SHOCKS IN ONE-DIMENSIONAL PROCESSES WITH DRIFT

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Abstract. The local structure of shocks in one-dimensional, nearest neighbor attractive systems with drift and conserved density is reviewed. The systems include the asymmetric simple exclusion, the zero range and the 'misanthropes' processes. The microscopic shock is identified by a 'second class particle' initially located at the origin. Second class particles also describe the behavior of the characteristics of the macroscopic equation related to the corresponding model when the hydrodynamic limit is performed. Law of large numbers and central limit theorems as well as the convergence of the system at the average position of the shock are reviewed.

Key words: Asymmetric simple exclusion, zero range process, second class particle, shock fluctuations, central limit theorem, dynamical phase transition, density fluctuation fields.

1. A Review

Second class particles appeared first as a tool to prove the ergodic properties of the simple exclusion and the zero range process (sep and zrp respectively). Then they were useful to show hydrodynamic limits and central limit theorems for tagged particles. It was also established that the process as seen from a single second class particle may present different asymptotic densities to the right and left of the second class particle. When this happens we say that the second class particle identifies a microscopic shock. Finally, when conveniently rescaled, the second class particle follows the shocks or the characteristics of the related hyperbolic equation resulting from the hydrodynamic limits.

Misanthropes are individuals that tend to avoid other persons. The sep and the zrp are particular cases of a general system called the misanthropes process for which only a few results were proven. However we prefer to present the known results in the frame of reference of the misanthropes to help the understanding of the difficulties to generalize the results. In the misanthropes process we consider here, a finite number of particles is allowed at each site $x \in \mathbb{Z}$. For simplicity we consider the *nearest neighbor totally asymmetric case*: the particles may jump only to the nearest neighbor site to its right with a rate that is a non-decreasing function of the number of particles in the departure site and a non-increasing function of the number of particles in the arrival site. The state space of the process is $\mathbf{X} = \mathbb{N}^{\mathbb{Z}}$. We use η, ζ, ξ to denote the configurations of $\mathbf{X}$. Let $b(n, m) \geq 0$ be the rate that a particle jumps from x to $x + 1$ when there are n particles at x and m particles at

$x+1$. The generator of the process is

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} b(\eta(x),\eta(x+1))[f(\eta^{x,x+1}) - f(\eta)] \tag{1}$$

where f is a cylinder function on $\mathbf{X}$. The configuration $\eta^{x,y}(z)$ is defined by

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x,y \\ \eta(x)-1 & \text{if } z = y \\ \eta(y)+1 & \text{if } z = x. \end{cases}$$

Let $S(t)$ denote the corresponding semigroup. The rates of jump b are assumed to satisfy:

(i) $b(0,.) \equiv 0$,
(ii) $b(n,m)$ is non decreasing in n and non decreasing in m,
(iii) there exist a bounded non-decreasing function g such that

$$\begin{aligned} b(n,m-1)g(m) &= b(m,n-1)g(n) \\ b(n,m) - b(m,n) &= b(n,0) - b(m,0). \end{aligned} \tag{2}$$

Condition (ii) guarantees that the process is atractive. Let $\rho \in [0,\infty)$ and ν_ρ be the product measure with marginals

$$\nu_\rho(\eta(x) = k) = \frac{1}{Z(\rho)} \frac{\rho^k}{g(1)\dots g(k)} \tag{3}$$

where $Z(\rho)$ is a normalizing constant. Condition (iii) implies that ν_ρ is invariant for $\rho \geq 0$ and that the reverse process with respect to this measure is a misanthropes process with the same rate b but with reversed jumps.

We obtain the simple exclusion process when $g(n,m) = 1\{n=1, m=0\}$, the indicator function that $n=1$ and $m=0$, and when the configuration space is restricted to $\{0,1\}^{\mathbb{Z}}$. The zero range process is obtained when there exists some function g such that $b(n,m) = g(n)$, that is, when the rate of jump does not depend on the destination site. If the number of particles allowed per site is bounded, then also blocking measures arise. In particular, for the simple exclusion process the measure concentrated on the configuration $\dots 000111\dots$ is invariant, as well as its translates. Indeed it is known that all invariant measures are convex combination of the product measures $\{\nu_\rho\}$ and the blocking measures. For the zrp with $\sup_k g(k) < \infty$, all the invariant measures are convex combinations of $\{\nu_\rho\}$. When the number of particles per site is not limited and the system is totally asymmetric as the one we study here, one expects that all invariant measures are translation invariant. It is known that for the partially asymmetric case (when jumps to the left are allowed) there are invariant measures that are not translation invariant. There are no complete results about the set of all invariant measures for the general misanthropes process.

Let $h(u) = \int d\nu_u(\eta) b(\eta(0),\eta(1))$. From now on we assume through the paper that $h(u)$ is strictly concave. This assumption is convenient to have a nice construction

of the entropic solutions of the related equation. It is easy to construct non-trivial examples of b that give rise to concave h. Let $u(r,t)$ denote the unique entropic solution of

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + \frac{\partial h(u)}{\partial r} = 0 \\ u(r,0) = u_0(r) \end{aligned}\right\} \tag{4}$$

where u_0 is a piecewise continuous function. Under the concavity condition on h, initial discontinuities where $u_0^- < u_0^+$ persist at later times, but the position may be translated in space (shocks). Initial discontinuities where $u_0^- > u_0^+$ disappear at time 0^+ (rarefaction).

In order to establish the hydrodynamic limit that relates the microscopic model with the pde, we consider a family of product measures $\nu_{u_0}^\varepsilon$ with marginals

$$\int d\nu_{u_0}^\varepsilon(\eta)\eta(\varepsilon^{-1}r) = u_0(r) \tag{5}$$

where by abuse of notation we do not write the integer parts. The next theorem gives the convergence of the distribution of the process as seen from a passenger travelling at constant velocity. It essentially says that in the continuity points of the solution of the equation the system looks asymptotically as in equilibrium with a parameter predicted by the equation.

Theorem 1. (Hydrodynamic limit) *Let $u_0(r)$ be an integrable uniformly bounded piecewise continuous function, and let $\nu_{u_0}^\varepsilon$ be a family of product measures with marginals $\nu_{u_0}^\varepsilon(\eta(\varepsilon^{-1}r)) = u_0(r)$. Then, for all cylinder function f*

$$\lim_{\varepsilon\to 0} \nu_{u_0}^\varepsilon S(\varepsilon^{-1}t)\tau_{\varepsilon^{-1}r} f = \nu_{u(r,t)} f \tag{6}$$

in the continuity points of $u(r,t)$, the solution of (4) with initial condition $u(r,0) = u_0(r)$.

Remark

(7) The integrability of the initial condition may be dropped in special cases like in the sep with asymptotic constant densities.

The equation (4) admits travelling wave solutions. In particular, if the initial condition is non-decreasing piecewise constant with only one discontinuity, then the solution is a translation of this initial condition. This puts the question of what happens at a microscopic level. Can one see the jump of the density in the particle system? The anwer is yes and one manner to see this is to look at the system as seen from a second class particle. The next theorem says that if the initial distribution of the system is a product measure with density to the left of the origin smaller than the density to the right of it, then the system as seen from a second class particle added at the origin at time zero will look very much the same way at later times.

The motion of a second class particle arises when a joint realization of the process with two different initial configurations is realized. The joint realization is called the basic coupling and the principle is that the jumps of particles sitting at the same site occur together for the two marginals, as much as possible. If the first initial

configuration is identical to the second but has one extra particle located at the origin, with the coupling at any later time t the first process will have an extra particle at position X_t. This is the position of the second class particle. The name comes from the fact that it gives priority to the other particles. The joint process $(\eta_t, X_t) \in \mathbf{X} \times \mathbb{Z}$ is Markovian and has generator

$$\begin{aligned}\bar{L}f(\eta, z) = &\sum_{x \neq z-1, z} b(\eta(x), \eta(x+1))[f(\eta^{x,x+1}, z) - f(\eta, z)] \\ &+ (b(\eta(z-1), \eta(z)) - b(\eta(z-1), \eta(z)+1))[f(\eta^{z,z-1}, z) - f(\eta, z)] \\ &+ (b(\eta(z)+1, \eta(z+1)) - b(\eta(z), \eta(z+1)))[f(\eta^{z,z+1}, z+1) - f(\eta, z)].\end{aligned}$$

The second class particle identifies the shock in the following sense. Let τ_x be the translation operator defined by $\tau_x \eta(z) = \eta(z+x)$. Let $\nu_{\rho,\lambda}$ be a product measure with marginals ν_ρ for $x \leq 0$ and ν_λ for $x > 0$. We say that $\mu \sim \nu_{\rho,\lambda}$ if $\lim_{x \to +\infty} \tau_x \mu = \nu_\lambda$ and $\lim_{x \to -\infty} \tau_x \mu = \nu_\rho$, where the limits are understood as weak limits. Let $\eta'_t = \tau_{X_t} \eta_t$ and let $S'(t)$ be the corresponding semigroup.

Theorem 2. (Microscopic interface) *Let the process be the sep and assume* $0 \leq \rho \leq \lambda \leq 1$. *The process as seen from the second class particle* η'_t *has an invariant measure* $\mu \sim \nu_{\rho,\lambda}$. *Furthermore, if* $\lambda > \rho$, $\lim_{t \to \infty} \nu_{\rho,\lambda} S'(t) = \mu$.

Remarks

(8) Theorem 2 is proven only for the simple exclusion process. I think however that Theorem 2 can be proven for the misanthropes using the same techniques, but I have not worked out the details.

(9) In the sep the invariant measure μ for the process as seen from the second class particle has been described explicitly for all $\lambda \geq \rho$. When $\lambda = \rho$ there is a reminiscence of a shock because the density to the right (respectively left) of the second class particle is bigger (respectively less) than ρ and the approach to ρ is slow (as an inverse power of the distance). When $\lambda > \rho$, the invariant measure μ is extremely close to $\nu_{\rho,\lambda}$: it is possible to couple μ and $\nu_{\rho,\lambda}$ in such a way that the number of sites where the corresponding configurations differ is a random variable with a positive exponential moment.

(10) *Open problem.* Does the process starting from ν_ρ as seen from a second class particle converge to the corresponding invariant measure? Conjecture: yes, but the technique used in the case of initial $\nu_{\rho,\lambda}$ for $\rho < \lambda$ does not work.

The second class particle moves along a characteristic of the macroscopic equation. This is the essential result of the next theorem. The characteristics $w(a,t)$ emanating from a corresponding to the equation (4) are the solutions of

$$\left.\begin{aligned}\frac{dw}{dt} + h'(u(w,t)) &= 0 \\ w(0) &= a.\end{aligned}\right\} \tag{11}$$

There is a unique characteristic emanating from a if the initial condition presents no decreasing discontinuity at a.

Theorem 3. *Let X_t^x be the position of a second class particle initially at x. Let $X^\varepsilon(a,t) = \varepsilon X^{\varepsilon^{-1}a}_{\varepsilon^{-1}t}$. If the characteristic emanating from a is unique, then for the sep, and the zrp with concave g,*

$$\lim_{\varepsilon\to 0} \varepsilon \mathbb{E}\left|X^\varepsilon(a,t) - w(a,t)\right| = 0. \tag{12}$$

Furthermore, if u_0 is piecewise constant, non-decreasing and has a finite number of discontinuities, then for the sep

$$\lim_{\varepsilon\to 0} X^\varepsilon(a,t) = w(a,t) \quad \mathbb{P}\text{-a.s.} \tag{13}$$

Remarks

(14) The weak law of large numbers (12) is proven for the sep and zrp with strictly concave h. The strong law (13) was proven for the sep with piecewise constant profiles presenting at most one increasing shock, but the proof easily extends to cases when a finite number of increasing shocks are present. I believe that the same techniques can be applied to the zero range process with concave jump rate g. The concavity of g guarantees that if at time zero the system is perturbed by adding a particle, then at later times there will be only one extra particle. A more complicated condition in this direction has been stated for the misanthropes process.

(15) *Open problem.* What happens when the concavity of g is violated and a perturbation added to the system will produce more perturbations? Is there still a shock? How can one describe it at a microscopic level?

(16) *Open problem.* Show Theorem 3 for the misanthropes process.

(17) An interesting question that arises here is: what happens when the second class particle is at a decreasing shock? In this case there are infinitely many characteristics emanating from that point (rarefaction front). It was recently proved for the sep that in this case the second class particle chooses uniformly among the characteristics emanating from that point.

The next series of theorems have been proven only for the simple exclusion process. Presumably using *ad hoc* couplings one can show analogous results for the zero range case. At this point it is not clear how to show them for the general misanthropes process. The next theorem gives the asymptotic variance of the second class particle in the case of an increasing shock. Moreover it establishes that in the scale $\sqrt{t}$ the fluctuations of the second class particle around its expected value at time t are determined by a function N_t of the initial configuration and not by the randomness due to the evolution. The function $N_t(\eta)$ is the number of empty sites of η between 0 and $(\lambda-\rho)t$ minus the number of particles of η between $-(\lambda-\rho)t$ and 0.

Theorem 4. *Let the process be the sep with initial distribution $\nu_{\rho,\lambda}$, with $0 \le \lambda < \rho \le 1$. Then the average position of the second class particle is given by*

$$\mathbb{E}X_t = (1-\lambda-\rho)t := vt \tag{18}$$

for all $t \ge 0$. The limiting variance (diffusion coefficient) exists and is given by

$$D := \lim_{t\to\infty} \frac{\mathbb{E}\,(X_t)^2 - (\mathbb{E}X_t)^2}{t} = \frac{\rho(1-\rho)+\lambda(1-\lambda)}{\lambda-\rho} \tag{19}$$

The position of the particle in the scale $t^{1/2}$ is determined by the initial configuration. Let

$$N_t(\eta) = \sum_{x=0}^{(\lambda-\rho)t} (1-\eta(x)) - \sum_{x=-(\lambda-\rho)t}^{0} \eta(x).$$

Then

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}\Big[X_t - (\lambda-\rho)^{-1}N_t(\eta_0)\Big]^2 = 0. \tag{20}$$

Remarks

(21) The interval determining the position of X_t can be obtained using the macroscopic equation. The left extreme of the interval is exactly the point where the leftmost characteristic arriving to vt at time t emanates, while the right extreme is the point where the rightmost characteristic arriving at vt at time t emanates.

(22) Noticing that $\mathbb{E}X_t = \mathbb{E}N_t = (1-\lambda-\rho)t$, it follows from (20) that $X_t - \mathbb{E}X_t$, the fluctuations around the mean of the second class particle at time t, are well approximated by $N_t(\eta_0) - \mathbb{E}N_t(\eta_0)$, the fluctuations of the density of the initial configuration on the interval $-(\lambda-\rho)t, (\lambda-\rho)t$. Since $N_{t+s} - N_t$ and N_t depend on disjoint sets and the initial distribution is product, we can conclude that $N_t(\eta_0)$ has independent increments. This is useful to show convergence of the rescaled position of the second class particle to the finite dimensional distributions of Brownian Motion.

(23) *Open problems.* One of the most important open questions in the field is the computation of the asymptotic behavior of the variance when $\lambda = \rho$. In other words, for which α, the limit as t goes to infinity of $t^\alpha(\mathbb{E}\,(X_t)^2 - (\mathbb{E}X_t)^2)$ exists and is non trivial? Physical arguments and simulations suggest $\alpha = -\frac{4}{3}$. Which is the right normalization in $d = 2$?

(24) In dimension $d \geq 3$ there is strong evidence that $\alpha = 1$. Let δ be the average vector jump for the underlying random walk. Consider a family of initial product distributions of first and second class particles such that (a) disregarding classes one has the equilibrium measure ν_ρ and (b) the distribution of second class particles is a product measure whose density at site $\varepsilon^{-1}r$ is $\varepsilon u_0(r)$ (so that the density of second class particles is of order ε). Then the suitable rescaled and translated density of second class particles converges to the solution of the viscous Burgers equation. More precisely, if $\nu^\varepsilon_{u_0}$ is a product measure such that

$$\int d\nu^\varepsilon_{u_0}(\eta)\eta(\varepsilon^{-1}r) = \rho - \varepsilon u_0(r),$$

then using the notation of Theorem 1,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\left(\rho - \int d\nu^\varepsilon_{u_0} S(\varepsilon^{-2}t)(\eta)\eta(\varepsilon^{-1}r + \varepsilon^{-2}(1-2\rho)t\delta)\right) = u(r,t) \tag{25}$$

where $u(r,t)$ is the solution of the d-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + \delta\cdot\nabla u^2 = \sum_{i,j=1}^{d} D_{i,j}\frac{\partial^2 u}{\partial z_i \partial z_j}. \tag{26}$$

Since the density of second classs particles goes to zero, this suggests that the diffusion coefficient of a single second class particle should be the same as the diffusion coefficient of the equation.

(27) *Open problem.* Show that the diffusion coefficient of a single second class particle in $d \geq 3$ is the diffusion coefficient of the density of second class particles in the equation (26).

(28) The dependence of the position of the second class particle on the initial configuration (20) for the case when there is only one shock suggests the following generalization. Define

$$N_t^\varepsilon(\eta) = \sum_{x=\varepsilon^{-1}a}^{y^+\varepsilon^{-1}} (1-\eta(x)) - \sum_{x=y^-\varepsilon^{-1}}^{\varepsilon^{-1}a} \eta(x). \tag{29}$$

where y^+ and y^- are the places where the rightmost and leftmost characteristics arriving at $w(a,t)$ emanate. (See also Remark (21).) Then one expects that the dependence on the initial configuration (20) holds also for this case.

(30) One of the key tools to show the dependence on the initial configuration (20) was the study of the variance of the current of particles through a passenger travelling at deterministic velocity r. In particular when the system is in equilibrium with initial distribution ν_ρ, and the passenger is travelling at the velocity of the characteristic $r = 1 - 2\rho$, the asymptotic variance of the current at time t divided by t converges to zero as t goes to infinity.

(31) *Open problem.* Can one say that the lines through which the current of particles have asymptotic variance zero in the correct scale are characteristics? This would give an intrinsic definition of the characteristics by means of a property that can not be obtained from the macroscopic equation. To start one can try to solve a simpler question. Assume that the initial distribution is $\nu_{\rho,\lambda}$. Does the asymptotic limiting variance of the current through a passanger travelling at deterministic velocity $(1-\lambda-\rho)$ vanish?

(32) *Open problem.* What is the behavior of the variance of the current of particles along the characteristics in the rarefaction front?

(33) An important step in proving Theorem 4 was to show that the exact computation of the diffusion coefficient (19) and the dependence of the initial configuration (20) are equivalent. Presumably the same techniques may be applied to show that with the generalized definition of N_t given by (29), the dependence on the initial configuration (20) is equivalent to the following identity for the limiting variance

$$\lim_{t\to\infty} \varepsilon\Big(\mathbb{E}\,(X_t^\varepsilon)^2 - (\mathbb{E}X_t^\varepsilon)^2\Big) = \frac{\int_{y^-}^{y^+} u_0(r)(1-u_0(r))dr}{u^+ - u^-} \tag{34}$$

where u^+ and u^- are the densities to the right and left of $w(a,t)$ respectively.

The following theorem establishes that the finite dimensional distributions of the position of the second class particle in the sep under shock initial conditions behave as those of Brownian motion. Its proof is a corollary to the dependence on the initial configuration (20) and the fact that N_t has independent increments as mentioned in Remark (22). It is not clear how to prove tightness.

Theorem 5. *Let the process be the sep. Let $W(t)$ be Brownian motion with diffusion coefficient D. Then if the process starts with either $\nu_{\rho,\lambda}$ or the invariant measure μ,*

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}(X_{\varepsilon^{-1}\cdot} - \mathbb{E}X_{\varepsilon^{-1}\cdot}) = W(.), \tag{35}$$

in the sense of the finite dimensional distributions.

In the case of an initial increasing shock, the hydrodynamic limit (6) means that under initial distribution $\nu_{\rho,\lambda}$, a traveller moving at deterministic velocity r observes asymptotically that the particles are distributed as ν_ρ for $r > v$ and ν_λ for $r < v$, where $v = (1 - \lambda - \rho)$. Indeed $u(r,t) = \rho 1\{r < vt\} + \lambda 1\{r > vt\}$ is the entropic solution of the Burgers equation when $u_0(r) = \lambda$ for $r > 0$ and ρ for $r \le 0$. When $r = v$ the system converges to a fair mixture of ν_ρ and ν_λ. This is the principal consequence of our next result. Its proof is based on the central limit theorem for X_t established in Theorem 5. Let

$$g(r,t) = \mathbb{P}(W(t) \le r) = \frac{1}{\sqrt{2\pi Dt}} \int_{-\infty}^{r} \exp\bigl(-s^2/(2Dt)\bigr) ds,$$

the normal distribution with variance Dt.

Theorem 6. (Dynamical phase transition) *Let $v = (1 - \lambda - \rho)$. Then*

$$\lim_{\varepsilon\to 0} \nu_{\rho,\lambda} S(t\varepsilon^{-1}) \tau_{vt\varepsilon^{-1} + a\varepsilon^{-1/2}} = (1 - g(a,t))\nu_\rho + g(a,t)\nu_\lambda. \tag{36}$$

Remark
(37) The case of two meeting shocks presents surprising features. Assume that the process at time zero has three densities: $u_0(r) = \rho_{-1} 1\{r < a\} + \rho_0 1\{a \le r < b\} + \rho_1 1\{r \ge b\}$; $\rho_{-1} < \rho_0 < \rho_1$. According to the Burgers equation, these shocks meet at time $\bar t = (b-a)/(\rho_1 - \rho_{-1})$ at the point $\bar r = [a\rho_1 + b\rho_{-1} + (b-a)(1-\rho_0)]/(\rho_1 - \rho_{-1})$. In this case the distribution at macroscopic time $\bar t$ at site $\bar r$ converges to a mixture of the three product distributions. More precisely, calling $\nu^\varepsilon_{u_0}$ the family of measures constructed as in Theorem 1, preliminary results indicate that

$$\lim_{\varepsilon\to 0} \nu^\varepsilon_{u_0} S(\varepsilon^{-1}\bar t) \tau_{\varepsilon^{-1}\bar r} = \tfrac{5}{16}\nu_{\rho_{-1}} + \tfrac{3}{8}\nu_{\rho_0} + \tfrac{5}{16}\nu_{\rho_1}. \tag{38}$$

Finally we turn over the fluctuation fields. The problem is to determine how the fluctuations around the initial density evolve with time. If the initial distribution is product with constant density ρ, then the fluctuation fields move deterministically along the corresponding characteristic $1 - 2\rho$. When the system starts with the increasing shock $\nu_{\rho,\lambda}$, a conflict arises because the characteristics to the right of the shock are slower than those to the left of it. The conflict is solved by the second class particle that moves to compensate the fluctuations changing in that way the position of the microscopic shock. In the space scale t, the one we use to study the fluctuations, the density fluctuations between y^- and y^+ concentrate at the point vt, where y^- and y^+ are the emanating points of the leftmost and rightmost characteristics arriving at vt at time t. This is essentially the content of Theorem

7 and (43) of Theorem 8. In (44) of Theorem 8 we look at point vt at time t in the scale $t^{1/2}$ to see how those fluctuations reflect on the position of the microscopic shock. The proofs of the theorems are based again on the fact that the limiting variance of the current along the characteristics vanishes.

Let Υ_t^ε be the fluctuations fields defined by

$$\Upsilon_t^\varepsilon(\Phi) = \varepsilon^{1/2} \sum_{x\in\mathbb{Z}} \Phi(\varepsilon x)[\eta_{\varepsilon^{-1}t}(x) - \mathbb{E}\,\eta_{\varepsilon^{-1}t}(x)], \tag{39}$$

for smooth integrable test functions Φ. For $t = 0$, if η_0 is distributed according to $\nu_{\rho,\lambda}$, then

$$\lim_{\varepsilon\to 0} \Upsilon^\varepsilon(\Phi) = \Upsilon(\Phi), \tag{40}$$

where $\Upsilon(\Phi)$ is Gaussian white noise with mean zero and covariance

$$\mathbb{E}\,(\Upsilon(\Psi)\Upsilon(\Phi)) = \int u_0(r)(1-u_0(r))\Psi(r)\Phi(r)dr. \tag{41}$$

where $u_0(r) = \lambda 1\{r \geq 0\} + \rho 1\{r < 0\}$.

Theorem 7. (Convergence of the fluctuation fields) *Assume that the initial distribution of the process is $\nu_{\rho,\lambda}$. Let $v = (1-\rho-\lambda)$. Let $u(r,t) = \lambda 1\{r > vt\} + \rho 1\{r \leq vt\}$. As $\varepsilon \to 0$, the fluctuation fields Υ_t^ε defined in (39) converge in a weak sense to the conservative solution Υ_t of the nonhomogeneous linear equation*

$$\frac{\partial}{\partial t}\Upsilon_t(r) = \frac{\partial}{\partial r}(1-2u(r,t))\Upsilon_t(r), \tag{42}$$

with initial condition Υ, the Gaussian field with zero mean and covariance given by (41).

Theorem 7 is a consequence of the L_2 convergence of the fluctuation fields established in the next theorem. The weak solutions of (42) present a singularity at the point (vt, t) due to the discontinuity of $u(r,t)$ at $r = vt$. For this reason there is no unique solution. However there is only one conservative solution. To better describe it let us introduce some notation. Assume that Φ is the indicator of the interval (a_1, a_2). For $i = 1, 2$ let $b_i(t)$ be the points where the characteristics arriving to a_i at time t emanate:

$$b_i(t) = \begin{cases} a_i - (1-2\rho)t & \text{if } a_i < vt \\ a_i - (1-2\lambda)t & \text{if } a_i > vt. \end{cases}$$

Then Υ_t, the solution of (42) is given by the following.

$$\int \Upsilon_t(r)\Phi(r)dr = \int_{a_1}^{a_2} \Upsilon_t(r)dr = \int_{b_1(t)}^{b_2(t)} \Upsilon_0(r)dr.$$

We can interpret this by saying that if $vt \in (a_1, a_2)$ then, the fluctuations present in the interval $(-(\lambda-\rho)t, (\lambda-\rho)t)$ at time zero concentrate in the point vt at time t. Formula (44) below says that these fluctuations are present in the scale $\sqrt{t}$. Indeed they reflect the shock fluctuations that occur in this scale.

Theorem 8. *Let* $A_\varepsilon = \mathbb{Z} \cap (\varepsilon^{-1}a_1, \varepsilon^{-1}a_2)$, $B_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}b_1(t), \varepsilon^{-1}b_2(t))$. *Then*

$$\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left(\sum_{x \in A_\varepsilon} [\eta_{\varepsilon^{-1}t}(x) - \mathbb{E}\, \eta_{\varepsilon^{-1}t}(x)] - \sum_{x \in B_\varepsilon(t)} (\eta_0(x) - \mathbb{E}\, \eta_0(x)) \right)^2 = 0. \tag{43}$$

Let $c > 0$, $C_\varepsilon(t) = \mathbb{Z} \cap (\varepsilon^{-1}vt - \varepsilon^{-1/2}c, \varepsilon^{-1}vt + \varepsilon^{-1/2}c)$ *and let* $K_\varepsilon(t) = \mathbb{Z} \cap (-\varepsilon^{-1}t(\lambda - \rho), \varepsilon^{-1}t(\lambda - \rho))$. *Then*

$$\lim_{\varepsilon \to 0} \varepsilon \mathbb{E} \left(\sum_{x \in C_\varepsilon(t)} [\eta_{\varepsilon^{-1}t}(x) - \mathbb{E}\, \eta_{\varepsilon^{-1}t}(x)] - T_{\varepsilon^{-1/2}c} \sum_{x \in K_\varepsilon(t)} (\eta_0(x) - \mathbb{E}\, \eta_0(x)) \right)^2 = 0, \tag{44}$$

where T_c *is truncation by* c*:*

$$T_c F(.) = \begin{cases} F(.) & \text{if } |F(.)| \le c \\ c & \text{if } F(.) > c \\ -c & \text{if } F(.) < -c. \end{cases}$$

Note that $C_\varepsilon(t)$ is an interval of length proportional to $\varepsilon^{-1/2}$ around the macroscopic point vt. When $c \to \infty$, (44) says that the fluctuations at time t in a region of length proportional to $\sqrt{t}$ around vt are given by the fluctuations at time 0 in a region of length proportional to t.

Remark

(45) *Open problem.* How do the fluctuation fields behave in the rarefaction front?

2. Final Remark

(46) We have described results characterizing the shocks at a microscopic level for one dimensional nearest neighbor processes. The results also hold for the partially asymmetric case, when jumps to the left are allowed at rate q ($< p$, the rate of jumps to the right). With the exception of the hydrodynamic limit of Theorem 1 that holds for asymmetric processes other than nearest neighbor and in greater dimensions for some initial conditions, the techniques used to prove the other results do not seem to work for a more general jump transition probability in one dimension or in more than one dimension.

3. Notes and References

The simple exclusion and the zero range processes were introduced by Spitzer (1970). The set of invariant measures for the sep was characterized by Liggett (1976, 1985) and the sets of invariant measures for the zrp were characterized by Andjel (1982). Cocozza (1985) gives conditions (i)–(iii) under which the product measures (3) are invariant for the misanthropes process.

The limit of Theorem 1 was first proven by Liggett (1975, 1977) for the case $r = 0$, before the conection between the process and the Burgers equation appeared. Rost (1982) established first the hydrodynamical properties of the equation for initial condition $\dots 111000 \dots$ (decreasing profile). Then Benassi and Fouque (1987) extended the result to decreasing one step profiles (the proof in the increasing case is incomplete) and Andjel and Vares (1987) did it for increasing profiles. Benassi, Fouque, Saada, and Vares (1991) worked out monotone initial profiles. The hydrodynamic limit for the zrp was studied by Andjel and Kipnis (1984). For the misanthropes process with general initial integrable profiles the hydrodynamic limit (6) follows now as a consequence of the law of large numbers of Rezakhanlou (1990) and the proof of local equilibrium of Landim (1992). Landim (1991) has some partial results in greater dimensions. Lax (1972) shows how shock waves appear in the related equations. Rezakhanlou (1993a, b) studies further questions about the equation. In particular it is shown there that if the initial condition presents no decreasing discontinuity at a, then there is only one characteristic emanating from a, as mentioned after (11).

Blocking measures were first noticed by Liggett (1976). This was the first example that show that this model presented microscopic shocks. Then the case $\rho = 0$, $\lambda < 1$ was studied by Ferrari (1986). The existence of the microscopic shock was supported by simulations of Boldrighini, Cosimi, Frigio, and Nunes (1989). Ferrari, Kipnis, and Saada (1991) show the existence of the microscopic shock in a non-Markovian way for the sep. However their construction is in the base of the Markovian characterization as well as most of the other results for the sep. Ferrari (1992) shows that an isolated second class particle describes the microscopic shock.

Remark (9) is a consequence of a result of Derrida, Janowsky, Lebowitz, and Speer (1993) who compute the invariant measure μ for finite systems and computed the asymptotic density of particles around the shock. Building on this computation Ferrari, Fontes, and Kohayakawa (1993) totally described μ. This description is important to show the equivalence between μ and $\nu_{\rho,\lambda}$ and the properties of μ.

The weak law of large numbers for the second class particle given by identity (12) of Theorem 3 was proven by Rezakhanlou (1993) who also established the conditions on the rates to obtain (12) for the misanthropes. Ferrari (1992) proved the strong law (13) for non-decreasing initial profile with at most one increasing shock. The proof works also when a finite number of increasing shocks is present in the (non-decreasing) initial profile. Rezakhanlou (1993) shows that in the decreasing case, the second class particle is concentrated in the set of characteristics emanating from the discontinuity point. Ferrari and Kipnis (1993) prove that in this case the second class particle chooses uniformly among those characteristics (Remark (17)).

Spohn (1991) proved that the expected position of the second class particle is given by the velocity predicted by the macroscopic equation. This is the content of (18). He also conjectured (19), the exact value of the asymptotic variance. Boldrighini et al. (1989) performed computer simulations that supported the conjecture. Gärtner and Presutti (1989) show that the position of the leftmost particle when the initial densities are $\rho = 0$ and $\lambda < 1$ depend on the initial configuration. Ferrari (1992) shows the equivalence between the exact value of the limiting variance (19) and the dependence on the initial distribution (20) and that the right-hand side of

(19) is a lower bound for D. Finally Ferrari and Fontes (1993b) settle this problem by computing explicitly the asymptotic variance (19).

The heuristics of the conjecture (23) about the fluctuations of a second class particle when $\lambda = \rho$ are in Spohn (1991). The diffusive limit in dimensions $d \geq 3$ as described by (25) was performed by Esposito, Marra, and Yau (1993), using the relative entropy method. The variance of the current was computed by Ferrari and Fontes (1993a). Remark (33) was inspired on a conjecture of Rezakhanlou (1993a) about the fluctuations of the second class particle about its mean, under general initial profiles. In that paper can be found heuristics leading to (34).

The central limit theorem for the leftmost particle when $\rho = 0$ is a consequence of Burke's Theorem (see Spitzer 1970, Liggett 1985, Kipnis 1986, Wick 1985 and De Masi, Kipnis, Presutti, and Saada 1988). This is also a special case of Theorem 5 which was proven by Ferrari and Fontes (1993b). The dynamical phase transition of Theorem 6 was proven first by Wick (1985) and De Masi, Kipnis, Presutti, and Saada (1988) for $\rho = 0$ and by Andjel, Bramson, and Liggett (1988) for $\lambda + \rho = 1$. The distribution of the process at the meeting place of two shocks given by (38) was found by Ferrari, Fontes, and Vares. The fact that the fluctuation fields move deterministically if the initial profile is constant was proven by Benassi and Fouque (1992) and also by Ferrari and Fontes (1993a). Theorems 7 and 8 about the convergence of the fluctuation fields when there is an increasing shock are proven by Ferrari and Fontes (1993b).

The behavior of a tagged particle for the nearest neighbor sep with jumps to the left and right with probability $q < p$ respectively has also been studied. The system starts with the invariant measure ν'_ρ, conditioned to have a particle at the origin. Kipnis (1986) proved a central limit theorem and law of large numbers for the position of the tagged particle. De Masi and Ferrari (1985) computed the variance of the limiting Gaussian distribution. Ferrari and Fontes (1993c) show that the position of the tagged particle is given by a Poisson process of rate $(1 - \rho)(p - q)$ plus a perturbation of order 1. Saada (1987) proved a law of large numbers for the process in dimensions greater than one. Rezakhanlou (1993b) shows that a tagged particle in a non-equilibrium system satisfies a law of large numbers. The macroscopic position of the tagged particle can be described as the solution of an equation related to the hydrodynamic limit. The result also holds for the zrp.

Bramson (1988), Lebowitz, Presutti, and Spohn (1988) and Spohn (1991) reviewed some of the previous results. A survey on the beginning of the hydrodynamic limits for particle systems is given by De Masi, Ianiro, Pellegrinotti, and Presutti (1984). The physical literature can be found in van Beijeren (1991).

Shocks in a cellular automaton introduced by Boghosian and Levermore (1987) can be found in Cheng, Lebowitz, and Speer (1990) and in Ferrari and Ravishankar (1993). In the last paper relations with the Automata 184 of Wolfram (1983) are established.

Finally we mention that Walker (1989) describes actual shocks in real highways which look very much as the shocks one find in the mathematical models studied here.

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