# An asymptotic vanishing theorem for generic unions of multiple points

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## 1 Introduction

This work is devoted to the following asymptotic statement:

**Theorem. 1.1** *Let $X$ be a projective geometrically reduced and irreducible scheme over a field $k$ of (arbitrary) characteristic $p$ and let $\mathcal{M}$, $\mathcal{L}$ be line bundles on $X$ with $\mathcal{L}$ ample. Suppose further that if $p > 0$ then $X$ is smooth in codimension one. For fixed $m \geq 0$ there exists $d_0 = d_0(m)$, depending only on $m, X, \mathcal{L}, \mathcal{M}$, such that for any $d \geq d_0$ and any generic union $Z$ of (fat) points of multiplicity $\leq m$ the canonical map*

$$H^0(X, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

*has maximal rank.*

Here, as usual, we call (fat) point of multiplicity $m$ in $X$, any subscheme defined by $\mathcal{I}_z^m$, where $\mathcal{I}_z$ is the ideal sheaf of a point $z$ in the smooth locus of $X$. The reader may prefer the following statement, which is more or less equivalent to the preceding one:

**Corollary. 1.2** *Let $X$, $\mathcal{M}$, $\mathcal{L}$, $m$ be as above. There exists $\chi$ such that for any generic union $Z$ of (fat) points of multiplicity at most $m$ and of total degree at least $\chi$, all the canonical maps*

$$H^0(X, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(Z, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

*have maximal rank.*

Note that the above statement applies as soon as the number of points is at least $\chi$.

To simplify the presentation and highlight the essential elements, a detailed proof will only be given in the case $\mathcal{M} = \mathcal{O}_X$. The easy modifications needed to prove the general result are then outlined in §7 along with another variant.

**Remark. 1.3** *The statement of the theorem is false for $p > 0$ if we allow $X$ to be singular in codimension one. This is illustrated in the example 7.6.*

These results are already new and significant in the case where $X$ is the projective plane (with $\mathcal{M} = \mathcal{O}$ and $\mathcal{L} = \mathcal{O}(1)$). Indeed, even in that case, the expected vanishing theorem for generic unions of fat points [S, Ha, Hi2] is still unproven, see a survey in [G] and more recent contributions in [Xu, ShT, CM]. Reformulations of the general problem and its relation to other topics have been considered at length in [N, I1, I2, MP]. Much attention has been paid to the "homogeneous" case on $\mathbb{P}^n$, namely when all the points have the same multiplicity $m$: see [AC] or [Hi1] for $n = m = 2$, [A, AHi1,2,3] for $m = 2$ and $n$ arbitrary, [Hi1] for $m = 3, n = 2, 3$, [LL2] for $m = 4, n = 2$ in perfectly adjusted cases, and finally [CM], where they have settled completely that case $m = 4, n = 2$ by a new and promising method.

In the heterogeneous plane case, at least formally, our result is new even for $m = 3$, although we have been aware for ten years that an easy (though possibly tedious) proof could be given for low $m$ and $n$, using only the "Horace" techniques developed in [Hi1] and [A]. Nonetheless, these techniques seem too weak to support a proof of a general asymptotic statement, even for projective spaces.

In [AHi2] we developed a new technique of a differential nature for the case $m = 2$ which, in that and later papers, made it possible not only to solve some delicate low-degree cases in [AHi2,3], but also to simplify the proof for the high-degree case [AHi4]. The main new ingredient in the proofs of the present paper is an extension of this technique applicable to higher-order fat points ($m > 2$), see lemma 2.3 and 10.2.

In §§2-7, we present the proof of the theorem. Sections §§8-10 are devoted to our differential Horace lemmas. Indeed, the results presented there (see 10.2) are substantially more general than 2.3. While for the present asymptotic statement, 2.3 is perfectly sufficient, the full strength of 10.2 will be much more efficient for concrete cases with small $n$ and $m$. The proof of 10.2 is achieved by an ideal theoretic arguement. We would like to point out that our original proof of the lemmas computed the first non-zero derivative of a determinant in a way which owed much to [LL1,2].

## 1.1 Outline of the proof of the theorem

In the remainder of this introduction we will try to illustrate the general ideas in the proof of the main theorem in the particular case of the projective plane. We start with a given maximum multiplicity $m$ and a sufficiently large degree $d$. We want to prove a maximal rank statement for a generic union $Z$ of multiple points, which, by adding simple points we can suppose to be of total degree at least $(d + 2)(d + 1)/2$. Horace's method amounts to specialising some of these points to the generic curve $\Gamma$ of some intermediate degree $\gamma$. Modulo an analogous maximal rank statement on $\Gamma$ which we suppose to hold inductively and a local to global condition that is easily verified, our differential lemma can then be applied under certain numerical conditions (holding for large $d$) and we reduce to a new subscheme $\mathfrak{D}(Z)$ (the *derivative* of $Z$, see §4) and a new degree $d - \gamma$. This can be safely applied as long as the current degree is not too small, say $d_c > \overline{d}$. But when $d_c$ becomes smaller than or equal to $\overline{d}$, we have to backtrack in order to complete the proof. Our trick is to modify the procedure early on so as to generate in the current subscheme $Z_c$ a sufficient number of unconstrained points of multiplicity at most $m - 1$ (of total degree at least $(\overline{d}+2)(\overline{d}+1)/2$). So that when the degree of the current subscheme has been lowered under $\overline{d}$, only points of multiplicity at most $m - 1$ remain. Having chosen $\overline{d}$ large enough (i.e. $\overline{d} \geq d_0(m - 1)$ in the notation of the theorem), we conclude by induction on $m$.

It remains to explain how we generate these free points (see §6): our differential lemma generates in $\mathfrak{D}(Z)$ a lot of points of multiplicity smaller than $m$, but all of them lie on the exploited divisor $\Gamma$ of degree $\gamma$. The trick here consists in specialising $\Gamma$ to the union of two generic divisors $\Gamma'$ and $\Gamma''$ of degrees $\gamma'$ and $\gamma''$, with the desired number of points specialised to say $\Gamma'$. If this number of points is sufficiently small with respect to $\gamma'$, these points suffer no constraint by being supported in a curve of degree $\gamma'$ and are thus freed. Of course, the points remaining on $\Gamma''$ should not be too numerous, and we have to find numbers $d_0, \gamma'$ and $\gamma''$ satisfying all the necessary inequalities.

A slight complication arises with the degree of the current divisor $\Gamma_c$. Indeed, the number of generated free points is computed in terms of the degree, say $\overline{\gamma}$, of the current divisor which must appear at the final stage of the procedure (this degree must be sufficiently large to comply with the induction hypothesis). On the other hand, the initial degree $\gamma$ of the current divisor must be significantly larger to allow the production of enough free points. This compels us to lower the degree of the current divisor, by specialization, at each stage of the procedure (see §5).

# 2 A differential lemma

All along the paper, $X$ stands for a quasi-projective variety which is geometrically reduced and irreducible, of dimension $n+1$ over a field $k$ of characteristic $p$. Since all statements are "generic" one can safely suppose $k$ algebraically closed. The hypothesis '$X$ is smooth in codimension one if $p > 0$' will not come into play until the proof of the theorem in §7.

Here we introduce the simplest technical lemma that suffices to prove the theorem. The proof of the lemma is postponed to §§8-10. This lemma, like most of the results that will be proven in the following sections, concerns unions of points and what we call *simple residues*. Before giving the definition we remind the reader of the meaning of the terms *trace* and *residual.*

**Definition. 2.1** *Let $H$ be a Cartier divisor on $X$ and let $W$ be a closed subscheme of $X$.*

*The schematic intersection*

$$W'' = H \cap W$$

*defined by the ideal $\mathfrak{I}_{H,W''} = (\mathfrak{I}_H + \mathfrak{I}_W)/\mathfrak{I}_H$ of $\mathcal{O}_H$ is called the trace of $W$ on $H$ and denoted by $Tr_H(W)$ or simply $W''$ if no confusion is possible.*

*The closed subscheme of $X$ defined by the conductor ideal $\mathfrak{I}_{W'} = (I_W : I_H)$ is called the residual of $W$ with respect to $H$ and denoted by $Res_H(W)$ or $W'$.*

*The canonical exact sequence*

$$0 \longrightarrow \mathfrak{I}_{W'}(-H) \longrightarrow \mathfrak{I}_W \longrightarrow \mathfrak{I}_{H,W''} \longrightarrow 0$$

*is called the residual exact sequence of $W$ with respect to $H$.*

**Definition. 2.2** *Let $H$ be a reduced Cartier divisor on $X$ and let $z$ be a non-singular point of $H$. We define the $m^{th}$ simple residue; denoted $D_{H,m}(z)$ or $D_m(z)$ if no confusion can arise; to be the trace of $z^m$ on $(m-1)H$;*

$$D_m(z) = z^m \cap H^{m-1}.$$

*We will say that $m$ is the* multiplicity *of the simple residue.*

Here is the announced lemma.

**Lemma. 2.3** *Suppose $X$ is projective and furnished with a line bundle $\mathfrak{L}$, and let $H$ be a reduced and irreducible effective Cartier divisor on $X$. Let $Z_0$ be a zero-dimensional subscheme of $X$, and let $a$, $d$ be positive integers. We suppose that*

$$r \;=\; h^0(H, \mathfrak{L}|_H) - deg(Tr_H(Z_0)) \geq 0$$

*and that $m_1, \ldots, m_r$ are positive integers satisfying*

$$deg(Z_0) + \sum_{i=1}^{r} \binom{m_i+n}{n+1} \;\geq\; h^0(X, \mathfrak{L}).$$

*Let $P_1, \ldots, P_r$ be generic points in $X$ and $Q_1, \ldots, Q_r$ be generic points in $H$. Suppose furthermore that*

$$(\star) \qquad H^1(X, \mathfrak{I}_{Q_1^{m_1}\cup\cdots\cup Q_r^{m_r}} \otimes \mathfrak{L}(-H)) \;=\; 0.$$

*In the notation of 2.1 and 2.2, set*

$$T = Z_0 \cup P_1^{m_1} \cup \cdots \cup P_r^{m_r} \quad ; \quad T'_\star = Z'_0 \cup D_{m_1}(Q_1) \cup \cdots \cup D_{m_r}(Q_r) \quad ; \quad T''_\star = Z''_0 \cup Q_1 \cup \cdots \cup Q_r.$$

*Then $H^0(X, \mathfrak{I}_T \otimes \mathfrak{L}) = 0$ if the following two conditions are satisfied :*

$$(dime) \qquad H^0(H, \mathfrak{I}_{T''_\star} \otimes \mathfrak{L}|_H) \;=\; 0$$

$$(degue) \qquad H^0(X, \mathfrak{I}_{T'_\star} \otimes \mathfrak{L}(-H)) \;=\; 0.$$

In proving our main theorem, we will be principally interested in the case where $\mathfrak{L} = \mathcal{O}(1)$ is ample and $H = G_a$ is the generic divisor in the linear system $|H^0(X, \mathcal{O}(a))|$ for suitably large $a$.

In this situation, the following lemma essentially says that the condition $(\star)$ of 2.3 is satisfied for $d \gg 0$. For $\boldsymbol{p} = (p_1, \ldots, p_r)$, we denote by $\max(\boldsymbol{p})$ the maximum among $p_1, \ldots, p_r$, by $\mathrm{lgth}(\boldsymbol{p})$ the number $r$ of components, and by $W_{\boldsymbol{p}}$ the union $Q_1^{p_1} \cup \cdots \cup Q_r^{p_r}$, where $Q_1, \ldots, Q_r$ are generic points in $G_a$.

**Lemma. 2.4** *Given integers $a, R, m$, there exists an integer $\mathbf{E}(a,R,m)$ such that for $d \geq \mathbf{E}(a,R,m)$ one has*

$$H^1(X, \mathfrak{I}_{W_{\boldsymbol{p}}}(d)) = 0$$

*for all $\boldsymbol{p}$ satisfying $lgth(\boldsymbol{p}) \leq R$ and $max(\boldsymbol{p}) \leq m$.*

**Proof.** For fixed $R$ and $m$, there are only finitely many choices for $W_{\boldsymbol{p}}$. In each case one has

$$H^1(X, \mathcal{I}_{W_{\boldsymbol{p}}}(d)) = 0$$

for $d \geq d(\boldsymbol{p})$ and it suffices to take the maximum. □

# 3 Configurations and candidates

Here we introduce the general class of subschemes of $X$ which we will be dealing with. From here on, $X$ is projective and furnished with an ample line bundle $\mathcal{O}(1)$ of degree $\nu$. We let $\alpha_0$ be the least integer such that $\mathcal{O}(a)$ is very ample for $a \geq \alpha_0$ and it will henceforth be understood that $a \geq \alpha_0$.

**Definition. 3.1** *Let $G_a$ be the generic effective divisor in the linear system $|H^0(X, \mathcal{O}(a))|$. A $G_a$-residue or just* **residue***, will be any point or simple residue (see 2.2) with support in $G_a$. The* **multiplicity** *of a residue will be its multiplicity as a point, or as a simple residue 2.2, respectively.*

*Given positive integers $a$ and $m$, an $(a, m)$-***configuration** *will be any subscheme $Z$ of $X$ which is the union of a generic set of points of multiplicity at most $m$ in $X$, called the free part of $Z$ and denoted Free($Z$), with a generic set of $G_a$-residues equally of multiplicity at most $m$, called the constrained part of $Z$ and denoted Const($Z$).*

*Given a positive integer $d$, we say that an $(a, m)$-configuration $Z$ is a $(d, m, a)$-***candidate** *if the following two conditions hold: $d$ is the greatest integer such that*

$$h^0(X, \mathcal{O}(d)) \leq deg(Z),$$

*and for this $d$ we have*

$$deg(Tr_{G_a}(Z)) \leq h^0(G_a, \mathcal{O}_{G_a}(d)).$$

*We consider a $(d, m, a)$-candidate $Z$ to be a candidate for the property $h^0(X, \mathcal{I}_Z(d)) = 0$ and we say that $Z$ is* **winning** *if this property holds.*

The following easy lemma says that for large $d$, candidates contain sufficiently many free points.

**Lemma. 3.2** *Let $m$ and $a$ be positive integers. For any $(d, m, a)$-candidate $Z$, we have*

$$deg(Free(Z)) \geq a\nu \tfrac{d^{n+1}}{(n+1)!} + O(d^n),$$

*where $\nu$ is the degree of $X$.*

For presentation purposes we introduce the

**Definition. 3.3** *Given a polarised pair $(V, \mathcal{O}(1))$ and $m > 0$ we define $\mathbf{d}(V, m)$ to be the least degree (a-priori possibly infinite, and a-posteriori finite by our theorem) such that for $d \geq \mathbf{d}(V, m)$ any $(d, m, 0)$-candidate is winning.*

# 4 Derivatives

In practice, when we apply lemma 2.3 we think of both conditions ($\star$) and (dime) as being satisfied. This is easily justified using 2.4 and an induction hypothesis on the dimension (i.e. precisely that $d(G_a, m)$ is finite) respectively. Lemma 2.3 is then a justification for replacing $T$ by $T'_\star$.

This leads us to introduce a formal operator $\mathfrak{D}$ sending one $(a, m)$-configuration to another which we call the derivative (see 4.1). Of course, we are especially interested in the case where this operator takes $(d, m, a)$-candidates to $(d - a, m, a)$-candidates. In the present section, we define the derivative and show that it behaves well for large $d$.

Here is the idea behind the definition of a derivative. Given a $(d, m, a)$-candidate $Z$, we wish to apply our lemma 2.3 as follows. We specialize the $s$ biggest free points of $Z$ on the divisor $G_a$, with $s$ as large as possible. Still a few conditions (say $r$) are missing in $G_a$, and we require that $r$ further free points are available in $Z$ so that we may apply 2.3. In that case, the derivative of $Z$ is the subscheme $T'_\star$ involved in the degue condition of 2.3.

**Definition. 4.1** *Let $Z$ be a $(d, m, a)$-candidate on $X$ with $t = t(Z)$ free points $P_1^{m_1}, \ldots, P_t^{m_t}$, where the multiplicities appear in non-decreasing order. Let $s = s(Z) \leq t$ be the greatest integer such that*

$$deg(Tr_{G_a}(Z)) + \binom{m_1 + n - 1}{n} + \cdots + \binom{m_s + n - 1}{n} \leq h^0(G_a, \mathcal{O}_{G_a}(d)),$$

*and set*

$$r = r(Z) = h^0(G_a, \mathcal{O}_{G_a}(d)) - deg(Tr_{G_a}(Z)) - \binom{m_1 + n - 1}{n} - \cdots - \binom{m_s + n - 1}{n}.$$

*We say that $Z$ is* **derivable** *with respect to $G_a$ if*

$$r \leq t - s.$$

*If $Z$ is a derivable $(d, m, a)$-candidate, its* **derivative** *with respect to $G_a$, denoted $\mathfrak{D}(Z)$, is defined to be the $(a, m)$-configuration*

$$\begin{aligned} \mathfrak{D}(Z) \;=\; & P_{s+r+1}^{m_{s+r+1}} \cup \cdots \cup P_t^{m_t} \cup Const(Z)' \cup \\ & Q_1^{m_1 - 1} \cup \cdots \cup Q_s^{m_s - 1} \cup \\ & D_{m_{s+1}}(Q_{s+1}) \cup \cdots \cup D_{m_{s+r}}(Q_{s+r}) \end{aligned}$$

*where $Q_1, \ldots, Q_{s+r}$ are generic points of $G_a$ and the notation is that of 2.1 and 2.2.*

Recall that $\alpha_0$ is the least integer such that $\mathcal{O}(a)$ is very ample. What we need to know on the derivative is the following :

**Lemma. 4.2** *Let $a \geq \alpha_0$ and $m$ be positive integers. Then there exists an integer* $\mathbf{der}(a, m)$ *such that for any $d \geq \mathbf{der}(a, m)$ and any $(d, m, a)$-candidate $Z$ on $X$:*

1. *$Z$ admits a derivative $\mathfrak{D}(Z)$;*

2. *for any $N$, if $Z$ has either no free point of multiplicity $m$ or at least $N$ free points of multiplicity less than $m$, then so does $\mathfrak{D}(Z)$;*

3. *The degree of the trace of $\mathfrak{D}(Z)$ satisfies the following estimate, where, as above, $\nu = deg(\mathcal{O}(1))$:*

$$deg\ Tr_{G_a}(\mathfrak{D}(Z)) = \left(\frac{(m-1)\,a\,\nu}{m+n-1}\right)\ \frac{d^n}{n!} + O(d^{n-1}) = h^0(G_a, \mathcal{O}_{G_a}(d-a)) - \left(\frac{n\,a\,\nu}{m+n-1}\right)\ \frac{d^n}{n!} + O(d^{n-1});$$

4. *$\mathfrak{D}(Z)$ is a $(d-a, m, a)$-candidate;*

5. *if $d(G_a, m)$ is finite and $\mathfrak{D}(Z)$ is winning, then so is $Z$.*

**Proof.**

For 1., it is enough to prove that the number of free points in $Z$ is larger than $2h^0(G_a, \mathcal{O}_{G_a}(d))$. The latter is bounded by $Cd^n$ for some constant $C$, so we may conclude by 3.2.

As for 2., it is an immediate consequence of the definition of the derivative.

For 3., let $r, s, t$ and $m_i$ be as in 4.1. Then

$$\frac{m+n-1}{n}\left(\sum_{i=1}^{s}\binom{m_i+n-2}{n-1}\right) \geq \sum_{i=1}^{s}\binom{m_i+n-1}{n} = h^0(G_a, \mathcal{O}_{G_a}(d)) - r$$

and

$$\begin{aligned}
\deg\ \mathrm{Tr}_{G_a}(\mathfrak{D}(Z)) &\leq \sum_{i=1}^{s}\binom{m_i+n-2}{n} + r\binom{m+n-1}{n} \\
&= \left(\sum_{i=1}^{s}\binom{m_i+n-1}{n} + r\right) - \sum_{i=1}^{s}\binom{m_i+n-2}{n-1} \\
&\quad -r + r\binom{m+n-1}{n} \\
&\leq h^0(G_a, \mathcal{O}_{G_a}(d)) - \frac{m-1}{m+n-1}\Big(h^0(G_a, \mathcal{O}_{G_a}(d)) - r\Big) \\
&\quad -r + r\binom{m+n-1}{n} \\
&\leq \frac{n}{m-1}\left(h^0(G_a, \mathcal{O}_{G_a}(d))\right) + \binom{m+n-1}{n}^2 \\
&= \left(\frac{(m-1)\,a\,\nu}{m+n-1}\right)\ \frac{d^n}{n!} + O(d^{n-1}).
\end{aligned}$$

Finally, we have

$$\begin{aligned}
h^0(G_a, \mathcal{O}_{G_a}(d-a)) - \deg\ \mathrm{Tr}_{G_a}(\mathfrak{D}(Z)) &\geq h^0(G_a, \mathcal{O}_{G_a}(d)) - \frac{m-1}{m+n-1}h^0(G_a, \mathcal{O}_{G_a}(d)) \\
&\quad - \binom{m+n-1}{n}^2 \\
&= \left(\frac{n\,a\,\nu}{m+n-1}\right)\ \frac{d^n}{n!} + O(d^{n-1}).
\end{aligned}$$

For 4., we first note that, when $\mathfrak{D}(Z)$ is defined and $H^1(X, \mathcal{O}(d-a)) = 0$, one has

$$h^0(X, \mathcal{O}(d-a)) \leq \deg(\mathfrak{D}(Z)) = \deg(Z) - h^0(G_a, \mathcal{O}_{G_a}(d)).$$

In view of the definition, this means that, for sufficiently large $d$, the $(a,m)$-configuration $\mathfrak{D}(Z)$ is a $(d-a, m, a)$-candidate, since by 3., its trace on $G_a$ has degree at most $h^0(G_a, \mathcal{O}_{G_a}(d-a))$.

For 5., using the notation of 4.1, we apply 2.3, with $Z_0$ the closed subscheme

$$\mathrm{Const}(Z) \cup Q_1^{m_1} \cup \cdots \cup Q_s^{m_s} \cup P_{s+r+1}^{m_{s+r+1}} \cup \cdots \cup P_t^{m_t}.$$

Let $W = Q_1^{m_{s+1}} \cup \cdots \cup Q_r^{m_{s+r}}$. We have

$$H^1(X, \mathcal{I}_W(d-a)) = 0$$

by 2.4, because $r = r(Z)$ is bounded by $\binom{m+n-1}{n}$. The dime of 2.3 holds for $d \geq d(G_a, m)$, while the degue of 2.3 is just the hypothesis that $\mathfrak{D}(Z)$ is winning, so the lemma follows from 2.3. □

# 5 Concentrated derivatives

If theorem 1.1 were known in small degrees, then repeated applications of lemma 2.3, hence of the derivative, would suffice to prove the theorem by induction on the degree. Instead one must modify the process and try to reduce the multiplicities of the free points, thus ending the proof by induction on the multiplicity. This is done using a specialisation of the second derivative (see 6.1): bearing in mind the semi-continuity of the cohomology, one easily sees that the (degue) of 2.3 holds if it holds for some specialisation of $T'_\star$. A complication arises with the degree of the base divisor $G_a$ which must be lowered during the induction on $d$ before an induction hypothesis on $m$ allows one to finish the proof. We get around this problem using a specialisation of the first derivative which we call a concentrated derivative. In this section we introduce this concentrated derivative and prove results analogous to those for derivatives.

**Definition. 5.1** *Let $d, m, a$ be positive integers with $a > 1$, and let $Z$ be a derivable $(d, m, a)$-candidate. We define the* **concentrated derivative** *of $Z$ with respect to $G_a$, denoted $\mathfrak{D}_c(Z)$, to be the $(a-1, m)$-configuration obtained from $\mathfrak{D}(Z)$ by degenerating $G_a$ to the generic union $G_1 + G_{a-1}$ and specialising all $G_a$-residues of $\mathfrak{D}(Z)$ to have generic support in $G_{a-1}$.*

What we need to know on the concentrated derivative is concentrated in the following:

**Lemma. 5.2** *Given $m > 0$ there exists an integer $A(m)$ such that for all $a \geq A(m)$ there exists an integer* **derc**$(a, m)$ *such that for any $d \geq$* **derc**$(a, m)$ *and any $(d, m, a)$-candidate $Z$:*

1. *$Z$ admits a concentrated derivative $\mathfrak{D}_c(Z)$ which is a $(d-a, m, a-1)$-candidate;*

2. *for any $N$, if $Z$ has either no free point of multiplicity $m$ or at least $N$ free points of multiplicity less than $m$, then so does $\mathfrak{D}_c(Z)$;*

3. *if $d(G_a, m)$ is finite and $\mathfrak{D}_c(Z)$ is winning, then so is $Z$.*

**Proof.**

For 1., let $A(m)$ be an integer $a$ satisfying

$$\tfrac{m-1}{m+n-1}A(m) < A(m) - 1.$$

Then, for $a \geq A(m)$ and $d$ sufficiently large, for any $(d, m, a)$-candidate $Z$, we have, by (4.2.3),

$$\deg \mathrm{Tr}_{G_{a-1}} (\mathfrak{D}_c(Z)) = \deg \mathrm{Tr}_{G_a} (\mathfrak{D}(Z)) \leq h^0(G_{a-1}, \mathcal{O}_{G_{a-1}}(d-a))$$

so that $\mathfrak{D}_c(Z)$ is a $(d-a, m, a-1)$-candidate.

As for 2., it follows from the similar statement for the derivative, since the derivative and the concentrated derivative have the same free points.

For 3., if $\mathfrak{D}_c(Z)$ is winning then so is $\mathfrak{D}(Z)$, since the former is a specialisation of the latter. As such $Z$ is winning for $d \geq \mathbf{der}(a, m)$ by (4.2.5). □

# 6 Special second derivative

In this section, we explain the construction which generates free points. This corresponds to a modified second derivative, which we denote by $\mathfrak{D}^2[\alpha]$, where $\alpha$ is an integer. As one may guess, we say that $Z$ is twice derivable if $Z$ and $\mathfrak{D}(Z)$ are derivable, and we write $\mathfrak{D}^2$ for $\mathfrak{D}\mathfrak{D}$.

**Definition. 6.1** *Let $m, a > 0$, and let $Z$ be a twice derivable $(d, m, a)$-candidate. Let $r^{(2)}(Z)$ be the number of residues of $\mathfrak{D}^2(Z)$ which are points, necessarily of multiplicity at most $m-1$. For $0 < \alpha < a$, we set*

$$r^{(2)}[\alpha](Z) = min\big(h^0(X, \mathcal{O}(\alpha)) - 1, r^{(2)}(Z)\big)$$

*and define $\mathfrak{D}^{(2)}[\alpha](Z)$ to be the specialisation of the second derivative $\mathfrak{D}^{(2)}(Z)$ obtained by degenerating $G_a$ and its residues to the generic union $G_\alpha + G_{a-\alpha}$ with $r^{(2)}[\alpha](Z)$ of the residues which are points specialised to have generic support in $G_\alpha$, and all other residues specialised to $G_{a-\alpha}$.*

Here is what we need to know about this construction.

**Lemma. 6.2** *Given $m, N$, let $\alpha = \alpha(N)$ be the least integer such that $h^0(X, \mathcal{O}(\alpha)) > N$. Then there exists $a_0 > \alpha$ such that for all $a \geq a_0$ there exists $d'_0 = d'_0(m, N, a)$ such that for $d \geq d'_0$ and any $(d, m, a)$-candidate $Z$:*

1. *$Z$ is twice derivable and $\mathfrak{D}^2[\alpha](Z)$ is a $(d-2a, m, a-\alpha)$-candidate having either no free point of multiplicity $m$ or at least $N$ free points of multiplicity at most $m-1$;*

2. *if $d(G_a, m)$ and $d(G_{a-\alpha}, m)$ are finite and $\mathfrak{D}^2[\alpha](Z)$ is winning, then so is $Z$.*

**Proof.** For 1., let $a_0 > \alpha$ be such that for $a - \alpha > \frac{m-1}{m+n-1} a$ for $a \geq a_0$. Then for $a \geq a_0$, and $d \gg 0$, we have

$$\deg \mathrm{Tr}_{G_a}\left(\mathfrak{D}^2(Z)\right) < h^0(G_{a-\alpha}, \mathcal{O}_{G_{a-\alpha}}(d-2a))$$

since by (4.2.3) we have

$$\deg \mathrm{Tr}_{G_a}\left(\mathfrak{D}^{(2)}(Z)\right) \leq \frac{m-1}{m+n-1} a\nu \frac{d^n}{n!} + O(d^{n-1})$$

and

$$h^0(G_{a-\alpha}, \mathcal{O}_{G_{a-\alpha}}(d-2a)) \geq (a-\alpha)\nu \frac{d^n}{n!} + O(d^{n-1}).$$

This implies that $\mathfrak{D}^{(2)}[\alpha](Z)$ is a $(d-2a, m, a-\alpha)$-candidate.

Now for $d \gg 0$, by (4.2.3), we have

$$h^0(G_a, \mathcal{O}_{G_a}(d-a)) - \deg \mathrm{Tr}_{G_a}\left(\mathfrak{D}(Z)\right) \geq N \binom{m+n-1}{n}$$

so that, in the notation of 4.1, $s(\mathfrak{D}(Z)) \geq N$. We first suppose that $\mathfrak{D}(Z)$ has at least $N$ free points of multiplicity $m$. Then $\mathfrak{D}^{(2)}[\alpha](Z)$ has $N$ free points of multiplicity $m-1$: indeed, in that case, $r^{(2)}[\alpha](Z)$ is equal to $N$, and the $N$ points specialised to $G_\alpha$ are without constraint, since any set of $N$ points lie on an effective divisor in the linear system $|H^0(X, \mathcal{O}(\alpha))|$. Now we turn to the case where $\mathfrak{D}(Z)$ has less than $N$ free points of multiplicity $m$. In that case, $\mathfrak{D}^2[\alpha](Z)$ has no more free points of multiplicity $m$.

For 2., we observe that the second derivative $\mathfrak{D}^2(Z)$ is also a winning candidate, and conclude by applying twice (4.2.5). □

# 7 Proof of the theorem

## 7.1 A proposition implying the theorem

The following proposition (which we prove below 7.2) sums up the efforts of the previous sections and, as we will now show, easily implies our theorem.

**Proposition. 7.1** *Let $X$ be a projective, geometrically reduced and irreducible variety of dimension $n+1$ over a field $k$ of arbitrary characteristic $p$. Suppose further that $X$ is smooth in codimension one if $p > 0$. Let $\mathcal{O}(1)$ be an invertible ample bundle on $X$. Given $m > 0$, there exists $a_0(m)$ such that for any $a \geq a_0(m)$ there exists $d_0(a, m)$ such that for all $d \geq d_0(a, m)$ any $(d, m, a)$-candidate is winning.*

**Proof of 1.1.**

We first handle the case where $\mathcal{M} = \mathcal{O}$.

We take $d_0(m) = d_0(a_0(m), m)$ and consider some $d \geq d_0$ and some generic union $Z$ of (fat) points of multiplicity $\leq m$. If the degree of $Z$ is smaller than $h^0(\mathcal{O}(d))$, we reduce to the case with equality by adding generic simple points. If the degree of $Z$ is not smaller than $h^0(\mathcal{O}(d+1))$, we reduce to the case where it is by increasing $d$. Summing up, we may suppose that $Z$ is a $(d, m, a)$-candidate, and conclude by 7.1.

Now, as announced, we will only gloss over the proof in the case where $\mathcal{M}$ is arbitrary. Letting $b$ be such that $\mathcal{M}(b)$ is very ample and writing $\mathcal{M}(d)$ as $\mathcal{M}(b)(d')$, one easily concludes that we can suppose $\mathcal{M}$ very ample. Let $\alpha_0'$ be the least integer such that $\mathcal{O}(a)$ is very ample for $a \geq \alpha_0$, so that $\mathcal{M}(a)$ is very ample too for $a \geq a_0$.

It suffices to show, for pairs $(W, d)$ where $W$ is a union of points of multiplicity at most $m$ and $d$ is the largest integer such that $\deg W \geq h^0(X, \mathcal{M}(d))$, that $H^0(X, \mathcal{I}_W \otimes \mathcal{M}(d)) = 0$ for $d \gg 0$.

In the first place, for any given $a \geq \alpha_0$, it is clear that for $d \gg 0$, $W$ can be decomposed into three disjoint subschemes $W_f$, $W_c$ and $P_1^{m_1} \cup \cdots \cup P_r^{m_r}$ such that, after specialising $W_c$ to have generic support in $G_a'$ and calling this $W_c''$, we have

$$\deg\ \mathrm{Tr}_{G_a'}(W_c'') + r = h^0(G_a', \mathcal{M}(d)|_{G_a'})$$

Now let $Q_1, \ldots, Q_r$ be generic simple points in $G_a'$. We clearly have the analogue of 2.4, so by proposition 2.3 it suffices, in view of the theorem applied to the dime, to show that $W_1 = W_f \cup \mathrm{res}_{G_a'}(W_c'') \cup D^{m-1}(Q_1) \cup \cdots \cup D^{m-1}(Q_r)$ satisfies $h^0(X, \mathcal{I}_{W_1} \otimes \mathcal{O}(d)) = 0$. An obvious analogue of (4.2.3) then shows that for a suitably large $a$ we have $\deg\ \mathrm{Tr}_{G_a'}(W_1) \leq h^0(G_a, \mathcal{O}_{G_a}(d-a))$ so that by degenerating $G_a'$ to $G_a + M$, where $M$ (resp. $G_a$) is the generic effective divisor in $|H^0(X, \mathcal{M})|$ (resp. $|H^0(X, \mathcal{O}(a))|$) , and specialising all residues of $W_1$ on the divisor $G_a'$ to have generic support in $G_a$, we obtain, as a specialisation of $W_1$, a $(d, m, a)$-candidate. One now applies the special case of the theorem. □

## 7.2 Proof of the proposition by induction

**Proof of 7.1.** To prove the proposition, we argue by induction on the dimension $n+1$. Note that in all characteristics, the generic effective divisor in a very ample linear system on a variety $X$ of dimension $> 1$, is a variety which is smooth outside the singular locus of $X$ (see [L] VII 13). Thanks to the initial cases 7.3 and 7.4 below, we may suppose that the proposition has been proven for multiplicity $m$ in dimension $n$ and for multiplicity $m-1$ in dimension $n+1$. This implies that $\mathbf{d}(G_a, m)$ is finite for all $a \geq 1$ and that there exists $a_0(m-1)$ such that for $a \geq a_0(m-1)$ there exists $d_0(a, m-1)$ such that for $d \geq d_0(a, m-1)$ any $(d, m-1, a)$-candidate is winning. We proceed in three steps.

**First step.** With the notation of 4.2 and 5.2, we define

$$b_0 = \max(A(m), a_0(m-1)),$$

$$\Delta = \max(\mathbf{der}(b_0, m), d_0(b_0, m-1) + b_0)$$

and

$$N = h^0(X, \mathcal{O}(\Delta + b_0)),$$

and prove by induction that, for any $d \geq \Delta$, any $(d, m, b_0)$-candidate with either no free point of multiplicity $m$ or at least $N$ free points of multiplicity less than $m$ is winning.

For $\Delta \leq d < \Delta + b_0$, we have to consider a $(d, m, b_0)$-candidate $Z$ with no free point of multiplicity $m$, since there is no room for $N$ free points of multiplicity less than $m$. Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, $Z$ has a first derivative $\mathfrak{D}(Z)$ which is a $(d-b_0, m-1, b_0)$-candidate. Thanks to $d - b_0 \geq d_0(b_0, m-1)$, this candidate is winning. Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2 again, $Z$ is winning too.

For $d \geq \Delta + b_0$ let $Z$ be a $(d, m, b_0)$-candidate having either no free point of multiplicity $m$, or at least $N$ free points of multiplicity at most $m - 1$. Thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, $Z$ has a first derivative $\mathfrak{D}(Z)$ which is a $(d - b_0, m, b_0)$-candidate having either no free point of multiplicity $m$ or at least $N$ free points of multiplicity at most $m - 1$. Thanks to the inductive assumption, $\mathfrak{D}(Z)$ is winning. Again thanks to $d \geq \mathbf{der}(b_0, m)$ and 4.2, $Z$ is winning too.

**Second step.** Here we prove that that for any $b \geq b_0$ there exists $\delta(b) = \delta(b, m)$ such that for $d \geq \delta$ any $(d, m, b)$-candidate having either no free point of multiplicity $m$, or at least $N$ free points of multiplicity at most $m - 1$ is winning.

The proof is by induction on $b$. The initial case $b = b_0$ is the previous step. For the induction step, we take $\delta(b) = max(\mathbf{derc}(b, m), \delta(b-1) + b)$. The statement then follows by 5.2, which applies because $b_0 \geq A(m)$.

**Final step.** Here we set $a_0 = a_0(m) = max(b_0 + \alpha(N), a_0(m, N))$ where $\alpha = \alpha(N)$ and $a_0(m, N))$ are given in 6.2, and, for $a \geq a_0$, $d_0 = d_0(a, m) = max(d_0'(m, N, a), \delta(a - \alpha, m) + 2a)$, and we prove the full statement, namely that, for $d \geq d_0(a, m)$, any $(d, m, a)$-candidate $Z$ is winning.

Indeed, by 6.2 applied to $n, N$, $Z$ is twice derivable and $\mathfrak{D}^2[\alpha](Z)$ is a $(d-2a, m, a-\alpha)$-candidate having either no free point of multiplicity $m$ or at least $N$ free points of multiplicity at most $m - 1$. Since $d - 2a \geq \delta(a - \alpha, m)$, this candidate is winning by the second step. This implies that $Z$ itself is winning by 6.2. □

**Remark. 7.2** *A further generalisation would be to take a fixed closed (zero-dimensional) subscheme $V_0$ and its unions with points of multiplicity $\leq m$. The union of $V_0$ with sufficiently many $k$ rational points has maximal rank in all degrees giving the analogous initial case for an induction on the multiplicity while the proof of the dimension one case is virtually unchanged.*

## 7.3 The proposition in dimension one

The initial case $n = 0$ can be deduced from the following general results for curves. We first treat the characteristic zero case with the

**Proposition. 7.3** *Let $C$ be a geometrically irreducible quasi-projective curve over a field $k$ of characteristic zero. Let $V \subset H^0(C, \mathcal{L})$ be a linear subspace of finite dimension $v$ of global sections of the invertible sheaf $\mathcal{L}$ on $C$. Let $x_1, \ldots, x_r$ be the generic set of $r$ closed points of $C$ defined over the function field $K$ of $C \times \cdots \times C$ ($r$ factors), and let $m_1, \ldots, m_r$ be positive integers. Let $D$ be the divisor $m_1x_1 + \cdots + m_rx_r$ on $C_K = C \times_k K$. Then the canonical map*

$$V \otimes K \longrightarrow H^0(C_K, \mathcal{O}_D \otimes \mathcal{L})$$

*has maximal rank.*

**Proof.** If $v \neq m = \sum_i m_i$ , one can either diminish the multiplicities or add (generic) free points and suppose that $v = m$. Since the property is open, we can specialise to the case of a single point $x$ and the divisor $D = mx$. In this case the proposition is equivalent to showing that the determinant of the canonical map

$$V \otimes \mathcal{O}_C \longrightarrow \mathrm{P}^v(\mathcal{L}) \tag{1}$$

is not identically zero, where $\mathrm{P}^v(\mathcal{L})$ is the sheaf of $v^{th}$ -order principal parts of $\mathcal{L}$. For this we can suppose that the base field is algebraically closed and, since this map commutes with localisation and the completion at a closed point of $C$, it is sufficient to show that the canonical map

$$V \otimes k[[t]] \longrightarrow k[[t, x]]/((x-t)^v)$$

$$f \mapsto f(t) + f'(t)(x-t) + f''(t)\frac{(x-t)^2}{2!} + \cdots + f^{(v-1)}(t)\frac{(x-t)^{v-1}}{(v-1)!}$$

has maximal rank. Choosing a basis $f_1, \ldots, f_v$ for $V$, the determinant of this map is just the Wronskian

$$W(f_1, \cdots, f_v) = \det \left[ \frac{\partial^i f_j}{\partial t^i} \right]$$

which, as is well known, has maximal rank for $f_1, \ldots, f_v$ linearly independent. □

We now give the initial case for smooth curves in arbitrary characteristic. This will be a corollary of the

**Proposition. 7.4** *Let $C$ be a smooth geometrically connected projective curve of genus $g$ over a field $k$ of arbitrary characteristic and let $\mathcal{L}$ be a non special invertible sheaf on $C$ of degree $d \geq g$. Let $n_1, \ldots, n_g$ be positive integers whose sum is $d$ and let $C^g$ be the product of $C$ with itself $g$ times, with projections $p_i : C^g \longrightarrow C$ for $i = 1, \ldots, g$. then the morphism*

$$\Theta : C^g \longrightarrow Pic^0(C) \quad ; \quad (x_1, \ldots, x_g) \mapsto \mathcal{L}(-n_1x_1 - \cdots - n_gx_g)$$

*is surjective. In particular, for $x_1, \ldots, x_g$ generic points on $C$, the divisor $n_1x_1 + \cdots + n_gx_g$ is linearly equivalent to the generic divisor of degree $d$.*

**Proof.** This is obvious if $g = 0$ so suppose henceforth that $g \geq 1$. Let $p, q$ be the first and second projections of $C^g \times C$ to $C^g$ and $C$ respectively. Let

$$x_i : C^g \longrightarrow C^g \times C$$

be the section induced by the pull-back by the $i^{th}$ projection $p_i : C^g \longrightarrow C$ of the diagonal section $\Delta$ of $C \times C$ over $C$. We will also write $x_i$ for the image $x_i(C^g)$ in $C^g \times C$. For each $i$, the relative Cartier divisor $n_i x_i$ on $C^g \times C$ over $C^g$ is then the pull-back via $p_i \times 1$ of $n_i\Delta$.

The morphism $\Theta$ is induced by the bundle $\mathcal{M} = q^\star(\mathcal{L}) \otimes \mathcal{O}_{C^g\times C}(-n_1x_1 - \cdots - n_gx_g)$ which is the kernel of the bundle morphism

$$(2) \quad q^\star\mathcal{L} \longrightarrow q^\star\mathcal{L} \otimes \mathcal{O}_{n_1x_1+\ldots+n_gx_g} = q^\star(\mathcal{L}) \otimes \Big((p_1 \times 1)^\star\mathcal{O}_{n_1\Delta} \times \cdots \times (p_g \times 1)^\star\mathcal{O}_{n_g\Delta}\Big).$$

What we must show is that for some $t \in C^g$, the sheaf $\mathcal{M}_t$ on $C_t$ has a non-zero section. Taking direct images of the morphism (2) gives the bundle map on $C^g$

$$\theta : \mathcal{O}_{C^g}^{d+1-g} \;=\; H^0(C,\mathcal{L}) \otimes \mathcal{O}_{C^g} \longrightarrow p_1^\star\Big(\mathrm{P}^{n_1-1}(\mathcal{L})\Big) \oplus \cdots \oplus p_g^\star\Big(\mathrm{P}^{n_g-1}(\mathcal{L})\Big) = \mathcal{E}$$

and we must show that the degeneracy locus $D_\theta$ of $\theta$ is non-empty.

Every irreducible component of $D_\theta$ has codimension at most $g$ and if every component has codimension at least $g$ then the class of $D_\theta$ in the Chow ring $A^g(C^g)$ of $C^g$ is the $g^{th}$ Chern class of the bundle $E$. Calculating the Chern polynomial

$$\begin{aligned} c_t(\mathcal{E}) &= c_t((p_1^\star(\mathrm{P}^{n_1-1}(\mathcal{L}))) \cdots c_t((p_g^\star(\mathrm{P}^{n_g-1}(\mathcal{L}))) \\ &= (1 + p_1^\star(c_1(\mathrm{P}^{n_1-1}(\mathcal{L})))t) \cdots (1 + p_g^\star(c_1(\mathrm{P}^{n_g-1}(\mathcal{L})))t) \end{aligned}$$

we see that

$$c_g(\mathcal{E}) = p_1^\star(c_1(\mathcal{P}^{n_1-1}(\mathcal{L}))) \cdots p_g^\star(c_1(\mathcal{P}^{n_g-1}(\mathcal{L}))).$$

Now one easily sees that

$$c_1(P^{n-1}(\mathcal{L})) = nD + \binom{n}{2}K = n(D + (n-1)K/2)$$

where $K$ is a canonical divisor on $C$ and $D$ is a divisor associated to $\mathcal{L}$. This is a very ample divisor on $C$. Now if $x_1, \ldots, x_g$ are distinct points on $C$ one has

$$\Delta_i = p_i^\star(x_i) = C \times \cdots \times C \times x_i \times C \times \cdots \times C$$

so that $\Delta_1 . \cdots . \Delta_g = [x_1 \times \cdots \times x_g]$. Applying the same arguement to smooth effective divisors $D_i \in |n_i(D + (n_i - 1)K/2)|$ for $i = 1, \ldots, g$ we conclude that

$$\deg(c_g(\mathcal{E})) = \deg(n_1(D + (n_1-1)K/2)) \cdots \deg(n_g(D + (n_g-1)K/2)) \;>\; 0$$

as required. □

**Corollary. 7.5** *Let $C$ be a smooth, geometrically connected, projective curve of genus $g$ over an arbitrary field. Let $\mathcal{M}$, $\mathcal{L}$ be line bundles on $C$ with $\mathcal{L}$ ample, let $m > 0$ be an integer and let $d_0(m)$ be the least integer $d$ such that $h^0(C, \mathcal{M} \otimes \mathcal{L}^d) \geq mg$ and $\mathcal{M} \otimes \mathcal{L}^d$ is non-special. Let $x_1, \ldots, x_r$ be generic points on $C$ and let $Z$ be the divisor $m_1x_1 + \cdots + m_rx_r$ where $0 < m_i \leq m$ for $i = 1, \ldots, r$. Then the canonical map*

$$H^0(C, \mathcal{M} \otimes \mathcal{L}^d) \longrightarrow H^0(C, \mathcal{O}_Z \otimes \mathcal{M} \otimes \mathcal{L}^d)$$

*has maximal rank for $d \geq d_0(m)$.*

**Proof.** Adding points if necessary, we can suppose $\deg(Z) \geq h^0(C, \mathcal{M} \otimes \mathcal{L}^d)$. We then have $r \geq g$. (This is the only place where we use the bound on the multiplicities.) Now replacing the hypothesis on the multiplicities by the hypothesis that $r \geq g$, we can specialise to the case where $r = g$. Finally, by the proposition, $\mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)$ is the generic sheaf in it's component of the Picard scheme of $C$, so that $h^0(C, \mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)) = 0$ or $h^1(C, \mathcal{L}^d \otimes \mathcal{M} \otimes \mathcal{O}(-Z)) = 0$. □

This completes the proof of the cases in dimension 1. We end with the following example showing that the 'smooth in codimension one' hypothesis cannot be dropped in characteristic $p > 0$.

**Remark. 7.6** *Let $C$ be the plane curve defined by the equation $y^2 - x^p$, where $p$ is an odd prime over an algebraically closed field of characteristic $p$. The tangent line at $z = (t^2, t^p)$, $t \neq 0$, is $(y - t^p)$ which has a contact of order $p$ with $C$ at $z$. It follows that for any $d$ and any choice $z_1, \ldots, z_d$ of points on the smooth locus of $C$, the divisor $Z = pz_1 + \cdots + pz_d$ is an effective divisor associated to $\mathcal{O}_C(d)$, so that the restriction map*

$$H^0(C, \mathcal{O}_C(d)) \longrightarrow H^0(C, \mathcal{O}_Z(d))$$

*is not of maximal rank.*

# 8 The formal lemma

We would like to point out that the original motivation and proof of the following results owed much to the work [LL1,2].

## 8.1 Preliminaries

Consider the algebra of formal functions $k[[\boldsymbol{x}, y]]$ where $\boldsymbol{x} = (x_1, \ldots, x_{n-1})$ which we furnish with an ideal $I$ of the form

$$I = I_0 \oplus I_1 y \oplus \cdots \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

where $I_\alpha \subset k[[\boldsymbol{x}]]$ is an ideal. We call such ideals *vertically graded ideals.* Note that

$$I_0 \subset I_1 \subset \cdots \subset I_{m-1}. \tag{3}$$

For any positive integer $r$, the ideal

$$I_t = I_0[[t]] \oplus I_1[[t]](y - t^r) \oplus \cdots \oplus I_{m-1}[[t]](y - t^r)^{(m-1)} \oplus ((y - t^r)^m)$$

in the algebra $k[[t, \boldsymbol{x}, y]]$ is called a *standard deformation* of the vertically graded ideal $I$. For $i \geq m$ we let $I_i = k[[\boldsymbol{x}]]$.

Given $F_0(\boldsymbol{x}, y) \in I$ and a standard deformation $F_0 + F_1 t^r + \cdots \in I_t$, the functions $F_i(\boldsymbol{x}, y)$ must satisfy certain residual conditions. If $r = 1$ and $I = (\boldsymbol{x}, y)^m$, the residual condition is just that $F_i(\boldsymbol{x}, y)$ must vanish to the order $m - i$ and can be compared with [Xu]. This is the sense of the following

**Proposition. 8.1** *Let $F_t = \sum_{\alpha\geq 0} F_\alpha(\boldsymbol{x}, y)t^\alpha$ be a function in $I_t$ and let $F_\alpha = \sum_{\beta\geq 0} F_{\alpha,\beta}(\boldsymbol{x})y^\beta$. Then*

$$F_{\alpha,\beta}(\boldsymbol{x}) \quad \in \quad I_{\beta+[\![\frac{\alpha}{r}]\!]}$$

*where $[\![\frac{\alpha}{r}]\!]$ is the greatest integer $\leq \frac{\alpha}{r}$. If $y$ divides $F_\alpha$ for $\alpha = 0, r, 2r, \ldots, pr$ then $F_0(\boldsymbol{x}, y)$ is in the ideal*

$$I_0 y \oplus I_1 y \oplus \cdots \oplus I_{p-1}y^p \oplus I_{p+1}y^{p+1} \oplus \cdots \oplus I_{m-1}y^{m-1} \oplus ((y^m))$$

**Proof.** Write $F_t$ in the following form

$$F_t = a_0(\boldsymbol{x}, t) + a_1(\boldsymbol{x}, t)(y - t^r) + \cdots + a_{m-1}(\boldsymbol{x}, t)(y - t^r)^{m-1} + a_m(\boldsymbol{x}, t)(y - t)^m + \cdots$$

with

$$a_i(\boldsymbol{x}, t) = \sum_{j\geq 0} a_{ij}(\boldsymbol{x})t^j$$

hence $a_{ij}(\boldsymbol{x}) \in I_i$. Developing out we find

$$\begin{array}{rcl} F_{\alpha,\beta} & = & \sum_{\nu=0}^{[\![\frac{\alpha}{r}]\!]} (-1)^\nu \binom{\beta+\nu}{\beta} a_{\beta+\nu,\, \alpha-\nu r}(\boldsymbol{x}) \\ & \in & I_{\beta+[\![\frac{\alpha}{r}]\!]} \end{array} .$$

This proves the first part.

Now suppose that $y$ divides $F_\alpha$ for $\alpha = 0, r, 2r, \ldots, pr$. Then for $\lambda = 0, 1, \ldots, p$ we have

$$0 = F_{\lambda r, 0} = a_{0,\lambda r} - a_{1,(\lambda-1)r} + \cdots + (-1)^{\lambda-1}a_{\lambda-1,r} + (-1)^p a_{\lambda,0}$$

so that $a_{0,0} = 0$ and $a_{\lambda,0} \in I_{\lambda-1}$ for $\lambda = 1, \ldots, p$ as one sees using $a_{\mu,\nu} \in I_\mu$ and (3). This gives the last part of the proposition. □

## 8.2 Statement and proof of the formal lemma

Throughout this subsection we will fix the following notation.

For $i = 1, \ldots, \ell$, let $B^{(i)} = k[[\boldsymbol{x}_i, y_i]]$ be an algebra of formal functions in $n$ variables where $\boldsymbol{x}_i = (x_{i,1}, \ldots, x_{i,n-1})$ and let

$$I^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{m_i-1}^{(i)} y_i^{m_i-1} \oplus (y_i^{m_i})$$

be a vertically graded ideal in $B^{(i)}$. Let

$$I = I^{(1)} \times \cdots \times I^{(\ell)} \subset B^{(1)} \times \cdots \times B^{(\ell)} = B$$

Let $k[[\boldsymbol{t}]] = k[[t_1, \ldots, t_\ell]]$ and let $I_{\boldsymbol{t}}$ in $B[[\boldsymbol{t}]]$ be the product of the ideals

$$I_{\boldsymbol{t}}^{(i)} = I_0^{(i)}[[\boldsymbol{t}]] \oplus I_1^{(i)}[[\boldsymbol{t}]](y_i - t_i) \oplus \cdots \oplus I_{m_i-1}^{(i)}[[\boldsymbol{t}]](y_i - t_i)^{m_i-1} \oplus ((y_i - t_i)^{m_i})$$

Let $y = (y_1, \dots, y_\ell)$ and for any linear subspace $V \subset B$, let $V_{\mathrm{res}(y)} = \{v \in B \mid vy \in V\}$. Since $y$ is a non-zero divisor, we get a residual exact sequence

$$0 \longrightarrow V_{\mathrm{res}(y)} \xrightarrow{\ y\ } V \longrightarrow V/V \cap (y) \longrightarrow 0 \tag{4}$$

The geometric setting can be understood by taking the two variables case $x, y$. If one thinks of the ideals $I^{(i)}$ as being the ideals of zero dimensional subschemes on the line $y = 0$ in the affine plane, the standard deformations simply translate these schemes vertically but not colinearly. Let $Z_0$ be the subscheme and $Z_{\boldsymbol{t}}$ be the family of translations. In the following proposition we have a space of functions $V \subset k[x, y]$ of dimension at most the total degree of the closed subscheme $Z_0$, and we want to show that none of these functions vanish on the union after the scheme is translated. To this end we impose conditions at the initial stage when all lie on $y = 0$, then, using the residual conditions appearing in the previous proposition, we obtain supplementary conditions which must be satisfied by any function vanishing on $Z_0$. The hypotheses then say more or less that there are no functions in $V$ satisfying the supplementary conditions.

The effect is as follows. We start with $Z_0$ whose trace on $y = 0$ is too big (greater than the dimension of $V/V \cap (y)$) and obtain as a derived condition something that is divided equitably between trace and residual. Instead of taking $I_0^{(i)}$ as the trace ideal, one can (1.) take $I_{p_i}^{(i)}$ for suitable $p_i$, provided the residual condition (2.) is satisfied.

In the proof we must reduce to a one variable deformation. In order to obtain supplementary conditions up to different orders $p_i$ at the points $z_i$ of the support of $Z_0$ using the previous proposition, the supplementary conditions must interact. For this to happen, we need to delay the departure of the points $z_i$ where $p_i$ is small and we do this by setting $t_i = t^{r_i}$ for suitable $r_i$.

**Proposition. 8.2** *Let $V \subset B$ be a $k$ linear subspace. Suppose that for $i = 1, \dots, \ell$ there exists integers $p_i$; $0 \leq p_i < m_i - 1$; such that the following two conditions are satisfied*

1. *the canonical map*

$$V/V \cap (y) \longrightarrow k[[\boldsymbol{x}_1]]/I^{(1)} \times \cdots \times k[[\boldsymbol{x}_\ell]]/I^{(\ell)}$$

   *is injective*

2. *The canonical map*

$$V_{res(y)} \longrightarrow B/J$$

   *is injective where $J = J^{(1)} \times \cdots \times J^{(\ell)}$ and*

$$J^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{p_i-1}^{(i)} y_i^{p_i-1} \oplus I_{p_i+1}^{(i)} y_i^{p_i} \oplus \cdots \oplus I_{m_i-1}^{(i)} y_i^{m_i-2} \oplus (y_i^{m_i-1})$$

*Then the canonical map*

$$\varphi_{\boldsymbol{t}} : V \otimes k[[\boldsymbol{t}]] \longrightarrow B_{\boldsymbol{t}}/I_{\boldsymbol{t}}$$

*is (generically) injective.*

**Proof.** Let
$$h = gcm(p_1 + 1, \dots, p_\ell + 1) = r_i(p_i + 1)$$
be the greatest common multiple of the $p_i + 1$ and consider the one parameter deformation obtained by setting $t_i = t^{r_i}$. Since the rank of $\varphi_{\boldsymbol{t}}$ is semi-continuous, we need only show that the canonical map
$$\varphi_t : V \otimes k[[t]] \longrightarrow B[[t]]/I_t$$
obtained by the formal base change $k[[t_1, \dots, t_\ell]] \longrightarrow k[[t]]$ sending $t_i$ to $t^{r_i}$ is injective.

Let
$$F_t = (F_t^{(1)}, \dots, F_t^{(\ell)}) \in \ker \varphi_t \;=\; V_t \cap I_t$$
be non zero, where $V_t$ is the image of $V \otimes k[[t]]$ and $I_t$ is the image of $I_{\boldsymbol{t}}$ in $B[[t]]$.

Dividing out any factor of $t$, we can suppose that $F_0 \neq 0$. We will argue by reduction to the absurd and show that $F_0 = 0$.

Since $F_t^{(i)} = \sum_{\alpha \geq 0} F_\alpha^{(i)}(\boldsymbol{x}_i, y_i)t^\alpha \in I_t^{(i)}$, where $I_t^{(i)}$ is the image of $I_{\boldsymbol{t}}^{(i)}$ in $B^{(i)}[[t]]$, it follows from the first part of proposition 8.1 that
$$(F_\alpha^{(1)}(\boldsymbol{x}_1, 0), \dots, F_\alpha^{(\ell)}(\boldsymbol{x}_\ell, 0)) \in I_{p_1} \times \dots \times I_{p_\ell}$$
for $\alpha = 0, 1, \dots, h$, then, by hypothesis 1. of the proposition, we conclude that $y$ divides $(F_\alpha^{(1)}(\boldsymbol{x}_1, y_1), \dots, F_\alpha^{(\ell)}(\boldsymbol{x}_\ell, y_\ell))$ for $\alpha = 0, 1, \dots, h$.

Now applying the second part of proposition 8.1 we conclude that
$$F_0^{(i)}(\boldsymbol{x}_i, y_i) = y_i G_0^{(i)}(\boldsymbol{x}_i, y_i)$$
where $G_0^{(i)}(\boldsymbol{x}_i, y_i) \in J_i$. Letting $G_0 = (G_0^{(1)}, \dots, G_0^{(r)})$ we see that $G_0 \in V_{\mathrm{res}(y)} \cap J$, but the second hypothesis of the proposition simply says that $V_{\mathrm{res}(y)} \cap J = 0$. This shows that $F_0 = 0$ as required. □

# 9 Translations and deformations

The essential notion of this section is that of the standard deformation of a zero-dimensional closed subscheme. This is given with the broader notion of coordinate translations which better illustrates the intuitional motivation based on affine translations. The definitions are all straightforward and well-known. They are only set forth here to justify the local to global arguement.

## 9.1 Coordinate translations

We fix an $n$-dimensional quasi-projective variety $X$ furnished with an effective Cartier divisor $H$ which is smooth in a neighbourhood of a $k$-rational point $z \in H$ and we fix a regular system of parameters $\boldsymbol{x} = x_1, \dots, x_n \in \mathfrak{m}_z \subset \mathcal{O}_{X,z}$, such that $x_n$ is a local equation for $H$.

A *coordinate chart* at $z$ will be an open affine neighbourhood $U$ of $z$ furnished with a lifting $\boldsymbol{x}' = x'_1, \dots, x'_n \in \mathcal{O}_X(U)$ of the regular system of parameters $\boldsymbol{x}$ such that the differentials $dx'_1, \dots dx'_n$ form a basis for $\Omega_{U/k}$. Since $z$ is a smooth point of $X$, this is always possible for sufficiently small $U$.

Given a coordinate chart $(U, \boldsymbol{x}')$, we have a smooth curve $T \subset U$, transverse to $H$ at $z$, given by the equations $x'_1 = \cdots = x'_{n-1} = 0$ and a regular parameter $t \in \mathfrak{m}_{T,z}$ induced by the parameter $x_n$. This gives a sequence

$$\operatorname{Spec} k[[t]] \longrightarrow T \longrightarrow U$$

and a diagram

$$\begin{array}{ccccc} \operatorname{Spec} k[[t]] \times_k X & \longrightarrow & T \times_k X & \longrightarrow & U \times_k X \\ p_1 \big\downarrow \big\uparrow \delta & & p_1 \big\downarrow \big\uparrow \delta & & p_1 \big\downarrow \big\uparrow \delta \\ \operatorname{Spec} k[[t]] & \longrightarrow & T & \longrightarrow & U \end{array}$$

where $\delta$ is the section induced by $\Delta_U : U \longrightarrow U \times_k U$.

Completing along this section in each case gives a sequence of algebras (of normal invariants)

$$\mathcal{O}_U[[S_1, \dots, S_n]] \longrightarrow \mathcal{O}_T[[S_1, \dots, S_n]] \longrightarrow k[[t, x_1, \dots, x_n]]$$

where $S_i$ is induced by $1 \otimes x'_i - x'_i \otimes 1 \in \mathcal{O}_U \otimes_k \mathcal{O}_U$ and the second map sends $S_i$ to $x_i$ for $i = 1, \dots, n-1$ and $S_n$ to $x_n - t$.

If $Z \subset X$ is a zero dimensional closed subscheme then the canonical map

$$\theta : \hat{\mathcal{O}}_{X,z} = k[[x_1, \dots, x_n]] \longrightarrow \mathcal{O}_U[[S_1, \dots, S_n]] \quad ; \quad x_i \mapsto S_i$$

allows one to define a closed subscheme

$$\begin{array}{ccccc} Z & \hookrightarrow & X \times_k U & \longrightarrow & X \\ & & \big\downarrow p_1 & & \\ & & U & & \end{array}$$

of $X \times_k U$, flat over $U$, corresponding to the ideal $\hat{\mathcal{I}}_{\mathbf{Z}} \subset \mathcal{O}_U[[S_1, \dots, S_n]]$ obtained as the completed extension of the ideal $\hat{\mathcal{I}}_{Z,z} \subset \mathcal{O}_{Z,z} = k[[x_1, \dots, x_n]]$ via $\theta$.

This deformation of $Z$ is called the *coordinate translation* (associated to the coordinate chart) and its inverse image over $T$, obtained via the analogous extension in $\mathcal{O}_T[[S_1, \dots, S_n]]$, is called the *transverse coordinate translation.* Restricting to $\operatorname{Spec} k[[t]]$, we get at formal scheme $Z_t$ given by the standard deformation of the ideal $\hat{\mathcal{I}}_{Z,z} \subset \hat{\mathcal{O}}_{X,z} = k[[x_1, \dots, x_n]]$ which we call the *standard deformation* of $Z$.

## 9.2 Deforming the interpolation map

Now with the same setup as in the previous section, let $V \subset H^0(X, \mathcal{L})$ be a finite dimensional subspace of global sections of the invertible sheaf $\mathcal{L}$ on $X$. Let

$$\varphi : V \subset H^0(X, \mathcal{L}) \longrightarrow \mathcal{L} \otimes \mathcal{O}_Z$$

be the canonical map.

Given a coordinate chart $(U, \boldsymbol{x}')$ at $z$, we get a deformation of $\varphi$

$$\boldsymbol{\varphi} : E \otimes U \longrightarrow p_{1\star}\left(p_2^\star \mathcal{L} \otimes \mathcal{O}_{\mathbf{Z}}\right) = \mathcal{O}_U[[S_1, \ldots, S_n]]/\hat{I}_{\mathbf{Z}} \otimes_{\mathcal{O}_U} \mathcal{L}$$

We call $\boldsymbol{\varphi}$ the coordinate deformation of $\varphi$. By restriction, we get a corresponding deformation over $T$ called the transverse coordinate deformation of $\varphi$.

Over Spec $k[[t]]$ this becomes the map

$$\varphi_t : V \otimes k[[t]] \longrightarrow k[[t, \boldsymbol{x}]] \widehat{\otimes}_{k[[\boldsymbol{x}]]} \hat{\mathcal{L}}_z$$

which we call the standard deformation of $\varphi$.

**Definition. 9.1** *Let $Y \subset X$ be a closed subscheme of the quasi-projective variety $X$ over $k$ and let $\boldsymbol{Y} \subset X \times_k S$ ba a closed subscheme flat over the irreducible $k$-scheme $S$ with a fiber $Y_0 = Y$. Let $E \subset H^0(X, \mathcal{L})$ be a finite dimensional subspace of the global sections of the invertible sheaf $\mathcal{L}$ on $X$. We say that the deformation $\boldsymbol{Y}$ attains the maximal rank with respect to $E$ if the linear map*

$$E \otimes k(\xi) \longrightarrow H^0(X_\xi, \boldsymbol{Y}_\xi \otimes \mathcal{L})$$

*has maximal rank where $\xi$ is the generic point of $T$.*

¿From this discussion we get the

**Proposition. 9.2** *If the standard deformation of $\varphi$ attains the maximal rank then so do the coordinate and transverse coordinate deformations.*

We finish this discussion by noting that the coordinate deformation is a deformation of the embedding $Z \hookrightarrow X$ since the subscheme corresponding to a $k$ rational point of $U$ is isomorphic to $Z$ by a simple change of coordinates.

## 9.3 The case of several points

We now rapidly extend all these definitions to the case of several points.

**Definition. 9.3** *Let $H$ be a reduced and irreducible Cartier divisor on the quasi-projective variety $X$ over $k$ and let $Z_1, \dots, Z_\ell$ be zero-dimensional closed subschemes of $X$, supported at the $k$-rational points $z_1, \dots, z_\ell$ in the smooth locus of $H$ (hence of $X$). Let $x_{i,1}, \dots, x_{i,n-1}, y_i$ be a regular system of parameters at $z_i$ ($i = 1, \dots, \ell$) such that $y_i$ is a local equation for $H$. Let $\mathfrak{U} = (U; (\boldsymbol{x}_i, y_i))$, be a coordinate chart around $z_i$ with smooth transverse curve $T_i$.*

*We define the coordinate translation (resp. the transverse coordinate translation, resp. the standard deformation) of $Z = Z_1 \cup \dots \cup Z_\ell$ to be the union of the pull backs of the corresponding deformation of each $Z_i$ to $U_1 \times_k \dots \times_k U_\ell$ (resp. $T_1 \times_k \dots \times_k T_\ell$ , resp. $k[[\boldsymbol{t}]] = k[[t_1, \dots, t_\ell]]$).*

**Remark. 9.4** *With the same terminology as in 9.3, let $E \subset H^0(X, \mathfrak{L})$ be a finite dimensional linear subspace of global sections of the invertible sheaf $\mathfrak{L}$ on $X$. Clearly, the coordinate and transverse coordinate deformations attain the maximal rank with respect to $E$ if the standard deformation does. The latter then concerns the canonical map of formal modules*

$$E \otimes k[[\boldsymbol{t}]] \longrightarrow \prod_i k[[\boldsymbol{t}]][[\boldsymbol{x}_i, y_i]]/I_t^{(i)} \tag{5}$$

*where $I_t^{(i)}$ is the canonical formal extension $I_{Z_i,z_i}[[t]]$ of the ideal $I_{Z_i,z_i} \subset k[[\boldsymbol{x}_i, y_i]]$ of $Z_i$ via the map*

$$k[[\boldsymbol{x}_i, y_i]] \longrightarrow k[[\boldsymbol{t}]][[\boldsymbol{x}_i, y_i]]$$

*sending $\boldsymbol{x}_i$ to itself and $y_i$ to $y_i - t_i$.*

We now introduce the class of subschemes for which our proofs work.

**Definition. 9.5** *Let $H$ be a reduced and irreducible Cartier divisor on the quasi-projective variety $X$ and let $z$ be a smooth point of $H$. A vertically graded subscheme of $X$ with base $H$ and support $z$, will be a zero-dimensional closed subscheme of $X$ with support $z$, furnished with a regular system of parameters $x_1, \dots, x_{n-1}, y \in \mathfrak{m}_z$ such that $y$ is a local equation for $H$ and the ideal of $Z$ in $k[[\boldsymbol{x}, y]] = \widehat{\mathcal{O}}_z$ is of the form*

$$\widehat{I}_Z = I_0 \oplus I_1 y \oplus \dots \oplus I_{m-1} y^{m-1} \oplus (y^m)$$

*where $I_i \subset k[[\boldsymbol{x}]]$.*

*In the same spirit, we say that such an ideal is vertically graded.*

# 10 The fundamental propositions

Throughout this section we fix the following : let $H$ be a reduced and irreducible Cartier divisor on the quasi-projective variety $X$ over the field $k$. Let $\mathfrak{L}$ be a line bundle on $X$ and let $V \subset H^0(X, \mathfrak{L})$ be a finite dimensional $k$-linear subspace.

Let $Z = Z_1 \cup \dots \cup Z_\ell \subset X$ be the disjoint union of the vertically graded subschemes $Z_i$ with base $H$ and support $z_i$.

For each $i \in \{1, \ldots, \ell\}$ let $p_i \geq 0$ be an integer. Let $Z'_{i,p_i} \subset H$ and $Z''_{i,p_i} \subset X$ be the closed subschemes defined respectively by the ideals

$$\begin{aligned} \mathcal{I}_{Z'_{i,p_i}} &= \mathcal{I}_{Z_i} + (\mathcal{I}_{Z_i} : \mathcal{I}_H^{(p_i+1)})\mathcal{I}_H^{p_i} \\ \mathcal{I}_{Z''_{i,p_i}} &= (\mathcal{I}_{Z_i} : \mathcal{I}_H^{p_i})/(\mathcal{I}_{Z_i} : \mathcal{I}_H^{(p_i+1)})\mathcal{I}_H \end{aligned}$$

and let

$$Z'_\star = \cup_{i=1}^{\ell} Z'_{i,p_i} \quad , \quad Z''_\star = \cup_{i=1}^{\ell} Z''_{i,p_i}.$$

**Proposition. 10.1** *Suppose that for some choice of the* $p_i$

1. **Dime** *the canonical map* $V'' = V_H \longrightarrow H^0(X, \mathcal{O}_{Z''_\star} \otimes \mathcal{L})$ *is injective*
2. **Degue** *the canonical map* $V' = V_{res} \longrightarrow H^0(X, \mathcal{O}_{Z'_\star} \otimes \mathcal{L}(-H))$ *is an injective,*

*then the standard deformation of* $Z$ *attains the maximal rank with respect to* $V$*.*

**Proof.** Firstly, let

$$I^{(i)} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{m_i-1}^{(i)} y_i^{m_i-1} \oplus (y_i^{m_i})$$

be the vertically graded ideal in $k[[\boldsymbol{x}_i, y_i]] \simeq \hat{\mathcal{O}}_{Z_i, z_i}$ defining $Z_i$. Then $Z''_{i,p_i}$ is given by the ideal $I_{i,p_i}$ of $k[[\boldsymbol{x}_i]]$ and $Z'_{i,p_i}$ is given by the ideal

$$I_{Z_{i,p_i}} = I_0^{(i)} \oplus I_1^{(i)} y_i \oplus \cdots \oplus I_{p_i-1}^{(i)} y_i^{p_i-1} \oplus I_{p_i+1}^{(i)} y_i^{p_i} \oplus \cdots \oplus I_{m_{i-1}}^{(i)} y_i^{m_i-2} \oplus (y_i^{m_i-1})$$

of $k[[\boldsymbol{x}_i y_i]]$.

We have a commutatif diagram with exact rows

(6)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \prod_{i=1}^{\ell} \widehat{\mathcal{L}}_{z_i} & \xrightarrow{x_n} & \prod_{i=1}^{\ell} \widehat{\mathcal{L}}_{z_i} & \longrightarrow & \prod_{i=1}^{\ell} \widehat{\mathcal{L}}_{Hz_i} & \longrightarrow & 0 \end{array}$$

where the completions are at $z_i$.

Choosing local trivialisations of $\mathcal{L}$ at each $z_i$ we can consider (6) as the residual exact sequence (see (4)) of a subspace $V \subset \prod_i k[[\boldsymbol{x}_i, y_i]]$. The proposition then follows from the formal case 8.2. □

**Corollary. 10.2** *Suppose that* $X$ *is projective, that* $|H|$ *is base point free and* $H^1(X, \mathcal{I}_Z \otimes \mathcal{L}(-H)) = 0$*. Let* $W \subset X$ *be a zero dimensional closed subscheme of* $X$ *not meeting the* $Z_i$*. We denote by* $W''$ *and* $W'$ *the trace and residual of* $W$ *with respect to* $H$*. Suppose that*

1. **Dime** $\qquad H^0(X, \mathcal{J}_{W'' \cup Z''_\star} \otimes \mathcal{L}) = 0$

2. **Degue** $\qquad H^0(X, \mathcal{J}_{W' \cup Z'_\star} \otimes \mathcal{L}(-H)) = 0$

*then* $H^0(X, \mathcal{J}_{W \cup Z} \otimes \mathcal{L}) = 0$.

**Proof.** Since $H^1(X, \mathcal{J}_Z \otimes \mathcal{L}(-H)) = 0$ and $Z$ is zero dimensional, we have $H^1(X, \mathcal{L}(-H)) = 0$. Noting that $Z'_\star \subset Z$ is a closed subscheme of $Z$, we then conclude that the following maps

$$H^0(X, \mathcal{L}(-H)) \twoheadrightarrow H^0(X, \mathcal{O}_Z \otimes \mathcal{L}(-H)) \twoheadrightarrow H^0(X, \mathcal{O}_{Z'_\star} \otimes \mathcal{L}(-H))$$

$$H^0(X, \mathcal{L}) \twoheadrightarrow H^0(H, \mathcal{L}|_H)$$

are all surjective. Since $H^0(X\mathcal{J}_{W'} \otimes \mathcal{L}(-H)) \longrightarrow H^0(X, \mathcal{O}_{Z'} \otimes \mathcal{L}(-H))$ is injective by the degue, it follows that we can choose a complement $V' \subset H^0(X, \mathcal{L}(-H))$ of $H^0(X, \mathcal{J}_{Z'} \otimes \mathcal{L}(-H))$ containing $H^0(X, \mathcal{J}_{W'} \otimes \mathcal{L}(-H))$, in particular the canonical map $V' \hookrightarrow H^0(X, \mathcal{J}_{Z'} \otimes \mathcal{L}(-H))$ is an isomorphism.

Now since $|H|$ is base point free, we can choose a section $s \in H^0(X, \mathcal{O}(H))$ which is invertible in a neighbourhood of $Z$. This gives a commutive square

$$\begin{array}{ccc} H^0(X, \mathcal{L}(-H)) & \twoheadrightarrow & H^0(X, \mathcal{O}_Z \otimes \mathcal{L}(-H)) \\ s \big\downarrow & & \simeq \big\downarrow s \\ H^0(X, \mathcal{L}) & \longrightarrow & H^0(X, \mathcal{O}_Z \otimes \mathcal{L}) \end{array}$$

where the left side vertical arrow is injective and the right side one is an isomorphism. Since $Z''_\star \subset Z$ is a closed subscheme, we conclude that the following maps

$$H^0(H, \mathcal{L}|_H) \twoheadrightarrow H^0(X, \mathcal{O}_Z \otimes \mathcal{L}|_H) \twoheadrightarrow H^0(X, \mathcal{O}_{Z''_\star, H} \otimes \mathcal{L}|_H)$$

are surjective. The map $H^0(H, \mathcal{J}_{W'', H} \otimes \mathcal{L}|H) \longrightarrow H^0(X, \mathcal{O}_{Z''} \otimes \mathcal{L}|_H)$ is injective by the dime so we can choose a complementary subspace $V'' \subset H^0(H, \mathcal{L}|_H)$ of $H^0(H, \mathcal{J}_{Z''} \otimes \mathcal{L}|_H)$, containing $H^0(H, \mathcal{J}_{W'', H} \otimes \mathcal{L}|H)$ and mapping isomorphically to $H^0(X, \mathcal{O}_{Z''} \otimes \mathcal{L}|_H)$.

Finally, since the sequence

$$0 \longrightarrow H^0(X, \mathcal{L}(-H)) \longrightarrow H^0(X, \mathcal{L}(-H)) \longrightarrow H^0(H, \mathcal{L}|_H) \longrightarrow 0$$

is exact, there is a well determined subspace $V \subset H^0(X, \mathcal{L})$ containing $H^0(X, \mathcal{J}_W \otimes \mathcal{L})$ and giving rise to a residual exact sequence

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0.$$

The dime and degue of the proposition 10.1 having been established, we conclude that $V$ maps isomorphically to $H^0(X, \mathcal{O}_Z \otimes \mathcal{L})$ and that $H^0(X, \mathcal{J}_W \otimes \mathcal{L})$ injects into $H^0(X, \mathcal{O}_Z \otimes \mathcal{L})$. This shows that $H^0(X, I_{W \cup Z} \otimes \mathcal{L}) = 0$. $\square$

The application we have in mind for the preceding statement is when the $Z_i$ are points (of various multiplicities). In this case, for any regular system of parameters $\boldsymbol{x} = x_1, \ldots, x_n$ at $z_i$ with $x_n$ a local equation for $H$, $Z_i$ is vertically graded and its ideal can be written in the form

$$\mathfrak{n}^m \oplus \mathfrak{n}^{m-1} x_n \oplus \cdots \oplus \mathfrak{n} x_n^{m-1} \oplus (x_n^m)$$

where $\mathfrak{n} = (x_1, \ldots, x_{n-1})$ and $m$ is the multiplicity of $Z_i$.

As a special case of corollary 12.2 we then have the

**Corollary. 10.3** *Let $X$ be a projective variety, let $|H|$ be a base point free linear system with $H$ a reduced and irreducible Cartier divisor on $X$. Let $\mathfrak{L}$ be an invertible sheaf on $X$ and let $W$ be a zero-dimensional closed subscheme of $X$. Let $P_1, \ldots, P_r$ be generic points of $X$ and $Q_1, \ldots, Q_r$ generic points in $H$. For a sequence of positive integers $m_1, \ldots, m_r$ let $Z = Q_1^{m_1} \cup \cdots \cup Q_r^{m_r}$.*

*We suppose that $H^0(X, \mathfrak{I}_Z \otimes \mathfrak{L}(-H)) = 0$. Then $H^0(\mathfrak{I}_{W \cup P_1^{m_1} \cup \cdots \cup P_r^{m_r}} \otimes \mathfrak{L}) = 0$ if the following two conditions are satisfied*

1. **Dime** $\quad H^0(X, \mathfrak{I}_{W'' \cup Q_1 \cup \cdots \cup Q_r} \otimes \mathfrak{L}|_H) = 0$
2. **Degue** $\quad H^0(X, \mathfrak{I}_{W' \cup D^{m_1}(Q_1) \cup \cdots \cup D^{m_r}(Q_r)} \otimes \mathfrak{L}(-H)) = 0$

**Proof.** This is just corollary 12.2 with $p_i = m_i - 1$. □

**Corollary. 10.4** *The corollary 10.2 implies the lemma 2.3.*

**Proof.** It suffices to take $H = G_a$ in the previous corollary. □

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