The displacement and split decompositions for a Q-polynomial distance-regular graph*

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Abstract

Let Γ denote a Q-polynomial distance-regular graph with diameter at least three and standard module V. We introduce two direct sum decompositions of V. We call these the *displacement* decomposition for Γ and the *split* decomposition for Γ. We describe how these decompositions are related.

1 Introduction

In this paper $\Gamma = (X, R)$ will denote a Q-polynomial distance-regular graph with diameter $D \geq 3$ and adjacency matrix A (see Section 2 for formal definitions). In order to describe our main results we make a few comments. Fix a vertex $x \in X$. For $0 \leq i \leq D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ which represents the projection onto the ith subconstituent of Γ with respect to x. Let $E_0, E_1, \ldots, E_D$ denote a Q-polynomial ordering of the primitive idempotents for A and let $A^* = A^*(x)$ denote the corresponding dual adjacency matrix. The *subconstituent algebra* $T = T(x)$ is the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A and A^*. Let W denote an irreducible T-module. By the *displacement* of W we mean $\rho + \tau + d - D$ where $\rho = \min\{i|E^*_i W \neq 0\}$, $\tau = \min\{i|E_i W \neq 0\}$, $d = |\{i|E_i W \neq 0\}| - 1$. We show the displacement of W is nonnegative and at most D. Let $V = \mathbb{C}^X$ denote the standard module. We show $V = \sum_{\eta=0}^{D} V_\eta$ (orthogonal direct sum) where V_η denotes the subspace of V spanned by the irreducible T-modules which have displacement η. This is the *displacement decomposition* with respect to x. For $-1 \leq i, j \leq D$ we define $V_{ij} = (E^*_0 V + \cdots + E^*_i V) \cap (E_0 V + \cdots + E_j V)$. We show $V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij}$ (direct sum) where $\tilde{V}_{ij}$ denotes the the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij} with respect to the Hermitean dot product. This direct sum is the *split decomposition* with respect to x. The above decompositions are related as follows. For $0 \leq \eta \leq D$ we show $V_\eta = \sum \tilde{V}_{ij}$ where the sum is over all ordered pairs i, j such that $0 \leq i, j \leq D$ and $i + j = D + \eta$. Using this we obtain the following results. For $0 \leq i, j \leq D$ we show $V_{ij} = 0$ if $i + j < D$. For $0 \leq i \leq D$ let θ_i (resp. θ^*_i) denote the eigenvalue of A (resp. A^*) for E_i (resp. E^*_i). For $0 \leq i, j \leq D$ we show $(A - \theta_j I)\tilde{V}_{ij} \subseteq \tilde{V}_{i+1,j-1}$ and $(A^* - \theta^*_i I)\tilde{V}_{ij} \subseteq \tilde{V}_{i-1,j+1}$, where $\tilde{V}_{rs} := 0$ unless $r, s \in \{0, 1, \ldots, D\}$. We finish with an application related to the work of Brouwer, Godsil, Koolen and Martin [4] concerning the dual width of a subset of X.

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2 Preliminaries concerning distance-regular graphs

In this section we review some definitions and basic concepts concerning distance-regular graphs. For more background information we refer the reader to [1], [3], [19] or [29].

Let X denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by X and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by X and whose entries are in $\mathbb{C}$. We observe $\mathrm{Mat}_X(\mathbb{C})$ acts on V by left multiplication. We call V the *standard module*. We endow V with the Hermitean inner product $\langle\, ,\, \rangle$ which satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where t denotes transpose and $-$ denotes complex conjugation. For all $y \in X$, let $\hat{y}$ denote the element of V with a 1 in the y coordinate and 0 in all other coordinates. We observe $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, with vertex set X and edge set R. Let ∂ denote the path-length distance function for Γ, and set $D := \max\{\partial(x, y) \mid x, y \in X\}$. We call D the *diameter* of Γ. We say Γ is *distance-regular* whenever for all integers h, i, j $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x, z) = i, \partial(z, y) = j\}|$$

is independent of x and y. The p^h_{ij} are called the *intersection numbers* of Γ.

For the rest of this paper we assume Γ is distance-regular with diameter $D \geq 3$.

We recall the Bose-Mesner algebra of Γ. For $0 \leq i \leq D$ let A_i denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$ with xy entry

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (x, y \in X).$$

We call A_i the ith *distance matrix* of Γ. We abbreviate $A := A_1$ and call this the *adjacency matrix* of Γ. We observe (i) $A_0 = I$; (ii) $\sum_{i=0}^D A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \leq i \leq D)$; (iv) $A_i^t = A_i$ $(0 \leq i \leq D)$; (v) $A_iA_j = \sum_{h=0}^D p^h_{ij}A_h$ $(0 \leq i, j \leq D)$, where I (resp. J) denotes the identity matrix (resp. all 1's matrix) in $\mathrm{Mat}_X(\mathbb{C})$. By these facts $A_0, A_1, \ldots, A_D$ form a basis for a commutative subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$. We call M the *Bose-Mesner algebra* of Γ. It turns out A generates M [1, p. 190]. By [3, p. 45], M has a second basis $E_0, E_1, \ldots, E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^D E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \leq i \leq D)$; (iv) $E_i^t = E_i$ $(0 \leq i \leq D)$; (v) $E_iE_j = \delta_{ij}E_i$ $(0 \leq i, j \leq D)$. We call $E_0, E_1, \ldots, E_D$ the *primitive idempotents* of Γ.

We recall the eigenvalues of Γ. Since $E_0, E_1, \ldots, E_D$ form a basis for M there exist complex scalars $\theta_0, \theta_1, \ldots, \theta_D$ such that $A = \sum_{i=0}^D \theta_iE_i$. Observe $AE_i = E_iA = \theta_iE_i$ for $0 \leq i \leq D$. By [1, p. 197] the scalars $\theta_0, \theta_1, \ldots, \theta_D$ are in $\mathbb{R}$. Observe $\theta_0, \theta_1, \ldots, \theta_D$ are mutually distinct since A generates M. We call θ_i the *eigenvalue* of Γ associated with E_i $(0 \leq i \leq D)$. Observe

$$V = E_0V + E_1V + \cdots + E_DV \qquad \text{(orthogonal direct sum).}$$

For $0 \leq i \leq D$, the space E_iV is the eigenspace of A associated with θ_i.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe $A_i \circ A_j = \delta_{ij} A_i$ for $0 \le i, j \le D$, so M is closed under $\circ$. Thus there exist complex scalars q^h_{ij} $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \le i, j \le D).$$

By [2, p. 170], q^h_{ij} is real and nonnegative for $0 \le h, i, j \le D$. The q^h_{ij} are called the *Krein parameters*. The graph Γ is said to be *Q-polynomial* (with respect to the given ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents) whenever for $0 \le h, i, j \le D$, $q^h_{ij} = 0$ (resp. $q^h_{ij} \neq 0$) whenever one of h, i, j is greater than (resp. equal to) the sum of the other two [1, 4, 5, 6, 9, 10, 14, 15, 23, 24]. From now on assume Γ is Q-polynomial with respect to $E_0, E_1, \ldots, E_D$.

We recall the dual Bose-Mesner algebra of Γ. Fix a vertex $x \in X$. We view x as a "base vertex". For $0 \le i \le D$ let $E^*_i = E^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry

$$(E^*_i)_{yy} = \begin{cases} 1, & \text{if } \partial(x, y) = i \\ 0, & \text{if } \partial(x, y) \neq i \end{cases} \qquad (y \in X). \tag{1}$$

We call E^*_i the ith *dual idempotent* of Γ with respect to x [29, p. 378]. We observe (i) $\sum_{i=0}^{D} E^*_i = I$; (ii) $\overline{E^*_i} = E^*_i$ $(0 \le i \le D)$; (iii) $E^{*t}_i = E^*_i$ $(0 \le i \le D)$; (iv) $E^*_i E^*_j = \delta_{ij} E^*_i$ $(0 \le i, j \le D)$. By these facts $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for a commutative subalgebra $M^* = M^*(x)$ of $\mathrm{Mat}_X(\mathbb{C})$. We call M^* the *dual Bose-Mesner algebra* of Γ with respect to x [29, p. 378]. For $0 \le i \le D$ let $A^*_i = A^*_i(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with yy entry $(A^*_i)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A^*_0, A^*_1, \ldots, A^*_D$ is a basis for M^* [29, p. 379]. Moreover (i) $A^*_0 = I$; (ii) $\overline{A^*_i} = A^*_i$ $(0 \le i \le D)$; (iii) $A^{*t}_i = A^*_i$ $(0 \le i \le D)$; (iv) $A^*_i A^*_j = \sum_{h=0}^{D} q^h_{ij} A^*_h$ $(0 \le i, j \le D)$ [29, p. 379]. We call $A^*_0, A^*_1, \ldots, A^*_D$ the *dual distance matrices* for Γ with respect to x. We abbreviate $A^* = A^*_1$ and call this the *dual adjacency matrix* for Γ with respect to x. The matrix A^* generates M^* [29, Lemma 3.11].

We recall the dual eigenvalues of Γ. Since $E^*_0, E^*_1, \ldots, E^*_D$ form a basis for M^*, there exist complex scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ such that $A^* = \sum_{i=0}^{D} \theta^*_i E^*_i$. Observe $A^* E^*_i = E^*_i A^* = \theta^*_i E^*_i$ for $0 \le i \le D$. By [29, Lemma 3.11] the scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are in $\mathbb{R}$. The scalars $\theta^*_0, \theta^*_1, \ldots, \theta^*_D$ are mutually distinct since A^* generates M^*. We call θ^*_i the *dual eigenvalue* of Γ associated with E^*_i $(0 \le i \le D)$.

We recall the subconstituents of Γ. From (1) we find

$$E^*_i V = \mathrm{span}\{\hat{y} \mid y \in X, \quad \partial(x, y) = i\} \qquad (0 \le i \le D). \tag{2}$$

By (2) and since $\{\hat{y} \mid y \in X\}$ is an orthonormal basis for V we find

$$V = E^*_0 V + E^*_1 V + \cdots + E^*_D V \qquad \text{(orthogonal direct sum).}$$

For $0 \le i \le D$, the space $E^*_i V$ is the eigenspace of A^* associated with θ^*_i. We call $E^*_i V$ the *ith subconstituent* of Γ with respect to x.

We recall the subconstituent algebra of Γ. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*. We call T the *subconstituent algebra* (or *Terwilliger algebra*) of Γ with respect to x [29, Definition 3.3]. We observe T is generated by A and A^*. We observe T has finite dimension. Moreover T is semi-simple; the reason is T is closed under the conjugate transpose map [12, p. 157]. See [7, 8, 11, 16, 17, 18, 20, 26, 29, 30, 31] for more information on the subconstituent algebra.

For the rest of this paper we adopt the following notational convention.

Definition 2.1 We assume $\Gamma = (X, R)$ is a distance-regular graph with diameter $D \geq 3$. We assume Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. We fix $x \in X$ and write $A^* = A^*(x)$, $E_i^* = E_i^*(x)$ $(0 \leq i \leq D)$, $T = T(x)$. We abbreviate $V = \mathbb{C}^X$. For notational convenience we define $E_{-1} = 0$, $E_{D+1} = 0$ and $E_{-1}^* = 0$, $E_{D+1}^* = 0$.

We have some comments.

Lemma 2.2 *[29, Lemma 3.2] With reference to Definition 2.1, the following (i), (ii) hold.*

(i) $AE_i^*V \subseteq E_{i-1}^*V + E_i^*V + E_{i+1}^*V$ $(0 \leq i \leq D)$.

(ii) $A^*E_iV \subseteq E_{i-1}V + E_iV + E_{i+1}V$ $(0 \leq i \leq D)$.

Lemma 2.3 *With reference to Definition 2.1, the following (i)–(iv) hold.*

(i) $A\sum_{h=0}^{i} E_h^*V \subseteq \sum_{h=0}^{i+1} E_h^*V$ $(0 \leq i \leq D)$.

(ii) $(A - \theta_i I)\sum_{h=0}^{i} E_hV = \sum_{h=0}^{i-1} E_hV$ $(0 \leq i \leq D)$.

(iii) $A^*\sum_{h=0}^{i} E_hV \subseteq \sum_{h=0}^{i+1} E_hV$ $(0 \leq i \leq D)$.

(iv) $(A^* - \theta_i^* I)\sum_{h=0}^{i} E_h^*V = \sum_{h=0}^{i-1} E_h^*V$ $(0 \leq i \leq D)$.

Proof: (i) Immediate from Lemma 2.2(i).
(ii) Recall $AE_j = \theta_j E_j$ for $0 \leq j \leq D$.
(iii) Immediate from Lemma 2.2(ii).
(iv) Recall $A^*E_j^* = \theta_j^* E_j^*$ for $0 \leq j \leq D$. □

3 The irreducible T-modules

In this section we recall some results on T-modules for later use.

With reference to Definition 2.1, by a *T-module* we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let W denote a T-module. Then W is said to be *irreducible* whenever W is nonzero and W contains no T-modules other than 0 and W. Let W, W' denote T-modules. By an *isomorphism of T-modules* from W to W' we mean an isomorphism of vector spaces $\sigma : W \to W'$ such that $(\sigma B - B\sigma)W = 0$ for all $B \in T$. The modules W, W'

are said to be *isomorphic as T-modules* whenever there exists an isomorphism of T-modules from W to W'. Any two nonisomorphic irreducible T-modules are orthogonal [7, Lemma 3.3].

Let W denote a T-module and let W' denote a T-module contained in W. Then the orthogonal complement of W' in W is a T-module [18, p. 802]. It follows that each T-module is an orthogonal direct sum of irreducible T-modules. In particular V is an orthogonal direct sum of irreducible T-modules.

Let W denote an irreducible T-module. By the *endpoint* of W we mean $\min\{i|0 \leq i \leq D,\ E_i^*W \neq 0\}$. By the *diameter* of W we mean $|\{i|0 \leq i \leq D,\ E_i^*W \neq 0\}| - 1$. By the *dual endpoint* of W we mean $\min\{i|0 \leq i \leq D,\ E_iW \neq 0\}$. By the *dual diameter* of W we mean $|\{i|0 \leq i \leq D,\ E_iW \neq 0\}| - 1$. The diameter of W is equal to the dual diameter of W [23, Corollary 3.3]. There exists a unique irreducible T-module with diameter D. We call this module the *primary* T-module. The primary T-module has basis $A_0\hat{x}, \ldots, A_D\hat{x}$ [29, Lemma 3.6].

Lemma 3.1 *[29, Lemma 3.4, Lemma 3.9, Lemma 3.12] With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then ρ, τ, d are nonnegative integers such that $\rho + d \leq D$ and $\tau + d \leq D$. Moreover the following (i)–(iv) hold.*

(i) $E_i^*W \neq 0$ *if and only if* $\rho \leq i \leq \rho + d, \quad (0 \leq i \leq D)$.

(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^*W \qquad$ *(orthogonal direct sum)*.

(iii) $E_iW \neq 0$ *if and only if* $\tau \leq i \leq \tau + d, \quad (0 \leq i \leq D)$.

(iv) $W = \sum_{h=0}^{d} E_{\tau+h}W \qquad$ *(orthogonal direct sum)*.

Lemma 3.2 *With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then the following (i), (ii) hold.*

(i) $AE_{\rho+i}^*W \subseteq E_{\rho+i-1}^*W + E_{\rho+i}^*W + E_{\rho+i+1}^*W \qquad (0 \leq i \leq d)$.

(ii) $A^*E_{\tau+i}W \subseteq E_{\tau+i-1}W + E_{\tau+i}W + E_{\tau+i+1}W \qquad (0 \leq i \leq d)$.

Proof: (i) Follows from Lemma 2.2(i) and since $E_j^*W = E_j^*V \cap W$ for $0 \leq j \leq D$.
(ii) Follows from Lemma 2.2(ii) and since $E_jW = E_jV \cap W$ for $0 \leq j \leq D$. □

Remark 3.3 With reference to Definition 2.1, let W denote an irreducible T-module. Then A and A^* act on W as a tridiagonal pair in the sense of [21, Definition 1.1]. This follows from Lemma 3.1, Lemma 3.2, and since A, A^* together generate T. See [22, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41] for information on tridiagonal pairs.

Lemma 3.4 *[6, Lemma 5.1, Lemma 7.1] With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then the following (i), (ii) hold.*

(i) $2\rho + d \geq D$.

(ii) $2\tau + d \geq D$.

Lemma 3.5 *With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Then*

$$W = \sum_{h=0}^{d} W_h \qquad (\textit{direct sum}), \tag{3}$$

where

$$W_h = (E^*_\rho W + \cdots + E^*_{\rho+h} W) \cap (E_\tau W + \cdots + E_{\tau+d-h} W) \qquad (0 \leq h \leq d). \tag{4}$$

Proof: Immediate from Remark 3.3 and [21, Theorem 4.6]. □

Remark 3.6 The sum (3) is not orthogonal in general.

4 The displacement decomposition

In this section we introduce the *displacement decomposition* for the standard module.

Definition 4.1 With reference to Definition 2.1, let W denote an irreducible T-module. By the *displacement* of W we mean the integer $\rho + \tau + d - D$, where ρ, τ, d denote, respectively, the endpoint, dual endpoint, and diameter of W.

Lemma 4.2 *With reference to Definition 2.1, let W denote an irreducible T-module with displacement η. Then $0 \leq \eta \leq D$.*

Proof: Let ρ, τ, d denote, respectively, the endpoint, dual endpoint, and diameter of W. By Lemma 3.4 we have $2\rho + d \geq D$ and $2\tau + d \geq D$; adding these inequalities we find $\rho + \tau + d \geq D$ so $\eta \geq 0$. By Lemma 3.1 we have $\rho \leq D$ and $\tau + d \leq D$. Combining these inequalities we find $\rho + \tau + d \leq 2D$ so $\eta \leq D$. □

Definition 4.3 With reference to Definition 2.1, For $0 \leq \eta \leq D$ we let V_η denote the subspace of V spanned by the irreducible T-modules which have displacement η. We observe V_η is a T-module.

Lemma 4.4 *With reference to Definition 2.1, we have*

$$V = \sum_{\eta=0}^{D} V_\eta \qquad (\textit{orthogonal direct sum}). \tag{5}$$

Proof: We mentioned earlier that V is spanned by the irreducible T-modules. By Lemma 4.2 and Definition 4.3, each of these modules is contained in one of $V_0, V_1, \ldots, V_D$. Therefore $V = \sum_{\eta=0}^{D} V_\eta$. We show this sum is orthogonal and direct. To do this, it suffices to show $V_0, V_1, \ldots, V_D$ are mutually orthogonal. For distinct integers i, j $(0 \leq i, j \leq D)$ observe V_i, V_j are orthogonal since the isomorphism classes of irreducible T-modules which span V_i are distinct from the isomorphism classes of irreducible T-modules which span V_j. We have now shown $V_0, V_1, \ldots, V_D$ are mutually orthogonal. It follows the sum $\sum_{\eta=0}^{D} V_\eta$ is orthogonal and direct. $\square$

Definition 4.5 We call the sum (5) the *displacement decomposition* of V with respect to x.

5 The split decomposition

In this section we introduce the *split decomposition* of the standard module.

Definition 5.1 *With reference to Definition 2.1, For $-1 \leq i, j \leq D$ we define*

$$V_{ij} = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_jV). \tag{6}$$

We observe $V_{ij} = 0$ if $i = -1$ or $j = -1$.

In the following three lemmas we make some observations concerning Definition 5.1. In each case the proof is routine and omitted.

Lemma 5.2 *With reference to Definition 2.1, For $0 \leq i, j \leq D$ the space V_{ij} consists of those vectors $v \in V$ such that $E_h^*v = 0$ for $i < h \leq D$ and $E_hv = 0$ for $j < h \leq D$.*

Lemma 5.3 *With reference to Definition 2.1, we have $V_{i-1,j} \subseteq V_{ij}$ and $V_{i,j-1} \subseteq V_{ij}$ for $0 \leq i, j \leq D$.*

Lemma 5.4 *With reference to Definition 2.1, the following (i)–(iii) hold.*

(i) $V_{iD} = E_0^*V + E_1^*V + \cdots + E_i^*V$ $(0 \leq i \leq D)$.

(ii) $V_{Dj} = E_0V + E_1V + \cdots + E_jV$ $(0 \leq j \leq D)$.

(iii) $V_{DD} = V$.

It will turn out that $V_{ij} = 0$ if $i + j < D$, $(0 \leq i, j \leq D)$. We will establish this later in the paper.

Definition 5.5 With reference to Definition 2.1, for $0 \leq i, j \leq D$ we let $\tilde{V}_{ij}$ denote the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij}. For notational convenience we define $\tilde{V}_{ij} := 0$ if $i < 0$ or $i > D$ or $j < 0$ or $j > D$.

Our next goal is to show $V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij}$ (direct sum) for $0 \leq r, s \leq D$. We will use the following lemma.

Lemma 5.6 *With reference to Definition 2.1, we have*

$$dim\,\tilde{V}_{ij} = dim\,V_{ij} - dim\,V_{i,j-1} - dim\,V_{i-1,j} + dim\,V_{i-1,j-1} \tag{7}$$

for $0 \le i, j \le D$.

Proof: Let z denote the dimension of $V_{i,j-1} + V_{i-1,j}$. The space $\tilde{V}_{ij}$ is the orthogonal complement of $V_{i,j-1} + V_{i-1,j}$ in V_{ij} so $\dim \tilde{V}_{ij} + z = \dim V_{ij}$. Using Definition 5.1 we find $V_{i,j-1} \cap V_{i-1,j} = V_{i-1,j-1}$ so $z + \dim V_{i-1,j-1} = \dim V_{i,j-1} + \dim V_{i-1,j}$. From these comments we routinely obtain (7). $\square$

Theorem 5.7 *With reference to Definition 2.1, for* $0 \le r, s \le D$ *we have*

$$V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij} \qquad (\textit{direct sum}).$$

Proof: We first show

$$V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \tilde{V}_{ij}. \tag{8}$$

The proof is by induction on $r + s$. The result is trivial for $r + s = 0$ so assume $r + s > 0$. Recall $\tilde{V}_{rs}$ is the orthogonal complement of $V_{r,s-1} + V_{r-1,s}$ in V_{rs}. Therefore

$$V_{rs} = \tilde{V}_{rs} + V_{r,s-1} + V_{r-1,s}. \tag{9}$$

By induction we have both

$$V_{r,s-1} = \sum_{i=0}^{r} \sum_{j=0}^{s-1} \tilde{V}_{ij}, \qquad\qquad V_{r-1,s} = \sum_{i=0}^{r-1} \sum_{j=0}^{s} \tilde{V}_{ij}. \tag{10}$$

Combining (9), (10) we routinely obtain (8). We now show the sum (8) is direct. From Lemma 5.6 we routinely obtain

$$\dim V_{rs} = \sum_{i=0}^{r} \sum_{j=0}^{s} \dim \tilde{V}_{ij}$$

and it follows the sum (8) is direct. $\square$

Corollary 5.8 *With reference to Definition 2.1, we have*

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \tilde{V}_{ij} \qquad (\textit{direct sum}). \tag{11}$$

Proof: Set $r = D$ and $s = D$ in Theorem 5.7. $\square$

Definition 5.9 We call the sum (11) the *split decomposition* of V with respect to x. This decomposition is not orthogonal in general.

6 The displacement and split decompositions

In this section we consider how the displacement decomposition is related to the split decomposition. Our main result is the following. With reference to Definition 2.1, for $0 \leq \eta \leq D$ we show $V_\eta = \sum \tilde{V}_{ij}$, where the sum is over all ordered pairs i, j such that $0 \leq i, j \leq D$ and $i + j = D + \eta$. We begin with a lemma.

Lemma 6.1 *With reference to Definition 2.1, let W denote an irreducible T-module with endpoint ρ, dual endpoint τ, and diameter d. Let the subspaces $W_0, W_1, \ldots, W_d$ be as in Lemma 3.5. Then $W_h \subseteq \tilde{V}_{\rho+h,\tau+d-h}$ for $0 \leq h \leq d$.*

Proof: Comparing (4) and (6) we find $W_h \subseteq V_{\rho+h,\tau+d-h}$. We show W_h is orthogonal to $V_{\rho+h-1,\tau+d-h} + V_{\rho+h,\tau+d-h-1}$. For $w \in W_h$ and for $v \in V_{\rho+h-1,\tau+d-h}$ we show $\langle w, v\rangle = 0$. Let $W^\perp$ denote the orthogonal complement of W in V. Observe $V = W + W^\perp$ (direct sum) and that $W^\perp$ is a T-module. Observe there exists $w_1 \in W$ and $v_1 \in W^\perp$ such that $v = w_1 + v_1$. By the construction $w \in W$ and $v_1 \in W^\perp$ so $\langle w, v_1\rangle = 0$. We show $w_1 = 0$. By Lemma 5.2 and since $v \in V_{\rho+h-1,\tau+d-h}$ we find $E^*_i v = 0$ for $\rho + h \leq i \leq D$ and $E_j v = 0$ for $\tau + d - h + 1 \leq j \leq D$. Since $V = W + W^\perp$ is a direct sum of T-modules we find $E^*_i w_1 = 0$ for $\rho + h \leq i \leq D$ and $E_j w_1 = 0$ for $\tau + d - h + 1 \leq j \leq D$. Since $w_1 \in W$ and since W has endpoint ρ we have $E^*_i w_1 = 0$ for $0 \leq i \leq \rho - 1$. Similarly since W has dual endpoint τ we have $E_j w_1 = 0$ for $0 \leq j \leq \tau - 1$. From these comments we find

$$w_1 \in (E^*_\rho W + \cdots E^*_{\rho+h-1} W) \cap (E_\tau W + \cdots E_{\tau+d-h} W). \tag{12}$$

Using (4) we find the intersection on the right in (12) is equal to $W_h \cap W_{h-1}$, where $W_{-1} = 0$. The sum (3) is direct so $W_h \cap W_{h-1} = 0$. We now see $w_1 = 0$. Now $v = v_1$ so $\langle w, v\rangle = 0$. We have now shown W_h is orthogonal to $V_{\rho+h-1,\tau+d-h}$. By a similar argument we find W_h is orthogonal to $V_{\rho+h,\tau+d-h-1}$. We conclude $W \subseteq \tilde{V}_{\rho+h,\tau+d-h}$. $\square$

Theorem 6.2 *With reference to Definition 2.1, the following (i)–(iii) hold.*

(i) For $0 \leq \eta \leq D$ we have $V_\eta = \sum \tilde{V}_{ij}$, where the sum is over all ordered pairs i, j such that $0 \leq i, j \leq D$ and $i + j = D + \eta$.

(ii) $\tilde{V}_{ij} = 0$ if $i + j < D$, $\qquad (0 \leq i, j \leq D)$.

(iii) $V_{ij} = 0$ if $i + j < D$, $\qquad (0 \leq i, j \leq D)$.

Proof: (i), (ii) For $-D \leq \eta \leq D$ we define $V'_\eta = \sum \tilde{V}_{ij}$ where the sum is over all ordered pairs i, j such that $0 \leq i, j \leq D$ and $i + j = D + \eta$. Using (11) we find $V = \sum_{\eta=-D}^{D} V'_\eta$ (direct sum). We show $V'_\eta = 0$ for $-D \leq \eta < 0$ and $V'_\eta = V_\eta$ for $0 \leq \eta \leq D$. Since the sums $V = \sum_{\eta=0}^{D} V_\eta$ and $V = \sum_{\eta=-D}^{D} V'_\eta$ are direct it suffices to show $V_\eta \subseteq V'_\eta$ for $0 \leq \eta \leq D$. Let η be given. Let W denote an irreducible T-module with displacement η. Combining Lemma 3.5 and Lemma 6.1 we find $W \subseteq V'_\eta$. The space V_η is spanned by the irreducible T-modules which have displacement η; therefore $V_\eta \subseteq V'_\eta$. We have now shown $V_\eta \subseteq V'_\eta$ for $0 \leq \eta \leq D$. We conclude $V'_\eta = 0$ for $-D \leq \eta < 0$ and $V'_\eta = V_\eta$ for $0 \leq \eta \leq D$. Lines (i), (ii) follow.

(iii) Combine (ii) above with Theorem 5.7. $\square$

We have some comments.

Theorem 6.3 *With reference to Definition 2.1, for* $0 \leq i, j \leq D$ *such that* $i + j \geq D$, *and for* $0 \leq \eta \leq D$,

$$V_{ij} \cap V_\eta = \sum \tilde{V}_{rs},$$

where the sum is over all ordered pairs r, s *such that* $0 \leq r \leq i$ *and* $0 \leq j \leq s$ *and* $r + s - D = \eta$.

Proof: Combine Theorem 5.7 and Theorem 6.2(i). $\square$

Corollary 6.4 *With reference to Definition 2.1, for* $0 \leq i, j \leq D$ *such that* $i + j \geq D$, *we have* $\tilde{V}_{ij} = V_{ij} \cap V_\eta$ *where* $\eta = i + j - D$.

Proof: Apply Theorem 6.3 with $\eta = i + j - D$. $\square$

7 The action of A and A^* on the split decomposition

In this section we describe how the adjacency matrix and the dual adjacency matrix act on the split decomposition.

Theorem 7.1 *With reference to Definition 2.1, the following (i), (ii) hold.*

(i) $(A - \theta_j I)\tilde{V}_{ij} \subseteq \tilde{V}_{i+1,j-1}$ $(0 \leq i, j \leq D)$.

(ii) $(A^* - \theta^*_i I)\tilde{V}_{ij} \subseteq \tilde{V}_{i-1,j+1}$ $(0 \leq i, j \leq D)$.

Proof: (i) Assume $i + j \geq D$; otherwise $\tilde{V}_{ij} = 0$ and the result is trivial. For convenience we treat the cases $i = D$ and $i < D$ separately. To obtain the result for the case $i = D$, we show $(A - \theta_j I)\tilde{V}_{Dj} = 0$. From Corollary 6.4 (with $i = D$ and $\eta = j$) we have $\tilde{V}_{Dj} = V_{Dj} \cap V_j$. Using Lemma 2.3(ii) and Lemma 5.4(ii) we find $(A - \theta_j I)V_{Dj} = V_{D,j-1}$. Therefore $(A - \theta_j I)\tilde{V}_{Dj} \subseteq V_{D,j-1}$. Recall V_j is a T-module so $(A - \theta_j I)V_j \subseteq V_j$. Therefore $(A - \theta_j I)\tilde{V}_{Dj} \subseteq V_j$. Now

$$\begin{aligned} (A - \theta_j I)\tilde{V}_{Dj} &\subseteq V_{D,j-1} \cap V_j \\ &= 0 \end{aligned}$$

in view of Theorem 6.3. We have now shown $(A - \theta_j I)\tilde{V}_{Dj} = 0$ so we are done for the case $i = D$. Next assume $i < D$. From Corollary 6.4 we have $\tilde{V}_{ij} = V_{ij} \cap V_\eta$ where $\eta = i + j - D$. Using Lemma 2.3 and (6) we find $(A - \theta_j I)V_{ij} \subseteq V_{i+1,j-1}$. Therefore $(A - \theta_j I)\tilde{V}_{ij} \subseteq V_{i+1,j-1}$. Recall V_η is a T-module so $(A - \theta_j I)V_\eta \subseteq V_\eta$. Therefore $(A - \theta_j I)\tilde{V}_{ij} \subseteq V_\eta$. Now

$$\begin{aligned} (A - \theta_j I)\tilde{V}_{ij} &\subseteq V_{i+1,j-1} \cap V_\eta \\ &= \tilde{V}_{i+1,j-1} \end{aligned}$$

in view of Corollary 6.4.
(ii) Similar to the proof of (i) above. $\square$

8 An application

In this section we give an application of Theorem 6.2(iii). We first give two definitions.

Definition 8.1 Let $\Gamma = (X, R)$ denote a distance-regular graph with standard module V. For $v \in V$, by the *support* of v we mean the subset of X consisting of those vertices y such that coordinate y of v is nonzero.

Definition 8.2 [4, Section 4] Let Γ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Let v denote a nonzero vector in the standard module V. By the *dual width* of v we mean

$$\max\{i | 0 \leq i \leq D,\ E_i v \neq 0\}.$$

Theorem 8.3 *Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D \geq 3$. Assume Γ is Q-polynomial with respect to the ordering $E_0, E_1, \ldots, E_D$ of the primitive idempotents. Let v denote a nonzero vector in the standard module V and let g denote the corresponding dual width from Definition 8.2. Then for all $x \in X$ there exists y in the support of v such that*

$$\partial(x, y) \geq D - g. \tag{13}$$

Proof: We assume the result is false and obtain a contradiction. Since the result is false, there exists $x \in X$ such that $\partial(x, y) < D - g$ for all vertices y in the support of v. Abbreviate $E_i^* = E_i^*(x)$ for $0 \leq i \leq D$. Then $v \in E_0^*V + \cdots + E_f^*V$ where $f = D - g - 1$. Using Definition 8.2 we find $v \in E_0V + \cdots + E_gV$. Now

$$\begin{aligned} v &\in (E_0^*V + \cdots + E_f^*V) \cap (E_0V + \cdots + E_gV) \\ &= V_{fg}. \end{aligned}$$

We mentioned $f = D - g - 1$ so $f + g < D$; combining this with Theorem 6.2(iii) we find $V_{fg} = 0$. Now $v = 0$ for a contradiction. The result follows. $\square$

Remark 8.4 Referring to Theorem 8.3, pick any $x \in X$. If v is not orthogonal to the primary module for $T(x)$ then (13) follows from [25, Equation (2.8)]. See also [4, Lemma 1].

9 Directions for further research

In this section we give some suggestions for further research.

Problem 9.1 With reference to Definition 2.1, recall that for $0 \leq i, j \leq D$ the space $\tilde{V}_{ij}$ depends on x. Does the dimension of $\tilde{V}_{ij}$ depend on x?

Problem 9.2 With reference to Definition 2.1, let W denote an irreducible T-module and consider the multiplicity with which W appears in V. In general this multiplicity is not determined by the intersection numbers of Γ [26]. Is this multiplicity determined by the intersection numbers of Γ and the scalars $\{\dim \tilde{V}_{ij} \mid 0 \leq i, j \leq D\}$?

Problem 9.3 Many Q-polynomial distance-regular graphs exist on the top fiber of a ranked poset [13], [27], [28]. For these graphs investigate how the poset structure is related to the split decomposition.

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