On the homotopy type of the subgroup complex

Ryousuke Fujita

General Education, Wakayama National College of Technology

(Transformation Group Symposium, Wakayama, Nov. 24, 2007)

The aim of this note is to introduce the latest results of research on the homotopy equivalence of a subgroup complex. Let P be a poset(= partially ordered set). The *order complex* of P is denoted by the symbol $\Delta(P)$; this is the simplicial complex whose k-dimensional simplices are the non-empty chains $x_0 < x_1 < \cdots < x_k$ of P. For a finite group G and a prime number p dividing its order, the *Brown complex* (respectively, *Quillen complex*) of G at p is defined as the order complex $\Delta(\mathcal{S}_p(G))$(resp. $\Delta(\mathcal{A}_p(G))$), where

$$\mathcal{S}_p(G) = \{\text{non-trivial } p\text{-subgroup of } G\},$$
$$(\text{resp. } \mathcal{A}_p(G) = \{\text{non-trivial elementary abelian } p\text{-subgroup of } G\},)$$

ordered by set inclusion. Hereafter, we often write P for $\Delta(P)$. For instance, $\mathcal{S}_p(G)$ means $\Delta(\mathcal{S}_p(G))$.

• In 1978, D. Quillen [12] proved that $\mathcal{A}_p(G)$ is homotopy equivalence to $\mathcal{S}_p(G)$. Moreover he stated that if G is a soluble group, $\mathcal{A}_p(G)$ is contractible complex if and only if G has a non-trivial normal p-subgroup.

• In 1991, J.Thévenaz and P.J.Webb [18] proved that $\mathcal{A}_p(G)$ is G-homotopy equivalence to $\mathcal{S}_p(G)$.

• In 1998, P. Symonds [16] proved that the orbit space of $\mathcal{S}_p(G)$ is contractible.

Let G be a finite group. As usual, we denote $\mathcal{S}(G)$ as the set of all subgroups of G. Then it is a poset by set inclusion. In particular, we consider the proper part $\mathcal{S}(G)^\circ = \mathcal{S}(G)\backslash\{1, G\}$.

• In 1985, J.Kratzer and J.Thévenaz [7] determined the homotopy type of $\mathcal{S}(G)^\circ$.

• In 1995, V.Welker [20] determined the G-homotopy type of $\mathcal{S}(G)^\circ$ and the homotopy type of of $\mathcal{S}(G)^\circ/G$ for a finite solvable group G.

For two simplicial complexes Δ_1, Δ_2, we define the *join* (respectively, *wedge*), $\Delta_1 * \Delta_2$ (resp. $\Delta_1 \vee \Delta_2$), as in [14]; in particular, $\Delta * \emptyset = \emptyset * \Delta = \Delta$. For a poset P and an element $r \in P$, we denote by $P_{\geq r}$ the poset $\{q \in P \mid q \geq r\}$. Similarly defined are the posets $P_>,\ P_\leq,\ P_<$.

We shall return to the subgroup complex $\mathcal{A}_p(G)$. If N is a normal p'-subgroup of G, we use the" bar "notation to denote to the quotient subgroups, namely we write $\overline{H}$ for the image HN/N of a subgroup H of G under the natural homomorphism $\pi : G \to G/N$.

Theorem 1.1. ([11] J. Pulkus, V.Welker 2000) Let G be a finite group and N a soluble normal p'-subgroup of G. Then $\Delta(\mathcal{A}_p(G))$ is homotopy equivalent to the wedge

$$\Delta(\mathcal{A}_p(\overline{G})) \vee \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \left(\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}})\right),$$

where for each $\overline{A} \in \mathcal{A}_p(\overline{G})$ an arbitrary chosen point $c_{\overline{A}} \in \Delta(\mathcal{A}_p(NA))$ is identified with $\overline{A} \in \Delta(\mathcal{A}_p(\overline{G}))$.

Lemma 1.2. Let G be a finite solvable group. For any $A \in \mathcal{A}_p(G)$, then the suspension $\mathbf{S}\left(\mathcal{A}_p(G)_{>A}\right)$ is homotopy equivalent to a wedge of spheres.

Theorem 1.3. ([5] F.Francesco 2005) Let G be a finite solvable group and p an odd prime number dividing the order of G. The Quillen complex of G at p is homotopy equivalent to a wedge of spheres.

Outline of proof of Theorem 1.3
In the case of $O_p(G) \neq 1$, the Quillen complex is contractible [12, 1.5]. Thus we can assume $O_p(G) = 1$. Denote with N the subgroup $O_{p'}(G)$ and call $\overline{G}$ the factor group G/N. By Theorem 1.1,

$$\Delta(\mathcal{A}_p(G)) \simeq \bigvee_{\overline{A} \in \mathcal{A}_p(\overline{G})} \left(\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}})\right),$$

where we used the fact that $\Delta(\mathcal{A}_p(\overline{G}))$ is contractible because $\overline{G}$ has a non-trivial normal p-subgroup. $\Delta(\mathcal{A}_p(NA)$ is a non-empty wedge of some spheres of dimension $\mathrm{rk}\,(A) - 1 \geq 0$ [12, 11.2]. It follows that

$$\begin{aligned}\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}}) &\simeq \left(\bigvee \left(S^{\mathrm{rk}\,(A)-2} * S^0\right)\right) * \Delta\left(\mathcal{A}_p(\overline{G})_{>\overline{A}}\right) \\ &\simeq \bigvee S^{\mathrm{rk}\,(A)-2} * \mathbf{S}\left(\mathcal{A}_p(\overline{G})_{>\overline{A}}\right),\end{aligned}$$

where, all the space on the right-hand side are wedged to a unique common point. By Lemma 1.2, the space $\Delta(\mathcal{A}_p(NA)) * \Delta(\mathcal{A}_p(\overline{G})_{>\overline{A}})$ is homotopy equivalent to a wedge of spheres.

Question 1.4. Let G be a non-solvable group. What homotopy type does $\mathcal{A}_p(G)$ have ?

If G is the symmetric group S_n, the homotopy type of $\mathcal{A}_p(S_n)$(or $\mathcal{S}_p(S_n)$) is not known.

• In 2003, R.Ksontini [8] investigated the homotopy type of the Quillen complex of the symmetric group. More precisely, he showed that $\mathcal{A}_p(S_n)$ is simply connected if p is odd and either $3p+2 \leq n \leq p^2$ or $n \geq p^2+p$.

• In 2004, R.Ksontini [9] determined that the homotopy type of the Quillen complex $\mathcal{A}_p(S_{3p})$ at the prime p.

It is well known that $\mathcal{S}\,(A_5)^0$ is homotopy equivalent to a wedge of 60 spheres S^1 [20].

Example 1.5. $\mathcal{A}_p(A_5)$ is not contractible, where $p = 2, 3, 5$.

References

[1] Aschbacher M and Smith S.D., *On Quillen's conjecture for the p-groups complex*, Ann. Math. **137** (1993), 473-529.

[2] Björner A., *Homotopy type of posets and lattice complementation*, J. Combinatorial Theory, Series A. **30**(1981), 90–100.

[3] Björner A and Walker J.W., *A homotopy complementation formula for partially ordered sets*, Europ. J. Combinatorics. **4**(1983),11-19.

[4] Bux K.U., *Orbit spaces of subgroup complexes, Morse theory, and a new proof of a conjecture of Webb*, Topology Proc. **24**(1999), 39–51.

[5] F. Francesco., *On the homotopy type of the Quillen complex of finite soluble groups*, J. Algebra. **283**(2005), 639–654.

[6] Hawkes T, Isaacs I.M. and Özaydin M., *On the Möbius function of a finite group*, Rocky Moutain J. Math. **19**(1989), 1003–1034.

[7] Kratzer C and Thévenaz J., *Type d'homotopie des treillis et treillis des sous-groupes d'un groupe fini*, Comment. Math. Helv. **60** (1985), 85–106.

[8] Ksontini R., *Simple connectivity of the Quillen complex of the symmetric group*, J. Combinatorial Theory, Series A. **103**(2003), 257–279.

[9] Ksontini R., *Hypergraph matching complexes and Quillen complexes of symmetric groups*, J. Combinatorial Theory, Series A. **106**(2004), 299–314.

[10] Lucido M.S., *On the partially ordered set of nilpotent subgroups of a finite group*, Comm. Algebra. **23(5)** (1995), 1825–1836.

[11] Pulkus J and Welker V., *On the homotopy type of the p-subgroup complex for finite solvable groups*, J. Austral. Math. Soc, Series A. **69**(2000), 212-228.

[12] Quillen D., *Homotopy properties of the poset of nontrivial p-subgroups of a group*, Advances in Math. **28**(1978), 101–128.

[13] Rival I., *A fixed point theorem for finite partially ordered sets*, J. Combinatorial Theory, Series A. **21**(1976), 309–318.

[14] Spanier E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[15] Stanley R.P., *Some aspects of groups acting on finite posets*, J. Combinatorial Theory, Series A. **32**(1982), 132–161.

[16] Symonds P., *The orbit space of the p-subgrpup complex is contractible*, Comment. Math. Helv. **73**(1998), 400–405.

[17] Thévenaz J., *Permutation Representations Arising from Simplicial Complexes*, J. Combinatorial Theory, Series A. **46**(1987), 121-155.

[18] Thévenaz J and Webb P.J., *Homotopy Equivalence of Posets with a Group Action*, J. Combinatorial Theory, Series A. **56**(1991), 173–181.

[19] Walker J.W., *Homotopy type and Euler characteristic of partially ordered set*, Europ. J. Combinatorics. **2**(1981), 373–384.

[20] Webb P.J., *Subgroup complexes*, Proc. Symp. in Pure Math. **47**(1987), the Arcata Conf. in Representation of Finite Groups, Vol I, 349–365.

[21] Welker V., *The Poset of Conjugacy Classes of Subgroups in a Finite Solvable Group*, J. Algebra. **148**(1992), 203-218.