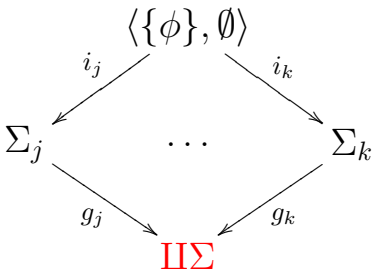
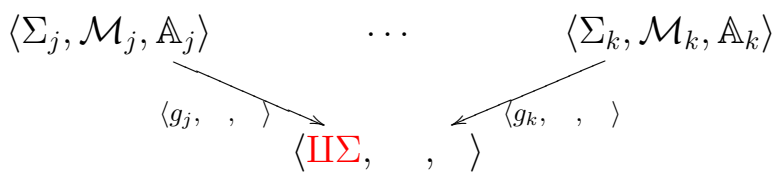
Sig



Coproducts in Crypt

Crypt



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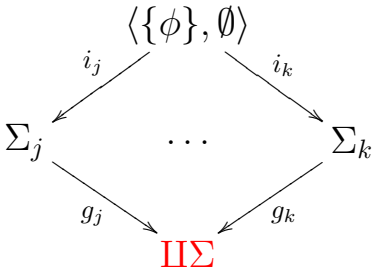

Coproducts in Crypt

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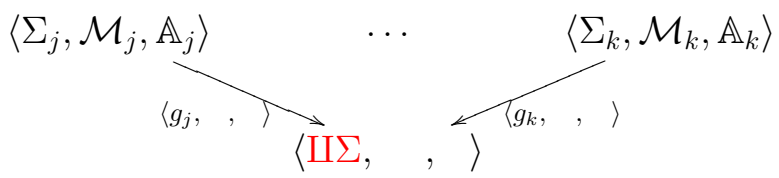

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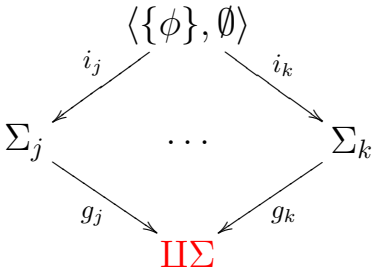
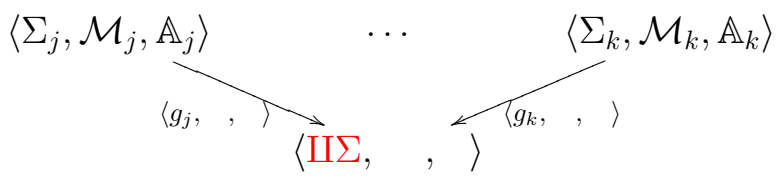


for each $\langle m_i \rangle \in \Pi_i \mathcal{M}_i$

Coproducts in Crypt

Crypt



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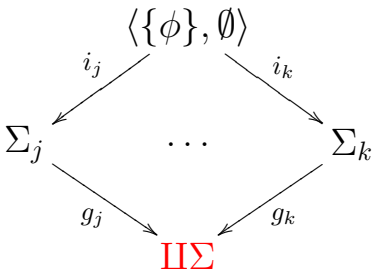
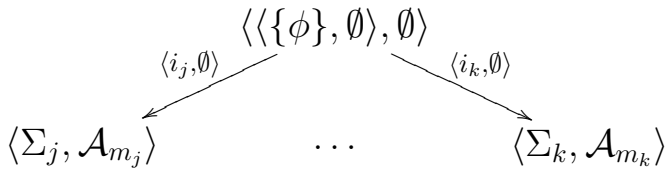


Alg



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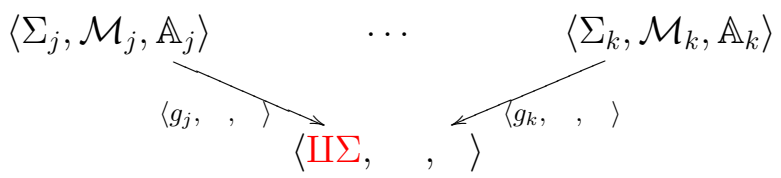
Coproducts in Crypt

Crypt



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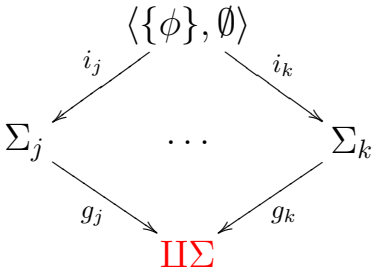

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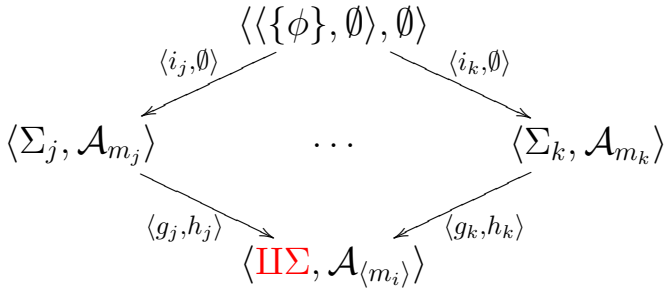

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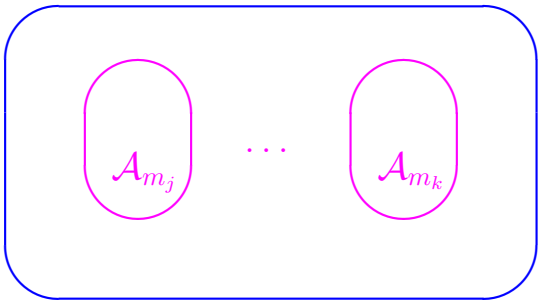
Coproducts in Crypt

$\langle \amalg\Sigma, \mathcal{A}_{\langle m_i \rangle} \rangle$ at ϕ



Coproducts in Crypt

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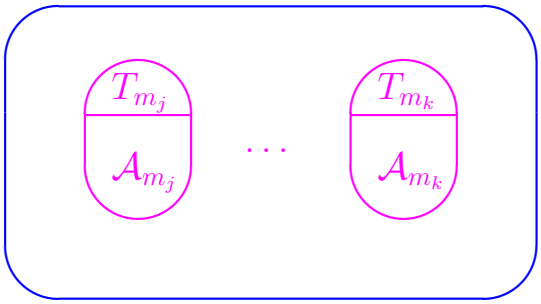

Coproducts in Crypt

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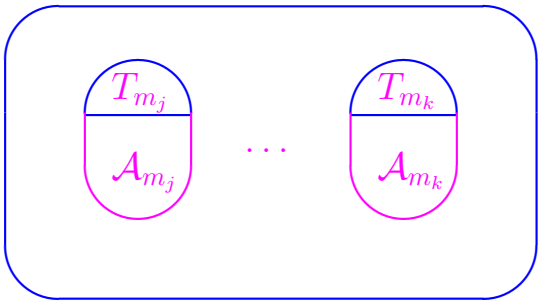
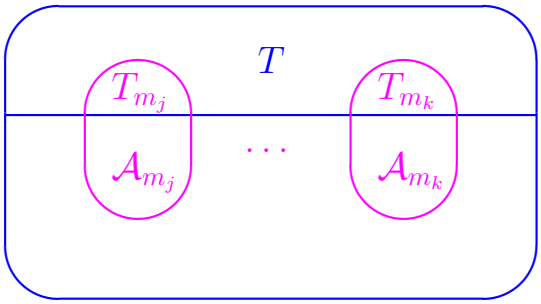


... not necessarily

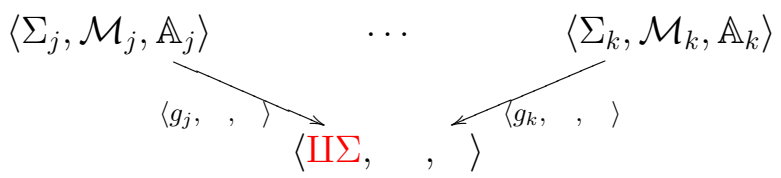
Coproducts in Crypt

$\langle \amalg\Sigma, \mathcal{A}_{\langle m_i \rangle} \rangle$ at ϕ

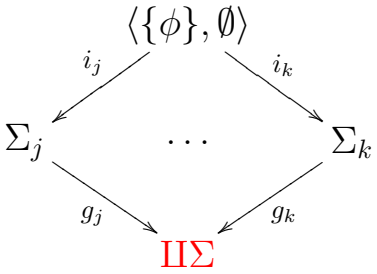


Coproducts in Crypt

Crypt



Sig



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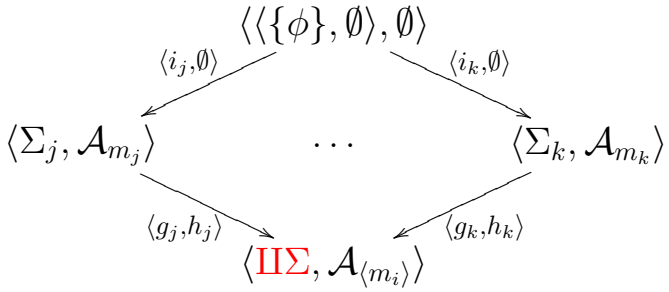
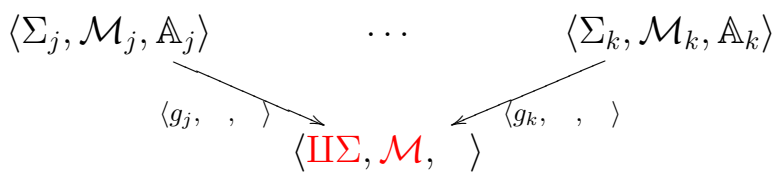


for each $\langle m_i \rangle \in \Pi_i \mathcal{M}_i$

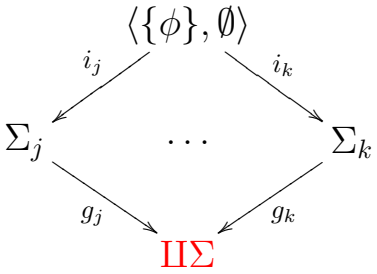
Coproducts in Crypt

Crypt

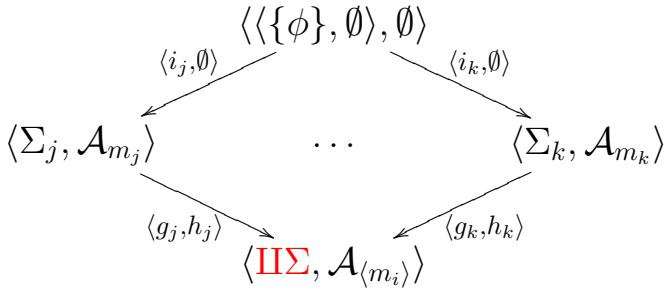


$\mathcal{M} = \{\langle \langle m_i \rangle, T \rangle : h_i^{-1}(T) = T_{m_i}\}$

Sig



Alg



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Coproducts in Crypt

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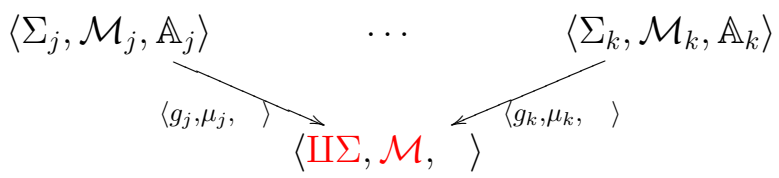
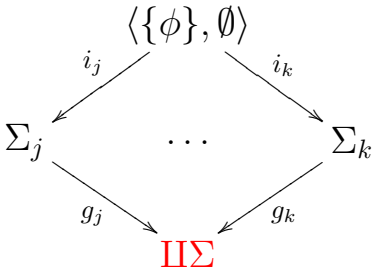
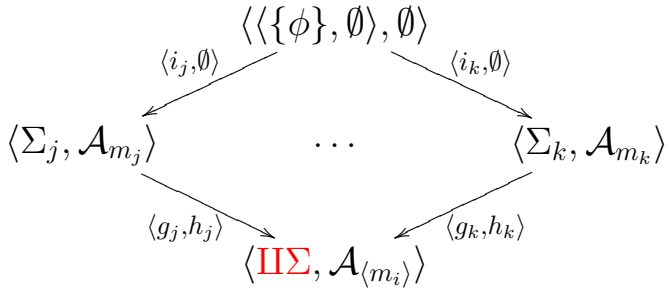


$\mathcal{M} = \{\langle \langle m_i \rangle, T \rangle : h_i^{-1}(T) = T_{m_i}\}$; $\mu_i(\langle m_i \rangle, T) = m_i$

Sig



Alg



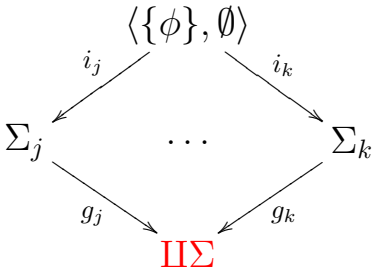
for each $\langle m_i \rangle \in \Pi_i \mathcal{M}_i$

Coproducts in Crypt!!!

Crypt

$$\mathcal{M} = \{\langle \langle m_i \rangle, T \rangle : h_i^{-1}(T) = T_{m_i}\} \;\; ; \;\; \mu_i(\langle m_i \rangle, T) = m_i \;\; ; \;\; \mathbb{A}(\langle m_i \rangle, T) = \langle \mathcal{A}_{\langle m_i \rangle}, T \rangle$$

Sig



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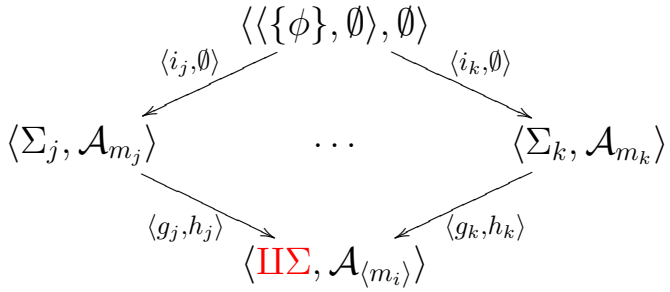


for each $\langle m_i \rangle \in \Pi_i \mathcal{M}_i$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\Longrightarrow}} \Sigma''$$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \quad \rangle]{} \langle \Sigma, \quad , \quad \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\rightrightarrows}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \ \ \rangle]{} \langle \Sigma, \quad , \quad \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\rightrightarrows}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \quad \rangle]{} \langle \Sigma, \quad , \quad \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\rightrightarrows}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

Alg

$$\langle \Sigma', \mathcal{A}_{m'} \rangle \underset{\langle g_2, h_2 \rangle}{\overset{\langle g_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{A}_{m''} \rangle$$

for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \ \ \rangle]{} \langle \Sigma, \quad , \quad \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\Longrightarrow}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

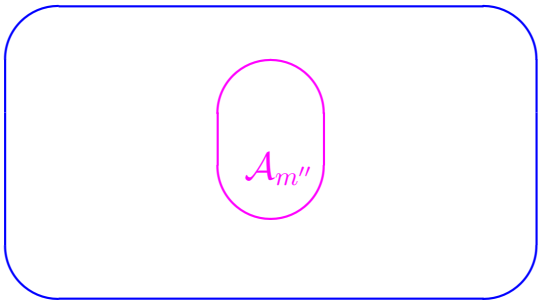
Alg

$$\langle \Sigma', \mathcal{A}_{m'} \rangle \underset{\langle g_2, h_2 \rangle}{\overset{\langle g_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{A}_{m''} \rangle \xrightarrow[\langle \equiv_g, \equiv_{gh} \rangle]{} \langle \Sigma, \mathcal{A} \rangle$$

for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

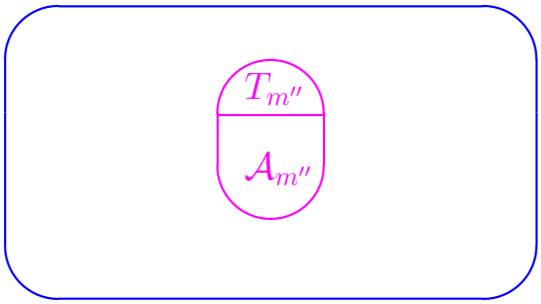
Coequalizers in Crypt

$\langle \Sigma, \mathcal{A} \rangle$ at ϕ



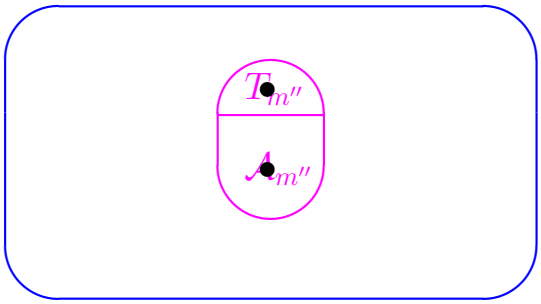
Coequalizers in Crypt

$\langle \Sigma, \mathcal{A} \rangle$ at ϕ



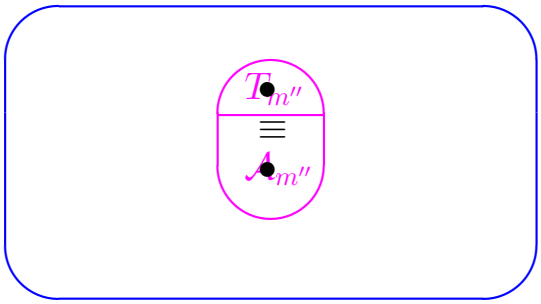
Coequalizers in Crypt

$\langle \Sigma, \mathcal{A} \rangle$ at ϕ



Coequalizers in Crypt

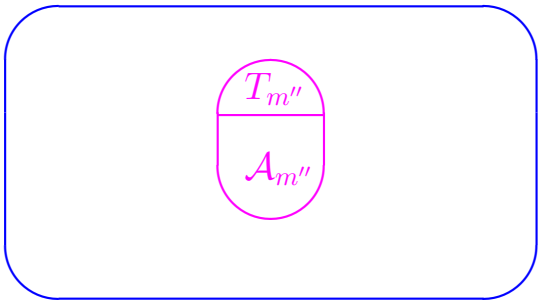
$\langle \Sigma, \mathcal{A} \rangle$ at ϕ



Forget m'' ...

Coequalizers in Crypt

$\langle \Sigma, \mathcal{A} \rangle$ at ϕ



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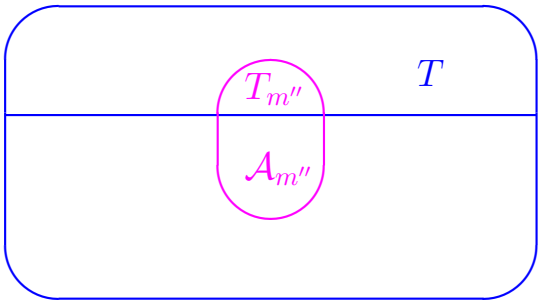

Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \ \ \rangle]{} \langle \Sigma, \quad , \quad \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\Longrightarrow}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

Alg

$$\langle \Sigma', \mathcal{A}_{m'} \rangle \underset{\langle g_2, h_2 \rangle}{\overset{\langle g_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{A}_{m''} \rangle \xrightarrow[\langle \equiv_g, \equiv_{gh} \rangle]{} \langle \Sigma, \mathcal{A} \rangle$$

for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

Coequalizers in Crypt

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$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \ , \ \rangle]{} \langle \Sigma, \mathcal{M}, \ \ \rangle$$

$$\mathcal{M} = \{ \langle m'', T \rangle : \equiv_{gh}^{-1} (T) = T_{m''} \}$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\Longrightarrow}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

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Coequalizers in Crypt

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\rightrightarrows}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \mu, \quad \rangle]{} \langle \Sigma, \mathcal{M}, \quad \rangle$$

$$\mathcal{M} = \{ \langle m'', T \rangle :\equiv_{gh}^{-1} (T) = T_{m''} \} \;\; ; \;\; \mu(m'', T) = m''$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\rightrightarrows}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

Alg

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for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

Coequalizers in Crypt!!!

Crypt

$$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \underset{\langle g_2, \mu_2, h_2 \rangle}{\overset{\langle g_1, \mu_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle \xrightarrow[\langle \equiv_g, \mu, \equiv_{gh} \rangle]{} \langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$$

$$\mathcal{M} = \{ \langle m'', T \rangle :\equiv_{gh}^{-1} (T) = T_{m''} \} \;\; ; \;\; \mu(m'', T) = m'' \;\; ; \;\; \mathbb{A}(m'', T) = \langle \mathcal{A}, T \rangle$$

Sig

$$\Sigma' \underset{g_2}{\overset{g_1}{\Longrightarrow}} \Sigma'' \xrightarrow[\equiv_g]{} \Sigma$$

Alg

$$\langle \Sigma', \mathcal{A}_{m'} \rangle \underset{\langle g_2, h_2 \rangle}{\overset{\langle g_1, h_1 \rangle}{\Longrightarrow}} \langle \Sigma'', \mathcal{A}_{m''} \rangle \xrightarrow[\langle \equiv_g, \equiv_{gh} \rangle]{} \langle \Sigma, \mathcal{A} \rangle$$

for each $m'' \in \mathcal{M}''$ with $\mu_1(m'') = \mu_2(m'') = m'$

Cryptofibring

Characterization

$\langle \Sigma', \mathcal{M}', \mathbb{A}' \rangle \circledast \langle \Sigma'', \mathcal{M}'', \mathbb{A}'' \rangle$ equivalent to $\langle \Sigma' \cup \Sigma'', \mathcal{M}_\circledast, \mathbb{A}_\circledast \rangle$

$\mathcal{M}_\circledast$ class of all 5-tuples $\langle \langle \mathcal{A}, T \rangle, m', m', h', h'' \rangle$ where

$$\langle \mathcal{A}, T \rangle \in \mathit{Str}(\Sigma' \cup \Sigma'')$$
$$m' \in \mathcal{M}' \text{ and } m'' \in \mathcal{M}''$$
$$h' : \mathbb{A}'(m') \to \langle \mathcal{A}, T \rangle \text{ an } i'\text{-cryptomorphism}$$
$$h'' : \mathbb{A}''(m'') \to \langle \mathcal{A}, T \rangle \text{ an } i''\text{-cryptomorphism}$$

$$\mathbb{A}_\circledast(\langle \mathcal{A}, T \rangle, m', m'', h', h'') = \langle \mathcal{A}, T \rangle$$

generalizing fibred semantics

CPL⊛IPL

No collapse

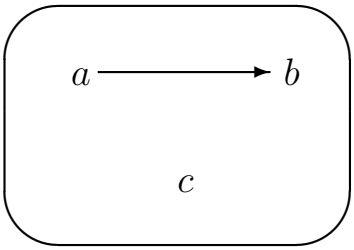
Cryptofibred semantics does not collapse $\rightarrow$ in $\Rightarrow$!

Find a cryptofibred model m of CPL⊛IPL
that satisfies all axioms and rules of CPL and IPL
such that $[\![((\mathbf{A} \rightarrow \mathbf{B}) \rightarrow \mathbf{A}) \rightarrow \mathbf{A}]\!]_m \notin T_m$ ...

CPL⊛IPL

No collapse

$\langle W, \leq \rangle$



$$\mathcal{U}_{\leq} = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$$

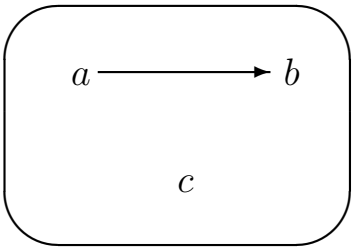
$$V : P \cup Q \to \mathcal{U}_{\leq}$$

$$V(P) \subseteq \{\emptyset, \{a, b, c\}\} \qquad V(q') = \{b\}; V(q'') = \emptyset$$

CPL⊛IPL

No collapse

$\langle W, \leq \rangle$



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$$\mathbb{A}(m) = \langle \mathcal{A}_m, T_m \rangle \text{ with } |\mathcal{A}_m| = \mathcal{U}_{\leq},\ T_m = \{\{a, b, c\}\}$$

$$p_m = V(p) \qquad q_m = V(q)$$

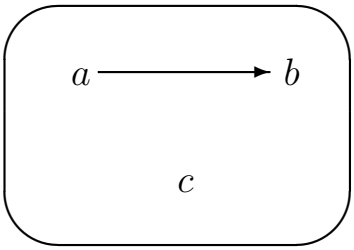
$$X \Rightarrow_m Y = (W \setminus X) \cup Y$$

$$X \to_m Y = \{w \in W : \{w' : w \leq w'\} \subseteq (W \setminus X) \cup Y\}$$

CPL⊛IPL

No collapse

$\langle W, \leq \rangle$



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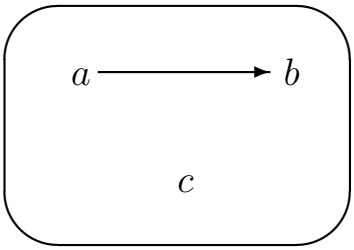
Watch out! For instance, $(\{b, c\} \Rightarrow_m \emptyset) = \{a\} \notin \mathcal{U}_{\leq}$

CPL⊛IPL

No collapse

$\langle W, \leq \rangle$



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No collapse

$$|\mathcal{A}_m| = \mathcal{U}_{\leq} = \{\emptyset, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,b,c\}\}$$

$\Rightarrow_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$
$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{b\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$

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$\{b\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$
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$\{a,b,c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$

$\rightarrow_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$
$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{b\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{a\}$	$\{a,b\}$	$\{a,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
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$\{a,b\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{a,b,c\}$	$\{b,c\}$	$\{a,b,c\}$
$\{b,c\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$

$\rightarrow_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a,b\}$	$\{b,c\}$	$\{a,b,c\}$
$\emptyset$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
$\{b\}$	$\{c\}$	$\{a,b,c\}$	$\{c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$
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$\{b,c\}$	$\emptyset$	$\{a,b\}$	$\{c\}$	$\{a,b\}$	$\{a,b,c\}$	$\{a,b,c\}$
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No collapse

$$V(q') = \{b\}; V(q'') = \emptyset$$

$$[\![((q' \to q'') \to q') \to q']\!]_m \notin T_m = \{\{a, b, c\}\}$$

$\to_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
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$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
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$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b\}$	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
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$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
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$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
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$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b\}$	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$

$$\begin{aligned} [\![((q' \to q'') \to q') \to q']\!]_m &= (((\{b\} \to_m \emptyset) \to_m \{b\}) \to_m \{b\} \\ &= (\{c\} \to_m \{b\}) \to_m \{b\} \end{aligned}$$

No collapse

$$V(q') = \{b\}; V(q'') = \emptyset$$

$$[\![((q' \to q'') \to q') \to q']\!]_m \notin T_m = \{\{a, b, c\}\}$$

$\to_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b\}$	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$

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$\to_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b\}$	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$

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$$V(q') = \{b\}; V(q'') = \emptyset$$

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$\to_m$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$
$\emptyset$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{b\}$	$\{c\}$	$\{a, b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{c\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b\}$	$\{c\}$	$\{b, c\}$	$\{c\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\{b, c\}$	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{a, b, c\}$
$\{a, b, c\}$	$\emptyset$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{b, c\}$	$\{a, b, c\}$

$$\begin{aligned}
[\![((q' \to q'') \to q') \to q']\!]_m &= (((\{b\} \to_m \emptyset) \to_m \{b\}) \to_m \{b\} \\
&= (\{c\} \to_m \{b\}) \to_m \{b\} \\
&= \{a, b\} \to_m \{b\} \\
&= \{b, c\}
\end{aligned}$$

Cryptofibring

Cryptomorphisms

Interpretation system

$\langle \Sigma, \mathcal{M}, \mathbb{A} \rangle$ based on logical matrices (Σ unsorted)

Cryptomorphism of interpretation systems

$\langle g, \mu, h \rangle : \langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \to \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle$ where $g : \Sigma_1 \to \Sigma_2$ is a signature morphism and $\mu : \mathcal{M}_2 \to \mathcal{M}_1$ is a map and $h = \{h_m\}_{m \in \mathcal{M}_2}$ such that $h_m : \mathbb{A}_1(\mu(m)) \to \mathbb{A}_2(m)$ is a g-cryptomorphism for each $m \in \mathcal{M}_2$

Cryptofibring

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that is

$h_m : \mathcal{A}_{\mu(m)} \to \mathcal{A}_m|_g$ is a homomorphism of Σ_1-algebras such that $h_m^{-1}(T_m) = T_{\mu(m)}$

Category Crypt (properly contains Int)

Cryptofibring

Unconstrained

$$\langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \circledast \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathcal{M}_1 \circledast \mathcal{M}_2, \mathbb{A}_{\circledast} \rangle$$

$\mathcal{M}_1 \circledast \mathcal{M}_2$ is the class of all tuples $(\langle \mathcal{A}, T \rangle, m_1, m_2, h_1, h_2)$ such that

$$\mathbb{A} \in \mathit{Str}(\Sigma_1 \cup \Sigma_2)$$

$$m_1 \in \mathcal{M}_1 \text{ and } m_2 \in \mathcal{M}_2$$

$$h_1 : \mathbb{A}_1(m_1) \to \langle \mathcal{A}, T \rangle \text{ is a } i_1\text{-cryptomorphism}$$

$$h_2 : \mathbb{A}_2(m_2) \to \langle \mathcal{A}, T \rangle \text{ is a } i_2\text{-cryptomorphism}$$

$$\mathbb{A}_{\circledast}(\langle \mathcal{A}, T \rangle, m_1, m_2, h_1, h_2) = \langle \mathcal{A}, T \rangle$$

Cryptofibring

Unconstrained

$$\langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \circledast \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathcal{M}_1 \circledast \mathcal{M}_2, \mathbb{A}_{\circledast} \rangle$$

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$$\mathbb{A}_{\circledast}(\langle \mathcal{A}, T \rangle, m_1, m_2, h_1, h_2) = \langle \mathcal{A}, T \rangle$$

thus generalizing fibred semantics

Cryptofibring

Amalgamated structures

$$\mathbb{A}_{\circledast}(\mathcal{M}_1 \circledast \mathcal{M}_2)$$

contains the fibred structures, but also amalgamated structures

$\langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$ where

$$\mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2} = \mathcal{W}_{\Sigma_1 \cup \Sigma_2}(|\mathcal{A}_{m_1}| \uplus |\mathcal{A}_{m_2}|)/_{\equiv}$$

with

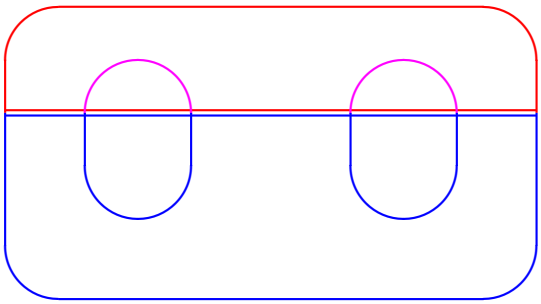
$$c_1(a_1^1, \ldots, a_1^k) \equiv \underline{c_1}_{m_1}(a_1^1, \ldots, a_1^k)$$
$$c_2(a_2^1, \ldots, a_2^k) \equiv \underline{c_2}_{m_2}(a_2^1, \ldots, a_2^k)$$

T such that $j_1^{-1}(T) = T_{m_1}$ and $j_2^{-1}(T) = T_{m_2}$

Cryptofibring

Amalgamated structures

$$\langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$$



Cryptofibring

Avoiding the collapse

CPL $\circledast$ **G3** includes models with the following structures

$$\langle \mathcal{A}_{v'} \oplus \mathcal{A}_{v''}, \{[1']_{\equiv}, [1'']_{\equiv}\}\rangle$$

and the result is distinct from **CPL**

taking v'' such that $v''(p'') = \frac{1}{2}''$

$$[\![(\neg''(\neg'' p'')) \Rightarrow'' p'']\!]_m = [\frac{1}{2}'']_{\equiv} \notin T_m$$

Conservativeness

$I' \circledast I''$ is absolutely free if it extends both I' and I'' conservatively, i.e.

$$\Gamma \cup \{\varphi\} \subseteq L_{\Sigma'}: \qquad \Gamma \models' \varphi \quad \text{iff} \quad \Gamma \models_{\circledast} \varphi$$

$$\Gamma \cup \{\varphi\} \subseteq L_{\Sigma''}: \qquad \Gamma \models'' \varphi \quad \text{iff} \quad \Gamma \models_{\circledast} \varphi$$

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Collapsing situations correspond to particular cases of failure of conservativeness

Conservativeness

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Collapsing situations correspond to particular cases of failure of conservativeness

Sufficient condition for conservativeness

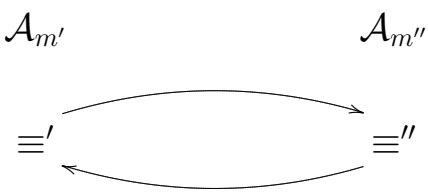
$I' \circledast I''$ is absolutely free if for each $m' \in \mathcal{M}'$ there exists $m'' \in \mathcal{M}''$ such that $\langle m', m'' \rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$, and vice versa

Conservativeness

Representation of $\langle m', m'' \rangle$ in $\mathcal{M}' \circledast \mathcal{M}''$

$\langle m', m'' \rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ if and only if

- $[\![\varphi]\!]_{m'} \in T'_m$ **iff** $[\![\varphi]\!]_{m''} \in T''_m$, **for every** $\varphi \in L_{\Sigma' \cap \Sigma''}$
- **$\equiv'$ is compatible with $T_{m'}$ and $\equiv''$ is compatible with $T_{m''}$**



$$[\![\varphi]\!]_{m'} = [\![\psi]\!]_{m'} \quad \longrightarrow \quad [\![\varphi]\!]_{m'} \equiv'' [\![\psi]\!]_{m'} \quad \text{with } \varphi, \psi \in L_{\Sigma' \cap \Sigma''}$$

$$\ldots \qquad \longleftarrow \qquad \ldots$$

Conservativeness

Representation of $\langle m', m'' \rangle$ in $\mathcal{M}' \circledast \mathcal{M}''$

$\langle m', m'' \rangle$ is represented in $\mathcal{M}' \circledast \mathcal{M}''$ if and only if

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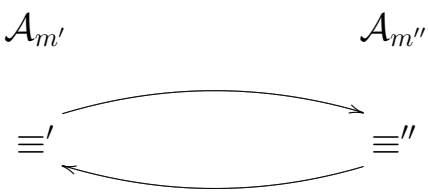

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$$\ldots \qquad \longleftarrow \qquad \ldots$$

Corollary: Unconstrained cryptofibring is absolutely free

Cryptofibring

Colimit?

$$\langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \oplus \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathcal{M}_1 \oplus \mathcal{M}_2, \mathbb{A}_\oplus \rangle$$

$\mathcal{M}_1 \oplus \mathcal{M}_2$ is the class of all tuples (m_1, m_2, T) such that

$$j_1^{-1}(T) = T_{m_1} \text{ and } j_2^{-1}(T) = T_{m_2}$$

$$\mathbb{A}_\circledast(m_1, m_2, T) = \langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$$

Cryptofibring

Colimit?

$$\langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \oplus \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, \mathcal{M}_1 \oplus \mathcal{M}_2, \mathbb{A}_{\oplus} \rangle$$

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$$\mathbb{A}_{\circledast}(m_1, m_2, T) = \langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$$

what can we do now?

Cryptofibring

Equivalent

$$int_1 \circledast int_2 \leftrightarrow int_1 \oplus int_2$$

therefore

$$\Phi \models_{\circledast} \psi \text{ iff } \Phi \models_{\oplus} \psi$$

Cryptofibring

Equivalent

$$int_1 \circledast int_2 \leftrightarrow int_1 \oplus int_2$$

therefore

$$\Phi \models_{\circledast} \psi \text{ iff } \Phi \models_{\oplus} \psi$$

is this good enough?

Cryptofibring

Entailment vs. schematic entailment

$\langle g, \mu, h\rangle : \langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1\rangle \to \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2\rangle$ cryptomorphism

For formulas in L_{Σ_1}: if $\Phi \models_1 \psi$ then $g[\Phi] \models_2 g(\phi)$

For schema formulas in $L_{\Sigma_1}(\Xi)$: $\Gamma \models_1 \delta$ does not imply $g[\Gamma] \models_2 g(\delta)$

Cryptofibring

Loss of soundness preservation

$\langle g, \mu, h\rangle : \langle \Sigma_1, \mathcal{M}_1, \mathbb{A}_1\rangle \to \langle \Sigma_2, \mathcal{M}_2, \mathbb{A}_2\rangle$ cryptomorphism

For formulas in L_{Σ_1}: if $\Phi \models_1 \psi$ then $g[\Phi] \models_2 g(\phi)$

For schema formulas in $L_{\Sigma_1}(\Xi)$: $\Gamma \models_1 \delta$ does not imply $g[\Gamma] \models_2 g(\delta)$

soundness is not expected to be preserved by cryptofibring

Soundness preservation

Fibred semantics always preserves soundness
If $\mathcal{D}'$ is sound for $\mathcal{I}'$ and $\mathcal{D}''$ is sound for $\mathcal{I}''$ then $\mathcal{D}' + \mathcal{D}''$ is sound for $\mathcal{I}' * \mathcal{I}''$

The same is not true, in general, for $\mathcal{I}' \circledast \mathcal{I}''$

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However, we can recover soundness preservation for cryptofibring by recasting cryptofibred semantics for logic systems $\mathcal{L} = \langle \mathcal{I}, \mathcal{D} \rangle$ with a soundness requirement $\vdash_{\mathcal{D}} \subseteq \models_{\mathcal{I}}$

We get rid in $\mathcal{L}' \circledast \mathcal{L}''$ of all cryptofibred models that make the rules unsound, but still all fibred models are kept since they are necessarily sound

This does also make cryptofibring sensitive to the deductive level, which is not a bad thing if we think of how many different deductive systems can present exactly the same consequence relation, but on the other hand makes it much harder to formulate sufficient conditions for conservativeness

Soundness preservation

Logic systems

Logic system

$\langle \mathit{dsy}, \mathit{int} \rangle$

Cryptomorphism of logic systems

$\langle g, \mu, h \rangle : \langle \mathit{dsy}_1, \mathit{int}_1 \rangle \longrightarrow \langle \mathit{dsy}_2, \mathit{int}_2 \rangle$ with $g : \mathit{dsy}_1 \longrightarrow \mathit{dsy}_2$ is a morphism of deduction systems and $\langle g, \mu, h \rangle : \mathit{int}_1 \longrightarrow \mathit{int}_2$ a cryptomorphism of interpretation systems such that $\mathbb{A}_2(m)$ is a structure for the g-image of the rules of dsy_1 whenever $\mathbb{A}_1(\mu(m))$ is a structure for the rules of dsy_1, for m any model of int_2

Category Log

Soundness preservation

Unconstrained cryptofibring

$$\langle \Sigma_1, R_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \circledast \langle \Sigma_2, R_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, R_1 \cup R_2, \mathcal{M}_1 ® \mathcal{M}_2, \mathbb{A}_® \rangle$$

$\mathcal{M}_1 ® \mathcal{M}_2$ is the class of all cryptofibred models $(\langle \mathcal{A}, T \rangle, m_1, m_2, h_1, h_2)$ such that

$\langle \mathcal{A}, T \rangle$ is a structure for R_1 if $\mathbb{A}_1(m_1)$ is
$\langle \mathcal{A}, T \rangle$ is a structure for R_2 if $\mathbb{A}_2(m_2)$ is

$$\mathbb{A}_®(\langle \mathcal{A}, T \rangle, m_1, m_2, h_1, h_2) = \langle \mathcal{A}, T \rangle$$

Soundness preservation

Amalgamated structures

$\mathbb{A}_{\circledR}(\mathcal{M}_1 \circledR \mathcal{M}_2)$

contains also amalgamated structures

$\langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$ as before, such that also

if $[\![\sigma[\Gamma]]\!]_m \subseteq T$ then $[\![\sigma(\delta)]\!]_m \in T$, for each $\langle \Gamma, \delta \rangle \in R_1 \cup R_2$ and ground σ

Soundness preservation

Colimit?

$$\langle \Sigma_1, R_1, \mathcal{M}_1, \mathbb{A}_1 \rangle \oplus_R \langle \Sigma_2, R_2, \mathcal{M}_2, \mathbb{A}_2 \rangle = \langle \Sigma_1 \cup \Sigma_2, R_1 \cup R_2, \mathcal{M}_1 \oplus_R \mathcal{M}_2, \mathbb{A}_{\oplus_R} \rangle$$

$\mathcal{M}_1 \oplus_R \mathcal{M}_2$ is the class of all tuples (m_1, m_2, T) such that

$$j_1^{-1}(T) = T_{m_1} \text{ and } j_2^{-1}(T) = T_{m_2}$$

T is closed for all ground instances of rules in $R_1 \cup R_2$

$$\mathbb{A}_{\oplus_R}(m_1, m_2, T) = \langle \mathcal{A}_{m_1} \oplus \mathcal{A}_{m_2}, T \rangle$$

Soundness preservation

Equivalent

$$log_1 ® log_2 \leftrightarrow log_1 \oplus_R log_2$$

therefore

$$\Phi \models_® \psi \text{ iff } \Phi \models_{\oplus_R} \psi$$

Completeness preservation

Completeness preservation carries over from fibring

All fibred models appear in cryptofibring

Completeness preservation

Completeness preservation carries over from fibring

All fibred models appear in cryptofibring

$\mathcal{L} = \langle \mathcal{I}, \mathcal{D} \rangle$ is full if every interpretation structure that makes all the rules of $\mathcal{D}$ sound is associated to some model of $\mathcal{I}$

Cryptofibred semantics preserves completeness of full logics
If $\mathcal{L}'$ is full and $\mathcal{L}''$ is full then $\mathcal{L}' \circledast \mathcal{L}''$ is complete

Completeness preservation

Completeness preservation carries over from fibring

All fibred models appear in cryptofibring

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Cryptofibred semantics preserves completeness of full logics
If $\mathcal{L}'$ is full and $\mathcal{L}''$ is full then $\mathcal{L}' \circledast \mathcal{L}''$ is complete

Still, the existence of many more cryptofibred models, when compared to fibring, opens the way to much better completeness preservation results, namely by weakening the very harsh assumption of fullness

Soundness preservation

$log = \langle dsy, int \rangle$

Soundness and completeness

log is sound if $\Phi \vdash \psi$ implies $\Phi \models \psi$

log is complete if $\Phi \models \psi$ implies $\Phi \vdash \psi$

Soundness preservation

if both log_1 and log_2 are sound

then $log_1 ® log_2$ is sound

Soundness preservation

$log = \langle dsy, int \rangle$

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log is sound if $\Phi \vdash \psi$ implies $\Phi \models \psi$
log is complete if $\Phi \models \psi$ implies $\Phi \vdash \psi$

Soundness preservation
if both log_1 and log_2 are sound
then $log_1 ® log_2$ is sound

Completeness preservation
even if log_1 and log_2 are complete,
$log_1 ® log_2$ is in general not complete

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log is sound if $\Phi \vdash \psi$ implies $\Phi \models \psi$
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if both log_1 and log_2 are sound
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Completeness preservation
even if log_1 and log_2 are complete,
$log_1 ® log_2$ is in general not complete

but comparing with fibring ...

Conclusion

- Combination of logics
- Fibring and the collapsing problem
- Cryptofibring

- Properties of CPL⊛IPL
  - Completeness, algebraization, topological semantics?
  - L. Humberstone and M. Spinks (BCSK)
    P. O'Hearn and D. Pym (BI)
- Completeness preservation
  - Specific and more powerful results for cryptofibring

The end

Center for Logic and Computation

clc.math.ist.utl.pt

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cs.math.ist.utl.pt/ccal.html

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Thank you!