Deformation techniques for counting the real solutions of specific polynomial equation systems*

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Abstract. We present a deformation technique which allows us to determine the number of real solutions of certain specific polynomial equation system. We apply this deformation technique in order to determine the number of stationary solutions of the semidiscretization of certain parabolic differential equations.

1 Introduction.

Multivariate real polynomial equation systems arise in connection with numerous scientific and technical problems (see e.g. [OR70] or [Mor90]). In order to solve such problems, one is usually led to consider the following two questions:

(*i*) Do solutions exist in a specified subset of $\mathbb{R}^n$?
(*ii*) How many solutions are there in such a set?

Usual numerical techniques cannot be used in order to answer questions (*i*) and (*ii*). Therefore, symbolic or seminumerical methods must be applied.

Symbolic methods based on rewriting techniques (see e.g. [GVRRT99]) usually rely on the computation of the signature of a suitable quadratic form (see e.g. [PRS93]), or reduce the question to the univariate case, by computing suitable projections (see e.g. [Rou97]). Unfortunately, the corresponding procedures require at least exponential time (see [Giu84], [Dub90]), and are therefore unsuitable for real–life purposes.

One may alternatively apply more efficient seminumerical procedures, based on geometric elimination procedures (see [BGHM97], [BGHM01]). The time complexity of these algorithms is polynomial in a certain geometric invariant of the input system, called its *degree*. Let us observe that the degree of the input system is always bounded by the Bézout–number of the input system and happens often

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to be considerably smaller. We call a system S *ill–conditioned*, from the point of view of symbolic or seminumeric solving, if the degree of the system S is close to Bézout number of the system S. Ill–conditioned systems in this sense are the system which cannot be efficiently solved by the procedures in [BGHM97] and [BGHM01].

The main purpose of this article is to exhibit a deformation technique for the determination of the number of real solutions of particular ill–conditioned polynomial equation systems. The systems we have in mind come from the discretization of operator equations (see e.g. [Rhe98]).

The computational solution of infinite–dimensional operator equations generally requires the construction of finite–dimensional approximations. It is reasonable to expect that these discretizations inherit any nonlinearities existent in the original operators. Here we consider a source for multivariate polynomial equations.

Let f, g, h denote C^∞ univariate functions, and let us consider the following problem in $[0,1] \times [0,T) \subset \mathbb{R}^2$:

$$\begin{cases} u_t = f(u)_{xx} - \lambda g(u) & \text{in } (0,1) \times [0,T), \\ f(u)_x(1,t) = h(u(1,t)) & \text{on } [0,T), \\ f(u)_x(0,t) = 0 & \text{on } [0,T), \\ u(x,0) = u_0(x) \geq 0 & \text{in } [0,1]. \end{cases}$$

These kind of equations appear in problems involving reaction terms in the boundary and absorption terms in the equation (see e.g. [Pao92]). In particular, the long time behaviour of these kind of equations has been intensively analyzed (see e.g. [CFQ91], [FF96], [Lev90], [SGKM95]). Usual numerical approximations of these problems consist in a first order finite element approximation on the variable x, with a uniform mesh, keeping the variable t continuous (see e.g. [BB98], [BHR96]). This semidiscretization in space leads to the following system:

$$\begin{cases} u_1' = 2n^2\big(f(u_2) - f(u_1)\big) - \lambda g(u_1), & \\ u_k' = n^2\big(f(u_{k+1}) - 2f(u_k) + f(u_{k-1})\big) - \lambda g(u_k) & (2 \leq k \leq n-1), \\ u_n' = 2n^2\big(f(u_{n-1}) - f(u_n)\big) - \lambda g(u_n) + 2n\, h(u_n), & \\ u_k(0) = u_0(x_k) & (1 \leq k \leq n), \end{cases} \tag{1}$$

where $x_1, \ldots, x_n$ define a uniform partition of the interval [0,1].

In order to describe the dynamic behaviour of the solutions of system (1) (e.g. for deciding whether long–time behaviour of the discrete solutions agrees with that of the corresponding continuous solutions), it is usually necessary to analyze the behaviour of the *stationary solutions* of system (1) (see e.g. [BR01], [FGR02], [GMW93]). These are the *positive* solutions of the following nonlinear equation system:

$$\begin{cases} 0 = 2n^2\big(f(u_2) - f(u_1)\big) - \lambda g(u_1), \\ \\ 0 = n^2\big(f(u_{k+1}) - 2f(u_k) + f(u_{k-1})\big) - \lambda g(u_k) \qquad (2 \leq k \leq n-1), \\ \\ 0 = 2n^2\big(f(u_{n-1}) - f(u_n)\big) - \lambda g(u_n) + 2n\, h(u_n), \end{cases} \quad (2)$$

Typical cases of study are $f(X) := X$, $g(X) := X^p$ and $h(X) := X^q$ (see [BR01], [CFQ91]).

Let us observe that (2)–like systems are usually ill–conditioned in the sense defined above, and therefore unsuitable for the application of the seminumerical procedures in [BGHM97] and [BGHM01].

In this article, we are going to exhibit symbolic *deformation techniques* which will allow us to determine the number of real positive solutions of certain (2)–like systems of interest. These deformation techniques can be roughly described as follows.

Suppose that there exists a *polynomial homotopy* $\Phi : [0,1] \to \mathcal{P}(\mathbb{R}^n)$ between the set of positive solutions $V \subset \mathbb{R}^n$ of the input system, and the (finite) set of positive solutions $W \subset \mathbb{R}^n$ of a suitably chosen n–variate polynomial equation system. Suppose furthermore that for any $t \in [0,1]$, the set $\Phi(t) \subset \mathbb{R}^n$ can be described as the solution set of an n–variate polynomial equation system *without singular solutions*. Then we will be able to conclude that the set V and W have the same number of elements. Therefore, knowing the number of elements of the set W will allow us to determine the number of elements of the set V *without explicitely computing it*.

2 Notions and Notations.

First we recall some standard notions and notations of commutative algebra and semi-algebraic geometry that we are going to use in the sequel.

We denote by $\mathbb{N}$ the set of natural numbers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{R}$ denotes the field of real numbers and $\mathbb{C}$ denotes the field of complex numbers. Let $X_1, \ldots, X_n$ be indeterminates over $\mathbb{Q}$. We denote by $\mathbb{Q}[X_1, \ldots, X_n]$ the ring of n–variate polynomials with rational coefficients in the indeterminates $X_1, \ldots, X_n$.

A set $V \subset \mathbb{R}^n$ is called an *algebraic set* if there exists a finite set of polynomials $f_1, \ldots, f_m \in \mathbb{Q}[X_1, \ldots, X_n]$ such that $V = \{x := (x_1, \ldots, x_n) \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0\}$ holds. We define the *ideal* $I(V)$ of an algebraic set $V \subset \mathbb{R}^n$ as the set of all polynomials $f \in \mathbb{Q}[X_1, \ldots, X_n]$ which vanish on any point of the set V. A set $V \subset \mathbb{R}^n$ is called a *semi-algebraic set* of $\mathbb{R}^n$ if it can be expressed as a finite union of subsets of $\mathbb{R}^n$ of the form $\{x := (x_1, \ldots, x_n) \in \mathbb{R}^n : f_1(x) = \cdots = f_m(x) = 0, f_{m+1}(x) > 0, \ldots, f_{m+k}(x) > 0\}$ with $f_1, \ldots, f_{m+k} \in \mathbb{Q}[X_1, \ldots, X_n]$.

An algebraic set $V \subset \mathbb{R}^n$ is called *irreducible* if it cannot be expressed as an irredundant decomposition $V = V_1 \cup V_2$, where V_1, V_2 are algebraic sets. Any algebraic set has a unique (up to reordering) irredundant decomposition as the

union of a finite set of irreducible algebraic sets $V = C_1 \cup \cdots \cup C_h$ (see e.g. [BCR98]). $C_1, \ldots, C_h$ are called the *irreducible components* of V.

For a given algebraic set $V \subset \mathbb{R}^n$, we define the *coordinate ring* $\mathbb{Q}[V]$ of V as the quotient ring $\mathbb{Q}[V] := \mathbb{Q}[X_1, \ldots, X_n]/I(V)$. If V is an irreducible algebraic set, the *dimension* of V is defined as the trascendence degree of the field extension $\mathbb{Q} \hookrightarrow \mathbb{Q}(V)$, where $\mathbb{Q}(V)$ denotes the fraction field of $\mathbb{Q}[V]$. The dimension of an arbitrary algebraic set $V \subset \mathbb{R}^n$ is defined as the maximum of the dimensions of all its irreducible components. An algebraic set is called *equidimensional* if all its irreducible components have the same dimension. An equidimensional algebraic set $C \subset \mathbb{R}^n$ of dimension 1 is called a *algebraic curve*.

In what follows, $\mathbb{R}^n$ will be considered endowed with its euclidean topology, unless otherwise stated. A semi-algebraic set $V \subset \mathbb{R}^n$ is called *semi-algebraically connected* if for any pair of semi-algebraic sets $C_1, C_2 \subset \mathbb{R}^n$ closed in V, which are disjoint and satisfy $C_1 \cup C_2 = V$, one has $V = C_1$ or $V = C_2$. Every semi-algebraic subset $V \subset \mathbb{R}^n$ can be decomposed in a unique way (up to reordering) into a disjoint union of a finite number of semi-algebraically connected semi-algebraic sets $C_1, \ldots, C_s$ which are both open and closed in V (see e.g. [BCR98]). The sets $C_1, \ldots, C_s$ are called the *semi-algebraic connected components* of V.

Let $V \subset \mathbb{R}^n$ be an algebraic set and let $I(V) = (f_1, \ldots, f_m)$ the ideal of V. Let $x \in V$. The *tangent space* $T_x(V)$ of V at the point x is defined as the linear subspace of $\mathbb{R}^n$ defined by $T_x(V) := \{z \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i}(x) z_i = 0 \text{ for } 1 \leq j \leq m\}$. A point $x \in V$ of an irreducible algebraic set $V \in \mathbb{R}^n$ is called *nonsingular* if the condition $dim\big(T_x(V)\big) = dim(V)$ is satisfied. An algebraic set is called nonsingular if all its points are nonsingular points.

Let $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ two semi–algebraic sets. A mapping $f : V \to W$ is a *semi-algebraic mapping* iff its graph is a semi-algebraic set of $\mathbb{R}^{n+m}$. In particular, $\mathbb{Q}$–definable polynomial mappings are semi-algebraic (see e.g. [BCR98]). Let $x \in V$ and let $y := f(x)$. The *differential mapping* of the mapping f at the point $x \in V$ is the linear mapping $d_x f : T_x(V) \to T_y(W)$ induced by the Jacobian matrix $J(f)(x)$. The point $x \in V$ is a *critical point* of f if the rank of the differential mapping $d_x f : T_x(V) \to T_y(W)$ is smaller than the dimension of V. A *critical value* of f is the image by f of a critical point.

Any polynomial mapping $f : V \to W$ between two algebraic sets $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ induces a ring homomorphism $f^* : \mathbb{Q}[W] \to \mathbb{Q}[V]$. A polynomial mapping $f : V \to W$ is called *finite* if the corresponding ring homomorphism $f^* : \mathbb{Q}[W] \to \mathbb{Q}[V]$ is injective and defines an integral extension of ring, i.e. if any element $f \in \mathbb{Q}[V]$ satisfies a monic polynomial equation with coefficients in $\mathbb{Q}[W]$. If a polynomial mapping $f : V \to W$ is finite, then the *fiber* $f^{-1}(y)$ of any point $y \in W$ is a finite set (see e.g. [Sha94]).

3 A Deformation Technique.

Let $\widetilde{f}_1, \ldots, \widetilde{f}_n$ be polynomials of $\mathbb{Q}[X_1, \ldots, X_n]$, and let $V \subset \mathbb{R}^n$ be the algebraic set defined by the polynomials $\widetilde{f}_1, \ldots, \widetilde{f}_n$. Let T be a new indeterminate. Suppose

that we are given polynomials $f_1, \ldots, f_n \in \mathbb{Q}[X_1, \ldots, X_n, T]$ such that the condition $\{x \in \mathbb{R}^n : f_1(x,0) = \cdots f_n(x,0) = 0\} = V$ holds. Let $W \subset \mathbb{R}^{n+1}$ be the algebraic set defined by $f_1, \ldots, f_n$. Let $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $\pi : W \subset \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be the semi-algebraic mappings defined by $f(x,t) := \big(f_0(x,t), \ldots, f_n(x,t)\big)$ and $\pi(x,t) := t$ respectively. Suppose that the condition $\#\big(\pi^{-1}(t)\big) < \infty$ holds. In this section we are going to exhibit a condition on the algebraic sets V and W which implies the identity $\#V = \#\big(\pi^{-1}(t)\big)$ for any $t \in [0,1]$.

Let $J_{(X,T)}(F) \in \mathbb{Q}[X_1, \ldots, X_n, T]^{n \times n+1}$ be the Jacobian matrix of the polynomials $f_1, \ldots, f_n$ with respect to the indeterminates $X_1, \ldots, X_n, T$. Let $J_X(F)$ denote the $(n \times n)$–submatrix of $J_{(X,T)}(F)$ consisting of the first n columns of $J_{(X,T)}(F)$. We start with a simple sufficient criterion to decide if a given point $(x,t) \in W$ is not a critical point of the mapping π.

Lemma 1. *Let $(x_0, t_0) \in W$ be a point such that the condition*

$$\det\big(J_X(F)\big)(x_0, t_0) \neq 0$$

holds. Then (x_0, t_0) is not a critical point of the mapping π.

Proof. By the Implicit Function Theorem (see e.g. [BCR98, Corollary 2.9.8]), there exist open neighborhoods $U_0 \subset \mathbb{R}^n$ of x_0 and U_1 of t_0, and a unique differentiable mapping $g : U_1 \to U_0$ such that for any $t \in U_1$ the conditions $\big(g(t), t\big) \in W$ and $\det\big(J_X(F)\big)\big(g(t), t\big) \neq 0$ hold. In particular, $g(t_0) = x_0$.

Since $f\big(g(t), t\big) = 0$ holds for any $t \in U_1$, from the Chain Rule we deduce the identity

$$J_{(X,T)}(F)\big(x_0, t_0\big) \cdot \big(g'(t_0), 1\big)^t = (0, \ldots, 0)^t. \tag{3}$$

Let $v \in \mathbb{R}^{n+1}$ be a nonzero tangent vector of the variety W at the point (x_0, t_0). Then v satisfies the identity

$$J_{(X,T)}(F)\big(x_0, t_0\big) \cdot v^t = (0, \ldots, 0)^t. \tag{4}$$

Since the matrix $J_{(X,T)}(F)\big(x_0, t_0\big)$ has rank n, from identities (3) and (4) we conclude that there exists a nonzero real number λ such that $v = \lambda\big(g'(t_0), 1\big)$ holds. Therefore, v is the tangent vector of a curve $C \subset W$, parametrized by the mapping $\widetilde{g} : \lambda^{-1}U_1 \to W$ defined by $\widetilde{g}(t) := \big(g(\lambda t), \lambda t\big)$, at the point t_0 . Since $(\pi \circ \widetilde{g})'(t_0) = \lambda$, we have λ belongs to the image of the mapping $d_{(x_0,t_0)}\pi$. As $T_{t_0}\mathbb{R}$ is an $\mathbb{R}$–dimensional vector space of dimension 1, we conclude that that the mapping $d_{(x_0,t_0)}\pi : T_{(x_0,t_0)}W \to T_{t_0}\mathbb{R}$ is surjective. This implies that the point (x_0, t_0) is not a critical point of the mapping π. □

In the next two lemmas we adapt the contents of [HRS91, Lemma 7 and Proposition 8] to our context.

Lemma 2. *Let $t \in [0,1]$ be a non–critical value of the mapping π, such that the fiber $\pi^{-1}(t)$ is non–empty and finite. Then there exists $\varepsilon > 0$ such that the semi-algebraically connected components $C_1, \ldots, C_s$ of the semi-algebraic set $\pi^{-1}(t-\varepsilon, t+\varepsilon)$ satisfy the following condition: for any $t' \in (t-\varepsilon, t+\varepsilon)$, the sets $C_1 \cap \pi^{-1}(t'), \ldots, C_s \cap \pi^{-1}(t')$ are nonempty and semi-algebraically connected.*

Proof. Let $\pi^{-1}(t) = \{(x^{(1)}, t), \ldots, (x^{(s)}, t)\}$. Following the proof of Lemma 1 we deduce that the algebraic set can be locally parametrized, in a neighborhood of the point $(x^{(j)}, t)$, by a differentiable curve for $1 \leq j \leq s$. Therefore, by considering the restriction of the mapping π to a set $W \cap \mathbb{R}^n \times [t - \widetilde{\varepsilon}, t + \widetilde{\varepsilon}]$ for $\widetilde{\varepsilon} > 0$ sufficiently small, we may assume without loss of generality that W is a compact subset of $\mathbb{R}^{n+1}$.

Since (x, t) is not a critical point of the mapping π, there exists $\varepsilon^* > 0$ such that the set $\pi^{-1}(t - \varepsilon^*, t + \varepsilon^*)$ contains no critical point of the mapping π. Therefore, proceeding as in the proof of Lemma 1, we deduce that for any point $(x, t') \in \pi^{-1}(t - \varepsilon^*, t + \varepsilon^*)$ there exists an open neighborhood $U_{(x,t')} \subset W$ of (x, t'), and a positive real number $\varepsilon_{(x,t')}$, such that $\pi : W \cap U_{(x,t')} \to \big(t' - \varepsilon_{(x,t')}, t' + \varepsilon_{(x,t')}\big)$ is an homeomorphism. In particular, we infer that for any $t' \in (t - \varepsilon^*, t + \varepsilon^*)$, the set $\pi^{-1}(t') \cap U_{(x,t')}$ is connected. Since the set $\pi^{-1}[t - \frac{\varepsilon^*}{2}, t + \frac{\varepsilon^*}{2}]$ is compact, there exist finitely many points $(x^{(1)}, t_1), \ldots, (x^{(s)}, t_s) \in \pi^{-1}[t - \frac{\varepsilon^*}{2}, t + \frac{\varepsilon^*}{2}]$ such that the open sets $U_{(x^{(1)},t_1)}, \ldots, U_{(x^{(s)},t_s)}$ cover the set $\pi^{-1}[t - \frac{\varepsilon^*}{2}, t + \frac{\varepsilon^*}{2}]$. Let $\varepsilon > 0$ satisfy $\varepsilon < \{\frac{\varepsilon^*}{2}, \varepsilon_{x_1}, \ldots, \varepsilon_{x_s}\}$. Therefore for any $t' \in (t - \varepsilon, t + \varepsilon)$ the set $\pi^{-1}(t')$ is contained in the set $U_{(x^{(1)},t_1)} \cup \ldots \cup U_{(x^{(s)},t_s)}$. Taking $C_j := U_{(x^{(j)},t_j)} \cap \pi^{-1}((t - \varepsilon, t + \varepsilon))$ for $1 \leq j \leq s$, the statement of Lemma 2 follows. □

Lemma 3. *Let $t_1 < t_2$ be two elements of $[0, 1]$ such that the interval (t_1, t_2) does not contain any critical value of the mapping π, and the fiber $\pi^{-1}(t)$ of any point $t \in (t_1, t_2)$ is finite. Let $C_1, \ldots, C_s$ be the semi-algebraically connected components of semi-algebraic set $\pi^{-1}(t_1, t_2)$. Then for any $t \in (t_1, t_2)$, the sets $C_1 \cap \pi^{-1}(t), \ldots, C_s \cap \pi^{-1}(t)$ are the semi-algebraically connected components of the algebraic set $\pi^{-1}(t)$. In particular, the number of semi-algebraically connected components remains constant when t ranges over the interval (t_1, t_2).*

Proof. First, let us consider arbitrary real numbers t_1', t_2' such that $t_1 < t_1' < t_2' < t_2$. Arguing in a similar way as at the beginning of the proof of Lemma 2, we deduce that the semi-algebraic set $\pi^{-1}([t_1', t_2'])$ is compact. Let $C_1', \ldots, C_{s'}'$ be the connected components of the set $\pi^{-1}([t_1', t_2'])$. We claim that for any $t \in [t_1', t_2']$ the sets $C_1' \cap \pi^{-1}(t), \ldots, C_{s'}' \cap \pi^{-1}(t)$ are the semi-algebraic connected components of the algebraic set $\pi^{-1}(t)$. In order to prove this assertion, we observe that the interval $[t_1', t_2']$ can be covered by finitely many open intervals with the properties stated in Lemma 2. These intervals can be arranged in a chain of successively overlapping members. Looking at the common values contained in any two of these overlapping intervals we infer our claim just by gluing connected sets together.

Now, the statement of Lemma 3 follows easily from our claim, by considering any ascending chain of closed intervals $[t_1', t_2'], [t_1'', t_2''], \ldots$ covering the interval (t_1, t_2). □

Now we can proceed to prove the main result of this section. This result asserts that, under the hypothesis that the mapping π does not have critical values in the interval [0,1], the number of points of the set $\pi^{-1}(t)$ remains constant

when t ranges over the interval [0,1]. In particular, the sets $\pi^{-1}(0)$ and $\pi^{-1}(1)$ have the same cardinality.

Theorem 1. *Let notations and assumptions be as above. Suppose that the mapping $\pi : W \to \mathbb{R}$ has non critical values in the interval [0,1], and has a finite fiber $\pi^{-1}(t)$ for any $t \in [0,1]$. Then there exists a positive integer s such that $\#\big(\pi^{-1}(t)\big) = s$ holds for any $t \in [0,1]$.*

Proof. Applying Lemma 1 for $t = 0, 1$ we deduce that there exists natural numbers s, s' and $0 < \varepsilon < 1$ such the set $\pi^{-1}(t)$ has s connected components for any $t \in [0, \varepsilon)$, and the set $\pi^{-1}(t)$ has s' connected components for any $t \in (1-\varepsilon, 1]$. In particular, we have $\#\big(\pi^{-1}(0)\big) = s$ and $\#\big(\pi^{-1}(1)\big) = s'$. Applying Lemma 3 in the interval $(0,1)$, we conclude that $\#\big(\pi^{-1}(t)\big) = \#\big(\pi^{-1}(t')\big)$ for any $t, t' \in (0,1)$. This implies $s = s'$ and the statement of Theorem 1. □

4 An Application Case.

In this section we are going to apply the deformation technique described in the previous section in order to find the number of positive solutions of some specific polynomial equation systems.

Let $\alpha \in \mathbb{R}_{>0}$ and let $f, g \in \mathbb{Q}[X]$ be polynomials which define increasing mappings in $\mathbb{R}_{>0}$. Let us write $g = g_1 - g_2$, where $g_1, g_2 \in \mathbb{Q}[X]$, g_1 contains the monomials of g with positive coefficient, and $g_2 := g_1 - g$. For the sake of simplicity of proofs, we are going to assume furthermore that $\deg g_1 = \deg g > \deg f$ and that $g(0) = f(0) = 0$ hold (these last hypotheses are nonessential and may be removed using a somewhat more complicated argumentation). For example, this conditions are satisfied if f, g are monomials with positive coefficient such that $\deg g > \deg f$. We are going to consider the following sub-family of the family of systems (2):

$$\begin{cases} 0 = n^2\big(f(U_2) - f(U_1)\big) - \frac{1}{2}g(U_1), \\ 0 = n^2\big(f(U_{k+1}) - 2f(U_k) + f(U_{k-1})\big) - g(U_k) \qquad (2 \leq k \leq n-1), \\ 0 = n^2\big(f(U_{n-1}) - f(U_n)\big) - \frac{1}{2}g(U_n) + n\alpha, \end{cases} \tag{5}$$

Particular instances of this family of systems have already been considered in e.g. [CFQ91].

Let us denote by $V \subset \mathbb{R}^n$ the set of solutions of system (5). It is easy to see that V is a finite set. In order to apply the deformation technique of the previous section, we are going to define a suitable algebraic curve $W \subset \mathbb{R}^n \times \mathbb{R}$, with the

property that $W \cap \mathbb{R}^n \times \{1\} = V$. Let T be a new indeterminate, and let

$$\begin{cases} 0 = n^2\big(f(U_2) - f(U_1)\big) - \frac{1}{2}g(U_1), \\ 0 = n^2\big(f(U_{k+1}) - (1+T)f(U_k) + Tf(U_{k-1})\big) - g(U_k) \quad (2 \leq k \leq n-1), \\ 0 = n^2T\big(f(U_{n-1}) - f(U_n)\big) - \frac{1}{2}\big(g_1(U_n) - Tg_2(U_n)\big) + 2n\alpha. \end{cases} \tag{6}$$

Let $\pi : W \subset (\mathbb{R}_{>0})^n \times \mathbb{R} \to \mathbb{R}$ the mapping induced by the projection on the last coordinate. We are going to prove that all conditions required to apply Theorem 1 are satisfied. Then, we shall be able to conclude that $\#\big(V \cap (\mathbb{R}_{>0})^n\big) = \#\big(\pi^{-1}(1)\big) = \#\big(\pi^{-1}(0)\big)$. As the number $\#\big(\pi^{-1}(0)\big)$ can be directly determined, we will be able to determine the number of positive solution of system (5).

Let us denote by $f_1, \ldots, f_n \in \mathbb{Q}[U_1, \ldots, U_n, T]$ the polynomials defining system (6), and let $F := (f_1, \ldots, f_n)$. Let us observe that the Jacobian matrix $J_U(F)$ of the mapping F with respect to the indeterminates $U_1, \ldots, U_n$ is the following tridiagonal matrix:

$$J_U(F) :=$$

$$\begin{pmatrix} -n^2f'(U_1) - \frac{1}{2}g'(U_1) & n^2f'(U_2) & & \\ n^2Tf'(U_1) & -n^2(1+T)f'(U_2) - g'(U_2) & \ddots & \\ & \ddots & \ddots & n^2f'(U_{n-1}) \\ & & n^2Tf'(U_{n-1}) & h(U_n) \end{pmatrix},$$

where $h(U_n) := -n^2Tf'(U_n) - \frac{1}{2}\big(g_1'(U_n) - Tg_2'(U_n)\big)$. From the definition of the polynomials f, g we easily deduce that $J_U(F)(u,t)$ is a strictly diagonally dominant matrix of $\mathbb{R}^{n\times n}$ for any point $(u,t) \in \mathbb{R}^{n+1}$. Therefore, Lemma 1 implies that no solution $(u,t) \in \mathbb{R}^{n+1}$ of system (6) is a critical point of the mapping π. Hence no value $t \in [0,1]$ is a critical value of the mapping π.

Now we show that the fiber $\pi^{-1}(t)$ is a finite set for any $t \in [0,1]$.

Lemma 4. *For any $t \in [0,1]$, the fiber $\pi^{-1}(t)$ is a finite set.*

Proof. We claim that the homomorphism of rings $\pi^* : \mathbb{Q}[T] \to \mathbb{Q}[W]$ induces an integral extension ring. This is equivalent to prove that $\mathbb{Q}[W]$ is a finitely generated $\mathbb{Q}[T]$–module (see e.g. [Sha94]). Let $u_1, \ldots, u_n$ be the coordinate functions of $\mathbb{Q}[W]$ induced by the indeterminates $U_1, \ldots, U_n$. Then any of the equations defining system (6) induces an equation in $\mathbb{Q}[W]$ involving the coordinate functions $u_1, \ldots, u_n$. Considering the right–hand side members of the equations defining

system (6) as elements of the polynomial ring $\mathbb{Q}[T][U_1, \ldots, U_n]$, we observe that the highest degree term (in the variables $U_1, \ldots, U_n$) of the right–hand side member of the k–th equation of system (6) is a monomial $c_k U_k^{N_k}$ with $c_k \in \mathbb{Q}\setminus\{0\}$ and $N_k \in \mathbb{N}$ for $1 \leq k \leq n$. We conclude that $\mathbb{Q}[W]$ is generated, as $\mathbb{Q}[T]$–module, by the monomials $u_1^{j_1} \cdots u_n^{j_n}$ with $j_k < N_k$ for $1 \leq k \leq n$. In particular, $\mathbb{Q}[W]$ is a finitely generated $\mathbb{Q}[T]$–module, and the mapping π is a finite morphism. This implies that the fiber $\pi^{-1}(t)$ is a finite set for any $t \in [0, 1]$. □

Finally, we determine the number of positive real of solutions system (5). In fact, we have:

Theorem 2. *System (5) has exactly one solution in* $(\mathbb{R}_{>0})^n$.

Proof. Lemma 4 and the above remarks show that system (6) satisfies the hypotheses of Theorem 1. We conclude that $\#\big(\pi^{-1}(1)\big) = \#\big(\pi^{-1}(0)\big)$ holds. Therefore, in order to prove the statement of Theorem 2 there remains to show that $\#\big(\pi^{-1}(0)\big) = 1$ holds. Observe that $\pi^{-1}(0) = \widetilde{V} \times \{0\}$, where $\widetilde{V} \subset (\mathbb{R}_{>0})^n$ is the semi-algebraic set defined by the solutions $(u_1, \ldots, u_n) \in (\mathbb{R}_{>0})^n$ of the following polynomial equation system:

$$\begin{cases} 0 = n^2\big(f(U_2) - f(U_1)\big) - \frac{1}{2}g(U_1), & \\ 0 = n^2\big(f(U_{k+1}) - f(U_k)\big) - g(U_k) \qquad (2 \leq k \leq n-1), & (7) \\ 0 = n\alpha - \frac{1}{2}g_1(U_n). & \end{cases}$$

Since g_1 is an injective increasing function in $\mathbb{R}_{>0}$ with $g_1(0) = 0$, there exists a unique positive real solution u_n of the equation $n\alpha - \frac{1}{2}g_1(U_n) = 0$. Now we show that for any $1 \leq k \leq n-1$, there exist unique values $u_k, \ldots, u_n \in \mathbb{R}_{>0}$ satisfying the last $n-k+1$ equations of system (7). We argue by induction on $n-k$. Let $k < n$ and suppose that our statement holds for $k+1$. Therefore, there exist unique values $u_{k+1}, \ldots, u_n \in (\mathbb{R}_{>0})^n$ satisfying the last $n-k$ equations of system (7). Then any possible value $u_k \in \mathbb{R}_{>0}$ must satisfy the equation $n^2 f(u_{k+1}) = n^2 f(U_k) + g(U_k)$. Since the polynomial $p(U_k) := n^2 f(U_k) + g(U_k)$ defines an increasing mapping on $\mathbb{R}_{>0}$ and satisfies $p(0) = 0$, there exists a unique value $u_k \in \mathbb{R}_{>0}$ satisfying the equation $n^2 f(u_{k+1}) = p(U_k)$. This completes our inductive argument and shows that system (7) has exactly one solution in $(\mathbb{R}_{>0})^n$. □

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