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Non-classical reductions of initial-value problems for a class of nonlinear evolution equations

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Abstract. We classify initial-value problems for a class of one-dimensional evolution equations, which can be reduced to Cauchy problems for some systems of first-order ordinary differential equations. The technique applied relies heavily on higher conditional symmetries of the equations under study, which means, in particular, that the obtained reductions cannot be derived within the framework of the standard Lie approach.

1. Introduction

The principal object of the study in the present paper is the problem of reduction of initial-value problems for nonlinear partial differential equations (PDEs) of the form

$$u_t = g(u)u_{xx} + f(u, u_x) \tag{1}$$

for the real-valued function u of two variables t, x. Provided $g(u) \neq$ constant, equation (1) can always be reduced to the form

$$u_t = uu_{xx} + F(u, u_x) \tag{2}$$

by the change of the dependent variable $u(t, x) = U(v(t, x))$ with an appropriately chosen function U. That is why, we will concentrate in the following on the case of the nonlinear PDE (2).

The underlying idea of our approach to the analysis of reducibility of the initial-value problems for equation (2) is to exploit their higher conditional symmetries as suggested in our recent paper [1]. The higher conditional symmetry is the generalization of the Lie–Bäcklund (higher Lie) symmetry (see, e.g., [2, 3]), on the one hand, and of the non-classical (conditional) symmetry, on the other hand.

The motivation for introducing the concept of higher conditional symmetry was a necessity to provide a symmetry setting for the 'nonlinear separation of variables', which is due to Galaktionov [4, 5] and the 'antireduction' [6, 7]. We have proved [8] that it is higher conditional symmetries that enable reductions of the corresponding evolution PDEs to systems of several ordinary differential equations (ODEs) (see, also, [9]). Note that using Lie and non-classical symmetries we can reduce a given PDE to one ODE. This is the reason why the phenomenon of nonlinear separation of variables cannot be completely understood within the framework of the theory of first-order symmetries.

Furthermore, we have proved in [1] that existence of non-trivial higher conditional symmetry is a necessary and sufficient condition for reducibility of a given evolution PDE to a system of ODEs. This statement is the generalization of the known relation between reduction of PDEs to ODEs and their non-classical symmetries (see, e.g., [10–16]).

An additional motivation for search for higher conditional symmetries of PDEs belonging to the class (2) is the fact that a particular equation of the form (2), namely, the porous medium equation,

$$u_t = uu_{xx} + u_x^2$$

does not admit non-classical symmetries [17]. So it would be natural to classify PDEs of a more general form that admit first- or higher-order conditional symmetries.

2. Some basic facts from the theory of higher Lie symmetries

When we talk about an (infinitesimal) Lie–Bäcklund transformation group, we mean the canonical representation of this group

$$\begin{aligned} u' &= u + \varepsilon\,\eta(t, x, u, u_1, \ldots, u_N) \\ u_1' &= u_1 + \varepsilon\, D_x\,\eta(t, x, u, u_1, \ldots, u_N), \ldots \\ u_k' &= u_k + \varepsilon\, D_x^k\,\eta(t, x, u, u_1, \ldots, u_N), \ldots . \end{aligned} \tag{3}$$

This group corresponds to the so-called Lie–Bäcklund vector field

$$Q = \sum_{k=0}^{\infty} \left(D_x^k\,\eta\right) \frac{\partial}{\partial u_k} \equiv \eta \frac{\partial}{\partial u} + (D_x\,\eta) \frac{\partial}{\partial u_1} + \left(D_x^2\,\eta\right) \frac{\partial}{\partial u_2} + \cdots . \tag{4}$$

In formulae (3) and (4) D_x is the total differentiation operator

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}$$

and $u_k = \frac{\partial^k u}{\partial x^k}$, $k = 0, 1, 2, \ldots$. The use of the canonical representation of the Lie–Bäcklund group is more convenient for computations, since the prolongation formulae simplify drastically. They are obtained by successive application of the total derivative operator D_x to η, as is readily seen from the formula (4).

The coefficient η entering into (3) and (4) may also depend on the derivatives of the function $u(t, x)$ with respect to the temporal variable t. However, we intend to consider the group (3) as a symmetry of the PDE (2) and, in view of this fact, we can express all the derivatives of u with respect to t in terms of $t, x, u, u_1, u_2, \ldots$ on the solution set of PDE (2). This yields (3) as the most general form of the infinitesimal Lie–Bäcklund group admitted by equation (2).

It is common knowledge that the Lie–Bäcklund infinitesimal transformation group is the generalization of the Lie infinitesimal transformation group. If the function η has the structure

$$\eta = \tilde{\eta}(t, x, u) - \xi_0(t, x, u)u_t - \xi_1(t, x, u)u_1 \tag{5}$$

then the Lie–Bäcklund vector field (4) is equivalent to the Lie vector field (for further details see [2, 3]):

$$Q = \xi_0(t, x, u)\frac{\partial}{\partial t} + \xi_1(t, x, u)\frac{\partial}{\partial x} + \tilde{\eta}(t, x, u)\frac{\partial}{\partial u}.$$

Note that relation (5) is the necessary and sufficient condition for (3) to be reducible to a Lie transformation group. Hence it follows, in particular, that if the order of the derivatives contained in (3) is higher than the order of PDE (2), than the corresponding symmetry cannot be reduced to a Lie symmetry.

Let us denote by the symbol $\mathcal{M}$ the surface in the space of variables $t, x, u, u_t, u_1, u_2, \ldots$ defined by the system of algebraic equations $u_t - uu_2 - F = 0$, $D_x^j(u_t - uu_2 - F) = 0$, $j = 1, 2, \ldots$. We say that PDE (2) is invariant under the Lie–Bäcklund vector field (4) if the condition

$$Q(u_t - uu_2 - F)\big|_{\mathcal{M}} = 0 \tag{6}$$

holds.

Relation (6) is the criterion for invariance of PDE (2) under the group (3) and as such it contains the four-step algorithm for calculation of Lie–Bäcklund symmetries admitted by (2). In the first step, we need to compute the result of the action of the operator Q (4) on $u_t - uu_2 - F$. The next step is to eliminate all the derivatives u_{tj}, $j = 0, 1, 2, \ldots$ with the use of the equations $D_x^j(u_t - uu_2 - F) = 0$, $j = 0, 1, 2, \ldots$. The obtained relation is split with respect to the variables u_{N+j}, $j > 0$, which yields the system of linear PDEs (determining equations) for the function η. Finally, in the fourth step, we solve the thus obtained system of PDEs and obtain the most general form of the infinitesimal Lie–Bäcklund group (3) admitted by equation (2).

Further details about higher Lie symmetries, as well as numerous examples of application of the above algorithm for calculating higher symmetries of specific PDEs can be found in [2, 3].

Denote by the symbol $\mathcal{L}_x$ the surface in the space of variables $t, x, u, u_t, u_1, u_2, \ldots$ defined by the system of algebraic equations $\eta(t, x, u, u_1, \ldots, u_N) = 0$, $D_x^j \eta(t, x, u, u_1, \ldots, u_N) = 0$, $j = 1, 2, \ldots$. We say that PDE (2) is invariant under the Lie–Bäcklund vector field (4) if the condition

$$Q(u_t - F)\big|_{\mathcal{M}\cap\mathcal{L}_x} = 0$$

holds.

The algorithm for computing higher conditional symmetries is very much the same as that for computing standard Lie–Bäcklund symmetries. The principal difference is the necessity to take into account not only differential consequences of the equation under study but the differential consequences of the side condition $\eta(t, x, u, u_1, \ldots, u_N) = 0$, as well. As a result, the number of determining equations decreases and new (non-Lie) symmetries arise. However, the price for this is the fact that the determining equations are no longer linear. Note that the same situation takes place for non-classical symmetries. To calculate these symmetries, one needs to solve nonlinear determining equations [10–16].

In full analogy with the classical symmetry reduction method, one can exploit conditional symmetries in order to perform the dimensional reduction of an invariant PDE. An invariant solution is looked for as the general solution of PDE

$$\eta(t, x, u, u_1, \ldots, u_N) = 0. \tag{7}$$

This equation is the criterion for invariance of the manifold $u - u(t, x) = 0$ with respect to the action of the infinitesimal group (3). PDE (7) contains no derivatives with respect to t and, consequently, can be regarded as the Nth-order ODE with respect to the variable x. Its general integral can be (locally) represented in the form

$$u(t, x) = U\left(t, x, \varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)\right) \tag{8}$$

where $\varphi_j(t)$ $(j = 1, \ldots, N)$ are arbitrary smooth functions. In the following, we call the expression (8) the ansatz invariant under the Lie–Bäcklund vector field (4). We have proved in [1] the assertion establishing the connection between reducibility of PDE of the form

$$u_t = F(t, x, u, u_1, \ldots, u_n) \tag{9}$$

to ordinary differential equations and its higher-order conditional symmetry.

Theorem. *Let equation (9) with $F \in C^{N+1}(\mathcal{D})$, where $\mathcal{D}$ is an open domain in $\mathbb{R}^{n+3}$, be conditionally invariant under the Lie–Bäcklund vector field (4) with $\eta \in C^2(\mathcal{D}')$, where $\mathcal{D}'$ is an open domain in $\mathbb{R}^{N+3}$ and $\partial\eta/\partial u_N \neq 0$ on $\mathcal{D}'$. Then ansatz (8) invariant under the Lie–Bäcklund vector field (4) reduces PDE (9) to a system of N ordinary differential equations for some functions $\varphi_j(t)$ $(j = 1, \ldots, N)$,*

$$\frac{\mathrm{d}\varphi_j}{\mathrm{d}t} = F_j(t, \varphi_1, \ldots, \varphi_N) \qquad j = 1, \ldots, N. \tag{10}$$

Now suppose the converse. Namely, that ansatz (8), where the function U and its derivatives $\partial U^{k+1}/\partial\varphi_j \partial x^k$ $(j = 1, \ldots, N,\ k = 0, \ldots, N)$ exist and are continuous on an open domain $\mathcal{D}_1$ in $\mathbb{R}^{N+2}$, reduces (9) to a system of ordinary differential equations (10) with $F_i \in C^1(\mathcal{D}_1')$, where $\mathcal{D}_1'$ is an open domain in $\mathbb{R}^{N+2}$. Then there exists a Lie–Bäcklund vector field (4) such that equation (9) is conditionally invariant with respect to it.

This theorem provides the connection between various direct methods for reducing a given evolution-type PDE to systems of ordinary differential equations and those methods which rely on its symmetries. It states that the direct and symmetry approaches, when taken in full generality, are, in some sense, equivalent. However, for the purposes of the present paper it is more convenient to apply the approach developed in [1, 8], that exploits the higher conditional symmetries of equation (2).

In order to be able to implement the above reduction algorithm efficiently, we have to integrate the nonlinear ODE (7) in a closed form. Evidently, this is not always possible. However, as our previous experience shows, there are sufficiently many nonlinear evolution equations such that their higher conditional symmetries are linear in the variables $u, u_1, \ldots, u_N$. So that, it would be natural to attempt classifying all the possible nonlinear PDEs (2), that admit higher conditional symmetries of the form

$$Q = \sum_{k=0}^{\infty} \left[D_x^k \left(u_N - \sum_{i=0}^{N-1} a_i(t, x) u_i \right) \right] \frac{\partial}{\partial u_k}. \tag{11}$$

If such conditional symmetries are found, then equation (7) is linear and can be integrated to yield an ansatz of the form

$$u = \sum_{i=1}^{N} f_i(t, x) \varphi_i(t). \tag{12}$$

There are numerous examples of such equations obtained both by the direct approach by Galaktionov [4, 5], Olver [19] and by the symmetry approach by Zhdanov [8], Fokas and Liu [9]. However, up to the best of our knowledge, the classification problem for PDE (2) admitting conditional symmetries of the form (11) has not been solved yet in full generality. One of the principal objectives of the present paper is to fill this gap. As shown in [18], the order of the operator Q satisfies the inequality $N \leqslant 5$. In what follows, we will study the cases $N = 3, 4, 5$ and describe all the possible inequivalent forms of the functions F such that nonlinear evolution equation (2) admits conditional symmetries (11). Another objective of the paper is to exploit higher conditional symmetries for reduction of initial-value problems for nonlinear PDEs (2) with the help of the reduction technique developed in [1, 20].

3. Classification of nonlinear equations (2) by their higher conditional symmetries

As we have already mentioned, the order of conditional symmetry N cannot be greater than 5. On the other hand, symmetry (11) with $N = 1, 2$ reduces to a first-order conditional symmetry. For these reasons, we restrict our considerations to the cases $N = 3, 4, 5$.

Theorem 1. *Let the nonlinear evolution equation (2) be conditionally invariant with respect to the Lie–Bäcklund operator (11) with $N \geqslant 3$. Then the function F is necessarily quadratic in u, u_x*

$$F = \lambda_0 u_x^2 + \lambda_1 u u_x + \lambda_2 u^2 + \mu_0 u_x + \mu_1 u + \mu_2 \tag{13}$$

where $\lambda_0, \lambda_1, \lambda_2, \mu_0, \mu_1, \mu_2$ are constants.

Proof. We give the proof of the assertion for the case $N = 3$, the remaining cases $N = 4, 5$ are handled in the same way.

Writing down the invariance condition (6) for the operator (11) under $N = 3$ we obtain

$$\begin{aligned}
F_{u_x u_x u_x} u_{xx}^3 &+ 3F_{u u_x u_x} u_{xx}^2 u_x + (3a_2 + 3F_{u u_x} + 2a_2 F_{u_x u_x}) u_{xx}^2 \\
&+ 3F_{u u u_x} u_{xx} u_x^2 + (6a_1 + a_2^2 + 3a_{1x} + 3F_{uu} + a_2 F_{u u_x} + 3a_1 F_{u_x u_x}) u_{xx} u_x \\
&+ (4a_0 + a_{2xx} + 2a_2 a_{2x} + 2a_{1x} + 3a_0 F_{u_x u_x}) u_{xx} u + (a_{2x} F_{u_x} - a_{2t}) u_{xx} \\
&+ F_{uuu} u_x^3 + (a_1 a_2 + 3a_{1x} + 3a_0 + 3a_1 F_{u u_x} - a_2 F_{uu}) u_x^2 \\
&+ (a_0 a_2 + 5a_{0x} + 2a_1 a_{2x} + a_{1\,xx} + 3a_0 F_{u u_x}) u_x u + (a_0 F_{u_x} + a_{1x} F_{u_x} - a_{1t}) u_x \\
&+ (2a_0 a_{2x} + a_{0xx}) u^2 + (a_0 F_u + a_{0x} F_{u_x} - a_{0\,t}) u - a_0 F = 0.
\end{aligned} \tag{14}$$

The above relation should hold identically with respect to the variables t, x, u, u_x, u_{xx}. Differentiating equation (14) three times with respect to the variable u_{xx}, we obtain $F_{u_x u_x u_x} = 0$, which means that the function F is a quadratic polynomial in u_x. With this fact in hand we can differentiate (14) twice with respect to u_{xx} and thus obtain that the coefficient of u_x^2 is a constant.

Next, differentiating (14) with respect to u_{xx} and twice with respect to u_x, we come to the conclusion that

$$F = \lambda_0 u_x^2 + \lambda_1 u u_x + \mu_0 u_x + F_1(u)$$

where $\lambda_0, \lambda_1, \mu_0$ are constants. Having substituted this function into (14) and differentiated three times with respect to u_x, we arrive at the desired form (13) for F. The theorem is proved. □

Before going any further we give a brief account of the equivalence transformations of the variables t, x, u that do not change the form of the whole class of PDEs

$$u_t = u u_{xx} + \lambda_0 u_x^2 + \lambda_1 u u_x + \lambda_2 u^2 + \mu_0 u_x + \mu_1 u + \mu_2 \tag{15}$$

or of some of its subclasses. These transformations will be used in the following in order to simplify the forms of evolution equations (15).

First of all, we note that in equation (15) we can cancel the coefficient μ_0 by transforming the spatial variable x

$$x' = x + \mu_0 t.$$

That is why, in what follows we will suppose that $\mu_0 = 0$.

Furthermore, given the condition $\lambda_1 \neq 0$, making the change of independent variables

$$x' = \lambda_1 x \qquad t' = \lambda_1^2 t$$

yields that $\lambda_1 = 1$. Next, provided $\lambda_1 = 0$, $\lambda_2 \neq 0$ we can rescale the variables t, x

$$x' = \sqrt{|\lambda_2|}\, x \qquad t' = |\lambda_2|\, t$$

in (15), thus getting $\lambda_2 = \pm 1$.

So, the class of PDEs (15) splits into the following three sub-classes:

$$u_t = uu_{xx} + \lambda_0 u_x^2 + uu_x + \lambda_2 u^2 + \mu_1 u + \mu_2$$

$$u_t = uu_{xx} + \lambda_0 u_x^2 \pm u^2 + \mu_1 u + \mu_2$$

$$u_t = uu_{xx} + \lambda_0 u_x^2 + \mu_1 u + \mu_2.$$

We will also use the fact, that the equation

$$u_t = uu_{xx} + \lambda_0 u_x^2 + uu_x + \frac{2+4\lambda_0}{(3+4\lambda_0)^2} u^2 + \mu_1 u \qquad \lambda_2 \neq -\tfrac{3}{4}$$

is transformed by the substitution

$$u(t,x) = \exp\left(-\frac{2x}{3+4\lambda_0}\right) v(t,y) \qquad y = \exp\left(\frac{x}{3+4\lambda_0}\right) \tag{16}$$

to become

$$v_t = vv_{yy} + \lambda_0 v_y^2 + \mu_1 v.$$

Given the condition $\lambda_0 = -\frac{3}{4}$, the substitutions

$$u(t,x) = (\sin x + 1)^2\, v(t,y) \qquad y = \frac{\sin x - 1}{\cos x}$$

$$u(t,x) = (\cosh x - 1)^2\, v(t,y) \qquad y = \frac{\cosh x + 1}{\sinh x}$$

reduce the equations

$$u_t = uu_{xx} - \tfrac{3}{4} u_x^2 \pm u^2 + \mu_1 u \tag{17}$$

to the form

$$v_t = vv_{yy} - \tfrac{3}{4} v_y^2 + \mu_1 v.$$

Note that the choice of one of the two transformations given above is implied by the sign at the term u^2 in equation (17).

Now we turn back to the symmetry classification of equations (15) that admit conditional symmetries (11) under $N = 3, 4, 5$. We consider in some detail the case $N = 3$. Inserting (13) into (14) yields the relation

$$\begin{aligned}
&(3a_2 + 3\lambda_1 + 4\lambda_0 a_2)u_{xx}^2 + (6a_1 + a_2^2 + 3a_{2x} + 6\lambda_2 + \lambda_1 a_2 \\
&\qquad +6\lambda_0 a_1)u_{xx}u_x + (4a_0 + a_{2xx} + 2a_2 a_{2x} + 2a_{1x} + 6\lambda_0 a_0)u_{xx}u \\
&\qquad +((2\lambda_0 u_x + \lambda_1 u)a_{2x} - a_{2t})u_{xx} + (a_1 a_2 + 3a_{1x} + 3a_0 + 3\lambda_1 a_1 \\
&\qquad -2\lambda_2 a_2)u_x^2 + (a_0 a_2 + 5a_{0x} + 2a_1 a_{2x} + a_{1xx} + 3\lambda_1 a_0)u_x u \\
&\qquad +((2\lambda_0 u_x + \lambda_1 u)(a_0 + a_{1x}) - a_{1t})u_x + (2a_0 a_{2x} + a_{0xx})u^2 \\
&\qquad +((\lambda_1 u_x + 2\lambda_2 u + \mu_1)a_0 + (2\lambda_0 u_x + \lambda_1 u)a_{0x} - a_{0t})u \\
&\qquad -(\lambda_0 u_x^2 + \lambda_1 uu_x + \lambda_2 u^2 + \mu_1 u + \mu_2)a_0 = 0.
\end{aligned}$$

As this relation must hold identically with respect to the variables u, u_x, we can split it by u, u_x and obtain the system of algebraic and differential equations for finding unknown functions a_0, a_1, a_2

$$
\begin{aligned}
&(3+4\lambda_0)a_2 + 3\lambda_1 = 0\\
&(3+2\lambda_0)a_{2x} + a_2^2 + \lambda_1 a_2 + 6(1+\lambda_0)a_1 + 6\lambda_2 = 0\\
&a_{2xx} + 2a_2 a_{2x} + \lambda_1 a_{2x} + 2a_{1x} + (4+6\lambda_0)a_0 = 0\\
&a_1 a_2 - 2\lambda_2 a_2 + (3+2\lambda_0)a_{1x} + 3\lambda_1 a_1 + (3+\lambda_0)a_0 = 0\\
&2a_1 a_{2x} + a_0 a_2 + a_{1xx} + \lambda_1 a_{1x} + (5+2\lambda_0)a_{0x} + 4\lambda_1 a_0 = 0\\
&2a_0 a_{2x} + a_{0xx} + \lambda_1 a_{0x} + \lambda_2 a_0 = 0\\
&a_{0t} = 0 \qquad a_{1t} = 0 \qquad a_{2t} = 0 \qquad \mu_2 a_0 = 0.
\end{aligned}
$$

Solving the above system yields an exhaustive description of all the possible Lie–Bäcklund operators (11) with $N = 3$. The results obtained are summarized below, where we give all inequivalent PDEs (2) admitting third-order conditional symmetries (11) and the corresponding Lie–Bäcklund operators. Note that we skip those third-order Lie–Bäcklund symmetries which lead to reductions that are particular cases of reductions through fourth- or fifth-order Lie–Bäcklund symmetries.

$$u_t = uu_{xx} + \lambda_0 u_x^2 + uu_x + \frac{2+4\lambda_0}{(3+4\lambda_0)^2}u^2 + \mu_1 u + \mu_2 \tag{18}$$

$$\left(\lambda_0 \neq -\tfrac{2}{3},\ \lambda_0 \neq -\tfrac{3}{4}\right)$$

$$Q = \left((3+4\lambda_0)^2 u_{xxx} + 3(3+4\lambda_0)u_{xx} + 2u_x\right)\frac{\partial}{\partial u} + \cdots$$

$$u_t = uu_{xx} - u_x^2 + \mu_1 u + \mu_2 \tag{19}$$

$$Q = (u_{xxx} - Bu_x)\frac{\partial}{\partial u} + \cdots \qquad B = \text{constant}$$

$$u_t = uu_{xx} + \lambda_0 u_x^2 + \epsilon(1+\lambda_0)u^2 + \mu_1 u + \mu_2 \tag{20}$$

$$\left(\lambda_0 \neq -\tfrac{3}{4},\ \lambda_0 \neq -1,\ \epsilon = 0, \pm 1\right)$$

$$Q = (u_{xxx} + \epsilon u_x)\frac{\partial}{\partial u} + \cdots.$$

Similarly, we obtain evolution equations that admit fourth- and fifth-order conditional symmetries of the form (11). The results obtained are listed below.

$$u_t = uu_{xx} - \tfrac{2}{3}u_x^2 + uu_x - 6u^2 + \mu_1 u + \mu_2 \tag{21}$$

$$Q = (u_{xxxx} + 6u_{xxx} - 9u_{xx} - 54u_x)\frac{\partial}{\partial u} + \cdots$$

$$u_t = uu_{xx} - \tfrac{2}{3}u_x^2 + \mu_1 u + \mu_2 \tag{22}$$

$$Q = (u_{xxxx})\frac{\partial}{\partial u} + \cdots$$

$$u_t = uu_{xx} - \tfrac{3}{4}u_x^2 + \epsilon u^2 + \mu_1 u + \mu_2 \qquad \epsilon = 0, \pm 1 \tag{23}$$

$$Q = \left(u_{xxxxx} + 5\epsilon u_{xxx} + 4\epsilon^2 u_x\right)\frac{\partial}{\partial u} + \cdots.$$

Remark 1. It is straightforward to apply the above results in order to describe PDEs of the more general form (1), that admit higher conditional symmetries. To this end, one has to look for a transformation $u(t,x) = g(v(t,x))$ reducing (1) to the form (15). As a result, one obtains the following generalization of PDE (2) admitting higher conditional symmetries:

$$v_t = g(v)v_{xx} + \left(g(v)\frac{\mathrm{d}^2 g(v)}{\mathrm{d}v^2}\left(\frac{\mathrm{d}g(v)}{\mathrm{d}v}\right)^{-1} + \lambda_0 g(v)\right)v_x^2 + \lambda_1 g(v)v_x$$

$$+\lambda_2 g^2(v)\left(\frac{\mathrm{d}g(v)}{\mathrm{d}v}\right)^{-1} + \mu_0 v_x + (\mu_1 g(v) + \mu_2)\left(\frac{\mathrm{d}g(v)}{\mathrm{d}v}\right)^{-1}$$

where $g(v)$ is an arbitrary smooth function. Conditional symmetries admitted by the above equation are given by formulae (18)–(23), where one has to change $u(t,x)$ to $g(v(t,x))$. Furthermore, the ansatz for the function $v(x)$ takes the form

$$g(v(t,x)) = \sum_{i=1}^{N} f_i(t,x)\varphi_i(t).$$

Remark 2. Higher conditional symmetries listed in (18)–(23) are not new (in the sense that the reductions corresponding to them are known). Our principal result is that they exhaust the set of all possible conditional symmetries (11) with $N = 3, 4, 5$ admitted by inequivalent PDEs of the form (2).

Remark 3. Equation (18) with $\mu_2 = 0$ possesses the additional conditional symmetry

$$Q = (u_{xxx} - 3u_{xx} - (B\exp(-2x) - 2)u_x + 2B\exp(-2x)u)\frac{\partial}{\partial u} + \cdots \qquad (24)$$

where B is an arbitrary real constant. We have not singled out equation (18) under $\mu_2 = 0$, since it is reduced to the form (20) under $\mu_2 = 0$, $\epsilon = 0$ with the help of transformation (16), and furthermore, conditional symmetry (24) is reduced to the conditional symmetry Q from (20) under $\epsilon = 0$. However, the ansätze

$$u = \exp(2x)\,(\varphi_1(t) + \varphi_2(t)\sinh(\beta\exp(-x)) + \varphi_3(t)\cosh(\beta\exp(-x)))$$

$$B = \beta^2 > 0$$

$$u = \exp(2x)\,(\varphi_1(t) + \varphi_2(t)\sin(\beta\exp(-x)) + \varphi_3(t)\cos(\beta\exp(-x)))$$

$$B = -\beta^2 < 0$$

$$u = \varphi_1(t) + \varphi_2(t)\exp(x) + \varphi_3(t)\exp(2x) \qquad B = 0$$

corresponding to (24) seem to be new. Note that hereafter we denote by the symbol $\varphi_i(t)$ an arbitrary smooth functions of t.

4. Reduction of initial-value problems for evolution equations (2)

Thus there exist six inequivalent classes of PDEs (2) that are invariant with respect to Lie–Bäcklund operators (11) under $N > 2$. Using their higher conditional symmetries we can reduce these equations to systems of three, four or five ODEs (the number of ODEs is determined by the order of the corresponding Lie–Bäcklund operator). Moreover, using the technique developed in [1] we can describe classes of initial-value problems for evolution equations (18)–(23), which can be reduced to Cauchy problems for the corresponding systems of ODEs.

Consider PDE (2) together with the initial-value condition

$$a(x)u_x(0,x) + b(x)u(0,x) = c(x) \tag{25}$$

where $a(x), b(x), c(x)$ are some smooth real-valued functions. According to [1], one can search for initial-value conditions, that are reducible with the help of higher conditional symmetry (4), using the following two-step algorithm:

- compute the maximal Lie invariance algebra of PDE (7) within the class of Lie vector fields

$$X = \xi(t,x)\frac{\partial}{\partial x} + (\zeta_1(t,x)u + \zeta_2(t,x))\frac{\partial}{\partial u} \tag{26}$$

- put

$$a(x) = \xi(0,x) \qquad b(x) = -\zeta_1(0,x) \qquad c(x) = \zeta_2(0,x).$$

As a result, we obtain the initial-value problem for the PDE (2) invariant with respect to the Lie–Bäcklund operator (4), which can be reduced to a Cauchy problem for some system of ODEs, provided the additional compatibility requirements are met (see, for further details, [20]).

4.1. Reduction of (18)

Let us apply the technique described to PDE (18). First, integrating equation

$$(3+4\lambda_0)^2 u_{xxx} + 3(3+4\lambda_0)u_{xx} + 2u_x = 0 \tag{27}$$

we obtain the ansatz

$$u(t,x) = \varphi_1(t) + \varphi_2(t)\exp\left(-\frac{x}{3+4\lambda_0}\right) + \varphi_3(t)\exp\left(-\frac{2x}{3+4\lambda_0}\right) \tag{28}$$

that reduces (18) to the system of three ODEs.

Next, we compute Lie symmetry of equation (27) and derive the following class of initial-value conditions:

$$\begin{aligned}
-(3+4\lambda_0)&\left(\alpha_1 + \alpha_2\exp\left(\frac{x}{3+4\lambda_0}\right) + \alpha_3\exp\left(-\frac{x}{3+4\lambda_0}\right)\right)u_x(0,x)\\
&+\left(\alpha_4 + 2\alpha_3\exp\left(-\frac{x}{3+4\lambda_0}\right)\right)u(0,x) = \beta_1\\
&+\beta_2\exp\left(-\frac{x}{3+4\lambda_0}\right) + \beta_3\exp\left(-\frac{2x}{3+4\lambda_0}\right).
\end{aligned} \tag{29}$$

Here $\alpha_1, \alpha_2, \ldots, \beta_3$ are arbitrary real constants.

Finally, inserting ansatz (28) into initial-value problem (18) and (29) yields the following Cauchy problem:

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= \frac{2+4\lambda_0}{(3+4\lambda_0)^2}\varphi_1^2 + \mu_1\varphi_1 + \mu_2\\
\frac{d\varphi_2}{dt} &= \frac{2+4\lambda_0}{(3+4\lambda_0)^2}\varphi_1\varphi_2 + \mu_1\varphi_2\\
\frac{d\varphi_3}{dt} &= \frac{2}{(3+4\lambda_0)^2}\varphi_1\varphi_3 + \frac{\lambda_0}{(3+4\lambda_0)^2}\varphi_2^2 + \mu_1\varphi_3
\end{aligned}$$

$$\begin{pmatrix} \alpha_4 & \alpha_2 & 0 \\ -2\alpha_3 & \alpha_1+\alpha_4 & 2\alpha_2 \\ 0 & -\alpha_3 & 2\alpha_1+\alpha_4 \end{pmatrix}\begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.$$

4.2. Reduction of (19)

Turn now to PDE (19). Integrating equation

$$u_{xxx} - Bu_x = 0 \tag{30}$$

yields the ansatz for $u(t, x)$, whose explicit form depends essentially on the sign of the parameter B, namely,

$$u(t, x) = \varphi_1(t) + \varphi_2(t)\sinh(\beta x) + \varphi_3(t)\cosh(\beta x) \qquad B = \beta^2 > 0 \tag{31}$$

$$u(t, x) = \varphi_1(t) + \varphi_2(t)\sin(\beta x) + \varphi_3(t)\cos(\beta x) \qquad B = -\beta^2 < 0 \tag{32}$$

$$u(t, x) = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 \qquad B = 0. \tag{33}$$

The forms of symmetry operators admitted by the differential equation (30) also depend on the sign of B. That is why, applying the above algorithm yields three different initial-value conditions for PDE (19)

$B = \beta^2 > 0$

$$(\alpha_1 + \alpha_2\sinh(\beta x) + \alpha_3\cosh(\beta x))\, u_x(0, x) + \beta\,(\alpha_4 - \alpha_2\cosh(\beta x) - \alpha_3\sinh(\beta x))\, u(0, x)$$
$$= \beta\,(\beta_1 + \beta_2\sinh(\beta x) + \beta_3\cosh(\beta x)) \tag{34}$$

$B = -\beta^2 < 0$

$$(\alpha_1 + \alpha_2\sin(\beta x) + \alpha_3\cos(\beta x))\, u_x(0, x) + \beta\,(\alpha_4 - \alpha_2\cos(\beta x) + \alpha_3\sin(\beta x))\, u(0, x)$$
$$= \beta\,(\beta_1 + \beta_2\sin(\beta x) + \beta_3\cos(\beta x)) \tag{35}$$

$B = 0$

$$\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2\right) u_x(0, x) + (\alpha_4 - 2\alpha_3 x)u(0, x) = \beta_1 + \beta_2 x + \beta_3 x^2. \tag{36}$$

Here $\alpha_1, \alpha_2, \ldots, \beta_3$ are arbitrary real constants.

Inserting (31) into the initial-value problem (19) and (34) yields the following Cauchy problem:

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= -\beta^2\varphi_2^2 + \beta^2\varphi_3^2 + \mu_1\varphi_1 + \mu_2 \\
\frac{d\varphi_2}{dt} &= \beta^2\varphi_1\varphi_2 + \mu_1\varphi_2 \\
\frac{d\varphi_3}{dt} &= \beta^2\varphi_1\varphi_3 + \mu_1\varphi_3 \\
&\begin{pmatrix} \alpha_4 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & \alpha_4 & \alpha_1 \\ -\alpha_2 & \alpha_1 & \alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.
\end{aligned} \tag{37}$$

Next, substituting ansatz (32) into initial-value problem (19) and (35) we obtain the Cauchy problem

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = -\beta^2\varphi_2^2 - \beta^2\varphi_3^2 + \mu_1\varphi_1 + \mu_2\\
&\frac{d\varphi_2}{dt} = -\beta^2\varphi_1\varphi_2 + \mu_1\varphi_2\\
&\frac{d\varphi_3}{dt} = -\beta^2\varphi_1\varphi_3 + \mu_1\varphi_3\\
&\begin{pmatrix} \alpha_4 & \alpha_3 & -\alpha_2\\ \alpha_3 & \alpha_4 & -\alpha_1\\ -\alpha_2 & \alpha_1 & \alpha_4 \end{pmatrix}\begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}.
\end{aligned} \tag{38}$$

Finally, inserting ansatz (33) into initial-value problem (19) and (36) we arrive at the following Cauchy problem:

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = 2\varphi_1\varphi_3 - \varphi_2^2 + \mu_1\varphi_1 + \mu_2\\
&\frac{d\varphi_2}{dt} = -2\varphi_2\varphi_3 + \mu_1\varphi_2\\
&\frac{d\varphi_3}{dt} = -2\varphi_3^2 + \mu_1\varphi_3\\
&\begin{pmatrix} \alpha_4 & \alpha_1 & 0\\ -2\alpha_3 & \alpha_2+\alpha_4 & 2\alpha_1\\ 0 & -\alpha_3 & 2\alpha_2+\alpha_4 \end{pmatrix}\begin{pmatrix}\varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\end{pmatrix}.
\end{aligned} \tag{39}$$

4.3. Reduction of (20)

Evidently, ansätze corresponding to conditional symmetry Q from (20) are obtained from formulae (31)–(33) under $B = \epsilon$. What is more, the initial-value conditions have the forms (34)–(36) with $B = \epsilon$. Inserting the ansätze (31)–(33) under $B = \epsilon$ into the corresponding initial-value problems for evolution equation (20) yields the following Cauchy problems:

$\epsilon = 1$

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = (1+\lambda_0)\varphi_1^2 + \lambda_0(-\varphi_2^2 + \varphi_3^2) + \mu_1\varphi_1 + \mu_2\\
&\frac{d\varphi_2}{dt} = (1+2\lambda_0)\varphi_1\varphi_2 + \mu_1\varphi_2\\
&\frac{d\varphi_3}{dt} = (1+2\lambda_0)\varphi_1\varphi_3 + \mu_2\varphi_3\\
&\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0) \text{ satisfy relations (37) with } \beta = 1
\end{aligned}$$

$\epsilon = -1$

$$\begin{aligned}
&\frac{d\varphi_1}{dt} = -(1+\lambda_0)\varphi_1^2 - \lambda_0(\varphi_2^2 + \varphi_3^2) + \mu_1\varphi_1 + \mu_2\\
&\frac{d\varphi_2}{dt} = -(1+2\lambda_0)\varphi_1\varphi_2 + \mu_1\varphi_2
\end{aligned}$$

$$\frac{d\varphi_3}{dt} = -(1+2\lambda_0)\varphi_1\varphi_3 + \mu_2\varphi_3$$

$\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0)$ satisfy relations (38) with $\beta = 1$

$\epsilon = 0$

$$\frac{d\varphi_1}{dt} = 2\varphi_1\varphi_3 + \lambda_0\varphi_2^2 + \mu_1\varphi_1 + \mu_2$$

$$\frac{d\varphi_2}{dt} = (2+4\lambda_0)\,\varphi_2\varphi_3 + \mu_1\varphi_2$$

$$\frac{d\varphi_3}{dt} = (2+4\lambda_0)\,\varphi_3^2 + \mu_2\varphi_3$$

$\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0)$ satisfy relations (39).

4.4. Reduction of (21)

Having calculated the Lie symmetry algebra of the equation

$$u_{xxxx} + 6u_{xxx} - 9u_{xx} - 54u_x = 0 \tag{40}$$

within the class of Lie vector fields (26), we obtain the following class of initial conditions for the evolution equation (21):

$$\begin{aligned}(\alpha_1 + \alpha_2\exp(3x) + \alpha_3\exp(-3x))\,u_x(0,x) + (\alpha_4 - 3\alpha_2\exp(3x) + 6\alpha_3\exp(-3x))\,u(0,x)\\ = \beta_1 + \beta_2\exp(3x) + \beta_3\exp(-3x) + \beta_4\exp(-6x)\end{aligned} \tag{41}$$

where $\alpha_1, \alpha_2, \ldots, \beta_4$ are arbitrary real constants.

Integrating (40) yields the ansatz

$$u(t,x) = \varphi_1(t) + \varphi_2(t)\exp(3x) + \varphi_3(t)\exp(-3x) + \varphi_4(t)\exp(-6x) \tag{42}$$

which reduces the initial-value problem (21) and (41) to the Cauchy problem

$$\frac{d\varphi_1}{dt} = 18\varphi_2\varphi_3 - 6\varphi_1^2 + \mu_1\varphi_1 + \mu_2$$

$$\frac{d\varphi_2}{dt} = \mu_1\varphi_2$$

$$\frac{d\varphi_3}{dt} = 54\varphi_2\varphi_4 - 6\varphi_1\varphi_3 + \mu_1\varphi_3$$

$$\frac{d\varphi_4}{dt} = 18\varphi_1\varphi_4 - 6\varphi_3^2 + \mu_1\varphi_4$$

$$\begin{pmatrix} \alpha_4 & 9\alpha_3 & -6\alpha_2 & 0 \\ -3\alpha_2 & 3\alpha_1+\alpha_4 & 0 & -3\alpha_2 \\ 6\alpha_3 & 0 & -3\alpha_1+\alpha_4 & -6\alpha_2 \\ 0 & 0 & 3\alpha_3 & 3\alpha_3+\alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

4.5. Reduction of (22)

Having calculated the Lie symmetry algebra of the equation $u_{xxxx} = 0$ we arrive at the following initial-value condition for evolution equation (22):

$$\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2\right) u_x(0, x) + (\alpha_4 - 3\alpha_3 x)\, u(0, x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 \tag{43}$$

where $\alpha_1, \alpha_2, \ldots, \beta_4$ are arbitrary real constants. Integrating the equation $u_{xxxx} = 0$ yields the ansatz

$$u(t, x) = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 + \varphi_4(t)x^3$$

which reduces the initial-value problem (22) and (43) to the Cauchy problem

$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = 2\varphi_1\varphi_3 - \tfrac{2}{3}\varphi_2^2 + \mu_1\varphi_1 + \mu_2$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = 6\varphi_1\varphi_4 - \tfrac{2}{3}\varphi_2\varphi_3 + \mu_1\varphi_2$$
$$\frac{\mathrm{d}\varphi_3}{\mathrm{d}t} = 2\varphi_2\varphi_4 - \tfrac{2}{3}\varphi_3^2 + \mu_1\varphi_3$$
$$\frac{\mathrm{d}\varphi_4}{\mathrm{d}t} = \mu_1\varphi_4$$
$$\begin{pmatrix} \alpha_4 & \alpha_1 & 0 & 0 \\ -3\alpha_2 & \alpha_2 + \alpha_4 & 2\alpha_1 & 0 \\ 0 & -2\alpha_3 & 2\alpha_2 + \alpha_4 & 3\alpha_1 \\ 0 & 0 & -\alpha_3 & 3\alpha_2 + \alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{pmatrix}.$$

4.6. Reduction of (23)

Consider the sixth class of evolution equations (23). Depending on ϵ, the general solution of the equation

$$u_{xxxxx} + 5\epsilon u_{xxx} + 4\epsilon^2 u_x = 0 \tag{44}$$

is given by one of the three formulae below

$$\epsilon = 1 \qquad u(t, x) = \varphi_1(t) + \varphi_2(t)\sin x + \varphi_3(t)\cos x + \varphi_4(t)\sin 2x + \varphi_5(t)\cos 2x \tag{45}$$

$$\epsilon = -1 \qquad u(t, x) = \varphi_1(t) + \varphi_2(t)\sinh x + \varphi_3(t)\cosh x + \varphi_4(t)\sinh 2x + \varphi_5(t)\cosh 2x \tag{46}$$

$$\epsilon = 0 \qquad u(t, x) = \varphi_1(t) + \varphi_2(t)x + \varphi_3(t)x^2 + \varphi_4(t)x^3 + \varphi_5(t)x^4. \tag{47}$$

Calculating the Lie symmetry algebra of equation (44) within the class of Lie vector fields (26) we obtain the following initial conditions for the corresponding evolution equations (23):

$\epsilon = 1$

$$(\alpha_1 + \alpha_2\sin x + \alpha_3\cos x)u_x(0, x) + (\alpha_4 - 2\alpha_2\cos x + 2\alpha_3\sin x)u(0, x) = \beta_1 + \beta_2\sin x + \beta_3\cos x + \beta_4\sin 2x + \beta_5\cos 2x \tag{48}$$

$\epsilon = -1$

$$(\alpha_1 + \alpha_2\sinh x + \alpha_3\cosh x)\, u_x(0, x) + (\alpha_4 - 2\alpha_2\cosh x - 2\alpha_3\sinh x)\, u(0, x) = \beta_1 + \beta_2\sinh x + \beta_3\cosh x + \beta_4\sinh 2x + \beta_5\cosh 2x \tag{49}$$

$\epsilon = 0$

$$\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2\right) u_x(0, x) + (\alpha_4 - 4\alpha_3 x)\, u(0, x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3 + \beta_5 x^4 \tag{50}$$

where $\alpha_1, \alpha_2, \ldots, \beta_5$ are arbitrary real constants.

Inserting ansätze (45)–(47) into the corresponding initial-value problems (23) with $\epsilon = 1$, (48); (23) with $\epsilon = -1$, (49); (23) with $\epsilon = 0$, (50); reduces them to the following Cauchy problems:

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= \varphi_1^2 - 3\varphi_4^2 - 3\varphi_5^2 - \tfrac{3}{8}\varphi_2^2 - \tfrac{3}{8}\varphi_3^2 + \mu_1\varphi_1 + \mu_2\\
\frac{d\varphi_2}{dt} &= -3\varphi_3\varphi_4 - 3\varphi_2\varphi_5 + \varphi_1\varphi_2 + \mu_1\varphi_2\\
\frac{d\varphi_3}{dt} &= -3\varphi_3\varphi_5 - 3\varphi_2\varphi_4 + \varphi_1\varphi_3 + \mu_1\varphi_3\\
\frac{d\varphi_4}{dt} &= \tfrac{3}{4}\varphi_2\varphi_3 - 2\varphi_1\varphi_4 + \mu_1\varphi_4\\
\frac{d\varphi_5}{dt} &= -\tfrac{3}{8}\varphi_2^2 + \tfrac{3}{8}\varphi_3^2 - 2\varphi_1\varphi_5 + \mu_1\varphi_5
\end{aligned}$$

$$\begin{pmatrix} \alpha_4 & \frac{3}{2}\alpha_3 & -\frac{3}{2}\alpha_2 & 0 & 0\\ -2\alpha_3 & \alpha_4 & -\alpha_1 & -2\alpha_2 & -2\alpha_3\\ -2\alpha_2 & \alpha_1 & \alpha_4 & 2\alpha_3 & -2\alpha_2\\ 0 & -\frac{1}{2}\alpha_2 & \frac{1}{2}\alpha_3 & \alpha_4 & -2\alpha_1\\ 0 & -\frac{1}{2}\alpha_3 & -\frac{1}{2}\alpha_2 & 2\alpha_1 & \alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\\ \varphi_5(0)\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}$$

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= -\varphi_1^2 - 3\varphi_4^2 + 3\varphi_5^2 - \tfrac{3}{8}\varphi_2^2 + \tfrac{3}{8}\varphi_3^2 + \mu_1\varphi_1 + \mu_2\\
\frac{d\varphi_2}{dt} &= 3\varphi_3\varphi_4 - 3\varphi_2\varphi_5 - \varphi_1\varphi_2 + \mu_1\varphi_2\\
\frac{d\varphi_3}{dt} &= 3\varphi_3\varphi_5 - 3\varphi_2\varphi_4 - \varphi_1\varphi_3 + \mu_1\varphi_3\\
\frac{d\varphi_4}{dt} &= -\tfrac{3}{4}\varphi_2\varphi_3 + 2\varphi_1\varphi_4 + \mu_1\varphi_4\\
\frac{d\varphi_5}{dt} &= -\tfrac{3}{8}\varphi_2^2 - \tfrac{3}{8}\varphi_3^2 + 2\varphi_1\varphi_5 + \mu_1\varphi_5
\end{aligned}$$

$$\begin{pmatrix} \alpha_4 & \frac{3}{2}\alpha_3 & -\frac{3}{2}\alpha_2 & 0 & 0\\ -2\alpha_2 & \alpha_4 & \alpha_1 & -2\alpha_2 & 2\alpha_3\\ -2\alpha_2 & \alpha_1 & \alpha_4 & 2\alpha_3 & -2\alpha_2\\ 0 & -\frac{1}{2}\alpha_2 & -\frac{1}{2}\alpha_3 & \alpha_4 & 2\alpha_1\\ 0 & -\frac{1}{2}\alpha_3 & -\frac{1}{2}\alpha_2 & 2\alpha_1 & \alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0)\\ \varphi_2(0)\\ \varphi_3(0)\\ \varphi_4(0)\\ \varphi_5(0)\end{pmatrix} = \begin{pmatrix}\beta_1\\ \beta_2\\ \beta_3\\ \beta_4\\ \beta_5\end{pmatrix}$$

$$\begin{aligned}
\frac{d\varphi_1}{dt} &= 2\varphi_1\varphi_3 - \tfrac{3}{4}\varphi_2^2 + \mu_1\varphi_1 + \mu_2\\
\frac{d\varphi_2}{dt} &= 6\varphi_1\varphi_4 - \varphi_2\varphi_3 + \mu_1\varphi_2\\
\frac{d\varphi_3}{dt} &= 12\varphi_1\varphi_5 + \tfrac{3}{2}\varphi_2\varphi_4 - \varphi_3^2 + \mu_1\varphi_3\\
\frac{d\varphi_4}{dt} &= 6\varphi_2\varphi_5 - \varphi_3\varphi_4 + \mu_1\varphi_4
\end{aligned}$$

$$\frac{d\varphi_5}{dt} = 2\varphi_3\varphi_5 - \tfrac{3}{4}\varphi_4^2 + \mu_1\varphi_5$$

$$\begin{pmatrix} \alpha_4 & \alpha_1 & 0 & 0 & 0 \\ -4\alpha_3 & \alpha_2+\alpha_4 & 2\alpha_1 & 0 & 0 \\ 0 & -3\alpha_3 & 2\alpha_2+\alpha_4 & 3\alpha_1 & 0 \\ 0 & 0 & -2\alpha_3 & 3\alpha_2+\alpha_4 & 4\alpha_1 \\ 0 & 0 & 0 & -\alpha_3 & \alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \\ \varphi_4(0) \\ \varphi_5(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix}.$$

Next, we give an example of using the above reduction technique for constructing the unique analytical solution of the initial-value problem for the evolution equation of the form (15). Consider the following problem:

$$\begin{aligned} &u_t = uu_{xx} - u_x^2 + \mu_1 u + \mu_2 \\ &\left(\alpha_1 + x + \alpha_3 x^2\right) u_x(0,x) - 2\alpha_3 x u(0,x) = \alpha_1 x + x^2 \end{aligned} \tag{51}$$

where $\mu_1, \mu_2, \alpha_3, \beta_1, \beta_2$ are arbitrary constants. Inserting (33) into (51) yields the Cauchy problem (39) under $\alpha_1 = \beta_2,\ \alpha_2 = \beta_3 = 1,\ \alpha_4 = \beta_1 = 0$. Integrating ODEs from (39) gives

$$\begin{aligned} \varphi_1 = \frac{1}{1 - C_1\exp(-\mu_1 t)}\Bigg(&\left(\mu_2 - \mu_1 C_2^2\right) t - C_2^2 \ln\left(1 - C_1\exp(-\mu_1 t)\right) \\ &+ \frac{\mu_2}{\mu_1} C_1 \exp(-\mu_1 t) + C_3\Bigg) \end{aligned}$$

$$\varphi_2 = \frac{C_2}{1 - C_1\exp(-\mu_1 t)} \qquad \varphi_3 = \frac{\mu_1}{2\left(1 - C_1\exp(-\mu_1 t)\right)}$$

where C_1, C_2, C_3 are arbitrary parameters (integration constants). They are specified by imposing the initial Cauchy data (the second equation from (51)) on the functions $\varphi_1, \varphi_2, \varphi_3$, whence we obtain the unique solution of (51)

$$\begin{aligned} u(t,x) = \frac{1}{1-(1-\mu_1)\exp(-\mu_1 t)} &\left(\mu_2 t + \left(\frac{\mu_2}{\mu_1} - \mu_2\right)(\exp(-\mu_1 t) - 1)\right) \\ &+ \frac{\mu_1}{2\left(1-(1-\mu_1)\exp(-\mu_1 t)\right)} x^2. \end{aligned}$$

It has been mentioned in the previous section that there are third-order conditional symmetries, which give rise to reductions obtainable with the aid of fifth-order conditional symmetries. However, these symmetries are useful within the context of initial-value problems, since they may yield new initial-value conditions, which cannot be derived with the aid of fifth-order conditional symmetries by the above described method. That is why, we give in the appendix additional third-order conditional symmetries and then present an example of the new initial-value problem that can be reduced to a Cauchy problem for a system of three ODEs.

5. Discussion

Application of the methods and ideas of the Lie theory of continuous groups to solving initial/boundary value problems for nonlinear PDEs still remains a great challenge for mathematicians. The principal reason for this situation is the fact that the class of initial or boundary conditions for PDEs of practical importance, that can be efficiently handled by the symmetry reduction routine, is too narrow compared with practical needs. This was one of the reasons why there were numerous attempts to generalize the notion of classical Lie symmetry in order to weaken constraints on the choice of initial/boundary conditions that are needed to secure invariance of the initial/boundary value problem under some prescribed symmetry group. In this way, a number of new efficient reduction techniques have been developed, like the non-classical [10], conditional symmetry [11–13, 15, 16], direct [14], nonlinear separation of variables [4, 5, 19], antireduction [6, 7] and higher conditional [7, 8] (or generalized conditional [9]) symmetry methods. These methods can be conventionally classified into two principal groups. The first group contains the direct methods (the ansatz method by Fushchych, the direct method by Clarkson and Kruskal, the antireduction method by Fushchych and Zhdanov and the method of nonlinear separation of variables, which is due to Galaktionov), relying upon a special *ad hoc* representation of the solution to be found in the form of an ansatz containing some arbitrary elements (functions) $f_1, f_2, \ldots, f_n$ and unknown functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ with fewer numbers of independent variables. Inserting the ansatz in question into the PDE under study and requiring for the obtained relation to be equivalent to a system of PDEs for the functions $\varphi_1, \varphi_2, \ldots, \varphi_m$ yields nonlinear determining equations for the functions $f_1, f_2, \ldots, f_n$. Having solved the latter yields a number of ansätze reducing a given PDE to one or several PDEs having fewer independent variables. The second group of methods (the non-classical method by Bluman and Cole, the method of conditional symmetries by Fushchych, the method of side conditions by Olver and Rosenau, the method of higher conditional symmetries by Zhdanov and Fokas and Liu) may be regarded as infinitesimal ones. They are in line with the traditional Lie approach to the reduction of PDEs, since they exploit symmetry properties of the equation under study in order to construct its invariant solutions. And again any deviation from the standard Lie approach requires solving an over-determined system of nonlinear determining equations.

The direct and infinitesimal approaches are equivalent in the sense that they yield the same invariant solutions. However, a proper choice of an approach when solving some specific problem may simplify essentially the calculations. In particular, when handling classification problems, it is definitely more convenient to use the infinitesimal approach. On the other hand, the direct approach is more flexible and is more straightforwardly generalized. No wonder that the first examples of reductions of nonlinear PDEs to several ordinary differential equations were found by direct methods by properly generalizing the form of similarity ansätze.

As shown in the present paper (see also [20]), higher conditional symmetries are the most efficient tool for solving the problem of dimensional reduction of initial-value problems for evolution-type PDEs in a purely algebraic way. Moreover, provided some reasonable smoothness conditions are met, reducibility of an initial-value problem for PDE (2) to a Cauchy problem for some system of ODEs is in one-to-one correspondence with higher conditional symmetries admitted by (2) [20].

Having reduced an initial-value problem (2) and (25) to a Cauchy problem for a system of ODEs extends the choice of methods for investigation of (2) and (25) substantially. First, we can try to solve the Cauchy problem in question analytically, thus obtaining an exact solution of the corresponding initial-value problem (2) and (25). However, even if we do not succeed in integrating it in a closed form, there is a broad choice of highly efficient routines for solving the Cauchy problem approximately, which, in turn, yields approximate solutions of the initial-value problem for the nonlinear partial differential equation (2).

Since higher Lie symmetries are particular cases of higher conditional symmetries, the above reduction technique can be applied to solitonic evolution equations, as well. So it might be of interest to investigate the problem of classifying initial-value conditions for solitonic equations such that the corresponding initial-value problems can be reduced to Cauchy problems for some systems of nonlinear ODEs.

The last remark concerns systems of evolution equations (which are not necessarily of parabolic type). Our approach can be easily modified in order to become applicable to systems of PDEs. This idea seems to be especially promising for systems of hydrodynamic-type equations which admit infinite-parameter Lie–Bäcklund groups. These and related problems are under study now and will be reported in our future publications.

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Appendix. Additional conditional symmetries for PDE (2)

Additional third-order conditional symmetries appears for the following PDE:

$$u_t = uu_{xx} - \tfrac{3}{4}u_x^2 + \epsilon u^2 + \mu_1 u + \mu_2$$

where $\epsilon = 0, \pm 1,\ \mu_2 \neq 0$. We give the list of these symmetries and the corresponding ansätze omitting the details of the derivation.

Case 1. $\epsilon = 1$

(1) $$Q = (u_{xxx} + 4u_x)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\cos 2x + \varphi_2(t)\sin 2x + \varphi_1(t)$$

(2) $$Q = \left(u_{xxx} + 3(\tan x)u_{xx} + \left(3\tan^2 x + 1\right)u_x\right)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\sin^2 x + \varphi_2(t)\sin x + \varphi_1(t).$$

Case 2. $\epsilon = -1$

(1) $$Q = (u_{xxx} - 4u_x)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\cosh 2x + \varphi_2(t)\sinh 2x + \varphi_1(t)$$

(2) $$Q = (u_{xxx} \mp 3u_{xx} + 2u_x)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\mathrm{e}^{\pm 2x} + \varphi_2(t)\mathrm{e}^{\pm x} + \varphi_1(t)$$

(3) $$Q = \left(u_{xxx} - 3(\tanh x)u_{xx} + \left(3\tanh^2 x - 1\right)u_x\right)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\sinh^2 x + \varphi_2(t)\sinh x + \varphi_1(t)$$

(4) $$Q = \left(u_{xxx} - 3(\coth x)u_{xx} + \left(3\coth^2 x - 1\right)u_x\right)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\cosh^2 x + \varphi_2(t)\cosh x + \varphi_1(t).$$

Case 3. $\epsilon = 0$

(1) $$Q = (u_{xxx})\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)x^2 + \varphi_2(t)x + \varphi_1(t)$$

(2) $$Q = (u_{xxx} - 3x^{-1}u_{xx} + 3x^{-2}u_x)\frac{\partial}{\partial u} + \cdots \qquad \text{(A1)}$$

$$u(t,x) = \varphi_3(t)x^4 + \varphi_2(t)x^2 + \varphi_1(t)$$

(3) $$Q = \left(u_{xxx} + \frac{6x}{x^2+\beta}u_{xx} - 6\left(\frac{1}{x^2+\beta} + 2\left(\frac{x}{x^2+\beta}\right)^2\right)u_x - \frac{24x}{\left(x^2+\beta\right)^2}u\right)\frac{\partial}{\partial u} + \cdots$$

$$u(t,x) = \varphi_3(t)\left(x^4+\beta^2\right) + \varphi_2(t)\left(x^3-\beta x\right) + \varphi_1(t)x^2.$$

Consider, for example, the conditional symmetry (A1). Calculating Lie symmetry of the equation

$$u_{xxx} - 3x^{-1}u_{xx} + 3x^{-2}u_x = 0$$

within the class of Lie vector fields (26), we arrive at the following initial-value problem:

$$\begin{aligned} &u_t = uu_{xx} - \tfrac{3}{4}u_x^2 + \mu_1 u + \mu_2 \\ &(\alpha_1 x + \alpha_2 x^{-1} + \alpha_3 x^3)u_x(0,x) + (\alpha_4 - 4\alpha_3 x^2)u(0,x) = \beta_1 + \beta_2 x^2 + \beta_3 x^4 \end{aligned} \qquad \text{(A2)}$$

where $\alpha_1, \alpha_2, \ldots, \beta_3$ are arbitrary real constants.

Comparing this problem to that corresponding to the fifth-order conditional symmetry $Q = (u_{xxxxx})\frac{\partial}{\partial u} + \cdots$, we conclude that (A2) cannot be reduced to the latter. Ansatz (A1) reduces (A2) to the Cauchy problem for the system of three ODEs

$$\begin{aligned} \frac{d\varphi_3}{dt} &= 2\varphi_3\varphi_2 + \mu_1\varphi_3 + \mu_2 \\ \frac{d\varphi_2}{dt} &= 12\varphi_3\varphi_1 - \varphi_2^2 + \mu_1\varphi_2 \\ \frac{d\varphi_1}{dt} &= 2\varphi_2\varphi_1 + \mu_1\varphi_1 \end{aligned}$$

$$\begin{pmatrix} \alpha_4 & 2\alpha_2 & 0 \\ -4\alpha_3 & 2\alpha_1+\alpha_4 & 4\alpha_2 \\ 0 & -2\alpha_3 & 4\alpha_1+\alpha_4 \end{pmatrix} \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \\ \varphi_3(0) \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.$$

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