Sequent Calculus for Justifications

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Abstract

We present a cut-free sequent calculus that can internalize its own proofs, providing a new justification system for modal logic.
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Introduction

The Logic of Proofs was introduced by Artemov in [1] (a survey article can be found at [2]). It is based on a notion of proof polynomial, which allows to talk about proofs and propositions in the same language. In a sense it is a refinement of modal logic — a formula $\Box A$, which is typically interpreted as *A is provable* or *A is known* is replaced by an expression of the form $t : A$, that can be read as *t is a formal proof of A* or *A is known because of the evidence provided by t*. Several variants of models and realization technics were later developed for different versions of this logic (see, for example [6], [7], [5]).

Since in these logics proof polynomial typically denote proofs in a Hilbert-style system, the sequent calculi developed for them do not internalize their own proofs and it is hard to describe and explore the relation between proof polynomials and cut elimination and other important properties of proofs in Gentzen-style systems. Therefore there is a need to for a sequent calculus that can internalize its own proofs. Some attempts in that direction were made in [3], but system presented there did not have the subformula property and did not include the function symbol for cut, and thus only could formalize cut-free proofs.

The system presented in this paper can formalize all its own proofs. The main difference from the previous approaches is that here the construction $t : A$ is not used. Instead, proof polynomial are considered to be formulas themselves. When we write a proof polynomial t in this system we intend it to be read as *t is a valid proof in our system.* Proof polynomials here are designed to contain all available information about the proof. In particular, it is always possible to determine which sequent is proved by a given proof polynomial, thus there is no need to specify it. That idea allows us to build a cut-free system that satisfy a version of the subformula property.

The calculus presented in this paper can realize the modal logic K45. As a part of realization procedure we translate our formulas into the signed nested sequent calculus — a modification of nested sequent calculus introduced in [4] and later used in [5]. The signed nested sequent calculus allows for more flexible

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approach to working with formulas while preserving all soundness and completeness properties the nested sequent calculus has.

Sequent Calculus for Classical Logic

In this section we describe a sequent calculus SC for classical propositional logic that will be further expanded to accomodate proof expressions.

First we have a set of propositional variables $\{x_1, x_2, \ldots\}$. We build formulas from them using standard classical connectives — binary $\vee$, $\wedge$, $\to$ and unary $\neg$. We denote the set of all formulas by Fm. A set of signed formulas Fm° is defined as $\{A^\circ \mid A \in \mathrm{Fm}\} \cup \{A^\bullet \mid A \in \mathrm{Fm}\}$. Sequents of this system are of the form $\vdash \Gamma$, where Γ is a sequence of signed formulas. The axiom and rules of the system are the following:

$$\frac{}{\vdash A^\circ, A^\bullet}\ Ax,$$

$$\frac{\vdash \Gamma}{\vdash A^*, \Gamma}\ \omega^0_{A^*}, \qquad \frac{\vdash A^*, A^*, \Gamma}{\vdash A^*, \Gamma}\ \gamma^0,$$

$$\frac{\vdash A^*, B^\star, \Gamma}{\vdash B^\star, A^*, \Gamma}\ \mu^0, \qquad \frac{\vdash A^*, \Gamma}{\vdash \Gamma, A^*}\ \nu^0,$$

$$\frac{\vdash A^\circ, B^\circ, \Gamma}{\vdash (A \vee B)^\circ, \Gamma}\ \beta^0, \qquad \frac{\vdash A^\bullet, \Gamma \quad \vdash B^\bullet, \Delta}{\vdash (A \vee B)^\bullet, \Gamma, \Delta}\ \times^0,$$

$$\frac{\vdash A^\circ, \Gamma \quad \vdash B^\circ, \Delta}{\vdash (A \wedge B)^\circ, \Gamma, \Delta}\ \times^0, \qquad \frac{\vdash A^\bullet, B^\bullet, \Gamma}{\vdash (A \wedge B)^\bullet, \Gamma}\ \beta^0,$$

$$\frac{\vdash A^\bullet, B^\circ, \Gamma}{\vdash (A \to B)^\circ, \Gamma}\ \beta^0, \qquad \frac{\vdash A^\circ, \Gamma \quad \vdash B^\bullet, \Delta}{\vdash (A \to B)^\bullet, \Gamma, \Delta}\ \times^0,$$

$$\frac{\vdash A^\bullet, \Gamma}{\vdash (\neg A)^\circ, \Gamma}\ \chi^0, \qquad \frac{\vdash A^\circ, \Gamma}{\vdash (\neg A)^\bullet, \Gamma}\ \chi^0.$$

Here $A^*, B^\star \in \mathrm{Fm}^\circ$. It can be easily shown that a formula A is a classical tautology if and only if A° is derivable in SC.

Proof Expressions

Now we want to extend our language to accommodate proof expressions.

For every sequent $\vdash \Gamma$ we add to our language a *proof constant* $p_{[\vdash\Gamma]}$.

Proof expressions are build from proof constants using *operators* from the set

$$Op = \{\omega^n_{A^*}, \gamma^n, \mu^n, \nu^n, \beta^n, \chi^n, !^n, ?^n, \pi^n \mid n \in \mathbb{N}, A^* \in \mathrm{Fm}^\circ\}$$

and connectives from the set $\mathrm{Co} = \{\times^n, \backslash^n, + \mid n \in \mathbb{N}\}$. If t and s are proof expressions, then so are αt and $(t \text{ con } s)$, where $\alpha \in \mathrm{Op}$ and $\mathrm{con} \in \mathrm{Co}$. Proof expressions are considered formulas and can be used to build more complex formulas. These new formulas add new operators $\omega^n_{A^\circ}$ and $\omega^n_{A^\bullet}$. Sequents using these formulas add new proof constants etc.

Formally, it can be described by the following context-free grammar:

$$
\begin{array}{rcl}
Fm & = & x_i \mid (Fm \vee Fm) \mid (Fm \wedge Fm) \mid (Fm \to Fm) \mid \neg Fm \mid Pr \\
Seq & = & \vdash \mid Seq, Fm^\circ \mid Seq, Fm^\bullet \\
Pr & = & p_{[Seq]} \mid \alpha Pr \mid (Pr \,\mathrm{con}\, Pr)
\end{array}
$$

Here $\alpha \in \mathrm{Op}$, $\mathrm{con} \in \mathrm{Co}$.

Axioms and Rules

Here are the axioms and rules of the calculus SJ:

Axioms. All axioms of SC.
If $\vdash \Gamma$ is an axiom, then $\vdash p^\circ_{[\vdash\Gamma]}$ is also an axiom.

Propositional Rules. All of the SC rules.

Proof Expression Rules.

$$
\frac{\vdash t^\circ, \Gamma}{\vdash (\alpha^n t)^\circ, \Gamma}\ \alpha^{n+1}, \qquad \frac{\vdash t^\circ, \Gamma \quad \vdash s^\circ, \Delta}{\vdash (t \times^n s)^\circ, \Gamma, \Delta}\ \times^{n+1}.
$$

Here $\alpha \in \{\omega_{A^*}, \gamma, \mu, \nu, \beta, \chi\}$.

Positive and Negative Introspection.

$$
\frac{\vdash t^\circ, \Gamma}{\vdash (!^n t)^\circ, \Gamma}\ !^{n+1},
$$

$$
\frac{\vdash t^\bullet, \Gamma}{\vdash (?^0 t)^\circ, \Gamma}\ ?^1, \qquad \frac{\vdash t^\circ, \Gamma}{\vdash (?^n t)^\circ, \Gamma}\ ?^{n+1}(n > 0).
$$

Cut.

$$
\frac{\vdash A^\circ, \Gamma \quad \vdash A^\bullet, \Delta}{\vdash \Gamma, \Delta}\ \backslash^0, \qquad \frac{\vdash t^\circ, \Gamma \quad \vdash s^\circ, \Delta}{\vdash \Gamma, \Delta, (t \backslash^n s)^\circ}\ \backslash^{n+1}.
$$

Plus.

$$
\frac{\vdash t^\circ, s^\circ, \Gamma}{\vdash (t + s)^\circ, \Gamma}\ \pi^0, \qquad \frac{\vdash t^\circ, \Gamma}{\vdash (\pi^n t)^\circ, \Gamma}\ \pi^{n+1},
$$

From now on we will omit superscripts in proof operators and connectives if they are 0.

Cut Elimination

The calculus SC is very similar to the standard two-sided sequent calculus for the classical propositional logic — the formulas with $\bullet$ correspond to the formulas in the antecedent of a two-sided sequent and formulas with $\circ$ correspond to the formulas in the succedent. So the cut rule $\backslash$ is admissible and the standard cut elimination procedure works for SC.

The rules that were added to get the system SJ deal with proof expressions and they only create positive instances of proof expressions. So the standard cut elimination procedure also work for SJ and we can eliminate all instances of the rule $\backslash$.

Forgetful Projection

The mapping *seq* maps proof expressions to sequents of our system.

- $seq(p_{[\vdash\Gamma]}) = \vdash \Gamma$,
- If $seq(t) = \vdash \Gamma$, then $seq(\omega_{A^*} t) = \vdash A^*, \Gamma$,
- If $seq(t) = \vdash A^*, A^*, \Gamma$, then $seq(\gamma t) = \vdash A^*, \Gamma$,
- If $seq(t) = \vdash A^*, B^\star, \Gamma$, then $seq(\mu t) = \vdash B^\star, A^*, \Gamma$,
- If $seq(t) = \vdash A^*, \Gamma$, then $seq(\nu t) = \vdash \Gamma, A^*$,
- If $seq(t) = \vdash A^\circ, \Gamma$, then $seq(\chi t) = \vdash (\neg A)^\bullet, \Gamma$,
- If $seq(t) = \vdash A^\bullet, \Gamma$, then $seq(\chi t) = \vdash (\neg A)^\circ, \Gamma$,
- If $seq(t) = \vdash A^\circ, B^\circ, \Gamma$, then $seq(\beta t) = \vdash (A \vee B)^\circ, \Gamma$,
- If $seq(t) = \vdash A^\bullet, B^\circ, \Gamma$, then $seq(\beta t) = \vdash (A \to B)^\circ, \Gamma$,
- If $seq(t) = \vdash A^\bullet, B^\bullet, \Gamma$, then $seq(\beta t) = \vdash (B \wedge A)^\bullet, \Gamma$,
- If $seq(t) = \vdash q^\circ, r^\circ, \Gamma$, then $seq(\pi t) = \vdash (q + r)^\circ, \Gamma$,
- If $seq(t) = \vdash q^\circ, \Gamma$, then $seq(\alpha^{n+1} t) = \vdash (\alpha^n q)^\circ, \Gamma$,
- If $seq(t) = \vdash A^\circ, \Gamma$ and $seq(s) = \vdash B^\circ, \Delta$, then $seq(t \times s) = \vdash (A \wedge B)^\circ, \Gamma, \Delta$,
- If $seq(t) = \vdash A^\circ, \Gamma$ and $seq(s) = \vdash B^\bullet, \Delta$, then $seq(t \times s) = \vdash (A \to B)^\bullet, \Gamma, \Delta$,
- If $seq(t) = \vdash A^\bullet, \Gamma$ and $seq(s) = \vdash B^\circ, \Delta$, then $seq(t \times s) = \vdash (B \to A)^\bullet, \Gamma, \Delta$,
- If $seq(t) = \vdash A^\bullet, \Gamma$ and $seq(s) = \vdash B^\bullet, \Gamma$, then $seq(t \times s) = \vdash (A \vee B)^\bullet, \Gamma$,
- If $seq(t) = \vdash q^\circ, \Gamma$ and $seq(s) = \vdash r^\circ, \Delta$, then $seq(t \times^{n+1} s) = \vdash (q \times^n r)^\circ, \Gamma, \Delta$,
- If $seq(t) = \vdash A^\circ, \Gamma$ and $seq(s) = \vdash A^\bullet, \Delta$, then $seq(t \backslash s) = \vdash \Gamma, \Delta$,
- If $seq(t) = \vdash q^\circ, \Gamma$ and $seq(s) = \vdash r^\circ, \Delta$, then $seq(t \backslash^{n+1} s) = \vdash (q \backslash^n r)^\circ, \Gamma, \Delta$,

- For every t, $seq(!t) =\vdash t^\circ$,
- If $seq(t) =\vdash q^\circ, \Gamma$, then $seq(!^{n+1}t) =\vdash (!^n q)^\circ, \Gamma$,
- For every t, $seq(?t) =\vdash t^\bullet$,
- If $seq(t) =\vdash q^\circ, \Gamma$, then $seq(?^{n+1}t) =\vdash (?^n q)^\circ, \Gamma$,
- If $seq(t) =\vdash \Gamma$ and $seq(s) =\vdash \Delta$, then $seq(t + s) = \Gamma, \Delta$.

If the sequents do not conform to any variants, then we put $seq(\alpha t) = seq(t)$ and $seq(t \operatorname{con} s) = seq(s)$.

The mapping *mod* maps sequents of our system to formulas of modal logic. The mapping $form$ maps signed formulas of our system to modal formulas. They are defined the following way:

- $form(x_i^\circ) = x_i$,
- $form((\neg A)^\circ) = \neg form(A^\circ)$,
- $form((A \vee B)^\circ) = form(A^\circ) \vee form(B^\circ)$,
- $form((A \wedge B)^\circ) = form(A^\circ) \wedge form(B^\circ)$,
- $form((A \to B)^\circ) = form(A^\circ) \to form(B^\circ)$,
- $form(t^\circ) = \Box(mod(seq(t)))$,
- $form(A^\bullet) = \neg form(A^\circ)$
- $mod(\vdash A_1^*, \ldots, A_m^\star) = (form(A_1^*) \vee \ldots \vee (form(A_m^\star))$.

Flat Sequent Calculus

The mapping $\flat$ maps formulas, signed formulas, sequences of formulas, and sequents to formulas, signed formulas, sequences of formulas, and sequents that do not contain operators and proof connectives.

- $\flat(x_i) = x_i$,
- $\flat(\neg A) = \neg\flat(A)$,
- $\flat(A \vee B) = (\flat A \vee \flat B)$
- $\flat(A \wedge B) = (\flat A \wedge \flat B)$
- $\flat(A \to B) = (\flat A \to \flat B)$,
- $\flat(A^*) = (\flat A)^*$
- $\flat(\Gamma, A^*) = \flat(\Gamma), \flat(A^*)$,
- $\flat(t) = p_{[\flat(seq(t))]}$,
- $\flat(\vdash \Gamma) =\vdash \flat(\Gamma)$.

We will call the set of all such formulas Fm$\flat$.

Lemma 1. *For every sequent* $\vdash \Gamma$, $mod(\vdash \Gamma) = mod(\flat(\vdash \Gamma))$.

Consider the following system. The language is the same as in SJ, but without operators and proof connectives. It has axioms of SJ plus the following rules:

$$\frac{\vdash \Gamma}{flat(seq(\alpha^n p_{[\vdash\Gamma]})} \; \alpha^n, \qquad \frac{\vdash \Gamma \quad \vdash \Delta}{flat(seq(p_{[\vdash\Gamma]} \times^n p_{[\vdash\Delta]}))} \; \times^n.$$

Here $\alpha \in \{\omega_{A^*}, \gamma, \mu, \nu, \beta, \chi, \pi\}$.

$$\frac{\vdash \Gamma}{flat(seq(!^{n+1} p_{[\vdash\Gamma]}))} \; !^{n+1} \qquad \frac{\vdash \Gamma}{flat(seq(?^{n+1} p_{[\vdash\Gamma]}))} \; ?^{n+1}$$

$$\frac{\vdash \Gamma \quad \vdash \Delta}{flat(seq(p_{[\vdash\Gamma]} \backslash^n p_{[\vdash\Delta]}))} \; \backslash^n.$$

We will call this system SJ$\flat$.

The cut rule ($\backslash$) is also eliminable in SJ$\flat$.

Lemma 2. *For every sequent* $\vdash \Gamma$ *provable in* SJ *the sequent* $\flat(\vdash \Gamma)$ *is provable in* SJ$\flat$.

Proof. If we take a proof of a sequent $\vdash \Gamma$ in SJ and apply the function $\flat$ to all sequents in the proof, then we will get a proof of the sequent $\flat(\vdash \Gamma)$ in SJ$\flat$. □

Definition 1. *A sequent* $\vdash \Gamma$ *is called clean if while evaluating* $mod(\vdash \Gamma)$ *in any proof expression* t*, for which we have to compute* $seq(t)$ *we never use proof operators and proof connectives that represent rules inapplicable to corresponding sequents. Similarly we define clean formulas and clean proofs.*

All formulas in the language of SJ$\flat$ are clean.

Lemma 3. *Every provable* SJ$\flat$ *sequent has a clean proof.*

From now on we presume all proofs to be clean.

Subformulas

We define two different notions of subformulas.

Definition 2. *For a formula* A *we define the set of strong subformulas* $S(A)$ *inductively as follows:*

- $S(x_i) = \{x_i\}$.
- $S(\neg A) = \{\neg A\} \cup S(A)$.
- $S(A \vee B) = \{A \vee B\} \cup S(A) \cup S(B)$.
- $S(A \wedge B) = \{A \wedge B\} \cup S(A) \cup S(B)$.
- $S(A \to B) = \{A \to B\} \cup S(A) \cup S(B)$.
- $S(p_{[\vdash\Gamma]}) = \{p_{[\vdash\Gamma]}\}$.
- $S(\alpha^n t) = \{\alpha^n t\} \cup S(t)$.

- $S(tcon^n s) = \{tcon^n s\} \cup S(t) \cup S(s)$.

For a sequent $\vdash A_1^*, \ldots, A_n^*$ *we define* $S(\vdash A_1^*, \ldots, A_n^*) = S(A_1) \cup \ldots \cup S(A_n)$.

The calculus SJ has strong subformula property.

For the calculus SJ♭ we define another notion of subformulas.

Definition 3. *For a formula* $A \in \mathrm{Fm}\flat$ *we define the set of subformulas* $s(A)$ *inductively as follows:*

- $s(x_i) = \{x_i\}$.
- $s(\neg A) = \{\neg A\} \cup s(A)$.
- $s(A \vee B) = \{A \vee B\} \cup s(A) \cup s(B)$.
- $s(A \wedge B) = \{A \wedge B\} \cup s(A) \cup s(B)$.
- $s(A \to B) = \{A \to B\} \cup s(A) \cup s(B)$.
- $s(p_{[\vdash\Gamma]}) = \{p_{[\vdash\Gamma]}\} \cup s(\Gamma)$.

For a sequent $\vdash A_1^*, \ldots, A_n^*$ *we define* $s(\vdash A_1^*, \ldots, A_n^*) = s(A_1) \cup \ldots \cup s(A_n)$.

Also we can define positive and negative subformulas in a natural way.

Annotated Sequents and Proofs

An *annotated* SJ♭ *sequent* is a SJ♭ sequent whose proof subformulas are labelled with SJ proof expressions.

A *realization* of an annotated sequent (or a subformula of an annotated sequent) is a SJ sequent (or formula) that is obtained by substituting all proof expressions that are strong subformulas with corresponding annotations.

We will say that annotation of a sequent is *coordinated*, if for every proof subformula $p_{[\vdash\Gamma]}$ annotated with t, t is clean and $seq(t)$ is a realization of annotated sequent $\vdash \Gamma$. Labeling each proof subformula with itself, for example, provides a coordinated annotation. A coordinated annotation of a sequent is defined by its realization of the sequent.

An *annotated proof* of a sequent is a clean proof of a sequent where all sequents in the proof are annotated.

Now we will define coordinated annotated proofs.

1. Annotations of all sequents in the proof must be coordinated.
2. Realization of an axiom instance must be an axiom instance in SJ.
3. Propositional rules do not change labels of corresponding subformulas.
4. If there is an application of a rule α^{n+1}, where

   $$\alpha \in \{\omega_{A^*}, \gamma, \mu, \nu, \beta, \chi, \pi\}$$

   and the first formula in the sequent is a proof expression labelled with t, then the first formula of the conclusion must be labeled with $\alpha^n t$.

5. If there is an application of the rule $\times^n$ where $n > 0$ and the first formulas in the premises are proof expressions that are labeled with t and s respectively, then the first formula of the conclusion must be labeled with $t \times^n s$.

6. If there is an application of the rule $\backslash^n$ where $n > 0$ and the first formulas in the premises are proof expressions that are labeled with t and s respectively, then the last formula of the conclusion must be labeled with $t\backslash^n s$.

7. If there is an application of a rule π and the first two formulas in the sequent are proof expressions labelled with t and s, then the first formula of the conclusion must be labeled with $t + s$.

8. If there is an application of a rule $!^1$ and the first formula in the sequent is a proof expression labelled with t, then the first formula of the conclusion must be labeled with $!t$.

9. If there is an application of a rule $?^1$ and the first formula in the sequent is a proof expression labelled with t, then the first formula of the conclusion must be labeled with $?t$.

10. If there is an application of a rule $!^n$ or $?^n$, where $n > 1$, and the first formula in the sequent is a proof expression labelled with t, then the last formula of the conclusion must be labeled with $!^n t$ or $?^n t$ respectively.

11. For applications of the rule γ^n the realizations of the contracted formulas must coincide.

12. For applications of the rule $\backslash^n$ realizations of the cut formula in the both premises must coincide.

Lemma 4. *If a* SJ$\flat$ *sequent* $\vdash \Gamma$ *has a coordinated annotated proof, then there is a provable* SJ *sequent* $\vdash \Gamma'$ *such that* $\flat(\vdash \Gamma') = \vdash \Gamma$.

Proof. Replacing all the formulas in the coordinated annotated proof by their realizations we get a valid proof in the calculus SJ. □

Affection Relation

Consider all the instances of proof subformulas in the whole proof. On this set we impose a relation called *affection*. We will talk how instances *affect* each other. The relation is defined by the following:

- Negative proof subformulas of the final sequent are not affected by anything.
- Positive subformulas of the axioms that do not have corresponding negative instances are not affected by anything.
- New positive subformulas, created by rules $!^n$ and $?^n$ or by weakening are not affected by anything.

- Positive subformulas of the axioms, that have corresponding negative instances are affected by those instances.
- If an instance of a positive subformula is not changed by given step of the proof this instance in the premise of the rule affects this instance in the conclusion. For example, when a contraction is used, a positive instance of a subformula in the contracted formula is affected by the two instances from the premise.
- If we apply a rule that changes a positive instance then this instance affects the resulting positive instance. Similarly, if the new instance is combined from two different instances, it is affected by both of them.
- Negative instances in the conlusion of a rule affect the same instances in the premises. Again, if it is a contraction, one instance affects two.
- In the premises of a cut rule the positive instances in one copy of the the cut formula affect corresponding negative instances in the other copy.

Lemma 5. *An instance of a formula cannot transitively affect its proper subformulas.*

We can think about a directed graph whose vertices are instances of proof subformulas in the SJ♭ proof, and whose edges connect vertices that affect one another. If that graph is acyclic we will call the proof acyclic.

Unfortunately this graph for a SJ♭ proof is not always acyclic. For example consider the following proof of the sequent $\vdash p^{\circ}_{[\vdash t^{\bullet}, t^{\circ}]}$, where t is some proof expression:

$$\dfrac{\dfrac{}{\vdash p^{\circ}_{[\vdash t_2^{\circ}, t_1^{\bullet}]}}\,Ax^1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p^{\circ}_{[\vdash t_7^{\circ}, t^{\bullet}]}, p^{\bullet}_{[\vdash t_6^{\circ}, t^{\bullet}]}}\,Ax \qquad \dfrac{\dfrac{}{\vdash p^{\circ}_{[\vdash t^{\circ}, t_9^{\bullet}]}, p^{\bullet}_{[\vdash t^{\circ}, t_{10}^{\bullet}]}}\,Ax}{\vdash p^{\circ}_{[\vdash t_8^{\bullet}, t^{\circ}]}, p^{\bullet}_{[\vdash t^{\circ}, t_{11}^{\bullet}]}}\,\mu^1}{\vdash p^{\circ}_{[\vdash t^{\bullet}, t^{\circ}]}, p^{\bullet}_{[\vdash t_5^{\circ}, t^{\bullet}]}, p^{\bullet}_{[\vdash t^{\circ}, t_{12}^{\bullet}]}}\,\backslash^1}{\vdash p^{\bullet}_{[\vdash t_4^{\circ}, t^{\bullet}]}, p^{\bullet}_{[\vdash t^{\circ}, t_{13}^{\bullet}]}, p^{\circ}_{[\vdash t^{\bullet}, t^{\circ}]}}\,\nu}{\vdash p^{\bullet}_{[\vdash t_3^{\circ}, t_{14}^{\bullet}]}, p^{\circ}_{[\vdash t^{\bullet}, t^{\circ}]}}\,\gamma}{\vdash p^{\circ}_{[\vdash t^{\bullet}, t^{\circ}]}}\,\backslash$$

Here affection relation creates a cycle $t_1 \to t_2 \to \ldots \to t_{14} \to t_1$.

In the present paper we will not discuss realization of proofs with cycles. We will however show that for a sufficiently large class of sequents there are cycle-free proofs.

Definition 4. *An instance of a cut rule in a proof is* good *if the affection relation restricted to the proof of one of the premises does not make one subformula of the corresponding copy of the cut formula affect another.*

Lemma 6. *If all cuts in the proof are good, then the proof is cycle-free.*

Proof. Suppose that the affection relation for a proof of a sequent $\vdash \Gamma$ contains cycles. Consider a subproof of this proof that is a minimal subproof that contains a cycle in the affection relation.

Since the subproof we chose is a minimal one, the last rule application in this subproof is necessarily a cut and the affection relation restricted to the proofs of the premises of this cut rule have no cycles. Without the connections provided by the last cut, instances of subformulas from one part cannot affect instances in the other. Therefore there are at least two connections participating in a cycle that come from this cut rule. This means that in both copies of the cut formula there are subformulas affecting one another. Thus the original proof contains an instance of a cut rule that is not good. ☐

Definition 5. *We define a* proof depth *of a* SJ♭ *formula or a sequent in the following inductive way:*

1. $d(x_i) = 0$;
2. $d(\neg A) = d(A)$;
3. $d(A \to B) = d(A \vee B) = d(A \wedge B) = max(d(A), d(B))$;
4. $d(\vdash A_1, \ldots, A_n) = max(d(A_1), \ldots, d(A_n))$;
5. $d(p_{[\vdash \Gamma]}) = 1 + d(\vdash \Gamma)$.

Lemma 7. *If* SJ♭ *sequent* $\vdash \Gamma$ *has proof that uses only good cuts, then it has a coordinated annotated proof.*

Proof. A *contraction explosion* is process of replacing a contraction in a SJ♭ proof with different proof parts. Our goal is to replace all the contractions in the proof with exploded ones. Let A_1 and A_2 refer to the two copies of the contracted formula A in the premise of the rule. If the proof depth of A is zero, then we do not change anything, and declare the contraction exploded. If it is not the case, then there are the following cases:

- Suppose that the contracted formula is of the form $(\neg A)^\circ$. In this case we replace

$$\dfrac{\ldots[\vdash (\neg A_1)^\circ, (\neg A_2)^\circ, \ldots]\ldots}{\ldots[\vdash (\neg A)^\circ, \ldots]\ldots}\,\gamma^n,$$

with

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash A_1^\circ, A_1^\bullet]\ldots}\,Ax^n}{\ldots[\vdash (\neg A_1)^\bullet, A_1^\bullet]\ldots}\,\chi^n \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash A_2^\circ, A_2^\bullet]\ldots}\,Ax^n}{\ldots[\vdash (\neg A_2)^\bullet, A_2^\bullet]\ldots}\,\chi^n \quad \dfrac{\ldots[\vdash (\neg A_1)^\circ, (\neg A_2)^\circ, \ldots]\ldots}{\ldots[\vdash (\neg A_2)^\circ, (\neg A_1)^\circ, \ldots]\ldots}\,\mu^n}{\ldots[\vdash A_2^\bullet, (\neg A_1)^\circ, \ldots]\ldots}\,\backslash^n}{\ldots[\vdash (\neg A_1)^\circ, A_2^\bullet, \ldots]\ldots}\,\mu^n}{}}{\ldots[\vdash A_1^\bullet, A_2^\bullet, \ldots]\ldots}\,\backslash^n}{\dfrac{\ldots[\vdash A^\bullet, \ldots]\ldots}{\ldots[\vdash (\neg A)^\circ, \ldots]\ldots}\,\chi^n}\,\gamma^n$$

The new contraction is considered not exploded. The cuts we used are good.

We treat contractions with the contracted formula of the form $(\neg A)^\bullet$ in a similar fashion.

- Suppose that the contracted formula is of the form $(A \vee B)^\circ$. In this case we replace

$$\dfrac{\dots[\vdash (A_1 \vee B_1)^\circ, (A_2 \vee B_2)^\circ, \dots]\dots}{\dots[\vdash (A \vee B)^\circ, \dots]\dots}\;\gamma^n$$

with

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dots[\vdash (A_1 \vee B_1)^\circ, (A_2 \vee B_2)^\circ, \dots]\dots \quad \dfrac{\mathcal{D}_1}{\dots[\vdash (A_1 \vee B_1)^\bullet, A_1^\circ, B_1^\circ]\dots}}{\dots[\vdash (A_2 \vee B_2)^\circ, \dots, A_1^\circ, B_1^\circ]\dots}\;\setminus^n \quad \dfrac{\mathcal{D}_2}{\dots[\vdash (A_2 \vee B_2)^\bullet, A_2^\circ, B_2^\circ]\dots}}{\dots[\vdash \dots, A_1^\circ, B_1^\circ, A_2^\circ, B_2^\circ]\dots}}{\cdots}}{\dots[\vdash A_1^\circ, A_2^\circ, B_1^\circ, B_2^\circ, \dots]\dots}}{\dots[\vdash A^\circ, B_1^\circ, B_2^\circ, \dots]\dots}\;\gamma^n}{\dots[\vdash B_1^\circ, B_2^\circ, \dots, A^\circ]\dots}\;\nu^n}{\dots[\vdash B^\circ, \dots, A^\circ]\dots}\;\gamma^n}{\cdots}}{\dots[\vdash A^\circ, B^\circ, \dots]\dots}}{\dots[\vdash (A \vee B)^\circ, \dots]\dots}\;\beta^n$$

Where $\mathcal{D}_{1,2}$ is the following proof:

$$\dfrac{\dfrac{\dfrac{}{\dots[\vdash A_{1,2}^\circ, A_{1,2}^\bullet]\dots}\;Ax^n}{\dots[\vdash A_{1,2}^\bullet, A_{1,2}^\circ]\dots}\;\mu^n \quad \dfrac{\dfrac{}{\dots[\vdash B_{1,2}^\circ, B_{1,2}^\bullet]\dots}\;Ax^n}{\dots[\vdash B_{1,2}^\bullet, B_{1,2}^\circ]\dots}\;\mu^n}{\dots[\vdash (A_{1,2} \vee B_{1,2})^\bullet, A_{1,2}^\circ, B_{1,2}^\circ]\dots}\;\times^n$$

The new contractions are considered not exploded. The cuts we used are good.

We treat contractions with the contracted formula of the forms $(A \wedge B)^\bullet$ and $(A \to B)^\circ$ in a similar fashion.

- Suppose that the contracted formula is of the form $(A \vee B)^\bullet$. In this case we replace

$$\dfrac{\dots[\vdash (A_1 \vee B_1)^\bullet, (A_2 \vee B_2)^\bullet, \dots]\dots}{\dots[\vdash (A \vee B)^\bullet, \dots]\dots}\;\gamma^n$$

with

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\dots[\vdash (A_2 \vee B_2)^\circ, (A \vee B)^\bullet]} \quad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\dots[\vdash (A_1 \vee B_1)^\circ, (A \vee B)^\bullet]} \quad \dots[\vdash (A_1 \vee B_1)^\bullet, (A_2 \vee B_2)^\bullet, \dots]\dots}{\dots[\vdash (A \vee B)^\bullet, (A_2 \vee B_2)^\bullet, \dots]\dots}\;\setminus^n}{\dots[\vdash (A_2 \vee B_2)^\bullet, (A \vee B)^\bullet, \dots]\dots}\;\mu_l^n}{\dots[\vdash (A \vee B)^\bullet, (A \vee B)^\bullet, \dots]\dots}\;\setminus^n}{\dots[\vdash (A \vee B)^\bullet, \dots]\dots}\;\gamma^n,$$

where $\mathcal{D}_1$ is the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\dots[\vdash A_1^\circ, A_1^\bullet]}\;Ax^n}{\dots[\vdash A_1^\bullet, A_1^\circ]}\;\mu^n}{\dots[\vdash A_2^\bullet, A_1^\bullet, A_1^\circ]}\;\omega_{A_2^\bullet}}{\dots[\vdash A_1^\bullet, A_2^\bullet, A_1^\circ]}\;\mu^n}{\dots[\vdash A^\bullet, A_1^\circ]}\;\gamma^n \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\dots[\vdash B_1^\circ, B_1^\bullet]}\;Ax^n}{\dots[\vdash B_1^\bullet, B_1^\circ]}\;\mu^n}{\dots[\vdash B_2^\bullet, B_1^\bullet, B_1^\circ]}\;\omega_{B_2^\bullet}}{\dots[\vdash B_1^\bullet, B_2^\bullet, B_1^\circ]}\;\mu^n}{\dots[\vdash B^\bullet, B_1^\circ]}\;\gamma^n}{\dfrac{\dfrac{\dots[\vdash (A \vee B)^\bullet, A_1^\circ, B_1^\circ]}{\dots[\vdash A_1^\circ, B_1^\circ, (A \vee B)^\bullet]}\;\nu^n}{\dots[\vdash (A_1 \vee B_1)^\circ, (A \vee B)^\bullet]}\;\beta^n}\;\times^n$$

and $\mathcal{D}_2$ is

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash A_2^\circ, A_2^\bullet]}\,Ax^n}{\ldots[\vdash A_2^\bullet, A_2^\circ]}\,\mu^n}{\ldots[\vdash A_1^\bullet, A_2^\bullet, A_2^\circ]}\,\omega_{A_1^\bullet}}{\ldots[\vdash A^\bullet, A_2^\circ]}\,\gamma^n \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash B_2^\circ, B_2^\bullet]}\,Ax^n}{\ldots[\vdash B_2^\bullet, B_2^\circ]}\,\mu^n}{\ldots[\vdash B_1^\bullet, B_2^\bullet, B_2^\circ]}\,\omega_{B_1^\bullet}}{\ldots[\vdash B^\bullet, B_2^\circ]}\,\gamma^n}{\ldots[\vdash (A \vee B)^\bullet, A_2^\circ, B_2^\circ]}\,\times^n}{\ldots[\vdash A_2^\circ, B_2^\circ, (A \vee B)^\bullet]}\,\nu^n}{\ldots[\vdash (A_2 \vee B_2)^\circ, (A \vee B)^\bullet]}\,\beta^n$$

The last contraction created (with the contracted formula $(A \vee B)^\bullet$) is considered already exploded. The other two created contractions are considered not. The formulas created by weakening (like A_2 and B_2 in $\mathcal{D}_1$) are considered 'copies' of 'original' A_2 and B_2 from the sequent $\ldots[\vdash (A_1 \vee B_1)^\bullet, (A_2 \vee B_2)^\bullet, \ldots]\ldots$. The cuts we used are good.

We treat contractions with the contracted formula of the forms $(A \wedge B)^\circ$ and $(A \to B)^\bullet$ in a similar fashion.

- Suppose that the formula A is of the form $p^\circ_{[\vdash A_1^{*1},\ldots,A_n^{*n}]}$. In this case we replace

$$\dfrac{\ldots[\vdash p^\circ_{[\vdash A_{1,1}^{*1},\ldots,A_{n,1}^{*n}]}, p^\circ_{[\vdash A_{1,2}^{*1},\ldots,A_{n,2}^{*n}]}, \ldots]\ldots}{\ldots[\vdash p^\circ_{[\vdash A_1^{*1},\ldots,A_n^{*n}]}, \ldots]\ldots}\,\gamma^n,$$

with

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\ldots[\vdash p^\circ_{[\vdash A_{1,1}^{*1},\ldots,A_{n,1}^{*n}]}, p^\circ_{[\vdash A_{1,2}^{*1},\ldots,A_{n,2}^{*n}]}, \ldots]\ldots}{\ldots[\vdash p^\circ_{[\vdash A_{1,1}^{*1},\ldots,A_{n,1}^{*n},A_{1,2}^{*1},\ldots,A_{n,2}^{*n}]}, \ldots]\ldots}\,\pi^n}{\cdots}}{\ldots[\vdash p^\circ_{[\vdash A_{1,1}^{*1},A_{1,2}^{*1},\ldots,A_{n,1}^{*n},A_{n,2}^{*n}]}, \ldots]\ldots}}{\ldots[\vdash p^\circ_{[\vdash A_1^{*1},\ldots,A_{n,1}^{*n},A_{n,2}^{*n}]}, \ldots]\ldots}\,\gamma^{n+1}}{\ldots[\vdash p^\circ_{[\vdash A_{2,1}^{*n},A_{2,2}^{*n},\ldots,A_{n,1}^{*n},A_{n,2}^{*n},A_1^{*1}]}, \ldots]\ldots}\,\nu^{n+1}}{\dfrac{\cdots}{\ldots[\vdash p^\circ_{[\vdash A_1^{*1},\ldots,A_n^{*n}]}, \ldots]\ldots}}$$

The new contractions created in the process are considered not exploded.

- Suppose that the formula A is of the form $p^\bullet_{[\vdash A_1^{*1},\ldots,A_n^{*n}]}$. In this case we replace

$$\dfrac{\ldots[\vdash p^\bullet_{[\vdash \Gamma_1]}, p^\bullet_{[\vdash \Gamma_2]}, \ldots]\ldots}{\ldots[\vdash p^\bullet_{[\vdash \Gamma]}, \ldots]\ldots}\,\gamma^n,$$

with

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_2}{\ldots[\vdash p^\circ_{[\vdash \Gamma_2]}, p^\bullet_{[\vdash \Gamma]}]\ldots} \quad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\ldots[\vdash p^\circ_{[\vdash \Gamma_1]}, p^\bullet_{[\vdash \Gamma]}]\ldots} \quad \ldots[\vdash p^\bullet_{[\vdash \Gamma_1]}, p^\bullet_{[\vdash \Gamma_2]}, \ldots]\ldots}{\ldots[\vdash p^\bullet_{[\vdash \Gamma]}, p^\bullet_{[\vdash \Gamma_2]}, \ldots]\ldots}\,\backslash^n}{\ldots[\vdash p^\bullet_{[\vdash \Gamma_2]}, p^\bullet_{[\vdash \Gamma]}, \ldots]\ldots}\,\mu_l^n}{\ldots[\vdash p^\bullet_{[\vdash \Gamma]}, p^\bullet_{[\vdash \Gamma]}, \ldots]\ldots}\,\backslash^n}{\ldots[\vdash p^\bullet_{[\vdash \Gamma]}, \ldots]\ldots}\,\gamma^n,$$

Here $\mathcal{D}_1$ is the following proof. We start with the axiom $\ldots[\vdash p^{\circ}_{[\vdash\Gamma]}, p^{\bullet}_{[\vdash\Gamma]}]$ and then if $*_1 = \circ$ we do this:

$$\dfrac{\ldots[\vdash p^{\circ}_{[\vdash A^{\circ}_1,\ldots,A^{*n}_n]}, p^{\bullet}_{[\vdash\Gamma]},\ldots]\ldots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash p_{[\vdash A^{\circ}_{1,1},A^{\bullet}_{1,1}]}]\ldots}\,Ax^{n+1}}{\ldots[\vdash p_{[\vdash A^{\bullet}_{1,1},A^{\circ}_{1,1}]}]\ldots}\,\mu^{n+1}}{\ldots[\vdash p_{[\vdash A^{\bullet}_{1,2},A^{\bullet}_{1,1},A^{\circ}_{1,1}]}]\ldots}\,\omega^{n+1}_{A^{\bullet}_{1,2}}}{\ldots[\vdash p_{[\vdash A^{\bullet}_{1,1},A^{\bullet}_{1,2},A^{\circ}_{1,1}]}]\ldots}\,\mu^{n+1}}{\ldots[\vdash p_{[\vdash A^{\bullet}_1,A^{\circ}_{1,1}]}]\ldots}\,\gamma^{n+1}}{\ldots[\vdash p^{\circ}_{[\vdash A^{*2}_2,\ldots,A^{*n}_n,A^{\circ}_{1,1}]}, p^{\bullet}_{[\vdash\Gamma]},\ldots]\ldots}\,\backslash^{n+1}$$

If $*_1 = \bullet$ we do this:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash p_{[\vdash A^{\circ}_{1,1},A^{\bullet}_{1,1}]}]\ldots}\,Ax^{n+1}}{\ldots[\vdash p_{[\vdash A^{\circ}_{1,2},A^{\circ}_{1,1},A^{\bullet}_{1,1}]}]\ldots}\,\omega_{A^{\circ}_{1,2}}}{\ldots[\vdash p_{[\vdash A^{\circ}_{1,1},A^{\circ}_{1,2},A^{\bullet}_{1,1}]}]\ldots}\,\mu^{n+1}}{\ldots[\vdash p_{[\vdash A^{\circ}_1,A^{\bullet}_{1,1}]}]\ldots}\,\gamma^{n+1} \qquad \ldots[\vdash p^{\circ}_{[\vdash A^{\bullet}_1,A^{*2}_2,\ldots,A^{*n}_n]}, p^{\bullet}_{[\vdash\Gamma]},\ldots]\ldots}{\ldots[\vdash p^{\circ}_{[\vdash A^{\bullet}_{1,1},A^{*2}_2,\ldots,A^{*n}_n]}, p^{\bullet}_{[\vdash\Gamma]},\ldots]\ldots}\,\backslash^{n+1}}{\ldots[\vdash p^{\circ}_{[\vdash A^{*2}_2,\ldots,A^{*n}_n,A^{\bullet}_{1,1}]}, p^{\bullet}_{[\vdash\Gamma]},\ldots]\ldots}\,\nu^{n+1}$$

We continue doing this until we get a proof of the sequent $\ldots[\vdash p^{\circ}_{[\vdash\Gamma_1]}, p^{\bullet}_{[\vdash\Gamma]}]\ldots$.

The proof $\mathcal{D}_2$ is a similar proof that proves the sequent $\ldots[\vdash p^{\circ}_{[\vdash\Gamma_2]}, p^{\bullet}_{[\vdash\Gamma]}]\ldots$.

The last contraction created (with the contracted formula $p^{\bullet}_{[\vdash\Gamma]}$) is considered already exploded. The other created contractions are considered not. The formulas created by weakening (like $A^{*_1}_{1,2}$ in $\mathcal{D}_1$) are considered 'copies' of 'original' $A^{*_1}_{1,2}$ from the sequent $\ldots[\vdash p^{\bullet}_{[\vdash\Gamma_1]}, p^{\bullet}_{[\vdash\Gamma_2]},\ldots]\ldots$. The cuts we used are good.

Since each step in the worst case takes one unexploded contraction and replaces it with several unexploded contractions with simpler contracted formulas, we can explode all contractions in the proof in a finite number of steps. For every contraction with non-zero proof depth of the contracted formula there are pairs of 'original' and 'copies' uniquely associated with this contraction.

If the original proof used only good cuts it will still be acyclic after contraction explosion, since all cuts we introduce in the process are good. For each contraction with non-zero proof depth of the contracted formula there are pairs of 'original' and 'copies' uniquely associated with this contraction. If we add to the definition of the affection relation a clause that positive proof subformulas of the original formula affect the corresponding positive subformula in the copy, the proof will still be acyclic.

We start with the labeling of the instances that are not affected by anything and do not have any own proof subformulas. We label them with themselves. Then we propagate the labeling to the instances affected by already labelled. Then we move on to instances that are not affected by anything, but have all their subformulas already labelled. We always need to have all instances that affect the current instance, and all its own proof subformulas to be already labelled, in order to label it. If the current instance is affected by a negative instance (it cannot be affected by more than one), then we label it with the same label. If a positive instance is affected by a single other positive instance, then it should be labelled either with the same label, or with the label that represent the effect of the rule used. Also a negative instance affected with a positive one (the cut was used) takes the same label. If a positive instance is built from two positive instances labelled t and s (and therefore is affected by them) with the rules $\times^n$, $\backslash^n$, or π^n, we label it with $t \times^n s$, $t \backslash^n s$, or $t + s$, respectively.

When labeling a unaffected negative instance $p_{[\vdash \Gamma]}$, whose subformulas were already labelled, we label it with $p_{[\vdash \Gamma']}$, where $\vdash \Gamma'$ is a realization of $\vdash \Gamma$ provided by already constructed parts of annotation. We do the same with unaffected positive instances from axioms. New instances that appear as a result of applying the rules $!^n$ and $?^n$ should be labelled with $!t$ and $?t$ respectively, where t is the label we already gave to the corresponding instance in the premise of the rule.

Because of lemma 5 and the acyclicity of the affection relation, we will always be able to label all the instances of proof subformulas in the whole proof.

When all instances are labelled, it can be easily checked that all conditions are satisfied. Due to the structure of contraction explosion all exploded contractions will satisfy condition (11) automatically. □

Lemma 8. *For every sequent $\Gamma \vdash \Delta$ that has a proof in* SJ$\flat$ *that uses only good cuts, there is a sequent $\Gamma' \vdash \Delta'$ provable in* SJ *such that* $\flat(\Gamma' \vdash \Delta') = \Gamma \vdash \Delta$.

This lemma follows directly from lemmas 7 and 4.

Nested Sequent Calculus

The nested sequent calculus is a deep inference system for modal logic that we will use to obtain cycle-free proofs of sequents in SJ$\flat$ that correspond to modal formulas.

The formulas in the nested sequent calculus are built from propositional variables $\{x_1, x_2, \ldots\}$ and their negations $\{\neg x_1, \neg x_2, \ldots\}$ using two binary connectives — $\vee$ (disjunction) and $\wedge$ (conjunction) and two modalities — $\Box$ and $\Diamond$. Nested sequents are defined the following way:

- $\varnothing$ is a nested sequent.
- If Σ is a nested sequent, then so is Σ, A.
- If Σ and Δ are nested sequents, then so is $\Sigma, [\Delta]$.

A *context* is a nested sequent with exactly one occurrence of the symbol $\{\}$, not inside formulas. A nested sequent can be put in a context, resulting in a new nested sequent. Example:

$$\Gamma\{\} = [A, [B], \{\}], \quad \Delta = C, [D], \quad \Gamma\{\Delta\} = [A, [B], C, [D]]$$

Axiom and rules of the nested sequent calculus:

$$\Gamma\{x_i, \neg x_i\}$$

$$\frac{\Gamma\{A, B\}}{\Gamma\{A \vee B\}}\ \vee \qquad \frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A \wedge B\}}\ \wedge$$

$$\frac{\Gamma\{A, A\}}{\Gamma\{A\}}\ ctr \qquad \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}}\ exch$$

$$\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}\ \Box \qquad \frac{\Gamma\{[A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ k$$

$$\frac{\Gamma\{[\Diamond A, \Delta]\}}{\Gamma\{\Diamond A, [\Delta]\}}\ 4 \qquad \frac{\Gamma\{\Diamond A, [\Delta]\}}{\Gamma\{[\Diamond A, \Delta]\}}\ 5a$$

$$\frac{\Gamma\{[\Delta], [\Diamond A, \Pi]\}}{\Gamma\{[\Diamond A, \Delta], [\Pi]\}}\ 5b \qquad \frac{\Gamma\{[[\Diamond A, \Pi], \Delta]\}}{\Gamma\{[\Diamond A, [\Pi], \Delta]\}}\ 5c$$

This system is sound and complete with respect to modal logic K45 with conjunction and disjunction.

Signed Nested Sequent Calculus

We can enrich the nested sequent calculus by using signed formulas. This will allow us to use formulas not necessarily in negation normal form without loss of soundness and completeness.

Formulas of signed nested sequent calculus are built from propositional variables $\{x_1, x_2, \ldots\}$ using one unary connective — $\neg$ (negation), three binary connectives — $\vee$ (disjunction), $\wedge$ (conjunction), and $\to$ (implication) and two modalities — $\Box$ and $\Diamond$. Signed nested sequents are defined the same way as nested sequents, except all the formulas are signed. Contexts are also defined the same way.

Axiom and rules of the signed nested sequent calculus:

$$\Gamma\{x_i^\circ, x_i^\bullet\}$$

$$\frac{\Gamma\{A^*, A^*\}}{\Gamma\{A^*\}}\ ctr \qquad \frac{\Gamma\{\Delta, \Sigma\}}{\Gamma\{\Sigma, \Delta\}}\ exch$$

$$\frac{\Gamma\{A^\bullet\}}{\Gamma\{(\neg A)^\circ\}}\ \neg^\circ \qquad \frac{\Gamma\{A^\circ\}}{\Gamma\{(\neg A)^\bullet\}}\ \neg^\bullet$$

$$\frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{(A \vee B)^\circ\}}\ \vee^\circ \qquad \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{(A \vee B)^\bullet\}}\ \vee^\bullet$$

$$\frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{(A \wedge B)^\bullet\}}\ \wedge^\bullet \qquad \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{(A \wedge B)^\circ\}}\ \wedge^\circ$$

$$\frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{(A \to B)^\circ\}}\ \to^\circ \qquad \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{(A \to B)^\bullet\}}\ \to^\bullet$$

$$\frac{\Gamma\{[A^\circ]\}}{\Gamma\{(\Box A)^\circ\}}\ \Box^\circ \qquad \frac{\Gamma\{[A^\bullet]\}}{\Gamma\{(\Diamond A)^\bullet\}}\ \Box^\bullet$$

$$\frac{\Gamma\{[A^\circ, \Delta]\}}{\Gamma\{(\Diamond A)^\circ, [\Delta]\}}\ k^\circ \qquad \frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{(\Box A)^\bullet, [\Delta]\}}\ k^\bullet$$

$$\frac{\Gamma\{[(\Diamond A)^\circ, \Delta]\}}{\Gamma\{(\Diamond A)^\circ, [\Delta]\}}\ 4^\circ \qquad \frac{\Gamma\{[(\Box A)^\bullet, \Delta]\}}{\Gamma\{(\Box A)^\bullet, [\Delta]\}}\ 4^\bullet$$

$$\frac{\Gamma\{(\Diamond A)^\circ, [\Delta]\}}{\Gamma\{[(\Diamond A)^\circ, \Delta]\}}\ 5a^\circ \qquad \frac{\Gamma\{(\Box A)^\bullet, [\Delta]\}}{\Gamma\{[(\Box A)^\bullet, \Delta]\}}\ 5a^\bullet$$

$$\frac{\Gamma\{[\Delta], [(\Diamond A)^\circ, \Pi]\}}{\Gamma\{[(\Diamond A)^\circ, \Delta], [\Pi]\}}\ 5b^\circ \qquad \frac{\Gamma\{[\Delta], [(\Box A)^\bullet, \Pi]\}}{\Gamma\{[(\Box A)^\bullet, \Delta], [\Pi]\}}\ 5b^\bullet$$

$$\frac{\Gamma\{[[(\Diamond A)^\circ, \Pi], \Delta]\}}{\Gamma\{[(\Diamond A)^\circ, [\Pi], \Delta]\}}\ 5c^\circ \qquad \frac{\Gamma\{[[(\Box A)^\bullet, \Pi], \Delta]\}}{\Gamma\{[(\Box A)^\bullet, [\Pi], \Delta]\}}\ 5c^\bullet$$

The rules marked with $\circ$ exactly correspond to the rules of nested sequent calculus.

Let $(\cdot)^\top$ and $(\cdot)^\perp$ be two mappings from formulas to formulas defined inductively by the following:

$$x_i^\top = x_i \qquad x_i^\perp = \neg x_i$$

$$(\neg A)^\top = A^\perp \qquad (\neg A)^\perp = A^\top$$

$$(A \vee B)^\top = A^\top \vee B^\top \qquad (A \vee B)^\perp = A^\perp \wedge B^\perp$$

$$(A \wedge B)^\top = A^\top \wedge B^\top \qquad (A \wedge B)^\perp = A^\perp \vee B^\perp$$

$$(A \to B)^\top = A^\perp \vee B^\top \qquad (A \to B)^\perp = A^\top \wedge B^\perp$$

$$(\Box A)^\top = \Box(A)^\top \qquad (\Box A)^\perp = \Diamond(A)^\perp$$

$$(\Diamond A)^\top = \Diamond(A)^\top \qquad (\Diamond A)^\perp = \Box(A)^\perp$$

The mapping $(\cdot)^\top$ maps a formula to its negation normal form.

Lemma 9. *A formula A is valid in the modal logic K45 if and only if $A^\top$ is derivable in the nested sequent calculus and if and only if A° is derivable in the signed nested sequent calculus.*

Proof. The first part is known. The equivalence between nested sequent calculus and signed nested sequent calculus can be done by a simple induction on the proofs. □

If the formula A does not have any $\Diamond$'s in it, then A° can be proved in the signed nested sequent calculus without using $\Diamond$'s anywhere in the proof.

Now we need to translate $\Diamond$-free signed nested sequents into SJ♭ sequents. It is done by the mapping $trf(\cdot)$ defined inductively by the following:

$$trf(x_i) = x_i;$$

$$trf(A \vee B) = (trf(A)) \vee trf(B); \qquad trf(A \wedge B) = (trf(A)) \wedge (trf(B));$$

$$trf(A \to B) = (trf(A)) \to (trf(B)) \qquad trf(\neg A) = \neg trf(A)$$

$$trf(\Box A) = p_{[\vdash trf(A)^\circ]};$$

$$trf(\Delta, A^*) = trf(\Delta), trf(A)^*; \qquad trf(\Delta, [\Gamma]) = trf(\Delta), trf([\Gamma]);$$

$$trf([\Gamma]) = p_{[\vdash trf(\Gamma)]}$$

Lemma 10 (Internalized Positive Introspection)**.** *For any $A \in \mathrm{Fm}\flat$ the following sequent is derivable in* SJ♭ *without using the rule* $!^n$*:*

$$\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}, p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}]}$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}]}, p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}}\,Ax}{\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}, p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}]}}\,\mu}{\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}, p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}]}}\,?^1}{\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}, p^\bullet_{[\vdash A^\circ]}]}}\,\pi}{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}, p^\bullet_{[\vdash A^\circ]}]}}\,?^2 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}, p^\bullet_{[\vdash A^\circ]}]}]}}\,Ax^2}{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}, p^\circ_{[\vdash A^\circ]}]}]}}\,\mu^2}{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}, p^\circ_{[\vdash A^\circ]}]}]}}\,?^3 \qquad \dfrac{}{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}, p^\bullet_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}]}}\,Ax^1}{\dfrac{\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}, p^\bullet_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}]}}{\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash p^\bullet_{[\vdash A^\circ]}]}]}, p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}]}}\,\mu^1}\,\backslash^2}{\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}, p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}]}}\,\backslash^1$$

□

Lemma 11. *If Γ is derivable in the signed nested sequent calculus, then $\vdash trf(\Gamma)$ is derivable in* SJ♭ *using only good cuts.*

Proof. First we should note, that structural boxes in the nested sequent Γ directly correspond to those proof subformulas in the sequent $\vdash trf(\Gamma)$ that by permutations can be put into position where we can apply rules directly to them and then put them back where they were before.

We prove the lemma by induction on the length of the derivation of Γ in the signed nested sequent calculus.

- Axiom: To prove $\vdash trf(\Gamma\{x_i^\circ, x_i^\bullet\})$, we start with an axiom $\vdash p_{[\ldots\vdash p_{[\vdash x_i^\bullet, x_i^\circ]}\ldots]}$ of appropriate depth, then apply some weakenings and permutations to get $\vdash trf(\Gamma\{x_i^\circ, x_i^\bullet\})$.
- The rule *ctr*: By permutations we can move the corresponding instances to the position where they can be contracted by the rule γ^n, and then move everything back.
- The rule *exch*: Our permutation rules can emulate any needed permutations.
- The rule $\Box^\circ$: This rule does not change our SJ♭ sequent at all.
- Rules $\neg^\circ$ and $\neg^\bullet$. We can move the formula that correspond to A^* to accessible position, apply the rule χ^n with appropriate n, and then move everything back.
- Rules $\vee^\circ$, $\wedge^\bullet$, and $\rightarrow^\circ$. We can move formulas that correspond to A^* and $B^\star$ to accessible position, apply the rule β^n with appropriate n, and then move everything back.

- Rules $\vee^\bullet$, $\wedge^\circ$, and $\to^\bullet$: We can move formulas that correspond to A^* and $B^\star$ to accessible positions, but so the rest of the sequents still coincide, apply the rule $\times^n$ with the appropriate n, and then we move everything back, while contracting duplicated formulas.

- The rule $k^\bullet$:

$$\frac{\Gamma\{[A^\bullet, \Delta]\}}{\Gamma\{(\Box A)^\bullet, [\Delta]\}}\ k^\bullet$$

  First we move the formula, corresponding to this structural box to accessible position, then we do the following:

$$\cfrac{\cfrac{}{\ldots[\vdash p^\circ_{[\vdash trf(A)^\circ]}, p^\bullet_{[\vdash trf(A)^\circ]}]\ldots}\ Ax^n \qquad \ldots[\vdash p^\circ_{[\vdash trf(A)^\bullet, trf(\Delta)]}\ldots]\ldots}{\cfrac{\ldots[\vdash p^\circ_{[\vdash trf(\Delta)]}, p^\bullet_{[\vdash trf(A)^\circ]}\ldots]\ldots}{\ldots[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, p^\circ_{[\vdash trf(\Delta)]}\ldots]\ldots}\ \mu_n}\ \backslash^{n+1}$$

  And then we move everything back.

  The cut we used is good.

- The rule $4^\bullet$:

$$\frac{\Gamma\{[(\Box A)^\bullet, \Delta]\}}{\Gamma\{(\Box A)^\bullet, [\Delta]\}}\ 4^\bullet$$

  First we move the formula, corresponding to this structural box to accessible position, then we do the following:

$$\cfrac{\cfrac{\cfrac{}{\ldots[\vdash p^\circ_{[\vdash trf(A)^\circ]}, p^\bullet_{[\vdash trf(A)^\circ]}]\ldots}\ Ax^n}{\ldots[\vdash p^\circ_{[\vdash p^\circ_{[\vdash trf(A)^\circ]}]}, p^\bullet_{[\vdash trf(A)^\circ]}]\ldots}\ !^{n+1} \qquad \ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, trf(\Delta)]}\ldots]\ldots}{\cfrac{\ldots[\vdash p^\circ_{[\vdash trf(\Delta)]}, p^\bullet_{[\vdash trf(A)^\circ]}\ldots]\ldots}{\ldots[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, p^\circ_{[\vdash trf(\Delta)]}\ldots]\ldots}\ \mu_n}\ \backslash^{n+1}$$

  And then we move everything back.

  The cut we used is good.

- The rule $5a^\bullet$:

$$\frac{\Gamma\{(\Box A)^\bullet, [\Delta]\}}{\Gamma\{[(\Box A)^\bullet, \Delta]\}}\ 5a$$

  First we move the formula, corresponding to this structural box to accessible position, then we do the following:

$$\cfrac{\cfrac{\ldots[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, p^\circ_{[\vdash trf(\Delta)]}\ldots]\ldots}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}]}, p^\circ_{[\vdash trf(\Delta)]}\ldots]\ldots}\ ?^{n+1}}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, trf(\Delta)]}\ldots]\ldots}\ \pi^n$$

  And then we move everything back.

- The rule $5b^\bullet$:

$$\dfrac{\Gamma\{[\Delta], [(\Box A)^\bullet, \Pi]\}}{\Gamma\{[(\Box A)^\bullet, \Delta], [\Pi]\}}\ 5b^\bullet$$

  Consider the following derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma\{[\Delta], [(\Box A)^\bullet, \Pi]\}}{\Gamma\{[\Delta], (\Box\Box A)^\bullet, [\Pi]\}}\ k^\bullet}{\Gamma\{(\Box\Box A)^\bullet, [\Delta], [\Pi]\}}\ exch}{\Gamma\{[(\Box\Box A)^\bullet, \Delta], [\Pi]\}}\ 5a^\bullet$$

  By the induction hypothesis and using constructions from previous cases we can construct a SJ♭ proof of the sequent

$$\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\circ_{[\vdash trf(A)^\circ]}]}, trf(\Delta)]}, p^\circ_{[\vdash trf(\Pi)]}\ldots]\ldots.$$

  We move the formula, corresponding to this structural box to accessible position, then we do the following:

$$\dfrac{\dfrac{\dfrac{}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}, p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}]}]\ldots}\ IPI^n}{\ldots[\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}, p^\bullet_{[\vdash A^\circ]}]}]\ldots}\ \mu^{n+1} \qquad \ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash p^\circ_{[\vdash trf(A)^\circ]}]}, trf(\Delta)]}, p^\circ_{[\vdash trf(\Pi)]}\ldots]\ldots}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, trf(\Delta)]}, p^\circ_{[\vdash trf(\Pi)]}\ldots]\ldots}\ \backslash^{n+1}$$

  Here IPI^n represents the proof from Lemma 10 adjusted for needed depth.

  And then we move everything back.

  The cuts we used are good.

- The rule $5c^\bullet$:

$$\dfrac{\Gamma\{[[(\Box A)^\bullet, \Pi], \Delta]\}}{\Gamma\{[(\Box A)^\bullet, [\Pi], \Delta]\}}\ 5c^\bullet$$

  First we move the formula, corresponding to this structural box to accessible position, then we do the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash A^\circ]}, p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}]}]\ldots}\ IPI^n}{\ldots[\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\circ_{[\vdash A^\circ]}]}, p^\bullet_{[\vdash A^\circ]}]}]\ldots}\ \mu^{n+1} \qquad \ldots[\vdash p^\circ_{[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, trf(\Pi)]}, trf(\Delta)]}, \ldots]\ldots}{\ldots[\vdash p^\circ_{[\vdash p^\circ_{[\vdash trf(\Pi)]}, p^\bullet_{[\vdash trf(A)^\circ]}, trf(\Delta)]}, \ldots]\ldots}\ \backslash^{n+2}}{\ldots[\vdash p^\circ_{[\vdash p^\bullet_{[\vdash trf(A)^\circ]}, p^\circ_{[\vdash trf(\Pi)]}, trf(\Delta)]}, \ldots]\ldots}\ \mu^{n+1}$$

  And then we move everything back.

  The cuts we used are good.

In every case we get a proof of $\vdash trf(\Gamma)$ in the calculus SJ♭ with only good cuts as a result.

□

Realization

Theorem 1 (Soundness.)**.** *If a sequent* $\vdash \Gamma$ *is derivable in* SJ *or in* SJ$\flat$*, then* $mod(\vdash \Gamma)$ *is valid in modal logic.*

Lemma 12. *Let* F *be a modal formula. Then* $F = form(trf(F^{\circ}))$*.*

Theorem 2. *If* F *is a valid modal formula, then the sequent* $\vdash trf(F)^{\circ}$ *is derivable in* SJ$\flat$ *and has a proof that uses only good cuts.*

Proof. By lemma 9 F° is derivable in the signed nested sequent calculus. By Lemma 11 $\vdash trf(F)^{\circ}$ is derivable in SJ$\flat$. □

Theorem 3. *If* F *is a valid modal formula of the logic K45, then there exists a* SJ *formula* F' *such that* $\vdash F'$ *is derivable in* SJ *and* $form(F') = F$*.*

This theorem is a direct consequence of Theorem 2 and Lemmas 8, 12, and 1.

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