LICENTIATE THESIS

Visualizing curved spacetime

RICKARD M. JONSSON

Department of Astronomy and Astrophysics
Centre for Astrophysics and Space Science
Chalmers University of Technology and Göteborg University
Göteborg, Sweden 2001

rijo@fy.chalmers.se

Visualizing curved spacetime

Rickard M. Jonsson

Department of Astronomy and Astrophysics
Center for Astrophysics and Space Science
Chalmers University of Technology and Göteborg University
Göteborg, Sweden

Abstract

In this thesis I derive two fundamentally different methods of visualizing curved spacetimes, the *dual* and the *absolute*. The images are nice pedagogical tools for teaching General Relativity. I also review the concept of space-eigentimes.

In the *dual* scheme, I start from the equations of motion in a 1+1 static, diagonal, Lorentzian spacetime, such as the Schwarzschild radial line element. I then find another metric, with *Euclidean* signature, which produces the same geodesics $x(t)$. This geodesically equivalent *dual* metric can be embedded in ordinary Euclidean space. On the embedded surface freely falling particles take the shortest paths. Thus, one can visualize how acceleration in a gravitational field is explained by particles moving on straight lines in a curved spacetime.

In the *absolute* scheme, I start from an arbitrary Lorentzian spacetime with a given field of timelike four-velocities u^μ. I then do a coordinate transformation to the local Minkowski system comoving with the given four-velocity at every point. In the local system, the sign of the spatial part of the metric is flipped to create a new metric of Euclidean signature. The new *absolute* metric can be covariantly related to the original one. This method is particularly well suited for visualizing gravitational time dilation and the horizon of Black Holes.

Acknowledgements

First of all I would like to thank my supervisor Marek Abramowicz for his support of myself and my ideas. I also thank Ulf Torkelsson and Martin Cederwall for their comments on my Paper.

I would like to salute all of the guys at the 6:th and 7:th floor, as well as the Onsala-people, for creating such a nice working environment. In particular I would like to thank Pontus, Martin and Alex for always being available objects for me to pester :) In Astrophysics I would like to mention Rim, Alessandro, Gustav, Tommy and Duilia for nice discussions on all sorts of topics.

Thank you Achim for being a good co-builder of longbows as well as a fellow rider. A special thanks also for giving a particular section of this thesis the most thorough read-through I ever saw.

Thank you Hans, as always, for your open mind and endless ongoing search for the Ultimate understanding.

Thank you Daniel for your contributions to my understanding of space-eigentimes.

Off the record I would also like to thank the girls at Salsa & Söndag for insuring that at least one evening per week I am not stuck in front of a computer. I am also grateful to the guys at the experimental workshop for their advice in the delicate business of making your own aluminum spacetime!

At last a very special thanks to all my students at the F-education for making not only teaching but also life outside the corridors so enjoyable.

April, nådens år 2001

Rickard Jonsson

P.S While I am not sure that they will get the message, I am very grateful to a certain family of laughing bottlenose dolphins in the Red Sea. They made a dream come true.

Appended Papers

Paper I

Embedding spacetime via a geodesically equivalent metric of Euclidean signature
Jonsson, R. (2001) - Accepted for publication in GRG, June 2001, Volume 33

Contents

1

Introduction

Ever since it was presented in 1916, Einstein's General theory of Relativity has been extremely successful in explaining all sorts gravitational phenomena. Two examples from the beginning of the century are the gravitational deflection of light from distant stars passing our sun and the precession of the perihelia of Mercury. More recent experiments involve gravitational red shifts, slowing down of atomic clocks and indirect measurements of gravitational waves, the latter matching the predictions of Einstein's theory to a fantastic precision.

Ever since the theory was launched it has also puzzled the minds of physicists and people in general. In fact it is rumored that in the beginning of the century a reporter asked Sir Arthur Eddington to comment on there only being three people in the world who understood Einstein's theory. The reply from Eddington was "Who's the third?". Today the number of physicists understanding General Relativity is quite large. For the more general audience the theory is however still clouded in mystique. In particular, people find the legendary Black Holes fascinating, and incomprehensible. I know, having explained the concepts so many times at the local pizza-place and other places.

Einstein's theory, which is a geometrical theory, is in many ways well suited to be explained by images. For instance the way a star is affecting the *space* around and inside it can easily be displayed by a curved surface. These kinds of images are invaluable in teaching General Relativity, but also for the seasoned relativist. For a proper understanding of General Relativity, one certainly needs the mathematical background, but it will always be accompanied by examples, mental images and even movies.

The very heart of Einstein's theory, *curved spacetime*, is however fundamentally difficult to display using curved surfaces. The reason is that in General Relativity there are negative (squared) distances, something that we do not have on ordinary curved surfaces.

1.1 This thesis

In this thesis I show how one can circumvent the problem of negative distances, and visualize curved spacetimes after all. I do this using two completely different methods. I also re-derive an already existing method.

Using the images of spacetime, one can explain how acceleration of particles here on Earth is caused by a curvature of spacetime rather than by a force. One can also, for instance, explain the gravitational slowing-down of clocks as a pure geometrical effect. The basic structure of the thesis is outlined below:

- **Chapter 2** is an introduction to General Relativity, aimed at a general audience. The main object of this chapter is to explain how spacetime *geometry* can explain why things are falling towards the Earth when we drop them. In General Relativity there is no such thing as a gravitational force. The underlying mathematics of this chapter is dealt with in Paper I, at the end of this thesis.

- **Chapter 3** deals with another idea for visualizing curved spacetime. This idea is more suited for understanding the slowing down of clocks close to gravitational sources. It is also well suited for understanding Black Holes and the event horizon. This idea is ideal for people who know Special but not General Relativity. It has however many virtues also for people who know nothing about General Relativity, as well as for seasoned General Relativists. This idea is covariant and applicable to any spacetime. I have yet to put this in a form more suited for a general audience.

- **Chapter 4** deals with some mathematical aspects of geodesics; it is in principle a comment to Paper I.

- **Chapter 5** is also a rather mathematical comment to Paper I, regarding the *shape* of the spacetime inside a star. I show that in the Newtonian limit it corresponds to an exact sphere.

- **Chapter 6** is a re-derivation of the basic ideas underlying the book by Epstein [2], which has a different way of displaying effects in curved spacetimes.

- **Paper I** Given the geodesic trajectories in a 1+1 static Lorentzian spacetime, such as the Schwarzschild line element, I find a new positive definite metric that has the *same* geodesics as the original metric. This new *dual* metric can be embedded as a rotational surface.

Apart from the three ways of displaying curved spacetimes dealt with in this thesis there is also another fundamentally different way invented by Marolf [1].

There are thus at least four, and probably more, fundamentally different methods of visualizing curved spacetimes by curved surfaces. They each have different virtues and explaining power. I believe Feynman once said that one doesn't have a proper understanding of a phenomena until one can explain it in at least five different ways. I think he had a good point.

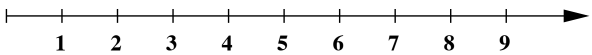
2

A geometrical introduction to gravitation

In this section I would like to give some intuition, without using mathematics, about the General Theory of Relativity, Einstein's theory of gravity. I will show how one can explain why things to fall to the ground when we drop them, not because a force is acting on them, but because the particles are moving straight in a *curved* spacetime. To understand curved spacetimes (don't worry, it's simple) we need first to understand what a *spacetime* is.

2.1 Spacetime, what is that?

Imagine a straight line drawn on the floor with meter marks on it:



We can describe all events that occur on the line, like an alarm bell ringing, in a *spacetime diagram*. See Fig 2.1

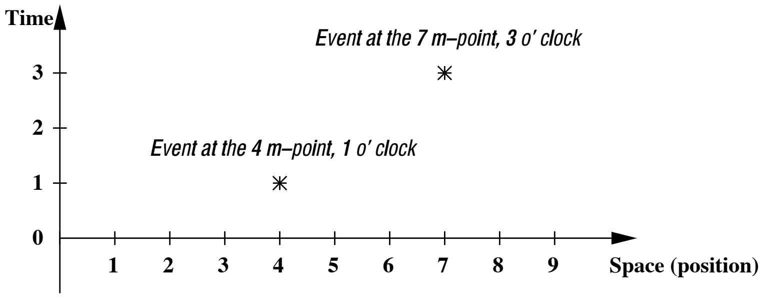


Figure 2.1: A diagram over all the events on the line

The later the event occurs, the higher up in the diagram it is marked. The more to the right the event occurs on the line, the more to the right it is marked. This graphical connection between space and time is called the *spacetime*. To be more strict one can say that spacetime is the collection of all events that occur on the line and the spacetime diagram is a map over all these events.

In a spacetime diagram it is easy to visualize how particles move on the line. See Fig 2.2. Notice that the more the trajectory of a particle is tilted, the faster it moves.

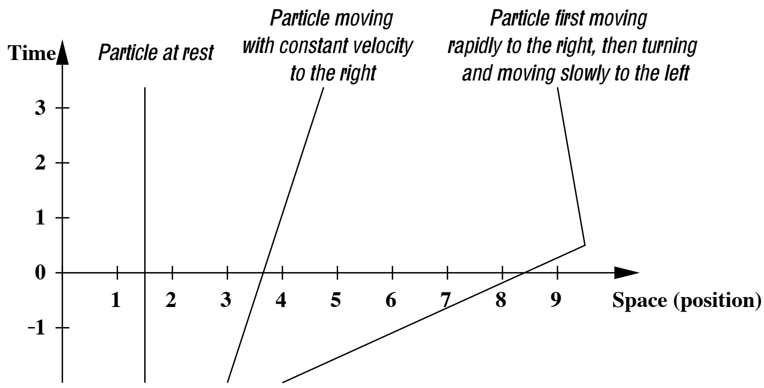


Figure 2.2: Particles in motion represented by lines in the spacetime diagram

Now imagine the line not drawn on the floor, but placed somewhere between the the stars, in empty space, where one doesn't feel any effects of gravity. In such a place, if you throw an apple to the right along the line, it will keep following the line moving at a constant speed without accelerating since there are no forces acting on it. In the same manner, if we release an apple on the line such that it was at rest when we let it go, it will remain at rest. In both cases the apple is moving with constant speed. This means that it follows a *straight line* in the spacetime. Thus we can explain the motion of particles that are not acted on by any forces, by saying that they move on straight lines in the spacetime, instead of saying that they do not accelerate.

Particles that *are* affected by forces however do accelerate. In the spacetime picture this corresponds to a curvature of the trajectory. In other words, the trajectory of a particle that is acted on by a force, is curving *away* from a straight line. Correspondingly, if the trajectory is curved, we know that there has to be a force acting on the particle. See Fig 2.3

On a flat surface there are many ways to define what we mean by a straight line. One way, which is also working on curved surfaces, is to say that a straight line is the path that a little toy car takes when we push it forward on the surface. We are then assuming that the little toy car is well manufactured and brand new so that it really goes straight forward :)

Hopefully we have now developed a feeling for how velocities and accelerations correspond to tilted and curved trajectories respectively in the spacetime picture. It will

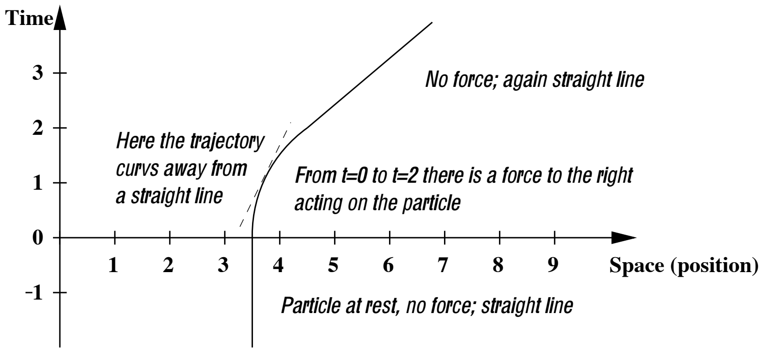


Figure 2.3: How a force gives a curvature of a spacetime trajectory

then be easy to understand how this is working also in a *curved* spacetime.

2.2 Gravity and curved spacetime

Suppose that we put our line outside the Earth.

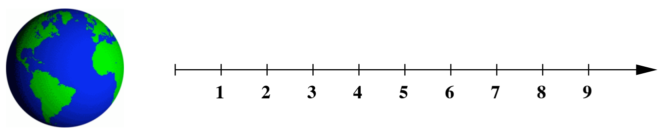


Figure 2.4: A line placed outside the Earth. The Earth is not in scale but put there to show in what direction the Earth lies

Now particles that are only affected by gravity will not move at a constant speed but will be accelerated towards the Earth as depicted in Fig 2.5.

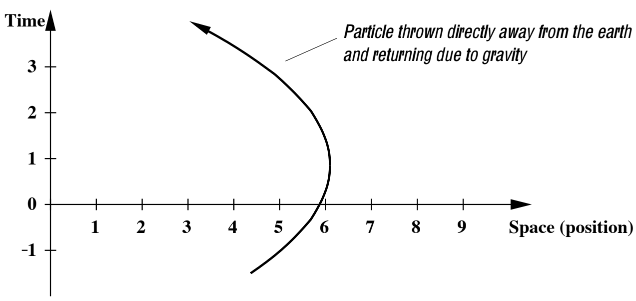


Figure 2.5: Particle accelerating towards the Earth due to gravity

Newton would explain the acceleration by saying that there is a gravitational force acting on the particle, causing it to accelerate towards the Earth.

Imagine however if the the mass of the Earth has curved the spacetime, so it really should look like that depicted in Fig 2.6.

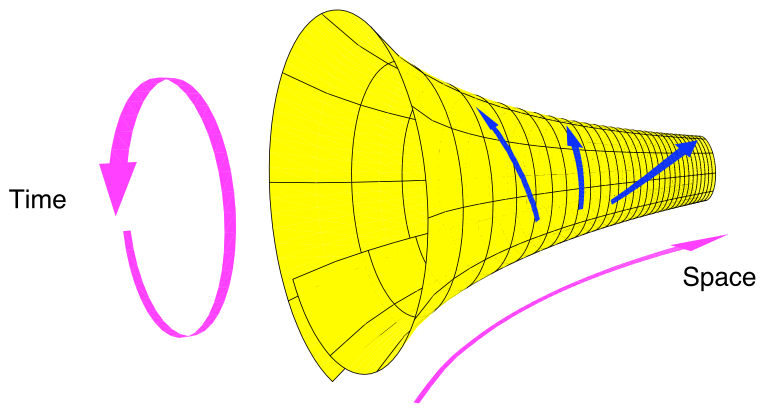


Figure 2.6: The curved spacetime

To create such a spacetime from the ordinary flat version, we can imagine that we roll it in the time direction, while also stretching it at the left end. Really the spacetime constructed this way should be like a toilet roll with infinitely many layers that are infinitely thin. When we walk one lap around the trumpet-like spacetime we thus come to a new point. From now on I will however suppress this in the images and let time close in on itself.

On the trumpet we can mark events, and motion, just like we did in the flat image. Still events that happen far to the right on the line, we mark far to the right on the trumpet, towards the narrow end, and vice versa for events that happened more to the left. However moving in time now means moving *around* the trumpet.

As a warm-up mental exercise, I have in the Fig 2.6 drawn the trajectories of three different particles. The one closest to the Earth (the leftmost one) corresponds to a particle traveling to the left, moving closer to the Earth. The middle trajectory corresponds to a particle at rest, moving only in *time*. The rightmost trajectory corresponds to a particle moving out from the Earth at high speed.

Imagine that we have a real curved surface as the one depicted in Fig 2.6, made of metal for instance. Suppose that we take a little toy car and put it on the outside of the trumpet. We direct it so that it is mostly pointing *around* the trumpet, but also a little towards the narrow end of the trumpet and push the car forward. We will find that the car first starts to move around, and to the right on the trumpet. Then it will reach a point where it is no longer moving to the right, but instead starts moving to the left. All this is due to the geometry of the trumpet. The path taken by the car will be as that depicted in Fig 2.7.

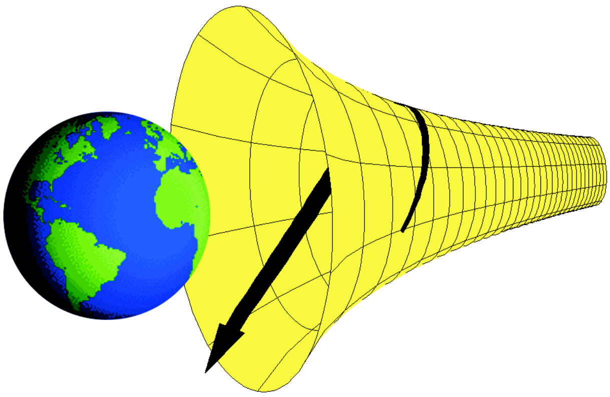


Figure 2.7: The trajectory of a particle in free fall as a straight line on a curved spacetime. Compare with Fig 2.5

The image of the Earth is included to remind the viewer in what direction the Earth lies. Strictly one shouldn't draw the spacetime in the same picture as the Earth however.

The trajectory in Fig 2.7 corresponds to a particle that is first moving away from the Earth, reaching a maximum and then falling back. Just like we we would expect an apple, thrown straight out from the Earth, to first move away from the Earth and then return due to gravity.

We see then the possibility to explain the acceleration caused by gravity as an effect of a spacetime *geometry* (or shape), rather than as an effect of a force.

A particle in free fall corresponds to a straight line in the spacetime

By 'free fall' we mean a particle that is not affected by anything except gravity. So it is just like in the preceding section where we had no gravity, we explain the motion of particles that are not affected by anything but gravity by saying that they move on a straight line (a line taken by a little toy car) in the spacetime. The difference is that the spacetime now is curved. Thus in geometrical gravity, there is no such thing as a *gravitational* force.

2.3 Forces and the acceleration paradox

Although gravity is not a force in the geometrical theory of gravity, there are still other forces that can affect particles. For instance there is the force with which we push our old car (Volvo Amazon) when we are trying to jump-start it. Just like in the case where we had no gravity, particles that are affected by a force will *not* follow a straight line, but will curve away from a straight line.

Consider for instance a particle that is held at rest at a certain position in space. Being at rest in space means moving around the trumpet on a circle. If we direct a toy car along such a circle and push it forward it will soon leave the circle and start moving to the left. We can however make the car stay on the circle by turning the wheels to the right a little bit. This means that the circular trajectory is bending away from what is straight, and there must therefore be a force acting on the particle corresponding to the circular trajectory. A particle at rest at fixed position outside the Earth must therefor constantly be pushed away from the Earth.

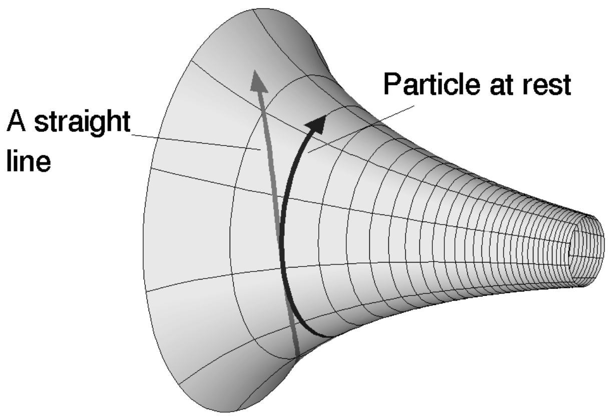


Figure 2.8: We see how the trajectory of a particle at *rest* is curving away from what is straight

Consider yourself, sitting on a chair. In Newtonian theory the gravitational force downwards on your body is exactly cancelled by an equally big force upwards from the chair. Therefore we do not accelerate. In Einstein's theory however there is only one force acting on us, namely the force from the chair. This means that we are accelerating upwards all the time. It may seem paradoxical how we can be accelerating upwards all the time, without going anywhere. It is however just like the car driving around on the spacetime. Although it is turning right all the time, it is never going to the right!

2.4 The full spacetime

So far we have described the spacetime of a line *outside* the Earth. Now let us study the spacetime of a line going *through* a hole in the Earth. See Fig 2.9.

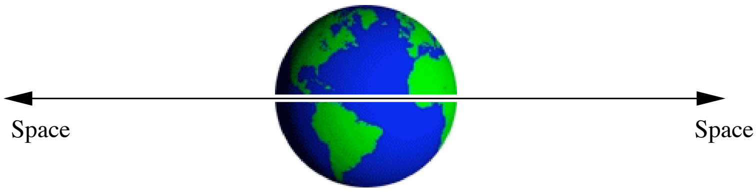


Figure 2.9: A line in a hole through the Earth

If we drop a particle outside the Earth on the line it will accelerate relative the earth towards the center, pass the center and start accelerating back from the other side. It will in this way oscillate back and forth. In the ordinary flat spacetime image this will look something like that depicted in Fig 2.10.

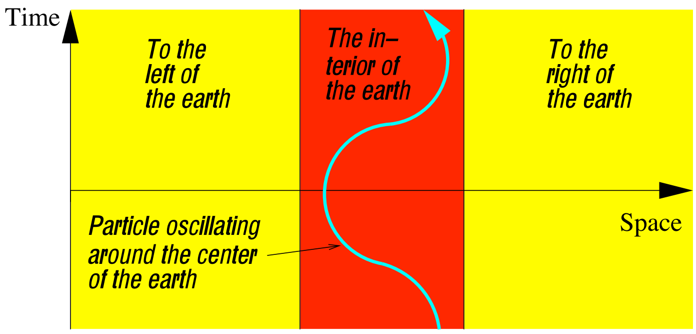


Figure 2.10: Particle oscillating around the center of the Earth

Newton would explain the motion by a gravitational force directed towards the center of the Earth. Now we will instead explain it by a spacetime geometry as that depicted in Fig 2.11.

Just like before, moving in time means moving around the spacetime whereas moving in space means moving to the right or to the left. In particular study a trajectory which is following the equator around the central bulge (the dark trajectory in Fig 2.11). This corresponds to a particle that is not moving in space, only in time, and is at rest at the center of the Earth. The trajectory is a also a straight line on the curved surface, since a little toy car pushed in the direction of the equator will remain on the equator. Thus the trajectory corresponds to a particle in free fall (a particle that is only affected by gravity). From Newtonian theory we would expect a particle that is placed at rest at

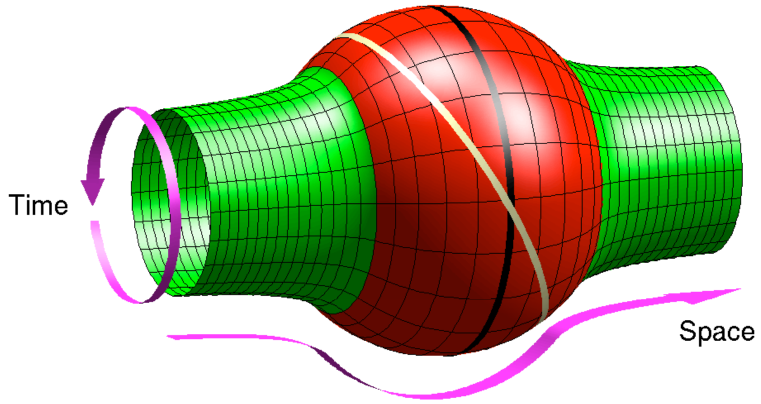


Figure 2.11: The curved spacetime of a central line through the Earth

the center of the Earth to remain at rest since there would be no gravitational forces in this point.

Next consider as a trajectory a tilted equator on the bulge (the light trajectory in Fig 2.11). Assuming the bulge to be exactly spherical this would also be a straight line and thus correspond to a particle in free fall. Here as we move in time the particle is first moving to the left, then reaching a maximum before moving back to the right, passing the ordinary equator, reaching a a maximum before moving back to to the left. This corresponds to a particle oscillating around the center of the Earth, just like we would expect from Newtonian theory. It completes one oscillation per lap around the spacetime. In the case of the Earth, one lap around the spacetime is roughly 84 minutes.

Notice that it does not matter how much the trajectory is tilted (i.e what amplitude the oscillation has), as long as it remains within the central bulge. Whatever the tilt, it always takes one lap around the spacetime for the trajectory to move away, pass and return again to the center of the Earth. The period time for an oscillation is thus independent of the amplitude. This is also expected from Newtonian theory.

2.5 Comments and conclusions

We have now seen a geometrical explanation of why things fall when we drop them, and why particles dropped in a hole through the Earth will oscillate around the center of the Earth. The explanation was not that the Earth was affecting the thing with a force but that the thing was traveling straight forward in a curved spacetime.

In a sense we have however only touched half of Einstein's theory of gravitation. Not only does the theory explain the motion of particles, as a geometrical effect, but it also predicts how *matter* generates this spacetime geometry. Suppose for instance that the core of the Earth had consisted of lead. Then there would have been a different matter distribution and we would have gotten a different *shape* of the spacetime. Matter curves the spacetime and the spacetime determines how matter moves.

We have seen how one can explain gravity in two ways (at least), and one may ask which one is the *correct* one. There is however no such thing as a correct theory. Every theory that we come up with to explain the nature around us, is a *model*. We can never know if our model is a fundamentally correct theory for the Universe (if even such a model exists). We can only, by measurements, prove that a theory is *wrong*, never that it is *right*. It may however be that some day we come up with a model that is so simple and beautiful, that explains everything we have so far tested and seen, that we *believe* it is a fundamental theory of the Universe.

Einstein's and Newton's theories for Gravity are both models, describing the world around us. There are however many effects that cannot be explained by Newtonian gravity, but that *can* be explained with Einstein's theory. The full theory of Einstein, which we have just tasted a sip of, is much richer than just explaining how particles move. It is also explaining why clocks slow down near gravitational sources, what gravitational waves are and how the Universe can be expanding. These are effects that we do see in nature directly or indirectly.

I would like to finish this introduction with an analogy between a curved spacetime and our own Earth. Think back to a time when man had not yet been convinced that the Earth was round (some still debate this) and we merely had ordinary flat maps of the world. Suppose that we did the experiment to send away the most straight-going vessel in the navy, perpendicular to the equator (towards the north). The cartographers would perhaps have noticed that relative their map the ship would turn to the right. Maybe they would have invented a theory of a special force guiding the vessel to the right. But then, they might also have gotten the idea that perhaps it was just the map that had the wrong shape and it should really have been round as depicted in Fig 2.12.

Figure 2.12: A flat map of an Earth that is really round

3

The absolute metric

The dual metric (see Paper I) is good for explaining acceleration of particles in free fall. To explain time dilation of for instance a star, with a rotational surface, we should however rather have a cylinder with a *dip* in the middle than a bulge as in the dual scheme. The Schwarzschild time should still be the azimuthal angle, and the proper time elapsed for an observer at rest should be the Euclidean distance traveled on the local circle (corresponding to a fixed position in space). We thus want a rotational body where the radius is proportional to $\sqrt{g_{tt}}$. We can also make the surface so that the proper distance (the distance to walk) between two adjacent spatial positions (i.e circles in the spacetime) is the correct one.

3.1 The absolute line element

We realize that the scheme outlined above corresponds to embedding the geometry we get if we simply flip the sign of the spatial part of the metric. Suppose that we have the line element of a central line through a star:

$$d\tau^2 = g_{tt}dt^2 + g_{xx}dx^2 \tag{3.1}$$

Here g_{xx} will be negative. Now we consider the metric:

$$ds^2 = g_{tt}dt^2 + |g_{xx}|dx^2 \tag{3.2}$$

An embedding of this metric for a constant proper density star is depicted in Fig 3.1.

In this picture the proper time traveled for an observer at rest is exactly the Euclidean distance on the surface. Also spatial distances as experienced by an observer at rest are exactly correct, except for a minus sign. This means that photons everywhere move at exactly 45° to the local spatial line.

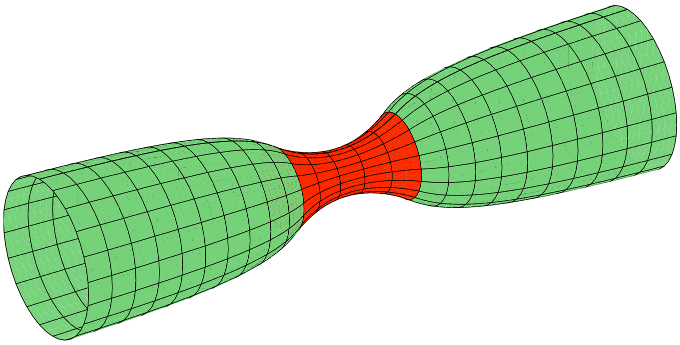


Figure 3.1: An hourglass embedding of the *absolute* spacetime of a central line through a star

We can also find the proper distance separating two infinitesimally displaced events on the surface. Given the direction of an arrow pointing from one point to the other, we know the gamma factor connected to the two events with respect to an observer at rest. Thus, knowing a little Special Relativity, we can evaluate any proper distance on the surface, just from the shape of the surface [1].

The key point is however that we *see* gravitational time dilation. The timelike distance traveled per lap around the spacetime is *shorter* in the center of the star than outside. To see what this means let an observer, far outside the star, send two photons separated by one lap, to the center of the star. They will arrive at the center of the star still separated by one lap. An observer in the center of the star, will however experience a shorter time separating the two photons than the observer at infinity. We understand that time inside the star runs slow relative time at infinity.

Alternatively, we see from Fig 3.1 that the lines of constant Schwarzschild time are lying closer to each other inside the star. An observer inside the star will therefor experience a local Schwarzschild clock to tic a lot faster than his own. In other words he will see the Universe outside the star evolving at a faster rate than that experienced outside the star. I have this vision of standing at some point with big time dilation and looking at galaxies flying by in the sky, like clouds in a storm[2].

3.1.1 Black hole embedding

Consider now a black hole rather than a star. On the outside we would expect the radius of the the rotational surface to go to zero at the horizon, corresponding to infinite time

[1] It is sufficient that we have the surface and that we know that moving in the azimuthal direction is timelike motion with proper time equaling the Euclidean distance and that moving along the surface is orthogonal to moving azimuthally and that proper spatial distance along the surface are equaling the Euclidean distance

[2] And then everything goes black - and you didn't get any action except on TV ...

dilation.

We can however also consider the inside of the black hole. Suppose that we create a new line element by taking the absolute value of both g_{tt} and g_{xx}:

$$ds^2 = |g_{tt}| \cdot dt^2 + |g_{xx}| \cdot dx^2 \tag{3.3}$$

This line element is time independent and positive definite. We should therefor be able to embed it as a rotational body where distances exactly along the surface or purely azimuthal distances are exactly right (up to a minus sign). Using the Schwarzschild black hole line element we easily get the embedding depicted in Fig 3.2.

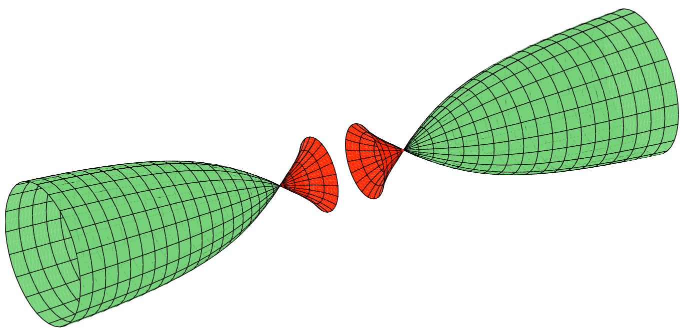


Figure 3.2: An embedding of the *absolute* spacetime of a central line through a black hole

Both on the inside and the outside, azimuthal angle means Schwarzschild time. Also photons move on 45 degrees both inside and outside. On the inside however moving *along* the surface is timelike motion contrary to the situation on the outside.

We notice that the singularity is not in the picture. While the distance to walk along the internal trumpet from the horizon to the singularity is finite [3] , the embedding radius is infinite at the singularity. Thus we stand no chance of embedding the singularity in this scheme.

Let us zoom in on the internal geometry as depicted in Fig 3.3.

From symmetry we know that (or *see* that) following a Schwarzschild time line inside the black hole is timelike geodesic motion. Study then the trajectories of two constant Schwarzschild time lines, starting close to each other near the horizon and extending towards the singularity. The corresponding two observers will to lowest order in the separation between them be at rest with respect to each other in the beginning. As they approach the singularity however they will drift further and further apart. At the singularity, where the embedding radius is infinite, they will be infinitely separated. It is then easy to imagine that the tidal forces between particles approaching the singularity will be infinite.

[3]Since we know that it takes a finite eigentime to reach the singularity once inside the horizon

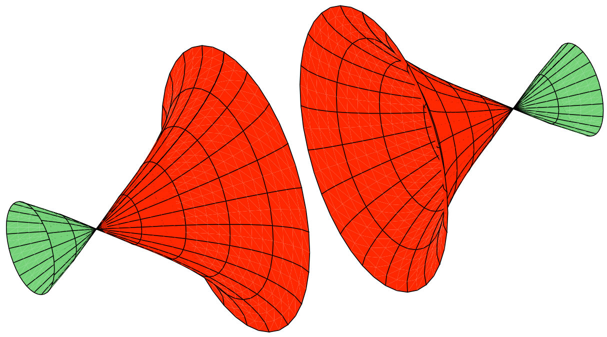


Figure 3.3: The *absolute* internal spacetime of a central line through a black hole

3.1.2 Comment regarding geodesics

Freely falling particles will in this scheme not follow the shortest Euclidean path on the surface. However, we know the true proper distance between points from the shape of the surface. We also know that freely falling particles will choose a way such that the eigentime experienced *maximized*. Consider then two events separated by some finite azimuthal angle only. It is easy to imagine that a particle traveling between the two events, will gain eigentime by moving out towards a bigger embedding radius, before moving back to the second event. On the other hand it cannot move out too fast since then special relativistic time dilation will slow down the internal clock. Since relativistic time dilation depends to second order on the velocity and the gravitational time dilation is proportional to the radius, it seems very plausible that the path that maximizes the eigentime will indeed be a trajectory that first moves out and then back, like we indeed see in physics.

3.2 Generalization to arbitrary spacetimes

The scheme used in the preceding section was quite specific for that particular metric written in those particular coordinates. There is however a way to generalize the absolute scheme. Given an arbitrary Lorentzian spacetime, we specify a field of four-velocities $u^\mu(x)$. We then make a coordinate transformation to a local Minkowski system comoving with the given four-velocity at every point. In the local system, we flip the sign of the spatial part of the metric to create a new *absolute* metric of Euclidean signature. Notice that the new metric will be highly dependent on our choice of generating four-velocities.

Together with the absolute metric, we still need the vector field $u^\mu(x)$ to keep track of what is timelike and what is not. Doing so, we maintain all information about the original spacetime. We can always do the backwards transform and flip Pythagoras into Minkowski.

Since u^μ is normalized we have $N-1$ degrees of freedom, where N is the total number of dimensions, in generating the absolute metric. In particular studying a 1+1 dimensional spacetime this means that the absolute metric may be specified by a single *function* of x and t.

In the preceding section the generators were simply the Schwarzschild observers at rest [4] . Notice that the observers standing just outside the horizon have infinite proper acceleration. It is then perhaps not surprising that the resulting embedding is singular at the horizon. In fact, by instead considering freely falling observers as generators, we could hope to better resolve the horizon.

3.2.1 A covariant approach

Suppose that with respect to a to a metric and a field of four-velocities, we want to find the absolute metric, from now on denoted $\bar{g}_{\mu\nu}$. We may then go to a system comoving with u^μ, insert a Euclidean metric and then transform the metric back to the original coordinates. This is how I first did it.

There is however a faster, more elegant way, using tensors. We know that the absolute metric is a tensor and in a comoving frame we have:

$$\bar{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.4}$$

We realize that we must have:

$$\bar{g}_{\mu\nu} = -g_{\mu\nu} + 2 \cdot u_\mu u_\nu \qquad\qquad u_\mu = g_{\mu\nu}\frac{dx^\mu}{d\tau} \tag{3.5}$$

Notice that both sides of the equality are tensors that equal each other in a certain system, thus they are equal in every system.

3.3 Freely falling observers as generators

As an example, consider freely falling observers initially at rest at infinity as generators. We also assume a Schwarzschild black hole 1+1 original metric. Using the squared Lagrangian formalism for the equations of motion we readily find the four-velocity of the generating freefallers:

$$u^\mu = \left(\frac{x}{x-1}, -\frac{1}{x}\right) \qquad \Rightarrow \qquad u_\mu = \left(1, \frac{1}{x-1}\right) \tag{3.6}$$

[4] Inside the horizon it was the $t=$ const-observers that were generators

The absolute metric is then according to Eq (3.5):

$$\bar{g}_{\mu\nu} = \begin{bmatrix} 1+\frac{1}{x} & \frac{2\sqrt{x}}{x-1} \\ \frac{2\sqrt{x}}{x-1} & \frac{x(x+1)}{(x-1)^2} \end{bmatrix} \tag{3.7}$$

To make an embedding of this metric we are wise to first diagonalize the metric by the coordinate transformation:

$$T = t + \phi(x) \qquad \text{where} \qquad \frac{d\phi}{dx} = \frac{\bar{g}_{tx}}{\bar{g}_{tt}} \tag{3.8}$$

This gives us the line element in the new coordinates as:

$$ds^2 = \bar{g}_{tt} \cdot dT^2 + \left(\bar{g}_{xx} - \frac{\bar{g}_{tx}^2}{\bar{g}_{tt}}\right) \cdot dx^2 \tag{3.9}$$

Inserting the explicit absolute metrical components we find:

$$ds^2 = \left(1+\frac{1}{x}\right) \cdot dT^2 + \left(1+\frac{1}{x}\right)^{-1} \cdot dx^2 \tag{3.10}$$

This metric is easy to remember since it happens to be exactly the Schwarzschild metric but with the minus signs exchanged to plus signs [5]. Notice in particular that nothing special happens with the metrical components at the horizon ($x = 1$). At the singularity ($x = 0$) however the absolute metric goes singular.

To produce a meaningful picture of this geometry we must include the worldlines of the freely falling observers used to generate the absolute geometry. This we do by taking the trajectories as calculated in the Schwarzschild metric, and coordinate transforming the trajectories using Eq (3.8):

$$\frac{dx}{dT} = \frac{\frac{dx}{dt}}{1 + \frac{\bar{g}_{tx}}{\bar{g}_{tt}} \cdot \frac{dx}{dt}} \tag{3.11}$$

We may then numerically solve for $x(T)$ or $T(x)$. In a similar manner we solve for the lines orthogonal to the freefallers, the local simultaneity lines of the generators (see Section3.6). In Fig 3.4 the freely falling absolute geometry is depicted.

Notice how the local Minkowski systems are twisted on the surface. The horizon lies exactly where the generating worldlines are at 45° to a purely azimuthal line.

Time dilation of observers at rest is now not solely determined by the local embedding radius, but also by the gamma factor of the observer at rest relative the generating observer. This can be calculated directly from what angle the world-line of the observer at rest makes with the generating world-line. For instance an observer at rest

[5] Except for the inversion minus sign on the dx^2-term

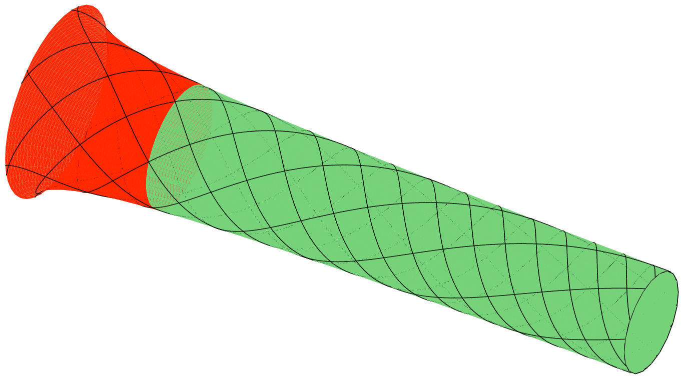


Figure 3.4: The absolute freefaller geometry. The darker area lies within the horizon.

at the horizon will be at 45° to the generating observer, corresponding to an infinite gamma factor, and his clock will therefor not tick at all during a Schwarzschild lap (one circumference), thus being infinitely time-dilated.

It turns out that the freely falling generators are geodesics also in the *absolute* space-time. This is generally true for freely falling generators as is proven in the following section.

3.4 On geodesics

Studying geodesics we may want an expression for the new *absolute* affine connection, i.e the affine connection for the absolute metric. For this we need the inverse absolute metric which is given by :

$$\bar{g}^{\mu\nu} = -g^{\mu\nu} + 2u^{\mu}u^{\nu} \tag{3.12}$$

Raised indexes on unbarred objects means that they are raised with the original metric. That this is indeed the inverse is easy to verify. It can also be derived straight-forwardly, using the contravariant analogue of the method used in Section 3.2.1.

It is a little surprising however that we get the inverse of the new metric by raising the indices with the original metric. This is a unique feature of this particular metrical construction. If we had instead considered a metric of the form $-g^{\mu\nu} + \alpha \cdot u^{\mu}u^{\nu}$, were α is some general number, the inverse would have been $-g^{\mu\nu} + \alpha/(\alpha - 1) \cdot u^{\mu}u^{\nu}$. It is only in the case $\alpha = 2$ that we can simply raise with the original metric to get the absolute metric.

Incidentally we may notice that also in the raising of u^{μ} the new and the old metrics make an equally good job:

$$\bar{g}^{\mu\nu}u_{\nu} = \left(-g^{\mu\nu} + 2u^{\mu}u^{\nu}\right)\, u_{\nu} = -u^{\mu} + 2\; u^{\mu} = u^{\mu} \tag{3.13}$$

Knowing the inverse absolute metric, the absolute affine connection is given by:

$$\begin{aligned}\bar{\Gamma}^{\mu}_{\alpha\beta} = &\tfrac{1}{2}\left(\text{-}g^{\mu\rho} + 2u^{\mu}u^{\rho}\right)\cdot \\ &\left(\partial_{\alpha}(\text{-}g_{\rho\beta} + 2u_{\mu}u_{\nu}) + \partial_{\beta}(\text{-}g_{\rho\alpha} + 2u_{\mu}u_{\nu}) - \partial_{\rho}(\text{-}g_{\alpha\beta} + 2u_{\alpha}u_{\beta})\right)\end{aligned} \tag{3.14}$$

From now on objects connected to the absolute metric will be barred. We may rewrite Eq (3.14):

$$\begin{aligned}\bar{\Gamma}^{\mu}_{\alpha\beta} = &\Gamma^{\mu}_{\alpha\beta} - 2u^{\mu}u^{\rho}\cdot\Gamma_{\rho\alpha\beta} \\ &+ \left(\text{-}g^{\mu\rho} + 2u^{\mu}u^{\rho}\right)\left(\partial_{\alpha}(u_{\rho}u_{\beta}) + \partial_{\beta}(u_{\rho}u_{\alpha}) - \partial_{\rho}(u_{\alpha}u_{\beta})\right)\end{aligned} \tag{3.15}$$

3.4.1 Proof regarding geodesic generators

In the new metric, proper distances will in general be very much different from those in the original metric. Distances measured along a generating congruence line will however be the same (these are unaffected by the sign-flip). In particular this means that:

$$u^{\mu} = \frac{dx^{\mu}}{d\tau} = \frac{dx^{\mu}}{d\bar{\tau}} = \bar{u}^{\mu} \tag{3.16}$$

Then the covariant four-acceleration of the generating congruence lines in the absolute metric can be expressed as:

$$\frac{\bar{D}\bar{u}^{\mu}}{\bar{D}\bar{\tau}} = \frac{du^{\mu}}{d\tau} + \bar{\Gamma}^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} \tag{3.17}$$

Assume now that the generating observers are geodesics in the original metric. We suspect that they are then geodesics in the absolute metric as well. To prove this it is sufficient to show that the absolute covariant four-acceleration is vanishing in one system (it then vanishes in all systems). Let us therefore evaluate Eq (3.17) in that particular *originally* freely falling system that is exactly [6] comoving with the reference observer u^{μ}. In this system the original affine connection vanishes. Also any derivative of the form $u^{\alpha}\partial_{\alpha}$ has to vanish since it equals $\frac{d}{d\tau}$ in the direction of the preferred observer, in which metrics and four velocities are constant, Minkowski and (1,0,0,0) respectively. [7] Thus forming $\bar{\Gamma}^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta}$ in this particular coordinate system we are left simply with:

$$\bar{\Gamma}^{\mu}_{\alpha\beta}u^{\alpha}u^{\beta} \stackrel{*}{=} g^{\mu\rho}u^{\alpha}u^{\beta}\partial_{\rho}(u_{\alpha}u_{\beta}) \tag{3.18}$$

[6]Meaning that we fasten the spatial origin of the freely falling system exactly on the freely falling observer

[7]If the generating observers had not been geodesics, and we had chosen that particular freely falling coordinate system that was locally comoving with the freely falling observer, $u^{\alpha}\partial_{\alpha}u^{m}$, would in general be non-zero

Here $\overset{*}{=}$ means that the equality holds in an originally freely falling system comoving with the generator [8] . Notice however that we have, in general coordinates:

$$u^\kappa u_\kappa = 1 \qquad \Rightarrow \qquad \partial_\rho(u_k u^\kappa) = 0 \qquad \Rightarrow \qquad u_k \partial_\rho u^\kappa + u^k \partial_\rho u_\kappa = 0 \tag{3.19}$$

Then we may write:

$$u_\kappa \partial_\rho u^\kappa = -u^\kappa \partial_\rho u_k = -u^\kappa \partial_\rho (g_{\kappa\lambda} u^\lambda) \overset{*}{=} -u^\kappa g_{\kappa\lambda} \partial_\rho u^\lambda \overset{*}{=} -u_\lambda \partial_\rho u^\lambda \tag{3.20}$$

Again using Eq (3.19) we have then:

$$u^\kappa \partial_\rho u_\kappa \overset{*}{=} 0 \tag{3.21}$$

Then we see trivially that the right hand side of Eq (3.18) vanishes in any originally freely falling system. Since also in these particular coordinates $\frac{du^\mu}{d\tau} = 0$ we find from Eq (3.17) that:

$$\frac{\bar{D}\bar{u}^\mu}{\bar{D}\bar{\tau}} \overset{*}{=} 0 \tag{3.22}$$

The left hand side being covariant we know that the equality hold also without the star over the equality sign.

So the covariant four-acceleration of a freely falling generating congruence line, vanishes also with respect to the absolute metric. It is thus a geodesic also in the absolute metric.

Incidentally, from Eq (3.21) we realize that:

$$u^\kappa \bigtriangledown_\rho u_\kappa = 0 \tag{3.23}$$

This implies that:

$$u^\kappa \partial_\rho u_\kappa = +u^\kappa {\Gamma^\lambda}_{\rho\kappa} u^\lambda \tag{3.24}$$

3.5 Covariant approach to photon geodesics

Out of curiosity I would like to see to what extent we can make photons move on geodesics in the absolute space. Of course we cannot simply make the congruence of observers (u^μ) follow the photon geodesics. Since photons do not experience eigentime, the resulting absolute metric would loose one dimension in a sense. All points along a certain photon congruence would be separated by a zero distance. However, maybe we can choose, an in general non-geodesic u^μ, that makes some photons retain their geodesic property. In particular I am interested in to what extent we can do this in 1+1 dimensions. Let us however start in a general dimensionality.

[8] I will later on use a star also when the equality holds in any originally freely falling system

In a freely falling system we may write, using Eq (3.15):

$$\bar{\Gamma}^{\mu}_{\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} f^{\alpha} f^{\beta} \left(\text{-}g^{\mu\rho} + 2u^{\mu}u^{\rho}\right) \left(\partial_{\alpha}(u_{\rho}u_{\beta}) + \partial_{\beta}(u_{\rho}u_{\alpha}) - \partial_{\rho}(u_{\alpha}u_{\beta})\right) \tag{3.25}$$

$$\stackrel{*}{=} f^{\alpha} f^{\beta} \left(\text{-}g^{\mu\rho} + 2u^{\mu}u^{\rho}\right) \cdot 2\left(u_{\beta}\partial_{\alpha}u_{\rho} + u_{\rho}\partial_{\alpha}u_{\beta} - u_{\beta}\partial_{\rho}u_{\alpha}\right) \tag{3.26}$$

I have deliberately violated *manifest* α, β-symmetry within the second parenthesis for compactness. We could expand this expression further, but we may as well leave it as it is since after all the raising metric in the above expression is trivial in a system comoving with u^{μ}. This means that $\bar{\Gamma}^{\mu}_{\alpha\beta}$ is numerically identical to $\bar{\Gamma}_{\mu\alpha\beta}$ in these coordinates. Thus let us write:

$$\bar{\Gamma}_{\mu\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} 2 \cdot \Big(f^{\beta}u_{\beta} f^{\alpha}\partial_{\alpha}u_{\mu} + u_{\mu} f^{\beta} f^{\alpha}\partial_{\alpha}u_{\beta} - f^{\beta}u_{\beta} f^{\alpha}\partial_{\mu}u_{\alpha}\Big) \tag{3.27}$$

In a comoving system photons will have an absolute four-velocity, in the point in question, given by:

$$f^{\alpha} = \frac{1}{\sqrt{2}}(1, \hat{n}) \tag{3.28}$$

Also, from the normalization of u^{μ} follows that:

$$0 = \partial_{\alpha}(g_{\mu\nu}u^{\mu}u^{\nu}) \stackrel{*}{=} \partial_{\alpha}((u^{t})^2 - (u^{x})^2) \tag{3.29}$$

$$\stackrel{*}{=} 2u^{t}\partial_{\alpha}u^{t} - 2u^{x}\partial_{\alpha}u^{x} \tag{3.30}$$

$$\stackrel{*}{=} 2\partial_{\alpha}u^{t} \tag{3.31}$$

Thus in a locally comoving freely falling system, all derivatives of u^{t} are zero (not so for u^{x} in general however). Remembering to henceforth treat f^{α} as a constant vector we may move it in over derivatives and rewrite Eq (3.27):

$$\bar{\Gamma}_{0\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} 2 \cdot \Big(f^{\alpha}\partial_{\alpha}(f^{i}u_{i}) - f^{0}\partial_{0} f^{i}u_{i}\Big) = f^{j}\partial_{j}(f^{i}u_{i}) \tag{3.32}$$

$$\bar{\Gamma}_{k\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} 2 \cdot \Big(f^{0} f^{\alpha}\partial_{\alpha}u_{k} - f^{0}\partial_{k} f^{i}u_{i}\Big) \tag{3.33}$$

Here Latin indices take the values $1, 2, 3$.

3.5.1 Photons in 1+1 dimensions

Specializing to 1+1 dimensions and $f^{\alpha} = 1/\sqrt{2} \cdot (1, 1)$ we find:

$$\bar{\Gamma}^{t}_{\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} \partial_{x}u_{x} \tag{3.34}$$

$$\bar{\Gamma}^{x}_{\alpha\beta} f^{\alpha} f^{\beta} \stackrel{*}{=} \partial_{t}u_{x} \tag{3.35}$$

We know that in a freely falling system the photon will to second order move on a straight line. This means that the absolute acceleration perpendicular to the photon trajectory vanishes in this system i.e.

$$\frac{d^2}{ds^2}(t - x) \stackrel{*}{=} 0 \tag{3.36}$$

I will from now on sometimes use ds rather than $d\bar{\tau}$ for absolute distance. For the photon to be following a local geodesic in the absolute spacetime, we must then have:

$$-\bar{\Gamma}^t_{\alpha\beta} f^\alpha f^\beta + \bar{\Gamma}^x_{\alpha\beta} f^\alpha f^\beta = 0 \qquad \Rightarrow \qquad \partial_x u_x - \partial_t u_x \stackrel{*}{=} 0 \tag{3.37}$$

So this has to hold in the comoving originally freely falling system. Now, can we make this into a covariant statement, that applies in any coordinate system? Indeed we can. Defining v^μ to be a vector orthogonal (both originally and absolutely) to u^μ and absolutely normalized ($\bar{g}_{\mu\nu} v^\mu v^\nu = 1$), we see that the covariant expression that reduces to Eq (3.37) is given by [9] :

$$v^\nu v^\mu \bigtriangledown_\mu u_\nu - v^\nu u^\mu \bigtriangledown_\mu u_\nu = 0 \tag{3.38}$$

In two dimensions v^μ is given up to a sign by the original metric and the generating four-velocities. The sign determines whether we are studying ingoing or outgoing photons. Notice that if we want also infalling photons to be moving on geodesics we would get the constraint:

$$\partial_x u_x + \partial_t u_x = 0 \tag{3.39}$$

This together with Eq (3.37) and $\partial_\alpha u^t = 0$ means that *all* the derivatives of the four-velocity of the generating observers relative the freely falling system would have to vanish. This would mean that all the derivatives of the absolute metric would vanish as well, making all original geodesics to absolute geodesics. My guess is that this can only happen at *every* spacetime point, if the spacetime is flat. Thus I draw the conclusion that in a non-trivial spacetime we cannot get all originally geodesic photons to move on absolute geodesics.

3.6 Photon geodesics in static spacetime

We found in Section (3.5) a covariant differential relation, Eq (3.38), usable to find generators such that outgoing photons move on geodesics in a general 1+1 dimensional spacetime. Assuming the original metric to be time independent, and letting the generators be time independent as well, it is however more clever to start over from the beginning. We instead use the power of the Lagrangian formalism, and constants of the motion. The goal is to find an embedding were *outgoing* photons move on geodesics, and were the horizon is included.

In general in $1+1$ dimensions we have only one freedom in u^μ since it must be normalized. A clever choice of parameter for this freedom may be to choose the velocity v of the generating observer relative a freely falling observer originally at rest at infinity, from now on denoted reference observer. That way we have a parameter that is ranging

[9] I have incidentally generalized this, in 1+1 dimensions, to apply to an arbitrary direction, not necessarily a photon: $(f^\alpha v_\alpha)^3 \cdot v^\nu v^\mu \bigtriangledown_\mu u_\nu - (f^\alpha u_\alpha)^3 \cdot v^\nu u^\mu \bigtriangledown_\mu u_\nu$

from -1 to 1, that hopefully has an absolute value smaller than 1 at the horizon [10]. For simplicity let us restrict ourselves to the case where the relative velocity v only depends on x.

3.6.1 The reference freefaller coordinates

For the task at hand it may be clever to express the Schwarzschild line element in coordinates in which it is possible to pass the horizon. There is a certain coordinate system which is ideally suited for reference observers. The line element in these coordinates is given by:

$$d\tau^2 = \left(1 - \frac{1}{x}\right) dT^2 - \frac{2}{\sqrt{x}}\, dxdT - dx^2 \tag{3.40}$$

These coordinates are related to the standard coordinates via:

$$T = t + \phi(x) \tag{3.41}$$

$$x = x \tag{3.42}$$

In principle it is just a resetting of the Schwarzschild clocks. The function $\phi(x)$ is such that we have the following nice features for reference observers:

- A constant T-line is a local simultaneity line for a reference observer
- The eigentime experienced by the reference observer between events on his world-line, is exactly the difference in T between the events
- The spatial distance experienced by the reference observer between infinitesimally displaced events separated only in x is exactly dx
- For the freefaller we have $\frac{dx}{dT} = -\frac{1}{\sqrt{x}}$
- An observer moving with velocity v relative the reference observer has $\frac{dx}{dT} = -\frac{1}{\sqrt{x}} + v$

In particular this means that the local gamma factor of a particle moving relative the local reference observer is simply $dT/d\tau$ which will be used below.

3.6.2 The absolute metric

Using the nice properties listed above we readily find the four-velocity of an observer moving with speed v relative a reference observer:

$$u^\mu = (\frac{dT}{d\tau}, \frac{dx}{d\tau}) \tag{3.43}$$

[10] As opposed to if we had chosen, as a parameter, the velocity relative an observer at *rest*

$$= \frac{dT}{d\tau}(1, \frac{dx}{dT}) \tag{3.44}$$

$$= \frac{1}{\sqrt{1-v^2}}(1, v - \frac{1}{\sqrt{x}}) \tag{3.45}$$

Lowering the four-velocities with the metric we find:

$$u_t = g_{tt}u^t + g_{tx}u^x = ... = \frac{1}{\sqrt{1-v^2}}(1 - \frac{v}{\sqrt{x}}) \tag{3.46}$$

$$u_x = g_{xt}u^t + g_{xx}u^x = ... = \frac{-v}{\sqrt{1-v^2}} \tag{3.47}$$

From this we may immediately verify that $u^\mu u_\mu = 1$. We have the general expression for the absolute metric:

$$\bar{g}_{\mu\nu} = -g_{\mu\nu} + 2u_\mu u_\nu \tag{3.48}$$

Then we can form the absolute metric as a function of the parameter v:

$$\bar{g}_{tt} = -\left(1 - \frac{1}{x}\right) + 2 \cdot \frac{1}{1-v^2}\left(1 - \frac{v}{\sqrt{x}}\right)^2 \tag{3.49}$$

$$\bar{g}_{tx} = \frac{1}{\sqrt{x}} + 2 \cdot \frac{1}{1-v^2}\left(1 - \frac{v}{\sqrt{x}}\right) \cdot (-v) \tag{3.50}$$

$$\bar{g}_{xx} = 1 + 2 \cdot \frac{1}{1-v^2} \cdot v^2 \tag{3.51}$$

Now remains to find v, and thus the absolute metric, such that the outmoving photon, whose trajectory is known, is a geodesic in the new absolute metric.

3.6.3 Equations of motion in a general 1+1, time independent metric

Suppose that we have a metric of the form:

$$ds^2 = a(x) \cdot dT^2 + 2b(x) \cdot dT\ dx + c(x) \cdot dx^2 \tag{3.52}$$

Using the squared Lagrangian formalism we find that geodesics obey:

$$a\frac{dT}{ds} + b\frac{dx}{ds} = K \tag{3.53}$$

Here K is a constant of the motion. Introducing $\sigma = 1/K^2$, and using Eq (3.52), we readily find:

$$\left(\frac{dx}{dT}\right)^2(\sigma b^2 - c) + \frac{dx}{dT} \cdot 2b(\sigma a - 1) + a(\sigma a - 1) = 0 \tag{3.54}$$

3.6.4 Outmoving photons

For an outmoving photon we have:

$$\frac{dx}{dT} = -\frac{1}{\sqrt{x}} + 1 \tag{3.55}$$

Inserting this into Eq (3.54) together with the explicit expressions for the metrical components as functions of v (Eq (3.49),Eq (3.50) and Eq (3.51)), we get a quite nasty expression. At first sight it appears to be a fourth order equation in v. The expression may however be drastically simplified, using Mathematica, and we can easily solve for v. The result is:

$$v = 1 - \frac{4x}{\sigma(1+\sqrt{x})^2 + 2x} \tag{3.56}$$

So, demanding outgoing photons to be geodesics yields v uniquely as a function of x up to a single parameter σ. This parameter determines the velocity at infinity through:

$$v_\infty = 1 - \frac{4}{\sigma + 2} \tag{3.57}$$

We see from Eq (3.56) that v lies in the range [-1,1] for all values of x and σ (sigma is positive since it is the square of a real number). Choosing to have $v = 0$ when x goes to infinity immediately yields $\sigma = 2$. Then we have v explicitly as a function of x which we may insert into the expressions for the absolute metric.

3.6.5 Embeddings

To easily embed the new metric we are wise to first diagonalize it. A general metric of the form Eq (3.52) is easily diagonalized using the coordinate transformation:

$$T' = T + f(x) \qquad \text{where} \qquad \frac{df}{dx} = \frac{b}{a} \tag{3.58}$$

The line element in the new coordinates is given by:

$$ds^2 = a(x) \cdot dT'^2 + \left(c(x) - \frac{b^2(x)}{a(x)} \right) \cdot dx^2 \tag{3.59}$$

With the metric in this form it is an easy task to embed it as a rotational surface. To plot the lines of the generators and photons moving on the surface we could use the same technique as that used for the freefallers Eq (3.11). From a numerical point of view however it is better to parameterize both x and t with s, the proper absolute distance along the curve [11]. The four-velocity transforms as a vector under the diagonalization. Thus we readily find:

$$u'^t = u^t + \frac{b}{a}u^x \quad (3.60)$$

$$u'^x = u^x \quad (3.61)$$

Knowing $u^t(x)$, $u^x(x)$, $b(x)$ and $a(x)$ we can easily numerically integrate these expressions to plot the various world lines.

To get a nice grid on the surface we need also to calculate the lines orthogonal to the generators, the lines that are local simultaneity lines of the generators. Letting v^μ be a vector determined by $g_{\mu\nu}v^\mu v^\nu = -1$ and $g_{\mu\nu}u^\mu v^\mu = 0$ we find:

$$v^\mu = (\pm)\frac{1}{\sqrt{1-v^2}} \cdot \left(v, 1 - \frac{v}{\sqrt{x}}\right) \quad (3.62)$$

This is a vector that is normalized to unity in the absolute metric (minus unity relative the original metric), and is orthogonal to the generator also in the absolute metric. It is thus the absolute four-velocity of the lines orthogonal to the generators and we may solve for these lines just like we solved for the generator lines above.

In Fig 3.5 we show the embedding with the lines of the generators together with outmoving photons.

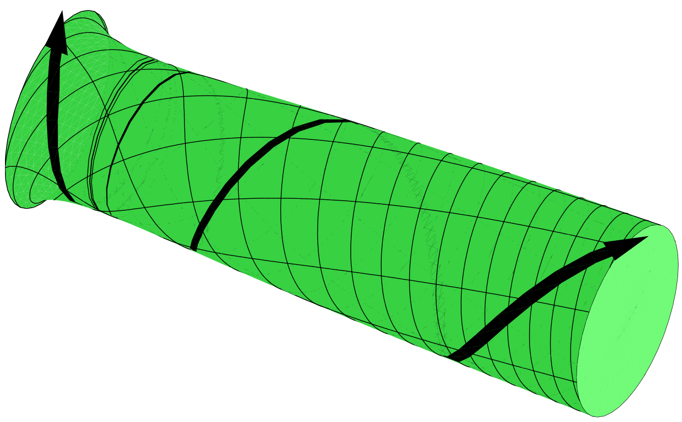


Figure 3.5: The absolute space using generators $u^\mu(x)$ such that outmoving photons move on geodesics. The embedding covers the range $x\epsilon[0.55, 4]$.

The thick lines with arrows correspond to photons moving *out*. The left one, being inside the horizon is however forced to follow the geometry into the singularity. Notice

[11] That way there is no problem in dealing with trajectories that are first increasing in t' and then decreasing in t' for instance

how the photons stay at exactly 45° to the generating grid. Notice also that the horizon is exactly where there is a minimum in the embedding radius. This is of course necessary if we want the photon to remain on a certain spatial position and at the same time move on a geodesic on the surface. Also we knew in advance that it would have to be a be a minimum lest a photon could oscillate back and forth around the horizon. It was however not obvious that the scheme would at all be successful.

Like before we can embed the internal black hole, but not all the way into the singularity. It is however a little bit unfortunate that the absolute spacetime for these particular generators has so little shape. Demanding photons to move at 45° at infinity we know from the equations of motion on a general rotational body (see Paper I) that the radius at the neck must be $1/\sqrt{2}$ times the radius at infinity. This seems to imply quite a lot of shape. The radius however turns out to increase quite slowly with increasing r, giving the embedding this, close to cylinder-like, appearance. It may be that we can create more shape by considering other generator-velocities at infinity. In fact we may realize that a positive v at infinity would decrease the radius of the neck.

Remember that the spacetime in these pictures is like a toilet roll. When you walk once around the roll you are on a new layer. The outgoing photon should therefore really just be seen in one layer, unless we rip the toilet paper in a clever way. In any case, we should not draw the conclusion that we have *three* photons moving outwards outside the horizon in the embedding.

3.7 Flat embeddings

Having seen one set of generators produce a spacetime where the radius increases towards the horizon and another where it decreases, it is natural to ask whether we can find generators such that the radius is constant. This is very easy to check, since it corresponds to having $\bar{g}_{tt}(v) = C$, where C is some constant. Using $\bar{g}_{tt}(v)$ from Eq (3.49) we may readily solve the second order equation in v to find:

$$v = \frac{2\sqrt{x} \pm \sqrt{-1 + 2x + x^2(C^2 - 1)}}{1 + x \cdot (1 + C)} \tag{3.63}$$

So there are two possibilities for every given C. Similar to when we demanded outgoing photons to be geodesics we have one free parameter determining the generator velocity at infinity. In this case we see that:

$$v_\infty = \pm\sqrt{C - 1} \tag{3.64}$$

We see from Eq (3.63) that if we choose $C < 1$ the root will go imaginary for sufficiently large x. Solving in x for when the root is zero yields:

$$x = \frac{1}{1 - C} \qquad \text{or} \qquad x = \frac{1}{1 + C} \tag{3.65}$$

I thus appears that choosing C sufficiently large the cylinder embedding will exist from infinity to arbitrarily close to the horizon. In particular however, choosing $\bar{g}_{tt} = 1$ at infinity ($C = 1$) the cylinder will exist into half the Schwarzschild radius.

We should however also verify that v stays in the interval [-1,1]. I haven't analyzed it properly but it appears to stay in this interval whichever x and C we have, given that the root is real.

3.7.1 Comments

Notice that for this kind of visualization the curvature of spacetime is manifested solely as a *twist* of the local Minkowski systems relative each other. In this picture we can of course also explain why particles in free fall move the way they do. In particular study a particle moving between two events separated only in the azimuthal direction on the cylinder. The task is to find a path that maximizes the eigentime. Moving out is of course good, because then you move to a region where the twist of the local Minkowski systems is smaller, and thus pure azimuthal motion means smaller gamma factor. On the other hand moving out means moving closer to the light-cone which increases the gamma factor. Which one of the effects that would win is perhaps not obvious in this case since the gamma factor is not quadratic in the change of velocity now (as opposed to the case in the hourglass embedding), but a first order effect. In any case, assuming the events to be separated by a vast distance in the azimuthal direction, we immediately understand that it's preferable to move out to a lower gamma factor before moving back, corresponding to the real physical effect.

Notice that the cylinder we are here considering is intrinsically flat. It is somewhat amusing that we are in a sense back to where we started, with a flat surface. Now the geometry is instead represented by a velocity. Unfortunately, just thinking in terms of degrees of freedom, this will not in general work in higher dimensions.

3.8 Spacelike generators?

Out of curiosity we may ask ourselves what would happen to the absolute metric if u^μ was spacelike? Assuming $|u^\mu u_\mu| = 1$ we can evaluate the absolute metric in that particular local Minkowski system that has u^μ directed along its x-axis. In this system we have:

$$u^\mu = (0, \frac{dx}{d\tau}, 0, 0) = (0, \frac{dx}{i \cdot ds}, 0, 0) \qquad \rightarrow \qquad u_\mu = (0, -\frac{1}{i}, 0, 0) \tag{3.66}$$

Then we find:

$$\bar{g}_{\mu\nu} = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} + 2\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.67}$$

In this case we thus get a metric with two time directions. In fact it is those two spatial directions that is orthogonal to the one prescribed by the spacelike generator, that becomes the new timelike directions. This is not particularly exiting but included for completeness.

3.9 A mathematical remark

Having seen the neat covariant expression of Eq (3.38), regarding photon geodesics, I start wondering whether there is a more general covariant relation that reduces to this one in the particular case of photons in two dimensions.

For a given particle trajectory of four-velocity q^μ, and a given generating field, we have a well defined *absolute* four-velocity $\bar{q}^\mu$. We know that $\frac{\bar{D}\bar{q}^\mu}{\bar{D}s}$ and $\frac{Dq^\mu}{D\tau}$ are both covariant objects, describing more or less the same thing. It seems plausible that we could relate the two quantities in some manifestly covariant way, probably involving covariant derivatives of the generating field. A good starting point for the search for such a relation is of course to first relate $\bar{q}^\mu$ to q^μ in a manifestly covariant way. This is straightforward as can be seen below.

$$\begin{aligned} \bar{q}^\mu = \frac{dx^\mu}{ds} = \frac{dx^\mu}{d\tau}\cdot\frac{d\tau}{ds} = q^\mu \cdot \sqrt{\frac{g_{\mu\nu}dx^\mu dx^\nu}{-g_{\mu\nu}dx^\mu dx^\nu + 2u_\mu u_\nu dx^\mu dx^\nu}} \\ = q^\mu \cdot \sqrt{\frac{1}{-1 + 2u_\mu u_\nu \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}} \\ = q^\mu \cdot \frac{1}{\sqrt{2(u_\mu q^\mu)^2 - 1}} \end{aligned} \tag{3.68}$$

So now we have a manifestly covariant expression relating the four-velocities. Notice that choosing $q^\mu = u^\mu$ yields $\bar{u}^\mu = u^\mu$ as we realized earlier. Also if we choose q^μ to be orthogonal to u^μ the four-velocities are related simply via an imaginary i, as expected. Evaluating $\frac{\bar{D}\bar{q}^\mu}{\bar{D}\bar{\tau}}$ in an *originally* freely falling system where the original affine connection vanishes, and then setting all unbarred derivatives to their covariant analogue we get (see Eq (3.26)):

$$\frac{\bar{D}\bar{q}^\mu}{\bar{D}\bar{\tau}} = \frac{1}{\sqrt{2(u_\mu q^\mu)^2 - 1}}\frac{D}{D\tau}\left(\frac{q^\mu}{\sqrt{2(u_\mu q^\mu)^2 - 1}}\right) \tag{3.69}$$

$$+2\frac{q^\alpha q^\beta}{2(u_\mu q^\mu)^2-1}\left(-g^{\mu\rho}+2u^\mu u^\rho\right)\left(u_\beta \bigtriangledown_\alpha u_\rho + u_\rho \bigtriangledown_\alpha u_\beta - u_\beta \bigtriangledown_\rho u_\alpha\right)$$

So here we have a manifestly covariant relation. The result is however not particularly illuminating, though it is possible that it can be simplified.

A small remark regarding the backwards transform from the absolute metric to the original metric. Notice that, since $\bar{u}^\mu = u^\mu$ and $\bar{u}_\mu = u_\mu$, we have a perfect symmetry in going from the Lorentzian system to the absolute system and vice versa:

$$\bar{g}_{\mu\nu} = -g_{\mu\nu} + 2u_\mu u_n \qquad \Rightarrow \qquad g_{\mu\nu} = -\bar{g}_{\mu\nu} + 2\bar{u}_\mu \bar{u}_n \tag{3.70}$$

Completely symmetric.

3.10 Comments

A nice application of the absolute metric scheme would be to apply it to a metric that is time dependent. For instance one may study a radial line through a spherical shell of matter that is collapsing. In fact the desire to see how this would look was the main reason for the generalization of the absolute scheme. I haven't done it yet. It would however be nice for then one would use more fully the freedoms of two-dimensional surfaces [12] . After all, the rotational bodies that I have produced so far may be described by a single one-dimensional function, namely the embedding radius. Another nice application would be the interior Kerr hole. There are some pretty interesting regions in there that may perhaps be nicely visualized using the absolute scheme.

In fact one of the major advantages of the absolute scheme, relative the dual scheme, is that we may visualize the structure of any two-dimensional spacetime slice taken in the full spacetime. With the dual metric scheme this was only meaningful if the slice could "contain" a set of geodesics. For a general slice, particles in free fall will in general not stay in the slice but will pass it, making the dual metric of the slice more or less meaningless.

Another advantage, as well as a disadvantage, of the absolute scheme relative the dual scheme is the big freedom in choosing the generators. Depending on what generators we choose, the embedded surface will be very much different. The *physics* of the surface however, will of course not depend the choice . Remember that the shape is not all, we need also the lines telling us the directions of the local Minkowski systems.

In the dual metric scheme we had three parameters determining the embedding while in the absolute scheme we have a whole function. This freedom is good since it increases our chances of producing esthetical pictures with explaining power, while it is also bad since it lacks absoluteness (pardon the small joke).

What I like the most with the absolute scheme is the scale invariance. Just like in the dual metric scheme we have local Minkowski systems sitting on the surface. In the

[12] I wonder what generators that would be clever to use in this case?

dual scheme however the scale of these systems was varying depending on position, and was also not "isotropic" (the opening angle of the light cone was not in general 90°). In the absolute scheme however all the local Minkowski systems have the same scale, we just need a direction to get all the local proper distances right.

The absolute scheme is also well suited to display cosmological concepts such as open and closed Universes, Big Bang and Big Crunch, preferred velocities and how it is possible that there was no time "before" the Big Bang. A more detailed discussion of this will however have to wait until I put the absolute scheme in a form more suited for a general audience.

Really the dual and the absolute schemes have different virtues depending who is looking at the respective embeddings. For the non-physicist, who doesn't know about negative distances and Minkowski systems, the dual metric is certainly the more powerful one when it comes to giving an understanding of gravitation as a geometrical feature rather than as a force. For the general relativist however the absolute scheme may be more interesting. If for nothing else, because it is covariant and is applicable to *any* original Lorentzian spacetime.

Lastly some *pure* speculation, mainly inspired from the dual scheme. Imagine if various particles would correspond to various geometrical signatures, so maybe fermions are in fact just *dual* particles! Typically the border of the particle would be defined by where the signature changes to Lorentzian (like it does in the artificial singularity in the dual scheme). In fact imagine if *all* physics is geometry, but not of a fixed signature necessarily?

4

Metrics, geodesics and affine connections

In this thesis and in particular in the dual scheme, geodesics are at the center of the discussion. In the dual metric scheme I am, given the geodesics, looking for possible metrics to create these. This I managed in the case of a 1+1 dimensional time independent metric, using constants of the motion. In this chapter I make a more general analysis, for arbitrary dimensions and metrics.

4.1 Finding the metric from the geodesics

The geodesic equations of motion on a manifold bestowed with a metric can be written (see D'Inverno [5] p. 100-101):

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = \frac{d^2\tau/d\lambda^2}{d\tau/d\lambda}\dot{x}^\mu \qquad \text{here} \qquad \dot{x}^\mu \equiv \frac{dx^\mu}{d\lambda} \tag{4.1}$$

Here λ is the parameter used to parameterize the worldline in question and τ is proper distance. We have $\Gamma^\mu_{\alpha\beta}$:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\alpha\rho}}{\partial x^\beta} + \frac{\partial g_{\beta\rho}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\rho}\right) \tag{4.2}$$

Suppose now that we have chosen some coordinates (imagine a Cartesian coordinate space). In these coordinates we have through every point and in every direction a geodesic. Suppose that there is a metric that generates these geodesics. Can we from the *geodesic* curves find out what metrics [1] that could generate them ?

[1] For any metric that produces the correct geodesics, a constant times this metric will also generate the right geodesics. The metric will thus not be unique.

4.1.1 Coordinate curvature

We thus want to find a metric that gives the right curvature for every geodesic at every point. How then do we specify the curvature of a geodesic line? Normally we would use $\frac{D^2x^\mu}{D\tau^2}$, the four-acceleration, but we do not know the metric so do not know the proper distance τ. Therefor this measure of the curvature is not applicable. A good way to specify the curvature of a particular geodesic in a point, would instead be to give a vector, coordinate-perpendicular (in the most ordinary sense, using a Pythagorean metric to do the dot product) to the geodesic, pointing in the direction towards which the geodesic curves, and with a length that is proportional to how fast it curves [2] Notice that this measure of the curvature is highly dependent on what coordinates that we use.

In the particular coordinates that we have chosen we introduce a Euclidean *coordinate metric* denoted $\delta_{\mu\nu}$, where $\delta_{\mu\nu} \equiv \mathrm{Diag}(1,1,1,1)$. Suppose that we parameterize the geodesic in question with, as I denote it, a coordinate-affine parameter λ:

$$d\lambda = K\sqrt{dx^2+dy^2+dz^2+dt^2} \equiv K\sqrt{\delta_{\mu\nu}dx^\mu dx^\nu} \tag{4.3}$$

Here K is any constant. Then we have the metric-independent (but coordinate dependent) measure of the curvature of a single geodesic , or any worldline, simply through:

$$\frac{d^2x^\mu}{d\lambda^2} \tag{4.4}$$

This has exactly the properties described above, when λ is given by Eq (4.4). I will call this the coordinate-curvature. Notice that this is not a tensor.

4.1.2 The geodesic equation using a coordinate affine parameter

If we use a coordinate-affine parameter, we *know* that there will be no part of $\frac{d^2x^\mu}{d\lambda^2}$ that is parallel to $\dot{x}^\mu$ since the (Euclidean) length of $\dot{x}^\mu$ is fixed along the curve for such a parameterization. Then we know that the right hand side of the equation of motion, which is manifestly parallel to $\dot{x}^\mu$, must be exactly canceled by the part of $\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$ that is parallel to $\dot{x}^\mu$. Thus we may write:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{\delta_{\rho\kappa}\dot{x}^\rho\Gamma^\kappa_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}{\delta_{\eta\xi}\dot{x}^\eta\dot{x}^\xi}\dot{x}^\mu = 0 \tag{4.5}$$

This we may rewrite into:

$$\frac{d^2x^\mu}{d\lambda^2} = \frac{1}{\delta_{\eta\xi}\dot{x}^\eta\dot{x}^\xi}\delta_{\rho\kappa}\dot{x}^\rho\dot{x}^\alpha\dot{x}^\beta\left(\Gamma^\kappa_{\alpha\beta}\dot{x}^\mu - \Gamma^\mu_{\alpha\beta}\dot{x}^\kappa\right) \tag{4.6}$$

[2] This is most easily visualized in three dimensions

Choosing a parameterization with $K = 1$ in Eq (4.3), giving $\delta_{\eta\xi}\dot{x}^\eta\dot{x}^\xi = 1$, would compactify the expression a little.

It may appear that Eq (4.6) is only a necessary condition for the affine connection to produce the right geodesics. We also have the restriction from the parallel part:

$$\frac{\dot{x}^\rho\delta_{\rho\kappa}\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}{\delta_{\eta\xi}\dot{x}^\eta\dot{x}^\xi}\dot{x}^\mu = \frac{d^2\tau/d\lambda^2}{d\tau/d\lambda}\dot{x}^\mu \tag{4.7}$$

This is however *automatically* satisfied for any affine connection obeying Eq (4.6), and every metric giving rice to this connection, assuming a coordinate affine parameterization. We should thus see this, not as a constraint, but as a piece of information and a reminder that we must still find a *metric* that produces the correct affine connection.

4.1.3 Equivalent affine connections

It is obvious from Eq (4.6) that all metrics that give rise to the same affine connection, will do equally well, or bad, in producing the right geodesics. It is however not obvious that we must have the same affine connection to still get exactly the same $\frac{d^2x^\mu}{d\lambda^2}$ as function of $\dot{x}^\mu$. In fact necessary and sufficient conditions for two different affine connections, Γ and $\tilde{\Gamma}$, to yield the same geodesics are:

$$\delta_{\rho\kappa}y^\rho y^\alpha y^\beta\left(\Gamma^\kappa_{\alpha\beta}y^\mu - \Gamma^\mu_{\alpha\beta}y^\kappa\right) = \delta_{\rho\kappa}y^\rho y^\alpha y^\beta\left(\tilde{\Gamma}^\kappa_{\alpha\beta}y^\mu - \tilde{\Gamma}^\mu_{\alpha\beta}y^\kappa\right) \tag{4.8}$$

This must hold for *all* values of the parameter y^μ, at every position x^μ for the two affine connections to be geodesically equivalent.

As an example of how different affine connections can give rise to the same geodesics, let us study a two-dimensional manifold. We also assume that the coordinate curvature, $\frac{d^2x^\mu}{d\lambda^2}$, vanishes everywhere and in every direction. From Eq (4.6) we find (after some calculus) that this will be satisfied for any affine connection of the form:

$$\Gamma^0_{11} = \Gamma^1_{00} \tag{4.9}$$

$$\Gamma^0_{00} = \phi_0(x^\mu) \qquad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\phi_0(x^\mu)}{2} \tag{4.10}$$

$$\Gamma^1_{11} = \phi_1(x^\mu) \qquad \Gamma^0_{01} = \Gamma^0_{10} = \frac{\phi_1(x^\mu)}{2} \tag{4.11}$$

Here ϕ_0 and ϕ_1 are arbitrary functions of spacetime position. So, at least in two dimensions, it is not necessary to have a vanishing affine connection to produce straight lines, and nothing but straight lines, as geodesics. In section 4.3, we will see an example of a non-constant *metric* that yields zero coordinate-curvature for every direction of the geodesics at a certain point.

4.2 On the construction of the dual metric

While I were constructing the dual scheme I used to think that what I did was *equivalent* to jumping to a freely falling system and doing a sign flip of the spatial part of the metric in this system. In this section we will understand that this is not the case [3].

4.2.1 Point-dual metrics

Suppose that we are in a freely falling coordinate system, where in one point, the metric reduces to Minkowski with vanishing derivatives [4] . Suppose that we just take this metric and flip the sign of every component except g_{tt}, to create a new metric. The new metric is Pythagorean with all derivatives still vanishing, in the point in question. When the derivatives of the metric vanishes, the affine connection also vanishes, see Eq (4.2). This means that both the ordinary and the new metric (if we construct it in this way) have all components of the affine connection equal to zero. This means that they produce the same geodesics, in the point in question. We may say that the new metric is point-dual to the original metric.

Notice that under coordinate transformations (Weinberg [3] p. 100), the affine connection transforms according to:

$$\Gamma'^{\lambda}_{\mu\nu} = \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial x^{\tau}}{\partial x'^{\mu}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} \Gamma^{\rho}_{\tau\sigma} + \frac{\partial x'^{\lambda}}{\partial x^{\rho}} \frac{\partial^2 x^{\rho}}{\partial x'^{\mu} x'^{\nu}} \tag{4.12}$$

Notice that the first term on the right hand side is zero when the affine connection is zero. The second part depends *only* on the coordinate transformation itself. This means that if we construct a point-dual metric according to the reasoning above, the point-dual affine connection would equal the ordinary affine connection in *any* coordinate representation.

In the dual metric scheme of Paper I, the dual affine connection does *not* equal the ordinary affine connection. The explanation is, that we do not get the dual metric by simply flipping the sign of the spatial part in the freely falling coordinates.

To get a dual affine connection which is different from the ordinary affine connection, the dual metric in the originally freely falling system *must* have non-vanishing derivatives. That this is actually the case is shown in Section 4.3.

4.3 On the dual metric in freely falling coordinates

We understood in the preceding section that the dual metric in freely falling coordinates does not equal Pythagoras with vanishing derivatives. Out of curiosity I would like to

[3] Having since then constructed also the absolute scheme, I know the limitations of this technique when it comes to geodesics.

[4] Compare with Chapter 3 regarding the absolute metric.

know what the dual metric, given by Eq (10) and Eq (11) in Paper I, *does* look like in a originally freely falling coordinate system. This will also serve as check of the correctness of the dual scheme. In a freely falling system the trajectories of freely falling particles will be straight lines (to second order) so the dual affine connection has to be of the form given by Eq (4.9), Eq (4.10) and Eq (4.11).

4.3.1 Finding the coordinate transformation to the freely falling coordinates

Around a point, $t = 0$ and $x = x_0$, we search for freely falling coordinates u, v where u is timelike [5] and v is spacelike. We assume that x and t depends at most to second order on u and v. Also we demand $\frac{\partial x}{\partial u} = 0$ and $\frac{\partial t}{\partial v} = 0$, that is we want the freely falling system which is at *rest*. Further on, we assume a diagonal, time independent, two-dimensional original metric (with positive a_0 and negative c_0). We have:

$$g'_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \tag{4.13}$$

Demanding that the coordinate transformed metric must reduce to Minkowski at $(u, v) = (0, 0)$ with vanishing u and v-derivatives yields (in two pages of calculation):

$$t = k_1 \cdot u + k_2 \cdot uv \tag{4.14}$$

$$x = x_0 + k_3 \cdot v + k_4 \cdot v^2 + k_5 \cdot u^2 \tag{4.15}$$

Here the constants are uniquely given by:

$$k_1 = \frac{1}{\sqrt{a_0}} \qquad k_2 = -\frac{a_0{}'}{2\sqrt{-c_0} \cdot a_0{}^{3/2}} \tag{4.16}$$

$$k_3 = \frac{1}{\sqrt{-c_0}} \qquad k_4 = \frac{c_0{}'}{4(-c_0)^2} \qquad k_5 = -\frac{a_0{}'}{4(-c_0)a_0} \tag{4.17}$$

Here prime denotes derivative with respect to x. The index $'0'$ means as usual the *original* Lorentzian metric. All the quantities are evaluated at the point x_0.

4.3.2 The dual metric in the freely falling coordinates

Knowing the coordinate transformation, we may calculate the transformation of the *dual* metric. The result is (after 4 pages of calculations), to first order in u and v:

$$\begin{bmatrix} g_{uu} & g_{uv} \\ g_{uv} & g_{vv} \end{bmatrix} = \begin{bmatrix} \frac{\alpha}{a_0 - \beta} - k_0 \cdot v & -\frac{k_0}{2} \cdot u \\ -\frac{k_0}{2} \cdot u & \frac{\alpha\beta}{(a_0 - \beta)^2} - \frac{2k_0\beta}{a_0 - \beta} \cdot v \end{bmatrix} \tag{4.18}$$

[5] Increasing u means moving timelike. Increasing v means moving spacelike

The constant k_0 is given by:

$$k_0 = \frac{\alpha \; {a_0}'}{(a_0 - \beta)^2 \sqrt{-c_0}} \tag{4.19}$$

Again all the quantities are defined at the point x_0. So a_0 is really $a_0(x_0)$ here. We may immediately check that setting $\alpha = -\beta = \infty$, corresponding to having the dual metric equal to the original metric, yields to first order exactly the Minkowski metric - which we demanded from the start.

The dual metric in the freely falling coordinates is of the form:

$$g_{\alpha\beta} = \begin{bmatrix} a + b \cdot v & c \cdot u \\ c \cdot u & d + e \cdot v \end{bmatrix} \tag{4.20}$$

Here a, b, c, d, e are constants. It is easy to show that the affine connection of such a metric obeys the necessary and sufficient conditions, Eq (4.9), Eq (4.10) and Eq (4.11) for coordinate straight geodesics at the point $(u, v) = (0, 0)$ if and only if:

$$c = \frac{b}{2} \qquad\qquad e = \frac{2bd}{a} \tag{4.21}$$

We see immediately that the freely falling dual metric, Eq (4.18), obeys these restrictions. We have thus verified that the dual metric gives the right equations of motion in the freely falling system.

5

On whether the interior dual metric is a sphere

We know that in Newtonian theory, a particle that is in free fall around the center of a spherical object of constant density, will oscillate with a frequency that is independent of the amplitude. This would fit well with an internal dual space that is *spherical*. We saw also in Paper I the possibility to choose parameters β and k that produces substantial curvature also in the case of the weak gravity outside our Earth. It is logical to ask whether we really can find parameters such that the internal case becomes spherical. In the following subsections we will see that it is not in general exactly spherical but one can choose parameters such that it *is* exactly spherical in the weak field limit.

5.1 Conditions for spheres

For a sphere of radius R, introduce definitions according to Fig 5.1:

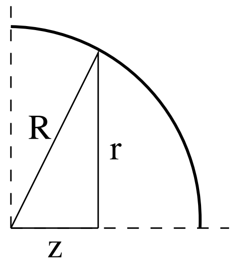


Figure 5.1: Definitions of variables for a sphere

It is easy to derive, from the Pythagorean theorem, that:

$$\frac{dr}{dz} = -\sqrt{\left(\frac{R}{r}\right)^2 - 1} \tag{5.1}$$

Assuming $z = z(x)$, where (see Paper I, Eq (23)):

$$\frac{dz}{dx} = \sqrt{c(x) - r'^2} \qquad \text{here} \qquad r' = \frac{dr}{dx} \qquad c = g_{xx} \tag{5.2}$$

We readily find:

$$R^2 = \frac{r^2}{1 - r'^2/c} \tag{5.3}$$

Also if we have a rolled up sphere such that $r = \gamma r_{\text{sphere}}$ and $r' = \gamma r'_{\text{sphere}}$ the formula is immediately modified to:

$$R^2 = \frac{r^2}{\gamma^2 - r'^2/c} \tag{5.4}$$

5.2 The dual interior metric

For the Schwarzschild interior dual metric we may show that [1] :

$$r'^2/c = \frac{\beta k^2 x^2}{4x_0^6(a_0 - \beta)} \qquad \text{and} \qquad r^2 = \alpha k^2 \frac{a_0}{a_0 - \beta} \tag{5.5}$$

Inserting this into Eq (5.4) we readily find that for the embedding to correspond to a sphere of radius R and rolling parameter γ we must have, for *all* $x < x_0$:

$$R^2\gamma^2(a_0 - \beta) - R^2\frac{\beta k^2 x^2}{4x_0^6} - \alpha k^2 a_0 = 0 \tag{5.6}$$

Notice that dividing the expression by αk^2 we see that any change in R^2 and γ can be cancelled by a corresponding change in α and k respectively.

For this expression to be true for all x it must be true for every power in x. a_0 has a lot of powers in x which would mean that the factors multiplying a_0 would have to vanish. Then the x^2-term in the middle cannot be canceled by anything. Thus the expression *cannot* be true for all values of x and thus the interior dual metric is not exactly spherical.

[1] The first relation can most rapidly be obtained re-juggling Eq (32) in Paper I (replace $<$ with $=$ and remove the "min"), look also at Eq (23) for understanding. The second is the square of Eq (21) in Paper I.

5.3 Approximative internal sphere

We *can* however produce something that is very similar to sphere. Set $\gamma = 1$, corresponding to a non-rolled sphere, and $\alpha = (1-\beta)/k^2$ meaning unit radius at infinity. We can expand the two occurrences of a_0 in Eq (5.6) to second order in x^2. Demanding the equation to hold to zeroth, first [2] and second order yields after some simplification (Mathematica does fine simplifications):

$$\beta = \frac{a_{00}\cdot(R^2-1)}{R^2-a_{00}} \quad \text{where} \quad a_{00} \equiv a_0(0) = \frac{1}{4}\left(3\sqrt{1-\frac{1}{x_0}}-1\right)^2 \tag{5.7}$$

$$k = \frac{2x_0^{7/4}}{\sqrt{3\sqrt{x_0-1}-\sqrt{x_0}}} \tag{5.8}$$

Notice that Eq (5.8), coming directly from the second order demand, is independent of β. The first relation is nothing but Eq (21) in Paper I, where one has demanded $r = R$ and $\alpha = (1-\beta)/k^2$, taken at $x = 0$.

Inserting β and k from Eq (5.7) and Eq (5.8) into my plotting program yields pictures with a very spherical appearance. The way it works is that the center of the bulge has the exact radius and curvature of a sphere, then the rest is not exactly spherical. In the Newtonian limit however, where $x_0 \to \infty$, we *do* get a perfect sphere. This is true whichever finite radius R we choose, as is explained below.

5.4 Spheres in the Newtonian limit

From Eq (5.3) we have the necessary relation for a sphere:

$$\frac{r'^2}{c} = 1 - \frac{r^2}{R^2} \tag{5.9}$$

Using the specific expressions for β and k given by Eq (5.7) and Eq (5.8) we may evaluate both the left and the right hand side of Eq (5.9) to lowest non-zero order [3] in $1/x_0$, to find that the equality holds *exactly* for all x to this order. Thus in this limit we get an exact sphere. In fact we knew in advance that the equality would hold. The reason is that, a_0 expanded to second order in x for which the equality holds, is the same (or in fact a bit more exact) as a_0 expanded to first order in $1/x_0$. If we instead would have had $a_0 = 1 - \sqrt{1 - x^3/x_0^4}$ which expanded in x to second order is zero, while *not* being zero expanded to first order in $1/x_0$ (remember that x^3/x_0^4 is of order $1/x_0$), we could not have used this little trick.

So, in the Newtonian limit the interior dual metric is isometric to a sphere for any value of β (so long as R remains finite).

[2] It is always satisfied to first order

[3] Terms like x^2/x_0^3 are of course treated as $1/x_0$-terms

An embedding diagram of the Earth spacetime is displayed in Fig 5.2. Here I have chosen the radius of the central bulge to be twice the embedding radius at infinity. The parameters β and k are given by Eq (5.7) and Eq (5.8) with $x_0 = 7.19 \cdot 10^8$, suitable for the Earth. As can be seen the bulge is quite spherical.

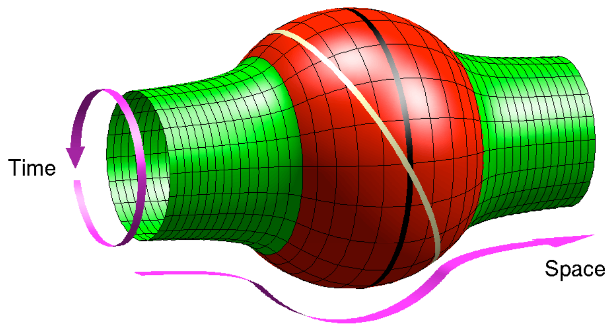


Figure 5.2: An embedding diagram for the dual spacetime of a central line through the Earth. The time per circumference is roughly 84 minutes.

Notice the two freefallers in the sphere, one static in the center, the other one oscillating around the Earth with a period time of roughly 84 minutes.

6

The Epstein-Berg way

There is yet another fundamentally different approach to embedding the spacetime, apart from the dual and the absolute schemes. In Epstein's book [2], 'Relativity Visualized', there are images which look a lot like my own dual metric embeddings. The exterior spacetime is represented by a trumpet-like shape and a star is a bulge on a cylinder. The interpretation of the diagram is however fundamentally different from my own.

In the book, which is a popular scientific book, there is no mathematical background to the images. For me it first appeared as a miracle that he could get the geodesics right. A good friend of mine, Daniel Berg, had however independently come up with an idea to create a positive definite metric. We realized that this idea exactly matched the Epstein view, and must have been how Epstein did it. The basic ideas are outlined below.

6.1 The main philosophy

Consider a time independent metric with no cross terms in the time component:

$$d\tau^2 = a \cdot dt^2 - h_{ij} \cdot dx^i dx^j \tag{6.1}$$

Here we assume that a is positive and h_{ij} is positive definite. The expression for the line element can be rewritten as:

$$dt^2 = \frac{1}{a} \cdot d\tau^2 + \frac{h_{ij}}{a} \cdot dx^i dx^j \tag{6.2}$$

This looks like a positive definite line element where τ is now a coordinate and dt is proper distance. Notice that if we would have considered an original line element that was time dependent we couldn't have done this little trick. Also if we had had off-diagonal terms we would have had to solve a second order equation to find dt. This would then not have been a quadratic form, and thus not a metric [1] .

6.1.1 Particle trajectories and mappings

To better understand the new line element we may consider a particle trajectory in the ordinary spacetime *and* in the new *space-eigentime*. See Fig 6.1.

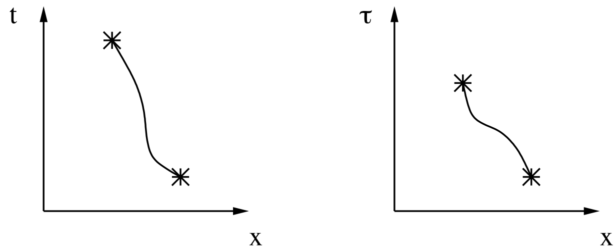


Figure 6.1: A trajectory in the original spacetime mapped to a corresponding trajectory in the space-eigentime

For a single particle there is a one-to-one mapping between points along the trajectory in the ordinary space and the trajectory in the space-eigentime. Notice however that *where* along the τ-axis in the space-eigentime we put the trajectory is completely arbitrary. There is in fact no general one-to-one mapping of events between the two spacetimes. To make this clear, consider two trajectories that have the same starting and ending point in the original spacetime. The eigentime elapsed along the two trajectories will in general not be the same. See Fig 6.2. We understand that in the space-eigentime we cannot then have both same starting and ending point for the two trajectories. Thus, unless we are specifying some special set of trajectories to use for the mapping, there is no unique mapping from the ordinary spacetime to the space-eigentime. There is however still a unique metric in both spaces. Notice however that the Epstein space-eigentime is not a spacetime in the ordinary sense since a single point in the space-eigentime can correspond to several physical events.

6.1.2 Intuition about geodesics

A trajectory that is a geodesic in the ordinary spacetime will also be a geodesic trajectory in the space-eigentime. This we may understand most easily using the variational principle. Consider first any trajectory in the ordinary space and a variation of this (with fixed end points), and how this is mapped to the space-eigentime. See Fig 6.2

[1]Of course, both off-diagonality and time independence depends on what coordinates we are using. I however refrain from a more general analysis for now.

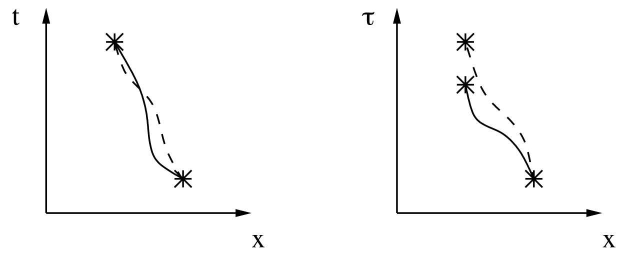


Figure 6.2: Varying the trajectories

Given that we start the trajectories at the same point in the space-eigentime, they will not in general end up in the same point, as discussed earlier. If we however assume the original trajectory in the ordinary space to be a geodesic, we know that for a small variation of the trajectory we do not change the integrated eigentime (to first order). This means that the varied path has the same endpoints in the space-eigentime also. What more is that the coordinate time t which is proper distance in the space-eigentime is exactly (to any order) the same for both the original and the variational trajectory, since it was so in the ordinary space. This would imply that the original trajectory in the space-eigentime must be a geodesic as well. Actually, for strictness, we should also argue that we can get *any* first order variation in the space-eigentime from first order variations in the original space. This we can probably do, but maybe we should prove the whole geodesic preservation mathematically in any case. This is done below.

6.1.3 Mathematics about geodesics

A geodesic in the original space obeys the Euler Lagrange equations, which for a time independent metric of the form Eq (6.1), becomes:

$$K = a\frac{dt}{d\tau} \tag{6.3}$$

$$0 = \partial_k a - \partial_k h_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} + \frac{d}{d\tau}\left(2h_{kj}\frac{dx^j}{d\tau}\right) \qquad \text{here} \qquad \partial_k \equiv \frac{\partial}{\partial x_k} \tag{6.4}$$

The index k takes the values 1, 2 or 3. Using $\frac{d}{d\tau} = \frac{K}{a}\cdot\frac{d}{dt}$, we may rewrite the spatial equations to a form which is more suitable for comparison with the equations of motion in the space-eigentime:

$$0 = \frac{K^2}{a}\cdot\left(\frac{1}{a}\partial_k a - \frac{1}{a}\partial_k h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} + \frac{d}{dt}\left(2h_{kj}\frac{dx^j}{dt}\right)\right) \tag{6.5}$$

The geodesic equations in the space-eigentime are equivalently:

$$\frac{1}{K} = \frac{1}{a}\frac{\partial\tau}{\partial t} \tag{6.6}$$

$$0 \quad = \quad -\frac{\partial_k a}{a^2} \cdot \left(\frac{d\tau}{dt}\right)^2 + \frac{a\partial_k h_{ij} - h_{ij}\partial_k a}{a^2}\frac{dx^i}{dt}\frac{dx^j}{dt} - \frac{d}{dt}\left(2h_{kj}\frac{dx^j}{dt}\right) \qquad (6.7)$$

Given that the equations of motion hold in the original spacetime, the first equation above is trivially satisfied and the spatial ones can be reduced to:

$$\frac{1}{a^2}\partial_k a \cdot \left(-\frac{a^2}{K^2} + a - h_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}\right) = 0 \qquad (6.8)$$

The expression within the parenthesis is however trivially zero, using the expression for the line element and the definition for the constant of the motion. Thus we find that a geodesic trajectory in the original spacetime is a geodesic also in the space-eigentime. Equivalently we may understand that a geodesic in the space-eigentime is a geodesic also in the original spacetime.

6.2 Embeddings

The Epstein line element Eq (6.2) in the case of a 1+1 Schwarzschild original spacetime can easily be embedded as a rotational surface. See Fig 6.3. Particles in free fall move on geodesics on the surface, just like in the dual metric scheme. Now however it is eigentime that is the azimuthal angle. This means for instance that photons move completely *along* the surface with no spiral what so ever in the azimuthal direction.

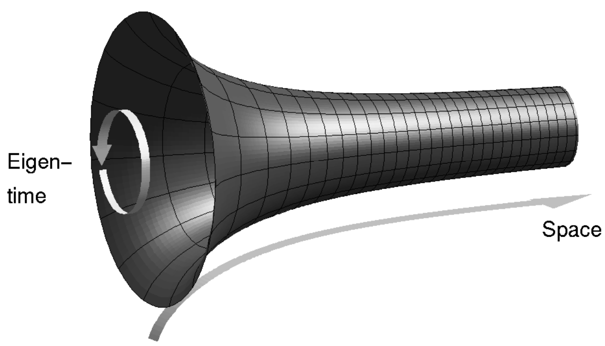


Figure 6.3: The Epstein embedding where x goes from 1.04 to 4. with embedding parameter $k = 0.5$. Notice that *eigen*time is the azimuthal angle.

The Schwarzschild time elapsed along a given trajectory on the surface is simply the length of the curve, as measured on the surface. Notice however again that this is not an image of the spacetime in the sense that every point on the surface corresponds to a unique event in the spacetime. For instance if two trajectories cross each other on the

Epstein surface that does not mean that they ever crossed in the ordinary spacetime. To determine whether two trajectories on the Epstein surface really meet in the ordinary spacetime, we need to specify the Schwarzschild time at an arbitrary point along each trajectory. Then we can take a ruler and measure the Schwarzschild time elapsed along each curve, to find the Schwarzschild time at any point along either curve. If ever we find that the Schwarzschild time along the two curves are the same, as the trajectories pass a certain spatial position - then they crossed each other in the real spacetime.

6.2.1 The Berg dynamical view

The procedure above is a little awkward. There is however another, more dynamical view on the scenario. Actually the idea was introduced to me by my friend, Daniel Berg. Take for instance two moving particles, corresponding to two trajectories in the original spacetime. Suppose that at a given slice $t = \text{const}$ in the original space, we have mapped the particles to two specific points on the Epstein trumpet, for instance to the same azimuthal angle. Then evolving things in Schwarzschild time, the two points will start moving on the surface. Notice that whichever directions the two particles take on the surface, which is in principle arbitrary so long as it is not *backwards* [2] in the azimuthal angle, the *speed* of the points will be the same and constant all the time. This is of course because the Schwarzschild time is measured as the distance on the surface.

Looking at the two points moving on the surface, we can see how the eigentime of the two particles evolve since it equals the azimuthal angle for each particle. In this dynamical view, whether the two particles collide or not is determined by if they are ever at the same spatial point (circle on the trumpet) at the same time! This is of course very close to the ordinary non-spacetime view of things. Notice that the azimuthal angle of the particles have nothing to do with whether they collide or not.

I find this latter view a quite intriguing. It is like a mixture of spacetime physics and dynamical physics. We are evolving things in 4 dimensions (in general) as opposed to the way one is doing things in numerical relativity, where one is evolving things in 3 dimensions. Of course in the Epstein-Berg dynamical we are not evolving the spacetime itself, we are merely studying particles, and also the spacetime has to be time independent. Still, I find it interesting.

6.3 The Epstein internal space

Just like in the dual scheme we may also consider the spacetime of a radial line through a spherically symmetric object such as a star. Like before we assume the object to be of constant proper density, so we can apply the standard Schwarzschild interior and exterior metrics to generate the Epstein metric. Looking at embeddings of the space-eigentime, the internal bulge looks quite spherical. In this section I investigate to what

[2] For antiparticles one might consider this after all!

extent it *is* spherical. We saw in Chapter 5 that an exact sphere has to obey Eq (5.3):

$$R^2 = \frac{r^2}{1 - r'^2/c} \tag{6.9}$$

In the Epstein case we have:

$$r = k \cdot \frac{1}{\sqrt{a0}} \quad \text{where} \quad a_0 = \frac{1}{4}\left(3\sqrt{1-\frac{1}{x_0}} - \sqrt{1-\frac{x^2}{x_0^3}}\right)^2 \tag{6.10}$$

$$c = \frac{-c_0}{a0} \quad \text{where} \quad c_0 = -\left(1 - \frac{x^2}{x_0^2}\right)^{-1} \tag{6.11}$$

Inserting this into the right hand side of Eq (6.9) we find quite easily that the equality cannot hold, to an arbitrary order in x. We may however, just like in the dual scheme, demand that the internal bulge shall be spherical to second order around the central star. This corresponds to a vanishing second order derivative of the right hand side of Eq (6.9), when $x = 0$. Demanding this, Mathematica yields that the embedding parameter k must obey:

$$k = \sqrt{x_0^3\left(3\sqrt{1-\frac{1}{x_0}} - 1\right)} \tag{6.12}$$

Incidentally, the third order derivative vanishes automatically (this is not in general true for the higher order derivatives). So, the interior bulge becomes quite spherical, as can be seen in Fig 6.4.

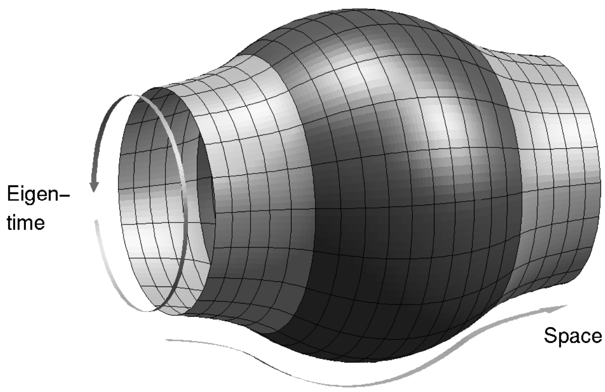


Figure 6.4: The Epstein embedding for a star with $x_0 = 2$ and with the embedding parameter chosen to make the interior as spherical as possible

6.3.1 Verification using the dual scheme

We may verify the correctness of Eq (6.12) using earlier derived formulas in the dual scheme. If we (for a given x_0) choose k according to Eq (5.8) and Eq (6.12) in the dual and Epstein scheme respectively, we know that we have (to second order) a perfect sphere in each case. Thus for small oscillations around the center, the orbital time will be independent of amplitude. In the dual scheme the, Schwarzschild time [3] per oscillation, will then equal the Schwarzschild time per circumference which is given by $2\pi \cdot k$. Analogously in the Epstein scheme the eigentime [4] per oscillation, will equal the eigentime per circumference which is given by $2\pi \cdot k$. Also, for very small oscillations, and thus very small local velocities we must have $d\tau = \sqrt{g_{00}} \cdot dt$. In other words we must have:

$$k_{\text{Epstein}} \stackrel{?}{=} \sqrt{a(0)} \cdot k_{\text{Dual}} \tag{6.13}$$

$$\sqrt{x_0^3\left(3\sqrt{1-\frac{1}{x_0}}-1\right)} \stackrel{?}{=} \frac{1}{2}\left(3\sqrt{1-\frac{1}{x_0}}-1\right)\cdot\frac{2x_0^{7/4}}{\sqrt{3\sqrt{x_0-1}-\sqrt{x_0}}} \tag{6.14}$$

Indeed we see that the equality holds, thus the dual and the Epstein scheme verify each other.

6.3.2 Comments

Notice that the interior bulge is not a perfect sphere in the Epstein view. This may also be easily seen from numerical embeddings in the limit $x_0 \to 1.125$. Also we lack the opportunity to produce shapefull images in the weak field limit. Whichever embedding constant we choose for the space-eigentime, of for instance a radial line through the Earth, the relative variations in the embedding radius will be of the order 10^{-9}.

6.4 Comments

One might think that there would be no use in having the Epstein trumpet layered, since we cannot distinguish events anyway. For every single particle it makes sense however, since eigentime is certainly not periodic.

Notice that time dilation is visualized in exactly the opposite way from that in the absolute scheme. In the Epstein scheme, time dilation is an *increase* of the embedding radius. In the absolute scheme, with observers at rest as generators, it is a *decrease* of the embedding radius.

A small note about spacelike distances and geodesics. In the Epstein embedding there is no such thing as spacelike distances. However, the equality Eq (6.2), which

[3] The rescaled and dimensionless Schwarzschild time

[4] The rescaled and dimensionless eigentime

is the basis for the embedding, holds also when $d\tau^2$ is negative. This would however correspond to an imaginary τ, and is thus not included in the embedding (where τ is taken as a real-valued angle).

Bibliography

[1] Marolf, D. (1999). *Gen. Rel. Grav.* **31**, 919

[2] Epstein, L. C. (1994). *Relativity Visualized*, (Insight Press, San Fransisco), ch. 10,11,12

[3] Weinberg, S. (1972). *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, (John Wiley & Sons, U.S.A), p. 77

[4] Kristiansson, S., Sonego, S., Abramowicz, M.A. (1998). *Gen. Rel. Grav.* **30**, 275

[5] D'Inverno, R. (1998). *Introducing Einstein's Relativity*, (Oxford University Press, Oxford), p. 99-101

Paper I