Integral Distances in Point Sets

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1. Introduction

First results on geometrical objects with integral sides go back to the time of the Pythagoreans. Many nonmathematicians like masons or bricklayers use a pythagorean triangle $(3, 4, 5)$ to check an angle whether it is a right one, that is, whether the two points on both legs being 60 and 80 units away from the vertex have a distance of 100 units. More general, Heron triangles were studied, that are triangles where all sides and the area are integers.

Also negative results are known. So the diagonal of an integer sided square cannot be rational. It may be mentioned that for finite sets of distances multiplication with an appropriate factor proves the rational and integral cases to be equivalent. As a second example the side length x of a cube with a volume which is twice the volume of an integer sided cube cannot be rational ($x^3 = 2a^3$).

Of course, there are many still open problems. Do Heron triangles exist with integral medians? Does a perfect box exist, that is, a cuboid with integral sides, face diagonals, and body diagonals? Do there exist points in the plane with integral distances to the four vertex points of an integral sided square?

Here it will be reported on some results and problems on integral distances which are of the last decades. For historical results on diophantine equations and on geometrical problems the references [4, 5, 7, 20, 21] can be used. Different applications are imaginable in radio astronomy (ware lengths), chemistry (molecules), physics (energy quantums), robotics, architecture, and other fields.

2. Pairwise integral distances

Do n points in R^d exist which have pairwise integral distances? In other words, can the complete graph K_n be geometrically realized with integral straight line edges? The answer is in the affirmative. This may be deduced from one of the following two constructions of infinite circular point sets with pairwise rational distances.

At first consider an angle α of a pythagorean triangle. Then $\cos\alpha$ and $\sin\alpha$ are rational and the points $P_1, P_2, \ldots$ on the unit circle for angles $\alpha, 2\alpha, 3\alpha, \ldots$ have pairwise rational distances (see Figure 1). It also can be proved that no period occurs and that this set of points is dense on the unit circle.

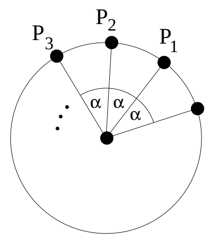

Figure 1.

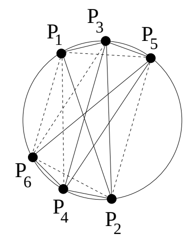

Figure 2.

The second construction starts with an integral triangle, say (1,2,2). On its circumcircle this triangle is used to find points $P_4, P_5, \ldots$ on the circle as indicated in Figure 2. Because of Ptolemy's theorem, where for a circular quadrangle the sum of the products of opposite pairs of sides equals the product of the diagonals, all additional distances easily are seen to be rational.

For higher dimensions add to a circular integral point set with radius r an integral $(d-2)$–dimensional tetrahedron of side length a such that a and r are legs of a pythagorean triangle.

What about the smallest examples of n points with integral distances, that is, where the largest distance, the diameter $D(d,n)$, is a minimum? In the plane the values $D(2,n) = 1, 4, 7, 8, 17, 21, 29$ are known for $3 \leq n \leq 9$ (see Figure 3 and [12]).

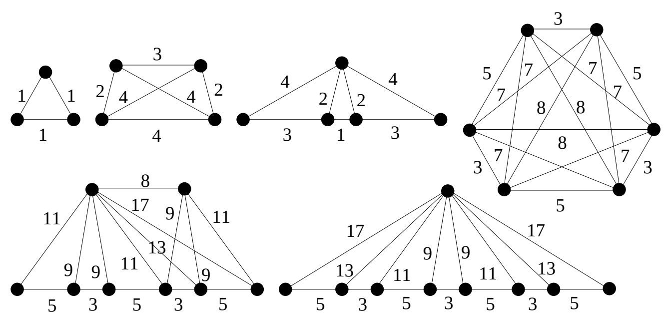

Figure 3.

In general, it holds $D(2,n) \leq n^{c \log \log n}$ for a constant c [14, 9]. If three points are not allowed to be collinear then the minimum distances $D_1(2,n) = 1, 4, 8, 8, 33, 56, 56$ are known for $3 \leq n \leq 9$ (see Figure 4) and $D_1(2,n) = 15$ is conjectured for $n = 10, 11, 12$.

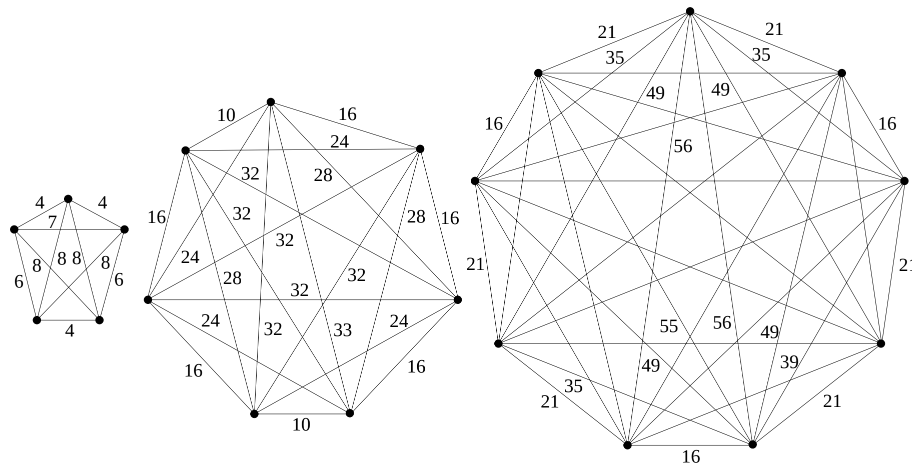

Figure 4.

For $d = 3$ the values $D(3, n) = 1, 3, 4, 8, 13, 17, 17$ for $4 \leq n \leq 10$ are known. The example for $n = 5$ in Figure 5 is a double pyramid. It may be asked whether this simple pyramid with height 1, edge lengths 2, and basis lengths 3 already was known to the Pythagoreans?

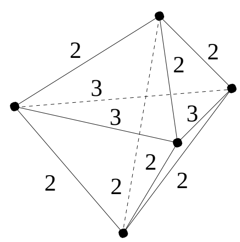

Figure 5.

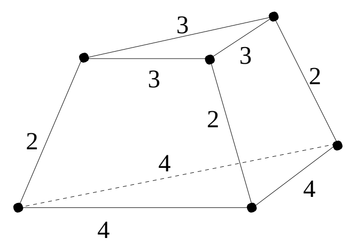

Figure 6.

For small numbers af points the unit tetrahedron or simplex proves $D(d, d+1) = 1$. Are $d + 2$ points in R^d possible such that only distances 1 and 2 occur? The answer is in the negative since $D(d, d + 2) \geq 3$ was proved in [13]. Two tetrahedra with side lengths 3 and 4 where corresponding vertices have distance 2 (see Figure 6 for $d = 3$) prove $D(d, n) \leq 4$ for $n \leq 2d$ and $D(d, 2d) = 4$ is known. Then for $d + 2$ points only $D(d, d + 2) = 3$ or 4 is in question. Up to $d = 10$ only for $d = 3, 6, 8$ the value 3 is attained for $D(d, d + 2)$. It is unknown whether there exist further dimensions d with $D(d, d + 2) = 3$.

It is an open problem of Ulam [24] whether other dense sets of points in the plane with pairwise rational distances do exist besides circular lines. In this context a nice result of Anning and Erdös [2] can be mentioned: Infinitely many points with pairwise integral distances are necessarily on a straight line. This leads to the question for the existence of n points, no $d+1$ in a hyperplane, no $d+2$ on a d–sphere, and all distances being integers.

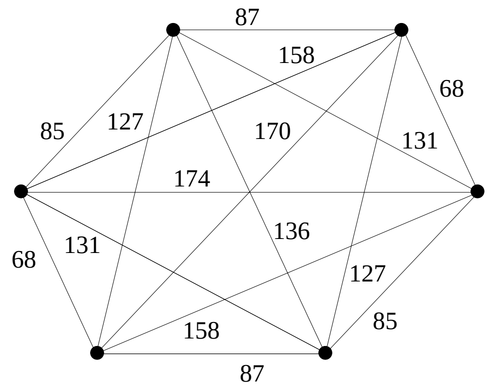

Figure 7.

For $d = 2$ integral point sets, no three in line, no four on a circle, are known for $n = 4, 5, 6$ with minimum diameters $D_2(2, n) = 8, 73, 174$ (see Figure 7 for $n = 6$ and [12, 19]). For $d = 3$ only $D_2(3, 5) = 3$ and $D_2(3, 6) = 16$ are known (see Figure 8 for $n = 6$).

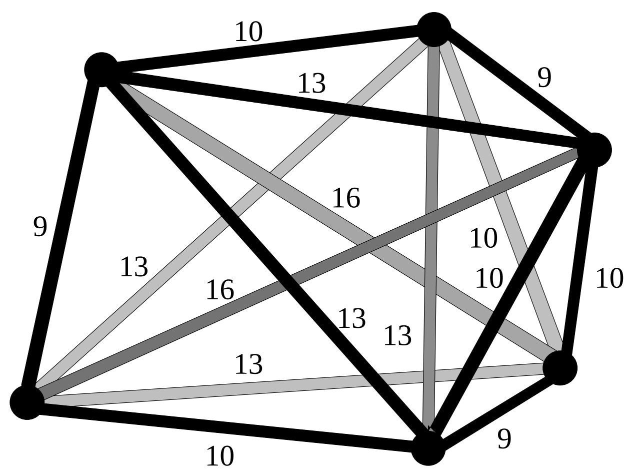

Figure 8.

This chapter may be finished with the result of Almering [1] that a dense set of points exists in the plane which have integral distances to the vertices of a given integral triangle, and with the open question of Besicovitch [3] whether every plane n–gon for $n \geq 5$ can be approximated by a rational n–gon.

3. Unit distances

Many distance problems only ask for one distance which then may be the unit. For the maximum number $f(n)$ of equal distances among n points in the plane it is only known $c_1 n^{1+\frac{c_2}{\log\log n}} < f(n) < c_3 n^{\frac{4}{3}}$. Extremal examples for $6 \leq n \leq 11$ are given in Figure 9.

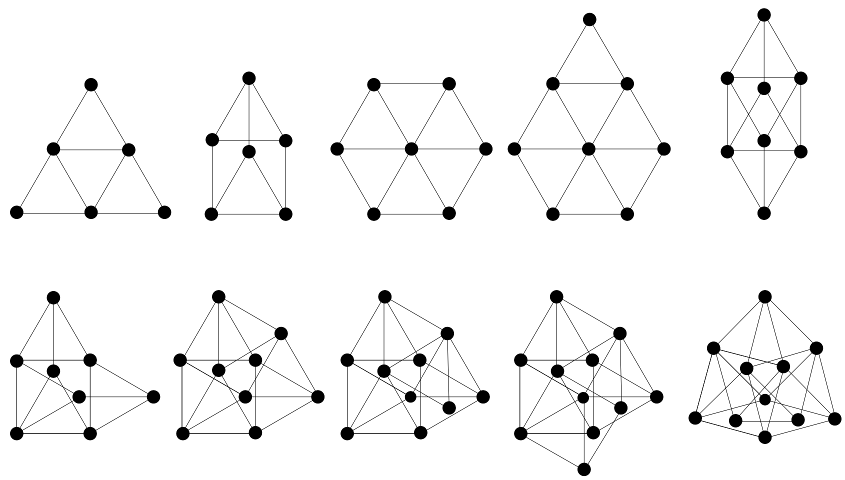

Figure 9.

The maximum numbers $M_1(n)$ and $M_2(n)$ of the largest and second largest distances among n points in the plane are known to be $M_1(n) = n$ and $M_2(n) = \lfloor \frac{3n}{2} \rfloor$. For the maximum numbers $m_1(n)$ and $m_2(n)$ of the smallest and second smallest distances among n points in the plane $m_1(n) = \lfloor 3n - \sqrt{12n-3} \rfloor$ and $m_2(n) \approx \frac{24n}{7}$ are known. References are given in [21].

In k–regular sets in the plane each of n points has distance one to exactly k other of the n points. If $p(k)$ denotes the minimum number of points of such a k–regular set then a projection of the k–dimensional unit cube proves $p(k) \leq 2^k$. For $k = 1, 2, 3, 4, 5$ the values $p(k) = 2, 3, 6, 9, 18$ are known (see Figure 10 for k=4) [8].

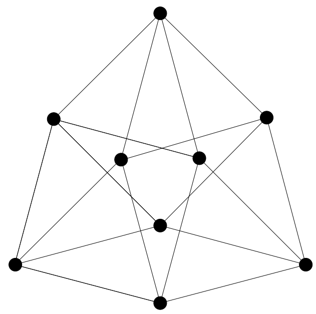

Figure 10.

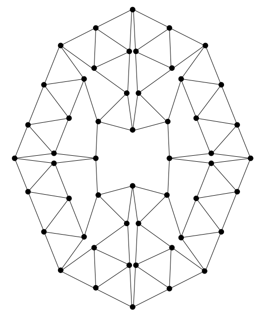

Figure 11.

If k–regular point sets of n points in the plane are considered where only one of two intersecting distances one are allowed then the minimum number $p_1(k)$ of points does not exist for $n \geq 5$, and $p_1(1) = 2$, $p_1(2) = 3$, $p_1(3) = 8$ are known [12]. The remaining case $p_1(4) \leq 52$ remains open (Figure 11).

In general, one can ask for the dimension of a graph G, that is, the smallest dimension such that a realization of G with unit distances for the edges of G is possible.

4. Different integral distances

Every graph can be represented by points for vertices such that adjacent vertices have integer distances since corresponding edges of an integral representation of the complete graph K_n can be deleted. It is, however, an open problem of Maehara whether only for edges of a graph integral distances can occur and all other distances are irrational. In the following four special problems are mentioned.

If only odd distances are considered then the maximum number of points with pairwise odd distances is $d+2$ if $d+2 \equiv 0 \pmod{16}$ [6]. Four points can have at most five odd distances. In [23] it is proved that n points can determine at most $\frac{n^3}{3} + \frac{r(r-3)}{6}$ odd distances, where $r = 1, 2, 3$ for $n \equiv r \pmod 5$.

Fibonacci triangles have Fibonacci numbers F_i as side lengths and an integral area. Only one Fibonacci triangle (5,5,8) is known so far [11, 16]. Because of $F_n = F_{n-1} + F_{n-2}$ only equalateral triangles are possible. Triangles (F_n, F_{n-1}, F_{n-1}) are impossible for $n \geq 7$ and (F_{n-k}, F_n, F_n) are impossible for $n \geq 5$. Do further Fibonacci triangles exist?

Two sequences of integral pentagons are defined by the use of Fibonacci numbers F_i (see Figures 12 and 13). Moreover, all segments of the diagonals are rational,

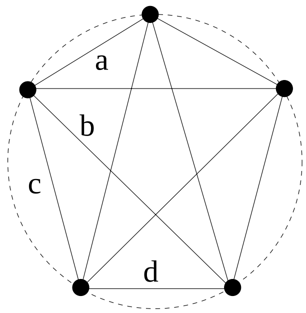

Figure 12.
$a = F_{n-1}(2F_n^2 - F_{n-1}^2)$, $b = F_n^3$,
$c = F_n(F_n^2 - F_{n-1}^2)$,
$d = F_{n-1}F_n^2$.

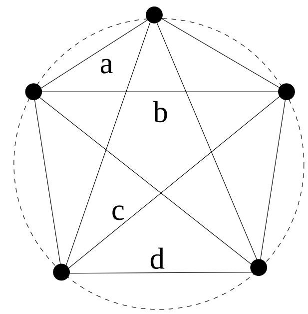

Figure 13. $a = F_n^3$, $b = F_{n+1}F_n^2$,
$c = F_n(F_{n+1}^2 - F_n^2)$,
$d = F_{n+1}(F_{n+1}^2 - 2F_n^2)$.

however, only for $n = 5$ in Figure 13 the areas of the pentagon and its central pentagon both are rational [18].

Combinatorial perfect boxes are constructed in [22], that are three opposite pairs of integral plane quadrilaterals which form a body with sides, face diagonals and body diagonals of integer length. The plane net of the smallest example in [22] is shown in

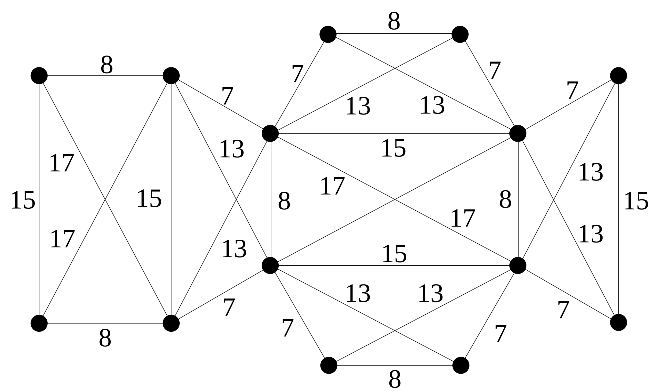

Figure 14.

Figure 14. That no smaller examples (with distances ≤ 17) occur is proved in [17]. It remains unknown whether infinitely many combinatorial perfect boxes do exist?

5. Platonic solid graphs

Planar graphs can be drawn in the plane without intersections of its edges. By results of Steinitz, Wagner or Fáry [25] this also is possible with straight line segments for all edges. Moreover, it is conjectured that the straight line segments can be chosen of integer lengths [15]. Although a general proof is missing for special graphs G the smallest diameter $D(G)$, that is, the smallest largest distance, of an integral representation of G can be asked for.

For the five platonic solid graphs, tetrahedron (T), cube (C), octahedron (O), dodecahedron (D), and icosahedron (I) these minimum diameters are $D(T) = 17$, $D(C) = 2$, $D(O) = 13$, $D(D) = 2$, and $D(I) = 159$ (see [10] and Figures 15 to 19).

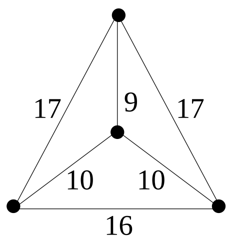

Figure 15.

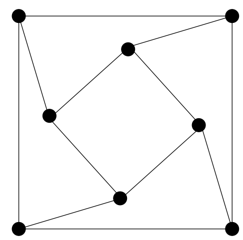

Figure 16.

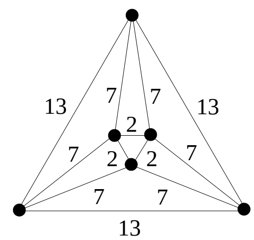

Figure 17.

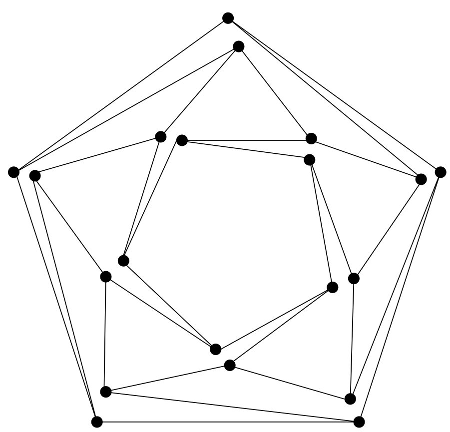

Figure 18.

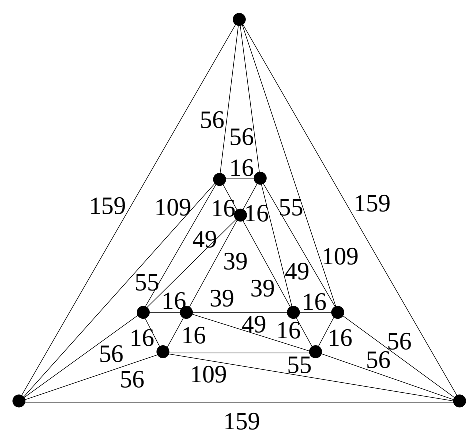

Figure 19.

If intersections of the edges are allowed in plane integral straight line representations then the minimum diameters $D_x(G)$ are $D_x(T) = 4$, $D_x(C) = 1$, $D_x(O) = 7$, $D_x(D) = 1$, and $D_x(I) = 8$ (see [15] Figures 20 to 24).

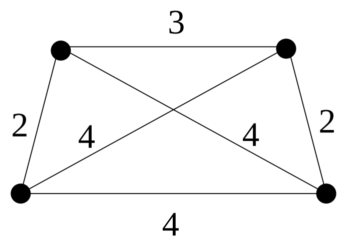

Figure 20.

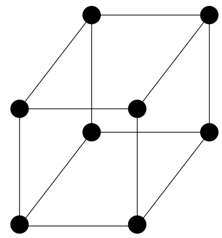

Figure 21.

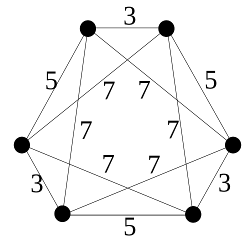

Figure 22.

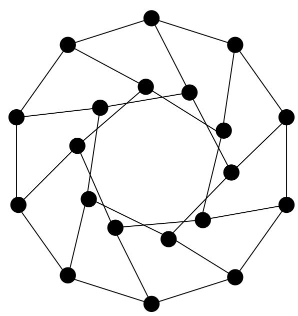

Figure 23.

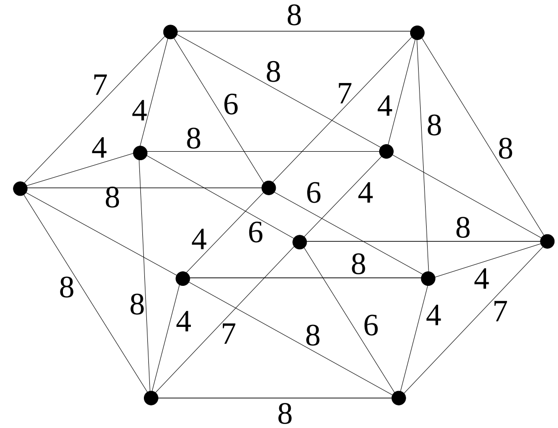

Figure 24.

Many other graphs wait for their integral realizations in the plane or in higher dimensions. Another problem arises if the number of different integral distances is restricted. The tetrahedron, for example, can be planar realized with only three different distances as in Figure 25.

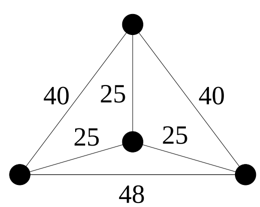

Figure 25.

Although problems on integral distances in point sets are discussed since centuries and many applications are imaginable only few results are known. One reason for this fact could be that difficult systems of diophantine equations are weaved into geometrical problems.

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