CHAIN DECOMPOSITION AND THE FLAG f-VECTOR

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Dedicated to the memory of Rodica Simion.

ABSTRACT. Ehrenborg introduced a quasi-symmetric function encoding, denoted F_P, for the flag f-vector of any finite, graded poset P with $\hat{0}$ and $\hat{1}$. Stanley observed that F_P is a symmetric function whenever P is locally rank-symmetric and asked for conditions under which F_P is Schur-positive. We provide formulas for F_P for three classes of locally rank-symmetric posets: graded monoid posets, generalized posets of shuffles and noncrossing partition lattices for classical reflection groups. Our flag f-vector expressions for generalized shuffle posets and noncrossing partition lattices exhibit Schur-positivity, while graded monoid posets do not always have Schur-positive flag f-vector.

Each of our flag f-vector expressions results from a poset chain decomposition. For the noncrossing partition lattices and shuffle posets, these also yield symmetric chain decompositions (by restriction to 1-chains), shellability and supersolvability results and combinatorial formulae including characteristic polynomial and zeta polynomial. Our (more complicated) flag f-vector expression for graded monoid posets involves Gröbner bases and a weighted notion of Möbius function for the poset of partitions of a multiset and related multiset intersection posets.

1. INTRODUCTION.

This paper gives chain decompositions and consequent flag f-vector formulas in terms of symmetric functions for several classes of posets: the noncrossing partition lattices for classical reflection groups, monoid posets and two generalizations of posets of shuffles. Recall that the **flag f-vector** of a finite poset P of rank n is a function α_P counting chains of rank set S for each $S \subseteq \{1, \ldots, n-1\}$, so it refines the f-vector for the order complex of P. The **order complex** of P, denoted $\Delta(P)$, is the simplicial complex whose r-faces are the chains $\hat{0} < u_0 < \cdots < u_r < \hat{1}$ of comparable poset elements.

For each $S = \{r_1, \ldots, r_k\} \subseteq \{1, \ldots, n-1\}$, $\alpha_P(S)$ is defined to be the number of chains in P consisting of elements at exactly the ranks in S. Ehrenborg introduced a quasi-symmetric function encoding for the flag f-vector, denoted by F_P, in [Eh, p. 9]. Summing over multichains in P containing exactly one copy of $\hat{1}$ and at least one copy of $\hat{0}$, F_P is defined by

$$F_P = \sum_{\hat{0}=t_0 \leq t_1 \leq \cdots \leq t_{k-1} < t_k = \hat{1}} x_1^{\rho(t_0,t_1)} x_2^{\rho(t_1,t_2)} \cdots x_k^{\rho(t_{k-1},t_k)},$$

letting $\rho(x, y)$ denote $\mathrm{rk(y)} - \mathrm{rk(x)}$ and letting k range over the positive integers.

1991 *Mathematics Subject Classification.* 05E25,06A07,05E05.

This work was supported by a Hertz Foundation Graduate Fellowship.

Section 2 makes explicit how the different symmetric function bases account for the contribution to F_P of different types of collections of poset chains, described by their structure within the order complex. Most of the posets we study have EL-labellings in which each decreasing chain indexes a Boolean algebra within the original poset. The order complex of any such Boolean algebra is the Coxeter complex of type A. The Schur function s_λ and the power-sum symmetric function p_λ each account for the contribution to F_P of a region of the type A Coxeter complex specified by λ. This will allow us to obtain expressions for F_P for various posets P from decompositions of the order complex into regions which each contribute a Schur function or power-sum symmetric function.

Lemma 2.2 shows that each skew-Schur function $s_{\lambda/\mu}$ accounts for a patch of the type A Coxeter complex bounded by hyperplanes $a_i = a_j$. More specifically, the region is the restriction to the type A Coxeter complex (which is a sphere) of an intersection of open and closed half-spaces, specified by the inequalities on the entries of a semi-standard Young tableau of shape λ/μ which force them to increase weakly in rows and strictly in columns. Elementary and complete homogeneous symmetric functions are treated as special cases of Lemma 2.2. On the other hand, each power-sum symmetric function will account for the restriction to the sphere of a particular subspace which belongs to the type A Coxeter arrangement.

The remainder of the section applies our approach to the noncrossing partition lattice NC_4 as an indication of how F_P is computed by chain decomposition and how this leads to a variety of applications. Recall that a partition of the set $\{1, \ldots, n\}$ is noncrossing if the numbers $1, \ldots, n$ may be placed on a number line with arcs above the line connecting numbers in the same component in such a way that all the arcs are noncrossing; the non-crossing partition lattice NC_n is the poset of noncrossing partitions of $\{1, \ldots, n\}$ ordered by refinement. NC_4 decomposes into five Boolean algebras specified by the trees in Figure 1. We let the arcs connect the numbers $1, 2, 3, 4$, listed sequentially. Any subset of the arcs in one of these trees yields a

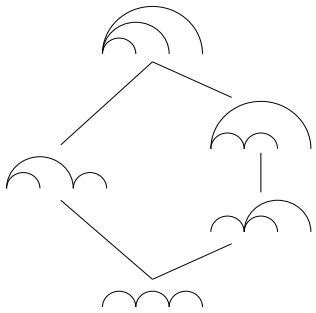


FIGURE 1. A partial order on Boolean algebras in NC_4

noncrossing partition; the noncrossing partitions that are achievable using only the arcs in a particular tree sit inside NC_4 as a Boolean algebra. We partially order the trees as in Figure 1 and then assign each chain to the earliest tree giving rise to it. For example, the chain $1|2|3|4 \to 13|2|4 \to 1234$ is assigned to the leftmost tree because this chain may be constructed using only the arcs in this tree, but cannot be constructed using only the arcs in the tree below it.

For NC_4, the chains in the earliest tree in Figure 1 contribute the elementary symmetric function e_1^3 to F_P while the three intermediate trees each add e_1e_2 to F_P and the last tree contributes e_3 to F_P. Summing these contributions yields $F_P = e_1^3 + 3e_1e_2 + e_3$. Stanley initiated an investigation of posets for which F_P is a symmetric function in [St2], and in particular he observed that F_P is a symmetric function whenever P is locally rank-symmetric; recall that a poset is **locally rank-symmetric** if each interval (u, v) is rank-symmetric, that is, if each interval has equal numbers of elements at rank r and at co-rank r for $0 \leq r \leq \mathrm{rk(v)} - \mathrm{rk(u)}$. The posets we study are all locally rank-symmetric. In Section 3, we recover flag f-vector formulas for the noncrossing partition lattices for classical reflection groups, by a chain decomposition into regions each contributing an elementary symmetric function to F_P. Section 4 uses skew-Schur functions of ribbon shape to count chains in generalized shuffle posets. Section 5 determines F_P for graded monoid posets in terms of power-sum symmetric functions, using Gröbner bases and weighted Möbius functions for the poset of partitions of a multiset and related posets.

Stanley conjectured in [St2, p. 6] that Schur-positivity of F_P for locally rank-symmetric posets should be related to Cohen-Macaulayness. He later observed in [St3, p. 5] that graded monoid posets can be shellable without having Schur-positive flag f-vector. One reason to be interested in when F_P is Schur-positive is the following observation of Stanley: whenever F_P is a symmetric function, the number of maximal chains in P is the dimension of the virtual symmetric group representation with F_P as Frobenius characteristic, so there could be a symmetric group action on maximal chains which has Frobenius characteristic equalling F_P or ωF_P. In [St2], Stanley introduced the notion of local symmmetric group actions on maximal chains, and then [SS] gave conditions under which local actions have Frobenius characteristic F_P or ωF_P. Our chain decompositions give constructive proofs of Schur-positivity for generalized shuffle posets and noncrossing partition lattices, which we hope provide some hint of what order complex structure translates to Schur-positivity.

Section 6 provides applications of the chain decompositions for generalized shuffle posets and noncrossing partition lattices, beginning with several combinatorial formulas. We then specialize chain decompositions to 1-chains to obtain symmetric chain decompositions and to maximal chains to obtain the orbits of local symmetric group actions. The chain decompositions also drew to our attention the existence of M-chains in the various types of shuffle posets and in the noncrossing partition lattices of type A, yielding short proofs of supersolvability in each of these cases. For example, the saturated chain $1|2|3|4 \to 12|3|4 \to 123|4 \to 1234$ turns out to be an M-chain in NC_4, essentially because it belongs to each of the boolean algebras shown in Figure 1. Figure 2 gives an example of a symmetric boolean decomposition for NC_4 which yields a symmetric chain decomposition. The noncrossing partitions assigned to any particular tree in a symmetric boolean decomposition are those given by all the solid arcs and any subset of the dashed arcs in that tree. Notice that the choice of which subset of the dashed arcs to include for each tree above determines a symmetrically placed boolean algebra.

2. Topological interpretations for symmetric function bases

Our choice of symmetric function basis for a particular class of posets will depend very much on the nature of the regions into which we decompose its order complex

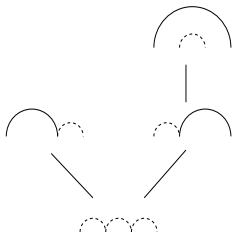


FIGURE 2. A symmetric boolean decomposition for NC_4

topologically, so let us first show how symmetric function bases naturally record the contribution to F_P by different types of regions in a type A Coxeter complex. Sections 2, 3 and 5 decompose several classes of posets into collections of chains each of which may be embedded via edge-labellings into Boolean lattices.

Definition 2.1. *Let $\mathcal{C}$ be a collection of chains in a poset P and let $\mathcal{M}$ be the set of multichains whose support are chains in C. Then $\mathcal{M}$* **contributes** *to F_P the quasi-symmetric function f if f is obtained by restricting F_P to a sum over multichains in $\mathcal{M}$.*

Let λ be a number partition into k parts. Recall that the complete homogeneous symmetric functions are defined by $h_n = \sum_{a_1 \leq \cdots \leq a_n} x_{a_1} \cdots x_{a_n}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$. Weak inequalities are replaced by strong ones in the elementary symmetric functions, where we have $e_n = \sum_{a_1 < \cdots < a_n} x_{a_1} \cdots x_{a_n}$ and $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$. The power-sum symmetric functions satisfy $p_n = \sum_{i>0} x_i^n$ and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$. We shall use the definition for the skew-Schur function $s_{\lambda/\mu}$ as the sum over semistandard Young tableaux (SSYT) of skew-shape λ/μ, where each summand is a monomial $x^{\nu(T)}$, letting $\nu(T)$ be the content of the SSYT T.

Ehrenborg observed that $F_{P\times Q} = F_P F_Q$ [Eh, p. 9-10]. Figure 3 gives an example

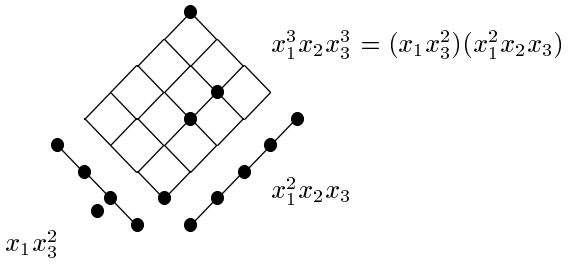


FIGURE 3. $F_{P\times Q} = F_P F_Q$

of how a multichain in P together with a multichain in Q give rise to a multichain in $P \times Q$ such that the corresponding monomials in F_P and F_Q are multiplied to get the corresponding monomial in $F_{P\times Q}$. When such a pair of multichains in P and Q differ in length, one must first augment the shorter one with copies of $\hat{1}$ before applying this construction to obtain a multichain in $P \times Q$.

Let us collect some useful facts about F_P.

(1) $F_{P\times Q} = F_P F_Q$

(2) $F_P = h_n$ when P is a chain C_{n+1}, since each possible monomial of degree n is contributed to F_P by a single multichain in C_{n+1}.
(3) $F_P = h_\lambda$ when P is the product of chains $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$, by 1 and 2.
(4) The collection of multichains in C_{n+1} which never jump in rank by more than one between consecutive multichain elements contributes e_n to $F_{C_{n+1}}$. Hence, e_λ accounts for exactly those multichains in $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$ which jump by at most one in each coordinate between consecutive multichain elements.
(5) Let $C_{\lambda/\mu}$ be the set of chains in a Boolean algebra B_n that correspond to the region of its order complex that is bounded by the constraints that entries of λ/μ must increase weakly in rows and strictly in columns. Then $C_{\lambda/\mu}$ contributes $s_{\lambda/\mu}$ to F_{B_n}, as is shown in Lemma 2.2.
(6) Let D_λ denote the chains in B_n corresponding to the type A Coxeter arrangement regions in a subspace of type λ. Then Proposition 2.4 will show that D_λ contributes p_λ to F_{B_n}.

Simion and Stanley expressed the above relationship between h_λ and e_λ in 3 and 4 in terms of labellings known as R^*S-labellings in [SS, p. 16-21]. We will use 5 and 6 together with other types of chain-labellings and chain decompositions to express flag f-vectors in terms of Schur functions and power-sum symmetric functions.

Now let us set up notation for Lemma 2.2 and Proposition 2.4. Each Boolean algebra multichain $\hat{0} = v_0 \leq v_1 \leq \cdots \leq v_{k-1} < v_k = \hat{1}$ gives a string of inclusions $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_1, \ldots, a_n\}$. Let V be a real vector space with coordinates $a_1, \ldots, a_n$, and let W be the subspace with the linear constraint $\sum_{i=1}^n a_i = 0$. Assign each multichain in B_n to an intersection of hyperplanes and half-spaces in W as follows. Partition $\{a_1, \ldots, a_n\}$ into ordered blocks $A_1, \ldots, A_k$ by letting $S_i \backslash S_{i-1} = A_i$ for $1 \leq i \leq k$. A multichain is then assigned to the intersection of all the hyperplanes $a_j = a_{j'}$ such that $a_j, a_{j'} \in A_i$ for some $1 \leq i \leq k$ and all the open half-spaces $a_j < a_{j'}$ such that $a_j \in A_i$ and $a_{j'} \in A_{i'}$ for some $1 \leq i < i' \leq k$.

Recall that the type A Coxeter arrangement is generated by the hyperplanes $a_i = a_j$ for $1 \leq i < j \leq n$. These hyperplanes decompose the W-slice of the unit sphere $\sum_{i=1}^n a_i^2 = 1$ into the type A Coxeter complex. We will exploit the fact that the type A Coxeter complex has the same face structure as the order complex of a Boolean algebra, by the map above which sends chains in a Boolean algebra to intersections of hyperplanes and half spaces in the Coxeter arrangement, i.e. to a region on the unit sphere. This point of view is informed by [HRW, p. 5-11].

In what follows below, let B_n consist of the subsets of $S = \{a_{i,j} | \mu_i < j \leq \lambda_i\}$ for some skew-shape λ/μ of size n. We get a system of inequalities, specifying a region in a subspace arrangement embedded in the type A Coxeter arrangement, from the requirements that entries in a semi-standard Young tableau of shape λ/μ increase weakly in rows and strictly in columns. Each pair of neighboring boxes in positions (i, j) and $(i, j+1)$ implies a weak inequality $a_{i,j} \leq a_{i,j+1}$, while each pair of neighboring boxes in positions (i, j) and $(i+1, j)$ yields a strict inequality $a_{i,j} < a_{i+1,j}$. Let $n = l - k$ for $\mu \subseteq \lambda$ satisfying $\mu \vdash k$ and $\lambda \vdash l$.

Lemma 2.2. *The collection of multichains in a Boolean algebra B_n satisfying the constraints described above for the skew-shape λ/μ contributes the skew-Schur function $s_{\lambda/\mu}$ to F_{B_n}.*

PROOF. This is a direct consequence of the combinatorial definition of skew-Schur function. Considering $s_{\lambda/\mu}$ as a sum over the semi-standard Young tableaux of shape λ/μ, we need only show that each such tableau giving rise to a monomial of content ν corresponds bijectively to a multichain in the bounded region which contributes x^ν to F_P. The bijection comes from placing the number d in the box at position (i,j) in a SSYT of shape λ/μ for $a_{i,j} \in S_d \setminus S_{d-1}$ in the corresponding multichain $\emptyset = S_0 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_{i,j} | \mu_i < j \leq \lambda_i\}$. The constraints on multichains in a region are designed to correspond naturally to the constraints on semi-standard Young tableaux entries so that legal multichains are mapped to legal SSYT. The monomials will agree because $|S_d| - |S_{d-1}|$ for a multichain will be the number of boxes labelled d in the corresponding SSYT. □

Example 2.3. *The SSYT in Figure 4 contributes the monomial $x_1 x_2^2 x_3 x_5 x_6^2$ to the*

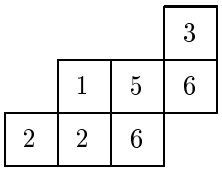


FIGURE 4. A semi-standard Young tableau of content $(1,2,1,0,1,2)$

skew-Schur function $s_{(4,4,3)/(3,1)}$. The multichain $\emptyset \subseteq \{a_{2,2}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}, a_{2,3}\} \subseteq \{a_{2,2}, a_{3,1}, a_{3,2}, a_{1,4}, a_{2,3}, a_{2,4}, a_{3,3}\}$ which corresponds via Lemma 2.2 to this SSYT contributes the same monomial $x_1 x_2^2 x_3 x_5 x_6^2$ to F_P.

A subspace S in the Coxeter arrangement corresponds to a partition π of $[n]$ by letting i,j belong to the same block in π for each i,j such that S satisfies $a_i = a_j$. The **type** of a subspace is the number partition of n which records the block sizes.

Proposition 2.4. *The multichains in B_n that lie in a subspace of type λ in $\Delta(B_n)$ contribute p_λ to F_{B_n}.*

PROOF. Recall that $x_1^{n_1} \ldots x_j^{n_j}$ accounts for the contribution to F_P of a multichain with jumps of sizes $n_1, \ldots, n_j$. Since $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$ and $p_r = \sum_{i>0} x_i^r$, we note that p_λ will account for all multichains with a particular collection of jumps of sizes $\lambda_1, \ldots, \lambda_k$ taken in any order or with any of these jumps merged together. These are exactly the multichains assigned to a subspace of type λ, because requiring coordinates to be equal amounts to insisting they occur in the same jump in a multichain. Merging jumps is taking intersections of subspaces, e.g. intersecting $a_1 = \cdots = a_{\lambda_1}$ with $a_{\lambda_1+1} = \cdots = a_{\lambda_1+\lambda_2}$. □

3. NONCROSSING PARTITION LATTICES FOR THE CLASSICAL REFLECTION GROUPS

Reiner generalized the noncrossing partition lattice NC_n (defined in the introduction) to the classical reflection groups in [Re], by viewing partitions as subspaces in a Coxeter arrangement and making a notion of noncrossing for subspaces of various types. Recall that the type B Coxeter arrangement is generated by the hyperplanes $x_i \pm x_j = 0$ for $1 \leq i \leq j \leq n$. Let us partition $\pm 1, \pm 2, \ldots, \pm n$ so that i is in the same component as j if and only if $-i$ is in the same component as $-j$.

To define type B noncrossing partitions, the numbers $1, 2, \ldots, n, -1, -2, \ldots, -n$ are placed clockwise about a circle with each i opposite $-i$. Let $i \lhd j$ if i is at most 180 degrees counterclockwise from j, noting that this is not a transitive relation. A type B partition is **noncrossing** if $a \lhd b \lhd c \lhd d$ and $a \lhd d$ for $a, c \in C_1$ and $b, d \in C_2$ implies $C_1 = C_2$. This is analogous to the type A condition because noncrossing arcs may be drawn inside the circle connecting the elements of each component. Note that the condition that $i, j \in C$ if and only if $-i, -j \in -C$ amounts to 180 degree rotational symmetry. For each component C there will be a component $-C$ such that $j \in C$ if and only if $-j \in -C$. In particular, one component may satisfy $C = -C$. When such a component exists, this is called the 0-component and is denoted by C_0. The noncrossing property precludes the existence of more than one 0-component.

The **type B noncrossing partition lattice** is the lattice of type B noncrossing partitions ordered by refinement. The **type D noncrossing partition lattice** is the restriction to noncrossing partitions with $C_0 \neq \{\pm i\}$ for $1 \leq i \leq n$, since type D excludes hyperplanes of the form $x_i = 0$. The **interpolating BD noncrossing partition lattices**, denoted NC_S^{BD}, are indexed by sets $S \subseteq \{1, \ldots, n\}$. NC_S^{BD} consists of those elements of the type B noncrossing partition lattice which have $C_0 \neq \{\pm i\}$ for $i \in S$.

To give a chain decomposition, we shall partially order a collection of Boolean algebras comprising a homology basis. We begin with type A. In this case, the Boolean algebras are indexed by the unlabelled, rooted planar trees on n vertices. We recover the following result of Stanley, before applying a similar chain decomposition to the noncrossing partition lattices of other types.

Theorem 3.1 (St3). *Let T be the set of unlabelled, rooted planar trees on n vertices, and for each $t \in T$, let $m_i(t)$ be the number of children the i-th tree element encountered in depth-first-search order has. Then*

$$F_{NC^A(n)} = \sum_{t \in T} e_{m_1(t)} e_{m_2(t)} \cdots e_{m_n(t)}.$$

PROOF. List the numbers $1, \ldots, n$ sequentially on a number line, and then $t \in T$ is specified by a collection of $n-1$ noncrossing arcs above the number line so that each number except 1 (which serves as tree root) is the right endpoint of exactly one arc, connecting the number to its parent in the tree. For any particular such tree, each subset of its arcs specifies a different noncrossing partition whose components are the graph components. Figure 5 gives an example. The noncrossing partitions given by the subsets of the arcs in any fixed tree sit inside the noncrossing partition lattice as a Boolean algebra.

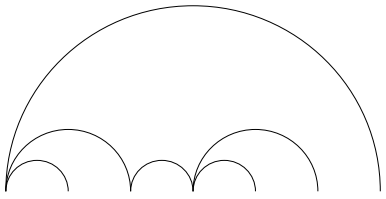


FIGURE 5. A boolean algebra in NC_n^A

Remark 3.2. *These Boolean algebras for type A are implicit to the EL-labelling of Björner-Gessel for NC_n^A given by $\lambda(u,v) = \max(\min B_1, \min B_2)$, where B_1, B_2 are the components of u to be merged in v.*

The covering relation merging B_1 and B_2 in a saturated chain may be represented by the tree arc connecting $\min B_1$ to $\min B_2$, to give a bijection between the decreasing chains in the EL-labelling of Björner-Gessel and the rooted, unlabelled planar trees. Our chain decomposition will result from an ordering on these Boolean algebras, each of which contains exactly one decreasing chain in the above EL-labelling. Let us now describe this partial order on trees. The tree comprised of arcs $i, i+1$ for each $1 \leq i \leq n-1$ will serve as the minimal element. If two trees u and v agree except that u has arcs i,j and j,k while v has arcs i,j and i,k for some $i<j<k$, then we have a covering relation $u \prec v$. These are all the covering relations.

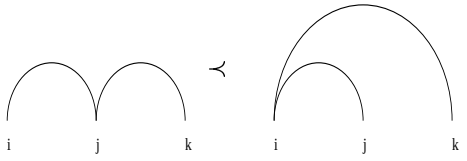


FIGURE 6. Covering relations for trees in NC_n^A

Now assign each chain to the earliest tree whose corresponding Boolean algebra contains it. This Boolean algebra is obtained by taking the lexicographically earliest extension of the chain to a saturated chain M, in terms of the EL-labelling of Björner-Gessel, and then drawing the tree which has arcs from $\min B_i$ to $\min B_j$ for each pair of blocks B_i, B_j that is merged by some covering relation of M.

For each tree, note that the multichains assigned to the corresponding Boolean algebra satisfy the following property: if two arcs have the same left endpoint, then the arc with right endpoint farther to the right must be inserted strictly earlier than the other arc. Otherwise, there would be an earlier tree which has these arcs ij and ik replaced by arcs ij and jk, as in Figure 6. Letting $a_{i,1}, \ldots, a_{i,m_i(t)}$ denote the multichain steps at which the $m_i(t)$ arcs with left endpoint i are inserted, multichains contributed by t to F_P must satisfy the strict inequalities

$$\begin{aligned} a_{1,1} &< \cdots < a_{1,m_1(t)} \\ a_{2,1} &< \cdots < a_{2,m_2(t)} \\ &\cdots \\ a_{n,1} &< \cdots < a_{n,m_n(t)}. \end{aligned}$$

We may now apply Lemma 2.2 to obtain the desired flag f-vector formula. □

See [St3, p. 6] and [Ed, p. 173-174] for earlier proofs of the above result by other methods. Next let us consider noncrossing partition lattices for the other classical reflection groups. One may define circular tree diagrams which preserve 180 degree rotational symmetry for type B. If there is an arc i,j with i no more than 180 degrees counterclockwise from j, then we consider i to be the parent of j in this treelike structure. Every node must have a parent, and we denote an arc by i,j where i is the parent of j.

Theorem 3.3. *If P is the type B noncrossing partition lattice $NC^B(n)$ and T is the collection of possible tree diagrams, then*

$$F_P = \sum_{t \in T} e_{m_1(t)} \cdots e_{m_n(t)} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ \alpha_1 + \cdots + \alpha_n = n}} e_\alpha.$$

PROOF. Tree diagrams give rise to Boolean algebras, by taking subsets of the arcs which preserve 180 degree rotational symmetry, such as the set of dashed arcs in Figure 7. These type B tree diagrams are partially ordered in a similar fashion to the type A situation: we say $u \prec v$ if u and v agree except that u has arcs i, j and j, k while v has arcs i, j and i, k. We get a system of strict inequalities, just as

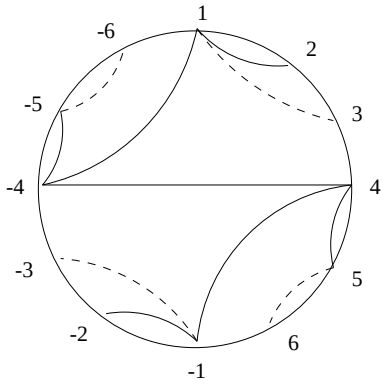


FIGURE 7. A Boolean algebra in NC_n^B

in the type A case. Summing F_P over pieces of a chain decomposition yields the desired flag f-vector formula, agreeing with the formula in [Re, p.13]. □

For example, $F_{NC^B(3)} = e_1^3 + 6e_1e_2 + 3e_3$. In the case of interpolating-BD noncrossing partitions, the allowable circular tree diagrams are those which do not include an arc from i to $-i$ for any $i \in S$. The chain decomposition for type B restricts to this case because the partial order on circular tree diagrams does not include any covering relations $u \prec v$ where u is forbidden and v is a legal circular tree diagram.

Corollary 3.4. *If P is the interpolating-BD noncrossing partition lattice NC_S^{BD}, then*

$$F_P = \sum_{t \in T} e_{m_1(t)} \cdots e_{m_n(t)} = \sum_{\substack{\alpha \in \mathbb{N}^n \cap PF_S \\ \alpha_1 + \cdots + \alpha_n = n}} e_\alpha$$

for $PF_S = \bigcap_{i \in S} \{\alpha | \alpha_i + \cdots + \alpha_j < j - i + 1$ for some $i \leq j \leq n$ or $\alpha_i + \cdots + \alpha_n + \alpha_1 + \cdots + \alpha_j < j + n - i + 1$ for some $j < i\}$.

4. GENERALIZED SHUFFLE POSETS

Greene defined and studied posets of shuffles in [Gr, p. 191-192]. We will provide flag f-vector formulas for two generalizations, but first we review Greene's original definition. Let $A_1 = \{a_1, \ldots, a_m\}$ and $A_2 = \{b_1, \ldots, b_n\}$ be two disjoint alphabets, let w_1 be the word $a_1 a_2 \cdots a_m$ and let w_2 be the word $b_1 b_2 \cdots b_n$. The traditional shuffle poset, denoted $W_{m,n}$, consists of all subwords of the words $w_1 \sqcup\!\sqcup w_2$ which

are obtained by interspersing the letters of w_1 with the letters of w_2. Thus, a shuffled word $w_1 \sqcup\!\sqcup w_2$ must satisfy $w_1 \sqcup\!\sqcup w_2|_{A_1} = w_1$ and $w_1 \sqcup\!\sqcup w_2|_{A_2} = w_2$. Denote the elements of $W_{m,n}$ by $u_1 \sqcup\!\sqcup u_2$ where $u_1 \sqcup\!\sqcup u_2|_{A_1} = u_1$ is a subword of w_1 and $u_1 \sqcup\!\sqcup u_2|_{A_2} = u_2$ is a subword of w_2. Let w_1 be the minimal element of $W_{m,n}$, let w_2 be the maximal element of $W_{m,n}$, and there is a covering relation $\mathbf{u} \prec \mathbf{v}$ whenever v is obtained from u by either deleting a letter belonging to A_1 or inserting a letter belonging to A_2 in a way that produces a poset element. It is implicit to this definition that each letter occurs with multiplicity one. Greene's original motivation was to provide a (very idealized) model for DNA mutation.

4.1. **Shuffle posets of multisets.** Stanley generalized shuffle posets to allow repetition of letters in the words to be shuffled, leading to posets called shuffle posets of multisets (personal communication). Let w_1 and w_2 be words on disjoint alphabets, with the constraint that identical letters must always appear consecutively within w_1 and w_2. We refer to the composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ as the **type** of the word $w = a_1^{\alpha_1} \cdots a_k^{\alpha_k}$. Suppose two such words w_1 and w_2 are of types α, β, respectively. These compositions α, β completely determine the shuffle poset of multisets given by w_1 and w_2 up to isomorphism, so we denote this poset $W_{\alpha,\beta}$. As in traditional shuffle posets, a word w is an element of $W_{\alpha,\beta}$ if w restricted to the alphabet A_1 is a subword of w_1 and w restricted to the alphabet A_2 is a subword of w_2, but with the additional requirement that identical letters must occur consecutively.

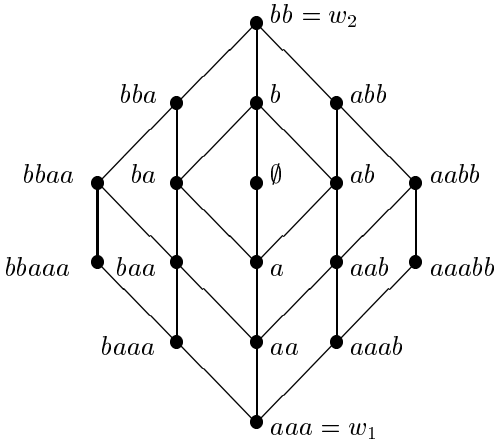


FIGURE 8. The shuffle poset of multisets $W_{3,2}$

For example, if $w_1 = aaab$ and $w_2 = c$, then $aacb$ is a valid poset element while $acab, ba$ and $aaaa$ are not. Figure 8 illustrates the poset $W_{3,2}$ with $w_1 = aaa$ and $w_2 = bb$. The traditional shuffle posets are usually denoted $W_{m,n}$, but unfortunately in the notation of shuffle posets of multisets this necessarily becomes $W_{1^m,1^n}$.

Theorem 4.1. *The flag f-vector $F_{W_{\alpha,\beta}}$ is*

$$\sum_{j=0}^{\min(l(\alpha),l(\beta))} \sum_{\substack{1 \le a_1 < \cdots < a_j \le l(\alpha) \\ 1 \le b_1 < \cdots < b_j \le l(\beta)}} \left(\prod_{i=1}^{j} s_{(\alpha_{a_i}+\beta_{b_i}-1,\beta_{b_i})/(\beta_{b_i}-1)} \right) \left(\prod_{i \notin \{a_1,\ldots,a_j\}} s_{\alpha_i} \right) \left(\prod_{i \notin \{b_1,\ldots,b_j\}} s_{\beta_i} \right).$$

This is a special case of the result for k-shuffle posets, so we will prove the more general result and simply state the special case above.

In the case of traditional shuffle posets, this becomes

$$F_{W_{1^m,1^n}} = \sum_{j=0}^{\min(m,n)} \binom{m}{j}\binom{n}{j} e_2^j e_1^{n+m-2j},$$

so we recover a result of [SS, p.21] for traditional shuffle posets. To see this, simply substitute e_2 for $s_{(1,1)}$ and e_1 for s_1 in Theorem 4.1 and group like terms.

Corollary 4.2. *$F_{W_{\alpha,\beta}}$ is Schur-positive.*

4.2. **k-shuffle posets.** More generally, posets for shuffling k words were introduced in [He2], answering a question of Stanley. A **k-shuffle poset** is specified by k words $w_1, \ldots, w_k$ on disjoint alphabets. Identical letters within any particular w_i are again required to occur consecutively. A k-shuffle poset is determined up to isomorphism by an ordered set of k compositions specifying the types of the words $w_1, \ldots, w_k$ to be shuffled. Let $\alpha^{(i)}$ be the type of the word w_i for $1 \leq i \leq k$, and let $W_{\alpha^{(1)},\ldots,\alpha^{(k)}}$ denote the k-shuffle poset specified by $w_1, \ldots, w_k$. Let u_i denote a subword of w_i and u_i^c be the complementary word within w_i. Poset elements will be consistent collections of pairwise shuffled words, in the following sense.

Definition 4.3. *Fix a choice of subwords $u_1, \ldots, u_k$ of the words $w_1, \ldots, w_k$. Then a collection $\{u_i^c \sqcup\!\sqcup u_j | i < j\}$ of pairwise shuffled words is* **consistent** *if there is some shuffled word $w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k$ which contains each $u_i^c \sqcup\!\sqcup u_j$ as a subword.*

Covering relations are defined in terms of an operation called del-sertion.

Definition 4.4. *Let C be a consistent collection $\{u_i^c \sqcup\!\sqcup u_j | i < j\}$ of pairwise shuffled words and let b be a letter belonging to some u_l^c. The letter b is* **del-serted** *by deleting b from each copy of u_l^c and at the same time inserting b in each copy of u_l in C.*

For v to cover u, v must be obtained from u by del-serting a letter which belongs to some u_i in such a way that the collections of pairwise shuffled words associated to u and v are consistent in the following sense.

Definition 4.5. *Let $\{u_i^c \sqcup\!\sqcup u_j | i < j\}$ and $\{v_i^c \sqcup\!\sqcup v_j | i < j\}$ be the collections of pairwise shuffled words in u and v, respectively. Then u and v are* **consistent** *if there exists some shuffled word $w_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup w_k$ which simultaneously contains each $u_i^c \sqcup\!\sqcup u_j$ and each $v_i^c \sqcup\!\sqcup v_j$ as a subword.*

It is shown in [He2] that edge-consistency implies chain consistency. Now let us turn to the chain decomposition. First we will need some additional notation.

Definition 4.6. *Given a letter a in a shuffled word $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$, let $w(a) = i$ for $a \in w_i$. A* **descent block** *in w is a maximal string $u_1 \ldots u_j$ of consecutive letters in w with $w(u_i) \geq w(u_{i+1})$ for $1 \leq i < j$.*

As an example, $w(u_1)w(u_2)\ldots w(u_{10}) = 3114214241$ has descent blocks $u_1u_2u_3$, $u_4u_5u_6, u_7u_8$ and u_9u_{10}. If B is the set of descent blocks for some P_w then the multichains within P_w which are assigned to it are the multichains which actually involve all the descents in the descent blocks of B.

Let $l(\mathbf{b})$ be the number of distinct letters in $\mathbf{b}$, let $m_1(\mathbf{b}), \ldots, m_{l(\mathbf{b})}(\mathbf{b})$ be the multiplicities of these letters and let $S(m_1(\mathbf{b}), \ldots, m_j(\mathbf{b}))$ denote the skew-Schur function of ribbon shape with $m_i(\mathbf{b})$ boxes in row $l(\mathbf{b}) - i + 1$.

Theorem 4.7. *Let $Shuf(w_1, \ldots, w_k)$ denote the collection of shuffled words of the form $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ and let $B(w)$ be the collection of descent blocks in a particular $w \in Shuf(w_1, \ldots, w_k)$. Then*

$$F_P = \sum_{w \in Shuf(w_1,\ldots,w_k)} \prod_{\mathbf{b} \in B(w)} S(m_1(\mathbf{b}), \ldots, m_{l(\mathbf{b})}(\mathbf{b})).$$

PROOF. We apply Lemma 2.2, as follows. Each shuffled word $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a product of chains subposet P_w comprised of all poset elements that are consistent with w. P_w takes the form C_α where α is the composition obtained by taking the union of the compositions $\alpha^{(1)}, \ldots, \alpha^{(k)}$ for w_i of type $\alpha^{(i)}$.

Partially order the subposets P_w using an interval in the weak order to order the words w. Let $l(w_i)$ denote the **length** of the composition $\alpha^{(i)} = type(w_i)$. Then we use the interval in the weak order from the permutation $(l(w_1), \ldots, 1, l(w_1) + l(w_2), \ldots, l(w_1)+1, \ldots, l(w_1)+\cdots+l(w_k), \ldots, l(w_1)+\cdots+l(w_{k-1})+1)$ to the longest element of S_n, namely $(l(w_1)+\cdots+l(w_k), \ldots, 1)$; by (non-standard) convention, our weak order covering relations come from swapping adjacent positions rather than values. This interval is chosen because it preserves the order of the letters within each word w_i. For example, Figure 9 gives the weak order interval for $w_1 = 112$,

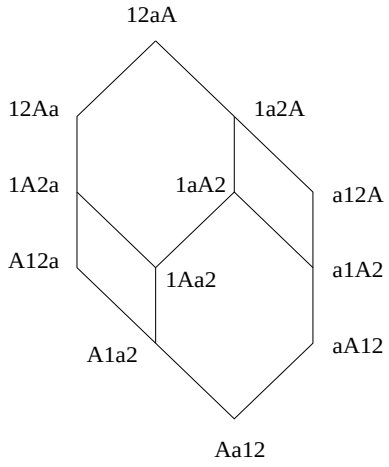


FIGURE 9. The partial order on sublattices indexed by shuffled words

$w_2 = aaa$ and $w_3 = A$.

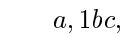
Each poset chain is assigned to the earliest P_w which is consistent with the chain. It is shown in [He2, Proposition 4.2] that every chain belongs to some P_w. We will use descent blocks to show that this choice of the earliest P_w containing a chain is well-defined.

We claim that the collection of multichains assigned to a product of chains with B as its set of descent blocks contributes

$$\prod_{\mathbf{b} \in B} S(m_1(\mathbf{b}), \ldots, m_{l(b)}(\mathbf{b}))$$

to F_P. For example, $S(3,1,2) = s$  would account for a descent block CBA in which $w(C) > w(B) > w(A), m_1(\mathbf{b}) = 3, m_2(\mathbf{b}) = 1$ and $m_3(\mathbf{b}) = 2$.

The multichains assigned to P_w are determined by the same system of weak and strong inequalities needed for the above product of skew-Schur functions of ribbon shape, so Lemma 2.2 applies. Thus, a descent block consisting of letters $a_1, \ldots, a_l$ with multiplicities $m_1, \ldots, m_l$ will contribute to F_P the skew-Schur function $S(m_1, \ldots, m_l)$. Simply note that we have weak inequalities on the order in which identical letters are del-serted and strict inequalities requiring that the last copy of a_i must be del-serted strictly after the first copy of a_{i+1} is del-serted. Otherwise, not all of the necessary descents would occur in the multichain. The identity $F_{P\times Q} = F_P F_Q$ together with the independence of our constraints on distinct descent blocks implies that the contribution of a particular P_w to F_P should be the product over descent blocks of the skew-Schur functions $S(m_1, \ldots, m_l)$. □

Example 4.8. *If $w_1 = aabbbcddddde, w_2 = ABBCDDD, w_3 = 1112344$, then the shuffled word $aa111bbbcA2BBddddCe344DDD$ contributes the product of skew-Schur functions* s  s  s s s s *to F_P. It has descent blocks $a, 1bc, A, 2Bd, Ce$ and $34D$.*

Corollary 4.9. *The flag f-vector formula F_P for k-shuffle posets is Schur-positive.*

PROOF. We have expressed F_P as a sum of products of skew-Schur functions. It is well-known that skew-Schur functions are Schur-positive, and that products of Schur functions are Schur-positive by the Littlewood-Richardson rule. □

5. Graded monoid posets

Any finite collection C of vectors in $\mathbb{N}^d$ gives rise to a **monoid** Λ endowed with a partial order as follows: let $\hat{0}$ be the 0-vector, let u be an element of Λ if u is a sum (i.e. with nonnegative coefficients) of elements in C, and let there be a covering relation $u \prec v$ if $v = u + w$ for some $w \in C$. Equivalently, Λ is the partial order by divisibility on the monomials in a semigroup ring $k[\Lambda]$, namely in a ring generated by a finite collection $t_1, \ldots, t_n$ of monomials in $k[z_1, \ldots, z_d]$ whose multidegrees are the vectors in C. When the vectors in C all lie in an affine hyperplane, then Λ is **graded**, and it makes sense to speak of its flag f-vector.

Let $\mathbf{a_1}, \ldots, \mathbf{a_n}$ be the generators for Λ, where $\mathbf{a}_i = (a_{i,1}, \ldots, a_{i,d}) \in \mathbb{N}^d$. Note that $k[\Lambda] \cong k[t_1, \ldots, t_n]/I_\Lambda$ with I_Λ denoting the kernel of the map

$$\phi : k[t_1, \ldots, t_n] \to k[z_1, \ldots, z_d]$$

which sends t_i to $z_1^{a_{i,1}} \ldots z_d^{a_{i,d}}$. Thus, I_Λ is the toric ideal of syzygies among the images under ϕ of the monomials $t_1, \ldots, t_n$. The monomials in $k[t_1, \ldots, t_n]/I_\Lambda$ form an (infinite) monoid poset, partially ordered by divisibility.

Monoid posets arose in work of Peeva, Reiner and Sturmfels [PRS] and subsequently Herzog, Reiner, Welker [HRW] as a tool for studying free resolutions of a field k as a $k[\Lambda]$-module.

Remark 5.1. *In [PRS] and [HRW], the variables $x_1, \ldots, x_n$ are used where we instead use $t_1, \ldots, t_n$. We break from the notation of [HRW], [PRS] in an effort to avoid confusion with the variables $x_1, x_2, \ldots$ appearing in the quasi-symmetric function encoding for the flag f-vector.*

Whenever a noncommutative ideal J_Λ closely related to I_Λ has a quadratic Gröbner basis, [PRS] used this to shell any interval of Λ and deduce in this case that $k[\Lambda]$ was Koszul. Motivated by their use of a monomial term order on $k[t_1, \ldots, t_n]$ and of the resulting Gröbner bases for I_Λ and J_Λ, we likewise use monomial term orders to obtain chain decompositions for Λ and use the resulting Gröbner bases on I_Λ to compute flag f-vectors.

It is observed in [PRS] that every interval $[u, v]$ in a monoid poset is isomorphic to an interval of the form $[\hat{0}, w]$, by letting $w = v - u$. Hence, it is completely general to compute flag f-vectors for intervals of the form $[\hat{0}, w]$. First we give a simple formula for the flag f-vector of an entire monoid poset Λ, which came out of a conversation with Bernd Sturmfels.

Proposition 5.2 (Hersh-Sturmfels)**.**

$$F_\Lambda = \prod_{i=1}^{\infty} Hilb(k[\Lambda], x_i)$$

PROOF. Each term $Hilb(k[\Lambda], x_i)$ records all possible jumps between the $(i-1)$-st and i-th elements of a multichain in the monoid poset Λ, because these jumps correspond naturally to monomials in $k[\Lambda]$. Furthermore, the rank differential of a jump equals the degree of the corresponding monomial. Thus, the right hand side can be viewed as the sum over all finite multichains in Λ that calculates F_Λ. □

Now we turn to finite intervals. For each monomial m in the image of the map $\phi : k[t_1, \ldots, t_n] \to k[z_1, \ldots, z_d]$, consider the interval $[\hat{0}, m]$. We will give a chain decomposition by using a monomial term order to order the monomials $f \in k[t_1, \ldots, t_n]$ which satisfy $\phi(f) = m$. Each such f specifies a product of chains subposet in $[\hat{0}, m]$. We refer to these monomials as the **factorizations** of m, and to the resulting product of chains subposets as the **fibres** of m.

Each saturated chain on $[\hat{0}, m]$ corresponds to a monomial in the ring $k\langle y_1, \ldots, y_n\rangle$ in noncommuting variables as follow: each covering relation $a \prec b$ in a saturated chain satisfies $b = a \cdot t_i$ for some $1 \leq i \leq n$, so label this covering relation y_i, and then associate to each saturated chain its label sequence, namely a monomial in $k\langle y_1, \ldots, y_n\rangle$. Equivalently, saturated chains are determined by a monomial in $k[t_1, \ldots, t_n]$ specifying the fibre along with a choice of saturated chain within that product of chains subposet.

Choose a monomial term order on $k[t_1, \ldots, t_n]$, and denote by B the resulting Gröbner basis for $I_\Lambda = ker(\phi)$. We first consider the case where the factorizations of m are all squarefree. For each such $f \in \phi^{-1}(m)$, let $L_{A(f,B)}$ be the intersection lattice of the subspace arrangement generated by subspaces $t_1 = t_2 = \ldots = t_k$ for each monomial $t_1 \ldots t_k$ which divides f and is the leading term of an element of B.

Theorem 5.3. *If P is a graded monoid poset interval $[\hat{0}, m]$ with $m \in im(\phi)$ such that the factorizations $f \in \phi^{-1}(m)$ are all squarefree, then*

$$F_P = \sum_{f \in \phi^{-1}(m)} \sum_{u \in L_{A(f,B)}} \mu(\hat{0}, u) p_{type(u)}(x_1, x_2, \ldots)$$

where μ is the Möbius function on the intersection lattice $L_{A(f,B)}$ defined above.

PROOF. Sum over factorizations f, with each summand accounting for the multichains in the fibre of f that are not in any earlier fibres. The inner summation will be obtained by inclusion-exclusion. First note that each factorization $f \in \phi^{-1}(m)$

gives rise to a Boolean algebra which contributes the power-sum symmetric function p_{1^n} to F_P, before we use Möbius functions to discount chains shared with earlier fibres.

Consider a chain

$$C = \hat{0} < m_1 < m_1 m_2 < \cdots < m_1 m_2 \cdots m_r = m$$

which is both in the fibre of f and also in the fibre of some earlier f'. At least one of the above monomials $m_i \in k[t_1, \ldots, t_n]/I_\Lambda$ recording jumps in C must have distinct factorizations $g, g' \in k[t_1, \ldots, t_n]$ such that g divides f and g' divides f' in the ring $k[t_1, \ldots, t_n]$, with $g - g' \in I_\lambda$ having leading term g. Thus, g is divisible by a leading term b in the Gröbner basis B which divides m, since b divides g and g divides m. Recalling from Section 2 the correspondence between chains in a Boolean algebra and regions in the type A Coxeter complex, notice that the chain C belongs to one of the subspaces generating $L_{A(f,B)}$, namely the subspace corresponding to b, since b divides g. Thus, any chain in the overlap of the f fibre with earlier fibres lies in a subspace in $L_{A(f,B)}$. On the other hand, chains not in any earlier fibre will not have any earlier factorizations for any jumps in the chain, so none of the jumps are divisible by the monomials corresponding to the subspaces that generate $L_{A(f,B)}$. Thus, the chains which do not belong to earlier fibres are those faces of the type A Coxeter complex lying in the complement of $L_{A(f,B)}$.

To show that $\sum_{u \in L_{A(f,B)}} \mu(\hat{0}, u) p_{type(u)}$ accounts for exactly the new chains in the fibre of f, recall from Proposition 2.4 that the restriction of a subspace u of type λ to the type A Coxeter complex contributes p_λ to F_P. Furthermore, multiplying by the Möbius function $\mu_{L_{A(f,B)}}(\hat{0}, u)$ ensures that each old chain is counted with multiplicity zero and each new chain with multiplicity one. This is because $p_{type(u)}$ counts chains with jumps given by u or with any of these jumps merged, so it counts the chains in each subspace v contained in u. Hence, the chains in each subspace $v \neq \hat{0}$ are counted with multiplicity $\sum_{0 \leq u \leq v} \mu(\hat{0}, u) = 0$, while the new chains are only counted in the summand $\mu(\hat{0}, \hat{0}) p_{1^n}$, so they are each counted exactly once. □

Example 5.4. *Let $a = uv, b = vx, c = yz, d = uy, e = xy$ and $f = vz$. Consider the poset interval from 1 to $uv^2xyz = abc = bdf = aef$, so I_Λ is generated by the Gröbner basis $\underline{df} - ac, \underline{bd} - ae, \underline{ef} - bc$ with leading terms underlined. For $[\hat{0}, uv^2xyz]$,*

$$F_P = p_1^3 + (p_1^3 - p_1 p_2) + (p_1^3 - p_1 p_2 - p_1 p_2 + p_3).$$

Each factorization contributes one copy of p_1^3. A copy of $p_1 p_2$ is subtracted for the intersection of the aef sphere with the earlier abc sphere in the $e = f$ hyperplane. Two copies of $p_1 p_2$ are subtracted to account for the $b = d$ and $d = f$ hyperplanes in the bdf sphere, but these hyperplanes share the subspace $b = d = f$, so p_3 must be added back.

Now let us more generally allow factorizations $f \in \phi^{-1}(m)$ which need not be squarefree. In this case, we use a variant on the Möbius function for the poset of partitions of a multiset, denoted μ', which was introduced in [HK]. The function μ' was defined to be multiplicative and consequently seems to have nicer closed formulas than μ. Letting P be the poset of partitions of the multiset $\{a_1^{n_1}, \ldots, a_k^{n_k}\}$, with elements reverse ordered by refinement, define $\mu'(\hat{0}, v)$ as follows: if v is a single block B_i, then let $\mu'(\hat{0}, v) = -\sum_{\hat{0} \leq u < B_i} \mu'(\hat{0}, u)$, and let $\mu'(\hat{0}, \hat{0}) = 1$; when v is a

partition into multiple blocks $B_1, \ldots, B_k$, then let

$$\mu'(\hat{0}, B_1|B_2|\ldots|B_k) = \prod_{1\leq i\leq k} \mu'(\hat{0}, B_i).$$

In [HK], it is shown that

$$\mu'(\hat{0}, ab_1^{n_1}\ldots b_k^{n_k}) = (-1)^{n-1}\frac{(n-1)!}{\prod_{i=1}^k n_i!}$$

and that $\mu'(\hat{0}, a^n) = 0$ unless n is a power of 2, in which case $\mu'(\hat{0}, a^n) = (-1)^{n-1}$. It remains open to give an explicit formula for $\mu'(\hat{0}, a_1^{n_1}\ldots a_k^{n_k})$ for general multiset type $(n_1, n_2, \ldots, n_k)$.

To extend Theorem 5.3 to graded monoid poset intervals in which monomials need not be square-free, we will use a weighted sieve argument, based on Lemma 5.5 below. For u a refinement of v, let $Sub(u, v)$ count the number of ways of choosing which types of blocks in u come from each actual block of v. That is, $Sub(u, v)$ counts the ways of refining each of the actual blocks in v (i.e. keeping track of block order, not just of block type) in such a way that together the resulting more refined blocks comprise the partition u. For example, if $u = a|a|aa$ and $v = aa|aa$, then $Sub(u, v) = 2$, but if v is replaced by $aaaa$ then $Sub(u, v) = 1$.

Lemma 5.5 will show that the matrices with entries $\mu'(u, v)$ and $Sub(u, v)$ are inverses to each other.

Lemma 5.5. *If $v \neq \hat{0}$, then*

$$\sum_{\hat{0}\leq u\leq v} \mu'(\hat{0}, u)Sub(u, v) = 0.$$

PROOF. If v consists of a single block B, then $\mu'(\hat{0}, B) = -\sum_{\hat{0}\leq u<B} \mu'(\hat{0}, u)$, and in this case $Sub(u, B) = 1$ for each $\hat{0} \leq u \leq B$. Suppose v has blocks $B_1, \ldots, B_k$ for $k > 1$.

Note that

$$\sum_{\emptyset\subsetneq S\subsetneq\{1,\ldots,k\}} (-1)^{|S|-1} \prod_{i\in S} \mu'(\hat{0}, B_i) \prod_{i\notin S} \left(\sum_{\hat{0}\leq u\leq B_i} \mu'(\hat{0}, u) \right) = 0,$$

because $\sum_{\hat{0}\leq u\leq B_i} \mu'(\hat{0}, u) = 0$ for each B_i. Thus, we may add the left side to another expression without changing its value, as is done in the third equality below. The fourth equality will involve substantial regrouping of terms. The summation

$$(-1)^k \sum_{\hat{0}\leq u<v} \mu'(\hat{0}, u)Sub(u, v)$$

comes from two facts: (1) that μ' is multiplicative, and (2) that each $u < B_1|B_2|\cdots|B_k$ appears as exactly one summand either in

$$\prod_{i=1}^k \Big(\sum_{\hat{0}\leq u<B_i} \mu'(\hat{0}, u) \Big)$$

or in

$$\sum_{\emptyset\subsetneq S\subsetneq\{1,\ldots,k\}} (-1)^{|S|-1} \prod_{i\in S} \mu'(\hat{0}, B_i) \prod_{i\notin S} \Big(\sum_{\hat{0}\leq u\leq B_i} \mu'(\hat{0}, u) \Big),$$

for each of the $Sub(u,v)$ ways of refining v to get u. The "error term"

$$(-1)^k \mu'(\hat{0}, B_1|\dots|B_k)\left(\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i}\right)$$

in the fourth equality comes from each $\emptyset \subsetneq S \subsetneq \{1,\dots,k\}$ contributing a summand $(-1)^{|S|-1}\mu'(\hat{0}, B_1|\dots|B_k)$. Thus,

$$\begin{aligned}
\mu'(\hat{0},v) &= \prod_{i=1}^{k}\mu'(\hat{0},B_i)\\
&= \prod_{i=1}^{k}\left(-\sum_{\hat{0}\le u<B_i}\mu'(\hat{0},u)\right)\\
&= (-1)^k\prod_{i=1}^{k}\left(\sum_{\hat{0}\le u<B_i}\mu'(\hat{0},u)\right)\\
&\quad + (-1)^k\sum_{\emptyset\subsetneq S\subsetneq\{1,\dots,k\}}(-1)^{|S|-1}\prod_{i\in S}\mu'(\hat{0},B_i)\prod_{i\notin S}\left(\sum_{\hat{0}\le u\le B_i}\mu'(\hat{0},u)\right)\\
&= (-1)^k\sum_{\hat{0}\le u<v}\mu'(\hat{0},u)Sub(u,v)\\
&\quad + (-1)^k\mu'(\hat{0},B_1|\dots|B_k)\left(\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i}\right)
\end{aligned}$$

If k is even, then we have $\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i} = 2$, which implies

$$\mu'(\hat{0},v) = \sum_{\hat{0}\le u<v}\mu'(\hat{0},u)Sub(u,v) + 2\mu'(\hat{0},B_1|\dots|B_k).$$

Subtracting $\mu'(\hat{0},v)$ from both sides yields the result.

For k odd, $\sum_{i=1}^{k-1}(-1)^{i-1}\binom{k}{i} = 0$, so we have

$$\mu'(\hat{0},v) = -\sum_{\hat{0}\le u<v}\mu'(\hat{0},u)Sub(u,v),$$

as desired. □

In Theorem 5.8, we will express F_P as a sum over symmetric functions rather than quasi-symmetric functions, which causes each u satisfying $u \le v$ to contribute $Sub(u,v)$ copies of its subspace v (in the order complex restricted to a particular fibre). In the non-square-free case, these incidence relation coefficients $Sub(u,v)$ are not always 1, necessitating the use of weighted Möbius functions.

Recall that a Gröbner basis together with a squarefree monomial $t_1\cdots t_m$ specify a subspace arrangement generated by the subspaces $t_{i_1} = \cdots = t_{i_k}$ for each Gröbner basis leading term $t_{i_1}\cdots t_{i_k}$ which divides $t_1\cdots t_m$.

We generalize this to the case in which $t_1\cdots t_m$ need not be squarefree, by formally defining a multiset analogue for intersection lattices of subspace arrangements embedded in the type A Coxeter arrangement. Fix a monomial $f = t_1^{m_1}\cdots t_l^{m_l}$ with

$m = \phi(f)$ and a Gröbner basis B for the ideal I_λ. Then the resulting multiset intersection poset, denoted $MP(f, B)$, is a subposet of the poset of partitions of the multiset $\{t_1^{m_1}, \ldots, t_l^{m_l}\}$, defined as follows. It has the partition of f into all singleton blocks as its minimal element, and its maximal element is the partition of f into a single block. The atoms are the **generating subspaces** of $MP(f, B)$; each atom has a single nontrivial block, whose content is that of a leading term in the Gröbner basis B which divides the monomial f.

Define the **lcm-join** of the atoms $b_1, \ldots, b_k$ to be the partition of f whose only nontrivial block has as its content the least common multiple (LCM) of the monomials $b_1, \ldots, b_k$.

Definition 5.6. *The* **multiset intersection poset** $MP(f, B)$ *has as its elements all partitions of m into blocks such that each nontrivial block is either the unique nontrivial block in an lcm-join of atoms or is obtained by merging blocks each of which is the unique nontrivial block in an lcm-join of atoms. $MP(f, B)$ has comparability relations $u < v$ whenever u is a refinement of v.*

Notice that multiset intersection posets need not always be lattices. For instance, if $f = a^3b^2$, then the atoms $a^2|a|b|b$ and $ab|a|a|b$ are both covered by both $a^2b|a|b$ and $a^2|ab|b$. In this case, $a^2b|a|b$ is the lcm-join of $a^2|a|b|b$ and $ab|a|a|b$, while $a^2|ab|b$ is the disjoint-block-join of these same two poset elements.

As before, a comparable pair of poset elements $u < v$ has **incidence multiplicity** $Sub(u, v)$, where $Sub(u, v)$ counts the number of ways the actual blocks of v are refined to obtained the set of blocks appearing in u. We again define the inverse, denoted μ', to the incidence algebra as follows. Let $\mu'_P(\hat{0}, B_1|\ldots|B_k) = \prod_{i=1}^k \mu'_{P_i}(\hat{0}, B_i)$, with covering relations in P_i coming from legal merge steps in the interval $[\hat{0}, B_i]$. Let $\mu'(\hat{0}, \hat{0}) = 1$ and $\mu'(\hat{0}, B) = -\sum_{\hat{0} \leq u < B} \mu'(\hat{0}, u)$ for a single block B, where u ranges over elements of the multiset intersection poset which has maximal element B. It remains open to find formulas for these weighted Möbius functions for multiset intersection posets, though perhaps they could be interpreted as reduced Euler characteristics for lexicographically shellable balanced complexes, in the sense of [He3].

Remark 5.7. *The proof of Lemma 5.5 generalizes immediately to arbitrary multiset intersection posets, by restricting to a multiset intersection poset within the multiset partition poset.*

Let $T(u)$ be the set of distinct blocks occurring in a multiset partition u, and let $mult(b)$ be the multiplicity with which any particular block $b \in T(u)$ occurs in a multiset partition u. Substituting p_n for $h_1(x_1^n, x_2^n, \ldots)$ and μ for μ' in the following formula allows one to recover Theorem 5.3.

Theorem 5.8. *If P is the graded monoid poset interval $[\hat{0}, m]$, then*

$$F_P = \sum_{f \in \phi^{-1}(m)} \sum_{u \in MP(f,B)} \mu'(\hat{0}, u) \prod_{b \in T(u)} h_{mult(b)}(x_1^{|b|}, x_2^{|b|}, \ldots).$$

PROOF. We use a chain decomposition and sieve argument similarly to the squarefree case, but with product of chains sublattices rather than Boolean algebras accounting for distinct factorizations. Each factorization $a_{i_1}^{\lambda_1} \ldots a_{i_k}^{\lambda_k}$ with $i_1 < \cdots < i_k$ gives rise to a product of chains sublattice $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$ which contributes h_λ to F_P. To account for overlap, these products of chains are partially ordered as

in the square-free case, using the Gröbner basis B on I_Λ. The symmetric function

$$\prod_{b\in T(u)} h_{mult(b)}(x_1^{|b|}, x_2^{|b|}, \dots)$$

accounts for all chains with jumps determined by the multiset partition u or with collections of these jumps merged into single jumps. The exponent $|b|$ keeps track of the size of the block b determining a jump while the subscript $mult(b)$ accounts for repetition of this block. Since F_P is multiplicative, we take a product over the different types of blocks occurring in the multiset partition u.

For any particular $v \in MP(f,B)$, consider the coefficient of v in

$$\sum_{u\in MP(f,B)} \mu'(\hat{0},u) \prod_{b\in T(u)} h_{mult(b)}(x_1^{|b|}, x_2^{|b|}, \dots).$$

The element v appears $Sub(u,v)$ times in each $\prod_{b\in T(u)} h_{mult(b)}(x_1^{|b|}, x_2^{|b|}, \dots)$ for $u \le v$, so altogether v is included $\sum_{\hat{0}\le u\le v} \mu'(\hat{0},u)Sub(u,v)$ times, so the immediate extension to $MP(f,B)$ of Lemma 5.5 tells us that v is counted 0 times in the f fibre when v is shared with an earlier fibre. On the other hand, each new chain is accounted for exactly once, namely in the term $\mu(\hat{0},\hat{0})h_{type(f)}$. Thus, a weighted sieve argument based on Lemma 5.5 shows that our formula accounts for each chain exactly once. □

For example, if $f = a^4$ and the only leading term in B which divides f is a^2, then the contribution to F_P from the product of chains C_5 specified by f is

$$h_4(x_1,x_2,\dots) - h_1(x_1^2,x_2^2,\dots)h_2(x_1,x_2,\dots) + h_2(x_1^2,x_2^2,\dots) - h_1(x_1^4,x_2^4,\dots).$$

The terms come from the jump collections $\{a,a,a,a\}$, $\{a,a,a^2\}$, $\{a^2,a^2\}$ and $\{a^4\}$, respectively.

Question 5.1. *Are there particular monomial term orders which give particularly nice expressions for this flag f-vector formula, at least in special cases?*

6. Combinatorial applications

We conclude with some applications of chain decompositions for generalized non-crossing partition lattices and shuffle posets. More details may be found in [He1], [He2], respectively, but not from the unified viewpoint of chain decomposition.

Whenever F_P is a symmetric function, one may obtain formulas for the Möbius function, zeta polynomial and characteristic polynomial by simple substitutions into the expression for F_P in the basis of complete homogeneous symmetric functions.

Proposition 6.1. *If P is a poset of rank n with flag f-vector expression F_P a symmetric function, then the coefficient of h_{1^n} in F_P is the Möbius function $\mu_P(\hat{0},\hat{1})$ multiplied by the sign $(-1)^n$.*

PROOF. Ehrenborg expressed $\mu_P(\hat{0},\hat{1})$ as a linear function of F_P in [Eh, p.17]; let g be this function. Note that $g((h_1)^{n+m}) = (-1)^{m+n}$ and $g(h_\lambda) = 0$ for all other λ because $g(h_\lambda) = \mu_P(\hat{0},\hat{1})$ for P the product of chains C_λ. Let $F_P = \sum_{\lambda\vdash n} c_\lambda h_\lambda$. Then $\mu_P(\hat{0},\hat{1}) = g(F_P) = g(\sum_{\lambda\vdash n} c_\lambda h_\lambda) = \sum_{\lambda\vdash n} c_\lambda g(h_\lambda) = (-1)^n c_{1^n}$. □

Ehrenborg expressed the zeta polynomial and characteristic polynomial as Hopf algebra homomorphisms in [Eh, p. 16-17]. We will use the complete symmetric function expression for F_P to give very explicit combinatorial formulas. Let $h_\lambda = h_{\lambda_1}\dots h_{\lambda_k}$, let M_P be the number of maximal chains in a poset P, let Ω_P be the

number of poset elements, let $\Omega_P(q)$ be the rank generating function, let $Z_P(s)$ be the zeta polynomial, counting multichains $\hat{0} \leq x_1 \leq \cdots \leq x_s = \hat{1}$ and let $\chi_P(t)$ be the characteristic polynomial $\sum_{u \in P} \mu(\hat{0}, u) t^{n - \mathrm{rk(u)}}$ In the expression for $\Omega_P(q)$, let $[n]_q = (1-q^n)/(1-q)$, namely the q-analogue of n.

Theorem 6.2. *If $F_P = \sum_{\lambda \vdash n} c_\lambda h_\lambda$, then*

$$\begin{aligned}
\Omega_P &= \sum_{\lambda \vdash n} c_\lambda (\lambda_1 + 1) \ldots (\lambda_k + 1) \\
\Omega_P(q) &= \sum_{\lambda \vdash n} c_\lambda [\lambda_1 + 1]_q \ldots [\lambda_k + 1]_q \\
M_P &= (\mathrm{rk(P)}!) \sum_{\lambda \vdash \mathrm{n}} \mathrm{c}_\lambda \frac{1}{\lambda_1!} \cdots \frac{1}{\lambda_\mathrm{k}!} \\
Z_P(s) &= \sum_{\lambda \vdash n} c_\lambda \binom{\lambda_1 + s - 1}{s-1} \cdots \binom{\lambda_k + s - 1}{s-1} \\
\chi_P(t) &= t^{\mathrm{rk(P)}} \sum_{\lambda \vdash n} c_\lambda \left(\frac{1}{1-t} \right)^k
\end{aligned}$$

PROOF. Linearity allows us to substitute $d+1$ for h_d to obtain Ω_P. This is because a chain of rank d satisfies $F_P = h_d$ and $\Omega_P = d+1$, and also because both F_P and Ω_P are multiplicative functions. Similarly, we substitute $[d+1]_q$ and $\binom{d+s-1}{s-1}$ for $d+1$ to obtain the rank generating function and zeta polynomial, respectively. In the formula counting maximal chains, M_P is not multiplicative, so $M_P/(\mathrm{rk(P)})!$ is used instead, introducing the factor of $(\mathrm{rk(P)})!$. Likewise, to obtain the characteristic polynomial, notice that $\sum_{u \in P} \mu(\hat{0}, u)(1/t)^{\mathrm{rk(u)}}$ is a multiplicative function equalling $\chi_P(t)/t^{\mathrm{rk(P)}}$, so an extra factor of $t^{\mathrm{rk(P)}}$ is introduced. □

Symmetric chain decompositions for generalized shuffle posets and for noncrossing partition lattices for the classical reflection groups are provided in [He1], [He2], respectively, by decomposing the posets into symmetrically placed product of chain subposets. These may alternatively be obtained by restricting our chain decompositions to 1-chains. Reiner originally provided symmetric chain decompositions for the noncrossing partition lattices for the classical reflection groups in [Re] by a different approach, relying on recursive structure in the poset.

Restriction of the chain decomposition for the noncrossing partition lattice to saturated chains gives a decomposition into the orbits of the local symmetric group action given in [St3], while similar restriction for the noncrossing partition lattices for the classical reflection groups gives the decomposition of [He1] into local action orbits for types B and D. Restriction of our chain decomposition for traditional shuffle posets to saturated chains gives the orbits of the local symmetric group action for shuffle posets defined in [SS]. Stanley introduced the notion of local symmetric group action on poset saturated chains in [St2]; further development of this notion may be found in [FS], [Gr], and [Ha].

Finally, let us relate chain decompositions to shellability and supersolvability. EL-labellings for the noncrossing partition lattice and traditional posets of shuffles specify Boolean algebras comprising homology bases, by taking the Boolean

algebras which contribute decreasing chains in the EL-labelling to our chain decomposition. The chain decompositions partially order these Boolean algebras, assigning each poset chain to the earliest of these Boolean algebras containing it.

Whenever a chain decomposition into product of chain subposets has a single saturated chain which belongs to all of these subposets, this chain must be an M-chain, implying supersolvability. In this manner, the chain decomposition for type A noncrossing partition lattices brought to our attention the existence of a saturated chain that belongs to each of the product of chains subposets in the decomposition and thereby turns out to be an M-chain. A similar argument in [He2] shows that shuffle posets of multisets and k-shuffle posets are supersolvable.

Theorem 6.3. *The noncrossing partition lattice $NC^A(n)$ is supersolvable.*

PROOF. Consider the saturated chain C in the type A noncrossing partition lattice which has the partition with one nontrivial block $\{1, \ldots, i+1\}$ as its element of rank i. This is an M-chain because it lies in each of the product of chain subposets giving rise to our chain decomposition; since each poset chain C' belongs to one of these product of chains subposets which also contains C, the claim follows from the modularity of this product of chains together with the fact that the chains C, C' together generate exactly this product of chains. □

In the noncrossing partition lattices for the other classical reflection groups types, the analogous chain no longer lies in all the product of chain subposets giving rise to our chain decomposition, and we are not aware of the existence of any M-chain.

7. Acknowledgments

The author thanks her thesis advisor Richard Stanley for suggesting flag f-vectors, generalized shuffle posets and noncrossing partition lattices as a possible area of study. She further thanks him as well as Paul Edelman, Vic Reiner and Rodica Simion for helpful discussions.

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