Controller Design for the Beam-and-Ball System

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Abstract

We design a controller for the Beam-and-Ball system that renders the origin semiglobally asymptotically stable (semi-GAS). First, we view the problem as designing a controller for a second-order subsystem of the system. Then, using a new backstepping procedure we reconstruct the control for the whole system. As our main result, we developed a new version of backstepping procedure for systems that do not have a triangular structure.

1 Introduction

The Beam-and-Ball system (BNB) which is shown in Figure (1), consists of a beam and a ball on it. There is an actuator on the joint connecting the beam to the base. The task is to bring the ball from any initial position with any initial speed to a desired position on the beam by applying an appropriate torque to the joint and changing the angle of the beam. The BNB system is a well-known example in nonlinear control. In [1], approximate input-output linearization of the BNB system was studied. In [3], a Lyapunov function for the BNB system was constructed by finding a Lyapunov function for a cross-term relating two subsystems of the system. In our approach, we design a controller for a second-order subsystem of the BNB system, then using backstepping procedure [2], we find the controller for the whole system. As our main result, we introduce a new version of the backstepping procedure for systems that do not have triangular structure. We believe that our approach works for more general classes of nonlinear systems as well.

Here is the dynamics of the BNB system ([3]):

$$\begin{cases} \ddot{r} + g\sin(\theta) + \beta\dot{r} - r\dot{\theta}^2 &= 0 \\ (r^2+1)\ddot{\theta} + 2r\dot{r}\dot{\theta} + gr\cos(\theta) &= \tau \end{cases} \tag{1}$$

where r is the position, θ is the angle of the beam, τ is the torque applied to the beam, g is the gravity constant (e.g. $g = 9.81$), and $\beta > 0$ is the friction coefficient.

Applying the feedback law

$$\tau = 2r\dot{r} + gr\cos(\theta) + (r^2+1)u \tag{2}$$

and taking $x_1 = r$, $x_2 = \dot{x}_1$, $x_3 = \theta$, and $x_4 = \dot{x}_3$, the dynamics of the BNB system can be written as the following:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - g\sin(x_3) + x_1 x_4^2 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= u \end{cases} \tag{3}$$

Now x_3 is a virtual control for the second-order sub-

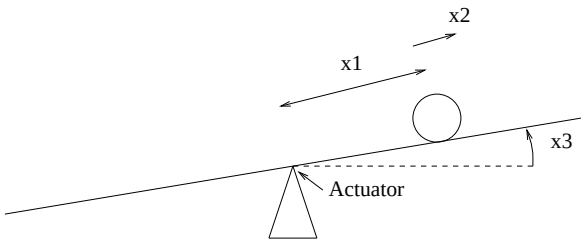

Figure 1: Beam-and-Ball System

system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - g\sin(x_3) + x_1\dot{x}_3^2 \end{cases} \tag{4}$$

Therefore, we would like to design a controller $x_3 = k_1(x_1, x_2)$ such that the following conditions are satisfied:

i) The equation

$$x_4 = \dot{x}_3 = \frac{\partial k_1}{\partial x_1}x_2 + \frac{\partial k_1}{\partial x_2}(-\beta x_2 - g\sin(k_1(x_1,x_2)) + x_1 x_4^2) \tag{5}$$

always has a solution $x_4 = k_2(x_1, x_2)$ for any x_1 and x_2.

ii) The origin for the second-order system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - g\sin(k_1(x_1,x_2)) + x_1 k_2(x_1,x_2)^2 \end{cases} \tag{6}$$

is semiglobally asymptotically stable.

Keep in mind that we only have to design $k_1(x_1, x_2)$. Then $k_2(x_1, x_2) = x_4$ is the solution of Equation (5) which is a second-order algebraic equation in x_4.

2 Controller design

Let us rewrite the dynamics of the second-order subsystem as

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x_1, x_2, x_3, x_4) \end{cases} \tag{7}$$

where

$$f(x_1, x_2, x_3, x_4) = -\beta x_2 - g\sin(x_3) + x_1 x_4^2 \tag{8}$$

and $x_3 = k_1(x_1, x_2)$ and $x_4 = k_2(x_1, x_2)$ are yet to be given. The idea behind the design of k_1 can be summarized in Figure (2).

There is one important fact that if x_3 is constant, then $x_4 = 0$ and the term $x_1 x_4^2$ in the equation of $\dot{x}_2$ plays no role. Therefore, the sign of $f(x_1, x_2, x_3, x_4)$ can be solely determined by x_3. Noting that for $\sigma(z) = \tanh(cz)$ where $c > 0$ is a constant, $|\sigma(z)| \approx 1$ and $d\sigma(z)/dz \approx 0$ if $|z| \gg 1/c$, one can come up with the following controller:

$$k_1(x_1, x_2) = c_1 \tanh(c_2(c_3 x_1 + x_2)); \quad c_1, c_2, c_3 > 0 \tag{9}$$

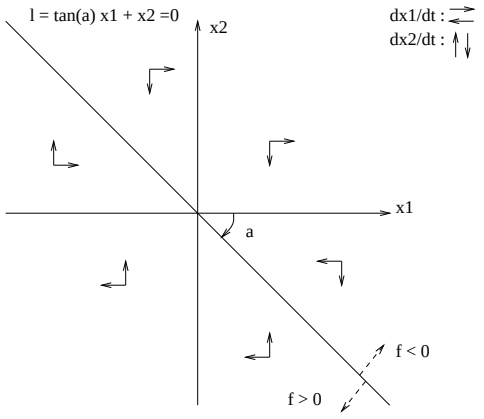

Figure 2: Desired directions of the vector field $(\dot{x}_1, \dot{x}_2)$ in the phase plane

where c_1 is the maximum swing-angle of the beam and $c_3 = \tan(\alpha)$ determines the slope of the line $l(x_1, x_2) = c_3 x_1 + x_2 = 0$. This controller gives the desired sign for $f(x_1, x_2, k_1(x_1, x_2), k_2(x_1, x_2))$ in the phase plane (x_1, x_2) as shown in Figure (2). It turns out that this controller satisfies both of the conditions stated in the previous section (the proof is given in the next section). Now using the backstepping procedure the controller for the third-order system can be given as the solution of the following equation in v:

$$\begin{aligned} v &= -c_4(x_3 - k_1(x_1, x_2)) + \tfrac{\partial k_1}{\partial x_1} x_2 \\ &+ \tfrac{\partial k_1}{\partial x_2}(-\beta x_2 - g\sin x_3 + x_1 v^2), \end{aligned} \tag{10}$$

we solve this equation by iterating on v starting with $v = -c_4((x_3 - k_1(x_1, x_2))$. Suppose $v^*(x_1, x_2, x_3)$ be the fixed point of Equation (10). Then we define the controller for the third-order subsystem as $v^*(x_1, x_2, x_3)$. Also, similarly the controller for the fourth-order system can be given as $u^*(x_1, x_2, x_3, x_4)$, where u^* is the fixed point of the following equation

$$\begin{aligned} u &= -c_5(x_4 - k_3(x_1, x_2, x_3, x_4)) + \tfrac{\partial k_3}{\partial x_1} x_2 + \tfrac{\partial k_3}{\partial x_2} f \\ &+ \tfrac{\partial k_3}{\partial x_3} x_4 + \tfrac{\partial k_3}{\partial x_4} u, \\ k_3 &= -c_4((x_3 - k_1(x_1, x_2)) + \tfrac{\partial k_1}{\partial x_1} x_2 + \tfrac{\partial k_1}{\partial x_1} f, \end{aligned} \tag{11}$$

where $c_5 > 0$.

3 Analysis for the second-order subsystem

We show that Equation (5) always has a solution $x_4 = k_2(x_1, x_2)$ for any x_1 and x_2 and controllers $x_3 = k_1(x_1, x_2)$ and $x_4 = k_2(x_1, x_2)$ render the origin, $(x_1, x_2) = 0$, semi-GAS for the second-order system

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - g\sin(x_3) + x_1 x_4^2 \end{cases} \tag{12}$$

by giving a radially unbounded Lyapunov function with a negative definite time derivative for this second-order system. For now we assume such a solution always exists if $x_3 = k_1(x_1, x_2)$ and present the existence proof later on.

To do the analysis, we approximate $\tanh(x)$ by $\mathrm{sat}(x)$, where $\mathrm{sat}(x) = x\min\{1, |x|\}$. Noting that $l(x_1, x_2) = c_3 x_1 + x_2$, we define the following areas of (x_1, x_2) plane:

$$\begin{aligned} A_1 &= \{(x_1, x_2) : l(x_1, x_2) > 1/c_2\} \\ A_2 &= \{(x_1, x_2) : l(x_1, x_2) < -1/c_2\} \\ A_3 &= \{(x_1, x_2) : |l(x_1, x_2)| \le 1/c_2\} \\ A_3^{++} &= \{(x_1, x_2) : 0 \le l(x_1, x_2) \le 1/c_2, x_1 > 0\} \\ A_3^{+-} &= \{(x_1, x_2) : -1/c_2 \le l(x_1, x_2) \le 0, x_1 > 0\} \\ A_3^{--} &= \{(x_1, x_2) : -1/c_2 \le l(x_1, x_2) \le 0, x_1 < 0\} \\ A_3^{-+} &= \{(x_1, x_2) : 0 \le l(x_1, x_2) \le 1/c_2, x_1 < 0\} \end{aligned}$$

Note that $A_3^{++} \cup A_3^{+-} \cup A_3^{--} \cup A_3^{-+} = A_3$. A_1, A_2, and A_3 are shown in Figure (3). We prove the following statements for the second-order system in Equation (12), under certain assumptions on c_1, c_2, and c_3 that will be stated later on:

i) Any solution of the system starting in $A_1 \cup A_2$ will enter A_3 in finite time.

ii) Any solution of the system starting in $A_3^{++} \cup A_3^{--}$ at $t = 0$, stays in A_3 for $t > 0$ and asymptotically converges to the origin.

iii) Any solution of the system starting in A_3^{+-} or A_3^{-+} at $t = 0$, either stays in A_3 for $t > 0$ and asymptotically converges to the origin, or converges to A_3^{--} or A_3^{++}, respectively, in finite time.

Because $A_1 \cup A_2 \cup A_3 = \Re^2$ the proof of these statements establishes global asymptotic stability (GAS) of the origin for this second-order system, provided that the given Lyapunov function is valid on $\Re^2$. But in the Lyapunov function that we give later, there is a parameter that depends on an upper bound on the norm of the initial state. Therefore, the initial state must belong to a compact set in $\Re^2$. This is equivalent to semiglobal asymptotic stability of the origin for the nonlinear second-order subsystem.

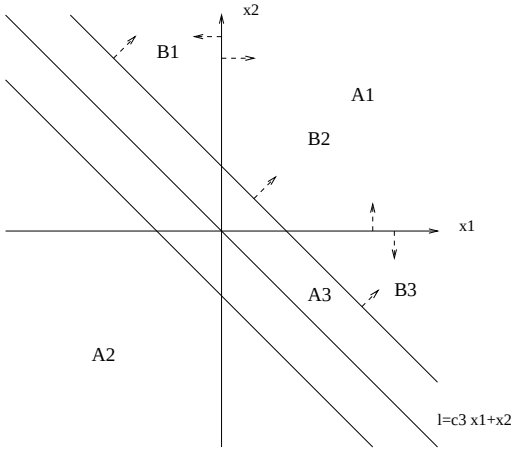

Figure 3: Areas A_1, A_2, A_3, and also B_1, B_2, and B_3.

Proof of i): Because of the geometric symmetry we prove statement (i) for the case where the solution starts at a point in A_1, the proof is similar for A_2. We divide A_1 into three areas:

$$\begin{aligned} B_1 &= \{(x_1, x_2) : l(x_1, x_2) > 1/c_2, x_1 < 0, x_2 > 0\} \\ B_2 &= \{(x_1, x_2) : l(x_1, x_2) > 1/c_2, x_1 \geq 0, x2 \geq 0\} \\ B_3 &= \{(x_1, x_2) : l(x_1, x_2) > 1/c_2, x_1 > 0, x2 < 0\} \end{aligned}$$

Because in A_1, $l(x_1, x_2) > 1/c_2$, we have $x_3 = c_1 \text{sat}(c_2 l(x_1, x_2)) = c_1$, and $x_4 = \dot{x}_3 = 0$. Therefore, the dynamics of the nonlinear second-order system in Equation (12) can be written as the following linear system:

$$\begin{cases} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\beta x_2 - g \sin(c_1) \end{cases} \tag{13}$$

The solution of this linear system is given by

$$\begin{aligned} x_1(t) &= \tfrac{1}{\beta}(x_2(0) + \tfrac{g \sin(c_1)}{\beta})(1 - \exp(-\beta t)) \\ &- \tfrac{g \sin(c_1)}{\beta} t + x_1(0), \\ x_2(t) &= (x_2(0) + \tfrac{g \sin(c_1)}{\beta}) \exp(-\beta t) - \tfrac{g \sin(c_1)}{\beta}, \end{aligned} \tag{14}$$

Note that $x_2(t) \to -g \sin(c_1)/\beta$ as $t \to \infty$. Therefore, in both B_1 and B_2, x_2 is strictly decreasing by time, and in B_3 depending on whether $x_2 \geq -g \sin(c_1)/\beta$ or $x_2 < -g \sin(c_1)/\beta$, x_2 is strictly decreasing or increasing by time, respectively. Also, because $\dot{x}_1 = x_2$ and in B_1 and B_2, $x_2 > 0$, x_1 is strictly increasing by time. This means that any solution starting at a point in B_1 either intersects $l_1 : l(x_1, x_2) = 1/c_2$ and goes to A_3^{-+}, or goes to B_2. Also, any solution starting at a point in B_2 either intersects l_1 and goes to A_3^{++}, or intersects $x_2 = 0$ and goes to B_3. In B_3, $\dot{x}_1 = x_2 < 0$, therefore x_1 is strictly decreasing by time. Now for any solution entering B_3 or starting at a point in B_3, either the solution intersects l_1 and goes to A_3^{++}, or the solution goes towards the asymptotic line $x_2 = -g \sin(c_1)/\beta$ and by decreasing of x_1 eventually intersects l_1 and goes to A_3^{++}. To summarize the argument, one notes that

- Any solution starting in B_1 either goes to A_3^{-+} or B_2.
- Any solution starting in B_2 either goes to A_3^{++} or B_3.
- Any solution starting in B_3 goes to A_3^{++}.

This means that any solution starting in $A_1 = B_1 \cup B_2 \cup B_3$ goes to A_3. Also, this transition from A_1 to A_3 occurs in finite time. Because in the worst case, for any point $(x_1, x_2) \in B_3$ on the solution of this system, $x_1 > 0$ decreases to zero in finite time, therefore the solution intersects l_1 in finite time.

Proof of ii): Now we show that any solution of the second-order system in Equation(12) starting in A_3^{++} at $t = 0$, stays in A_3 for $t > 0$ and asymptotically goes to the origin. Defining an affine change of coordinates

$$\begin{cases} z_1 &= x_1 \\ z_2 &= c_3 x_1 + x_2 \end{cases} \tag{15}$$

Equation (12) can be rewritten in the following form in new coordinates

$$\begin{cases} \dot{z}_1 &= -c_3 z_1 + z_2, \\ \dot{z}_2 &= -c_3(c_3 - \beta) z_1 + (c_3 - \beta) z_2 \\ &- g \sin(c_1 \tanh(c_2 z_2)) + \lambda(z_2)(c_1 c_2)^2 z_1 \dot{z}_2^2, \end{cases} \tag{16}$$

where $\lambda(z_2) = (1 - \tanh^2(c_2 z_2))^2$. Note that $0 \leq \lambda(z2) \leq 1$. We take the following Lyapunov function

$$V(z_1, z_2) = \frac{1}{2} z_1^2 + \frac{1}{2} \eta z_2^2 \tag{17}$$

where $\eta > 0$. We try to find an upper bound on η such that $\dot{V}(z_1, z_2) < 0$. Note that the origin is invariant under the affine change of coordinates given in Equation (15) (e.g. if $(x_1, x_2) = 0$, then $(z_1, z_2) = 0$).

By taking c_1 relatively small we apply the following approximations (only for analysis purposes)

$$\begin{aligned} \sin(c_1 \tanh(c_2 z_2)) &\approx c_1 \tanh(c_2 z_2) \\ \tanh(c_2 z_2) &\approx \text{sat}(c_2 z_2) \end{aligned}$$

In new coordinates, we have

$$A_3 = \{(z_1, z_2) : |z_2| < 1/c_2\}.$$

Therefore, in A_3, $\text{sat}(c_2 z_2) = c_2 z_2$ and the approximate dynamics of the system can be written as the following form

$$\left\{ \begin{array}{lcl} \dot{z}_1 &=& -c_3 z_1 + z_2 \\ \dot{z}_2 &=& -c_3(c_3-\beta) z_1 - (g c_1 c_2 + \beta - c_3) z_2 \\ &+& \lambda(z_2)(c_1 c_2)^2 z_1 \dot{z}_2^2 \end{array} \right. \quad (18)$$

First, we need to make some assumptions and prove a lemma.

Assumption 1: We assume $g c_1 c_2 > (5/4)(c_3 - \beta) \geq 0$.

Assumption 2: $\dot{z}_2$ takes its maximum absolute value when $z_2 = 0$.

Lemma 1: $\dot{z}_2$ is bounded. Specifically, $\dot{z}_2^2 < c_3(c_3 - \beta)/(c_1 c_2)^2$.

Proof: Based on Assumption 2, we know that $\dot{z}_2$ takes it's maximum absolute value when $z_2 = 0$. But from Equation (16), $\dot{z}_2$ is the solution of the following equation

$$\begin{aligned} \dot{z}_2 &= -c_3(c_3-\beta) z_1 + (c_3 - \beta) z_2 \\ &- g \sin(c_1 \tanh(c_2 z_2)) + \lambda(z_2)(c_1 c_2)^2 z_1 \dot{z}_2^2 \end{aligned}$$

setting $z_2 = 0$, we get

$$\dot{z}_2 = -c_3(c_3-\beta) z_1 + (c_1 c_2)^2 z_1 \dot{z}_2^2.$$

Because $\Delta = 1 + 4(c_1 c_2)^2 c_3(c_3 - \beta) z_1^2 > 0$, this second-order equation always has two solutions $\dot{z}_2^{(1)}$ and $\dot{z}_2^{(2)}$ with different signs. We define $\dot{z}_2$ as the solution of the above equation with the smaller absolute value. Thus, we have

$$\dot{z}_2^2 < |\dot{z}_2^{(1)} \dot{z}_2^{(2)}| = \frac{c_3(c_3-\beta)}{(c_1 c_2)^2}$$

therefore,

$$|\dot{z}_2| < \frac{\sqrt{c_3(c_3-\beta)}}{c_1 c_2}$$

and $\dot{z}_2$ is bounded.

Now for two different areas where $z_1 z_2 \geq 0$ and $z_1 z_2 < 0$ we try to find upper and lower bounds on η such that $\dot{V}(z_1, z_2) < 0$ for $(z_1, z_2) \neq 0$.

Case 1, $z_1 z_2 \geq 0$: In this case, we have

$$\begin{aligned} \dot{V}(z_1, z_2) &= -c_3 z_1^2 - \eta(g c_1 c_2 + \beta - c_3) z_2^2 \\ &+ (1 - \eta c_3(c_3 - \beta)) z_1 z_2 \\ &+ \eta \lambda(z_2)(c_1 c_2)^2 z_1 z_2 \dot{z}_2^2 \\ &= -(\sqrt{c_3} z_1 - \sqrt{\eta(g c_1 c_2 + \beta - c_3)} z_2)^2 \\ &+ (1 - \eta c_3(c_3 - \beta)) z_1 z_2 \\ &+ (\eta \lambda(z_2)(c_1 c_2)^2 \dot{z}_2^2 \\ &- 2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)}) z_1 z_2 \end{aligned}$$

in the last equation, the first term is negative definite. Because $z_1 z_2 \geq 0$, by taking $1 - \eta c_3(c_3 - \beta) \leq 0$, or $\eta \geq 1/c_3(c_3 - \beta)$, the second term becomes negative semidefinite. For the last term to be negative definite, η must satisfy the following inequality

$$\eta \lambda(z_2)(c_1 c_2)^2 \dot{z}_2^2 \leq 2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)}$$

but $\lambda(z_2) \leq 1$ and based on Lemma 1, $\dot{z}_2^2 < c_3(c_3 - \beta)/(c_1 c_2)^2$, thus

$$\eta \lambda(z_2)(c_1 c_2)^2 \dot{z}_2^2 \leq \eta c_3(c_3 - \beta)$$

therefore, we must have

$$\eta c_3(c_3 - \beta) \leq 2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)}$$

or

$$\eta \leq \frac{4(g c_1 c_2 + \beta - c_3)}{c_3(c_3 - \beta)^2}. \quad (19)$$

Case 2, $z_1 z_2 < 0$: In this case, we have

$$\begin{aligned} \dot{V}(z_1, z_2) &= -c_3 z_1^2 - \eta(g c_1 c_2 + \beta - c_3) z_2^2 \\ &+ (1 - \eta c_3(c_3 - \beta)) z_1 z_2 \\ &+ \eta \lambda(z_2)(c_1 c_2)^2 z_1 z_2 \dot{z}_2^2 \\ &= -(\sqrt{c_3} z_1 + \sqrt{\eta(g c_1 c_2 + \beta - c_3)} z_2)^2 \\ &+ (2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)} \\ &+ 1 - \eta c_3(c_3 - \beta)) z_1 z_2 \\ &+ \lambda(z_2)(c_1 c_2)^2 z_1 z_2 \dot{z}_2^2 \end{aligned}$$

In the last equation, because $z_1 z_2 < 0$, the first term and the last term are negative definite. If

$$2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)} + 1 - \eta c_3(c_3 - \beta) \geq 0$$

then the second term is negative definite as well and therefore $\dot{V}(z_1, z_2) < 0$ for $(z_1, z_2) \neq (0, 0)$. To satisfy this condition, we take $\eta \geq 1/c_3(c_3 - \beta)$, and solve the following inequality for η

$$\eta c_3(c_3 - \beta) - 1 \leq 2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)}$$

because $c_3(c_3 - \beta) - 1 < c_3(c_3 - \beta)$, we need to solve

$$\eta c_3(c_3 - \beta) \leq 2\sqrt{\eta c_3(g c_1 c_2 + \beta - c_3)}$$

or

$$(\eta c_3(c_3 - \beta))^2 \leq 4\eta c_3(g c_1 c_2 + \beta - c_3)$$

thus

$$\eta \leq \frac{4(g c_1 c_2 + \beta - c_3)}{c_3(c_3 - \beta)^2}. \quad (20)$$

Comparing Equations (19) and (20), one notes that both of the upper bounds on η are equal to η_{max}, where

$$\eta_{max} = \frac{4(gc_1c_2 + \beta - c_3)}{c_3(c_3 - \beta)^2}. \quad (21)$$

Also, the lower bounds on η are equal as well. The lower bound on η is given by

$$\eta_{min} = \frac{1}{c_3(c_3 - \beta)}. \quad (22)$$

for η_{min} and η_{max} to be well-defined we must have $\eta_{min} < \eta_{max}$, or $gc_1c_2 > (5/4)(c_3 - \beta)$, which is equivalent to Assumption 1.

Now to show that the solution of the second-order system that starts in (or enters) $A_3^{++} \cup A_3^{--}$, never leaves A_3, first we need to prove the following lemma.

Lemma 2: Any solution of the system that enters to A_3^{++}, or starts in A_3^{++} intersects $z_2 = 0$ in finite time.

Proof: We show that in $A_3^{++} = \{(z_1, z_2) : z_1 \geq 0, 0 < z_2 \leq 1/c_2\}$, $\dot{z}_2 < 0$. This proves that z_2 is strictly decreasing by time and goes to zero. Note that $A_3^{++} \subset A_3$ and in A_3, we have

$$\dot{z}_2 = -c_3(c_3-\beta)z_1-(gc_1c_2+\beta-c_3)z_2+\lambda(z_2)(c_1c_2)^2z_1\dot{z}_2^2$$

or

$$\dot{z}_2 = -(c_3(c_3-\beta)-\lambda(z_2)(c_1c_2)^2\dot{z}_2^2)z_1-(gc_1c_2+\beta-c_3)z_2$$

but based on Lemma 1, we have

$$c_3(c_3 - \beta) > (c_1c_2)^2\dot{z}_2^2 > \lambda(z_2)(c_1c_2)^2\dot{z}_2^2$$

Also, because in A_3^{++}, $z_1 \geq 0$ and $z_2 > 0$, we get $\dot{z}_2 < 0$.

Now, we try to show how large η_{max} can be. Take $c_3 = \beta + \epsilon$, where $\epsilon > 0$. We have

$$\eta_{max}(\epsilon) = \frac{4(gc_1c_2 - \epsilon)}{(\beta + \epsilon)\epsilon^2}.$$

Therefore, by taking ϵ to be relatively small, $\eta_{max}(\epsilon)$ can be made relatively large. Intuitively, this means that after the solution intersects line $z_2 = 0$ based on Lemma 2, it stays very close to this line until it converges to the origin and it never leaves $A_R = \{(z_1, z_2) : (z_1^2 + \eta_{max}z_2^2) \leq R^2\}$, where $R = z_1^*$ and $(z_1^*, 0)$ is the point that the solution intersects $z_2 = 0$. But for a relatively large η, $A_R \subset A_3$, and the solution never leaves A_R. Therefore the solution never leaves A_3 and asymptotically goes to the origin.

Proof of iii): For any solution starting in A_3^{-+}, we have $\dot{z}_1 = -c_3z_1 + z_2 > 0$. Therefore, z_1 is strictly increasing. Depending on the sign of $\dot{z}_2$, the solution either goes towards the line $z_2 = 0$ and asymptotically goes to the origin (like in case (ii)) , or leaves A_3^{-+} and goes to A_3^{++} directly, or first goes to B_1 and then based on the arguments in the proof of (ii) eventually goes to A^{++}. By a similar argument, one can show that any solution starting in A_3^{+-}, either asymptotically goes to the origin, or eventually goes to A_3^{--}.

The following Lemma proves that the second-order algebraic equation given in Equation (5) always has a continuous solution $x_4 = k_2(x_1, x_2)$ for any x_1 and x_2.

Lemma 3: The following second-order equation in y always has a continuous solution $y = \kappa(z_1, z_2)$ for any z_1 and z_2.

$$\begin{aligned} y &= -c_3(c_3 - \beta)z_1 + (c_3 - \beta)z_2 \\ &- g\sin(c_1\tanh(c_2z_2)) + \lambda(z_2)(c_1c_2)^2z_1y^2 \end{aligned} \quad (23)$$

Proof: First, we prove the existence of the solution. There are two cases:

Case 1, $|z_2| > 1/c_2$: In this case, $\lambda(z_2) = 0$ and

$$y = -c_3(c_3 - \beta)z_1 + (c_3 - \beta)z_2 - g\sin(c_1)$$

is the solution of this equation. It is clear that y is a continuous function of z_1 and z_2.

Case 2, $|z_2| \leq 1/c_2$: In this case, the equation can be written in the following form

$$\lambda(z_2)(c_1c_2)^2z_1y^2-y-c_3(c_3-\beta)z_1-(gc_1c_2+\beta-c_3)z_2 = 0$$

for this equation we have

$$\Delta = \frac{1}{4}+\lambda(z_2)(c_1c_2)^2z_1(c_3(c_3-\beta)z_1+(gc_1c_2+\beta-c_3)z_2)$$

If $z_1z_2 \geq 0$, then

$$\text{sign}\{\lambda(z_2)(c_1c_2)^2z_1\} = \text{sign}\{(c_3(c_3-\beta)z_1+(gc_1c_2+\beta-c_3)z_2)\}$$

therefore, $\Delta \geq 1/4 > 0$, and the equation has two solutions.

Now suppose $z_1z_2 < 0$, then we have

$$\begin{aligned} \Delta &= \tfrac{1}{4} + \lambda(z_2)(c_1c_2)^2c_3(c_3 - \beta)z_1^2 \\ &+ \lambda(z_2)(c_1c_2)^2(gc_1c_2 + \beta - c_3)z_1z_2) \end{aligned}$$

because the second term is positive semidefinite to make $\Delta \geq 0$ we must have

$$\frac{1}{4} + \lambda(z_2)(c_1c_2)^2(gc_1c_2 + \beta - c_3)z_1z_2 \geq 0$$

Take $c_1c_2 = \epsilon$ and $c_3 = \beta + n\epsilon$, where $\epsilon > 0$ is relatively small and $0 < n \leq g - 1$. We have

$$\begin{aligned} |\lambda(z_2)(c_1c_2)^2(gc_1c_2 + \beta - c_3)z_1z_2)| &= \lambda(z_2)\epsilon^3(g - n)|z_1z_2)| \\ &\leq \epsilon^3(g - n)|z_1z_2)| \\ &\leq \frac{\epsilon^3}{c_2(g - n)}|z_1| \\ &= \frac{c_1^3c_2^2}{g - n}|z_1| \end{aligned}$$

defining

$$z_{max} = \frac{1}{4c_1^3 c_2^2}, \tag{24}$$

for $|z_1| \leq z_{max}$ we have

$$|\lambda(z_2)(c_1c_2)^2(gc_1c_2 + \beta - c_3)z_1z_2)| \leq \frac{1}{4(g-n)} \leq \frac{1}{4}.$$

Thus

$$\frac{1}{4} + \lambda(z_2)(c_1c_2)^2(gc_1c_2 + \beta - c_3)z_1z_2 \geq 0$$

and therefore $\Delta \geq 0$. Note that c_1 is relatively small. Therefore, by taking c_2 relatively small, based on Equation (24), z_{max} can be made relatively large. Therefore, in the limit, when $z_{max} \to \infty$, Equation (23) always has a solution for any z_1 and z_2 belong to a compact set in $\Re^2$. It is straightforward to show that the solution is continuous.

4 Analysis for the fourth-order system

Now suppose that $V(x_1, x_2)$ be a Lyapunov function for the second-order subsystem. Then, using backstepping procedure, we can construct a control Lyapunov function and its associated control law for the fourth-order system. Based on the arguments for the controller design in section (2), the control Lyapunov function for the fourth-order system is in the following form

$$\begin{aligned} V_4(x_1, x_2, x_3, x_4) &= V(x_1, x_2) + \tfrac{1}{2}(x_3 - k_1(x_1, x_2))^2 \\ &+ \tfrac{1}{2}(x_4 - k_3(x_1, x_2, x_3, x_4))^2, \end{aligned} \tag{25}$$

We expect that after a finite *transition period*, $x_4 \to k_3(x_1, x_2, x_3, x_4)$, or $x_4 \to v^*$ and $x_3 \to k_1(x_1, x_2)$. But if $x_3 \to k_1(x_1, x_2)$ then $v^*(x_1, x_2, x_3) \to k_2(x_1, x_2)$. Now, if $x_3 = k_1(x_1, x_2)$ and $x_4 = k_2(x_1, x_2)$, then $\dot{V}(x_1, x_2) < 0$. Therefore

$$\begin{aligned} \dot{V}_4(x_1, x_2, x_3, x_4) &= \dot{V}(x_1, x_2) - c_4(x_3 - k_1(x_1, x_2))^2 \\ &- c_5(x_4 - k_3(x_1, x_2, x_3))^2 < 0, \end{aligned} \tag{26}$$

for $x \neq 0$ and the origin is semi-GAS for the fourth-order system.

5 Results

For $\beta = 0.1$ and $g = 9.81$, by taking $c_1 = \pi/10$, $c_2 = 0.5$, $c_3 = \tan(\pi/4) = 1$, and $c_4 = 1$, $k_3(x_1, x_2, x_3)$ renders the origin GAS for the third-order subsystem. The simulation results are in agreement with the analysis. Figure (4) shows the variations of position, speed, angle of the beam, and the angular velocity of the beam.

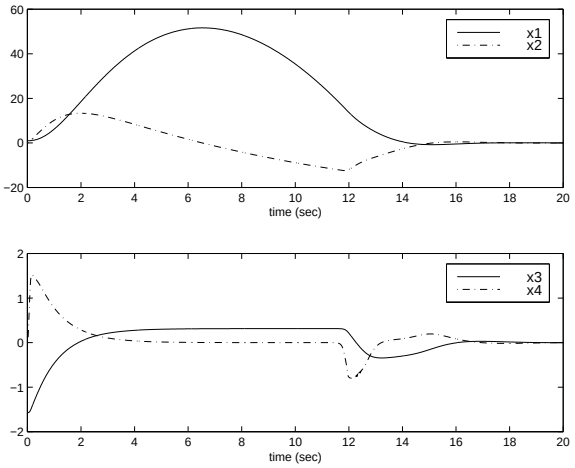

Figure 4: The variation of the state vector for the Beam-and-Ball System with initial state $(1, 0, -\pi/2, 0)$.

6 Conclusions

We designed a controller for the Beam-and-Ball system that renders the origin semiglobally asymptotically stable. For analysis, we approximated the controller by a saturation function and gave a quadratic Lyapunov function for the second-order subsystem of the system when the solution enters the linear area of the saturation function. We derived upper and lower bounds on a parameter η which determines the shape of the level sets of the Lyapunov function. We showed that the origin is globally asymptotically stable for the second-order system. Then, using a new version of the backstepping procedure, we obtained the controller for the whole fourth-order system. Simulation results approve the analysis. Also, the controller manages to bring the ball from any (bounded) initial position and speed to the origin efficiently without doing unnecessary movements.

References

[1] J. Hauser, S. Sastry, and P. Kokotović. Nonlinear control via approximate input-output linearization. *IEEE Trans. on Automatic Control*, vol.37:pp. 392–398, 1992.

[2] M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. John Wiley & Sons, 1995.

[3] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control*. Springer-Verlag, 1997.