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STOCHASTIC MODEL FOR ULTRASLOW DIFFUSION

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Abstract. Ultraslow diffusion is a physical model in which a plume of diffusing particles spreads at a logarithmic rate. Governing partial differential equations for ultraslow diffusion involve fractional time derivatives whose order is distributed over the interval from zero to one. This paper develops the stochastic foundations for ultraslow diffusion based on random walks with a random waiting time between jumps whose probability tail falls off at a logarithmic rate. Scaling limits of these random walks are subordinated random processes whose density functions solve the ultraslow diffusion equation.

1. Introduction

The classical diffusion equation $\partial c/\partial t = \partial^2 c/\partial x^2$ governs the scaling limit of a random walk where IID particle jumps have zero mean and finite variance. The probability density $c(x,t)$ of the Brownian motion scaling limit $B(t)$ solves the diffusion equation, and represents the relative concentration of a cloud of diffusing particles. Self-similarity $B(ct) \stackrel{d}{=} c^{1/2}B(t)$ implies that particles spread at the rate $t^{1/2}$ in this classical model. In many practical applications the diffusion is anomalous: spreading rate is slower (subdiffusion) or faster (superdiffusion) than the classical model predicts, and/or plume shape is non-Gaussian. Anomalous superdiffusion can be modeled using infinite variance particle jumps that lead to space-fractional derivatives in the governing partial differential equation [6, 10, 25, 26]. Anomalous subdiffusion can be modeled using IID infinite mean waiting times between particle jumps, leading to a fractional time derivative in the governing partial differential equation [29, 36, 41, 48]. Continuous time random walks (CTRW) with IID waiting times between IID particle jumps were introduced in [32, 39]. Some recent surveys of their wide application in physics and connections with fractional governing equations are given in [18, 23, 31, 46].

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Ultraslow subdiffusion occurs when the spreading rate of a plume is logarithmic. Several examples from polymer physics, particles in a quenched force field, random walks in random media, and nonlinear dynamics are given in [19, 15, 34, 40, 43]. Recently a connection has been established between ultraslow kinetics and distributed-order time-fractional derivatives in the diffusion equation [11, 12, 13, 45]. In this model, the first time derivative in the classical diffusion equation is replaced by a fractional derivative of order $0 < \beta < 1$ as in the usual subdiffusive model, and then the order β of the fractional time derivative is randomized according to some probability density $p(\beta)$ on $0 < \beta < 1$. When β is fixed and nonrandom, the relevant CTRW model has waiting times in the domain of attraction of a β-stable subordinator, and CTRW scaling limits involve subordination to the inverse stable subordinator [29, 28]. Randomizing β leads to waiting times with a slowly varying probability tail. Limit theorems for these random walks were developed in [30] using nonlinear scaling, the usual approach for slowly varying tails [16, 22, 47].

In this paper, using a triangular array approach instead of the nonlinear scaling used in [30], we give a more detailed description of possible scaling limits together with asymptotic behavior of moments. Furthermore we show that our approach actually gives a stochastic solution to the distributed-order time-fractional diffusion equations and we provide explicit formulas for the solutions of those equations. Those solutions are density functions of subordinated stochastic process, where the subordinator is the inverse of the limit process of the triangular array that governs waiting times between particle jumps. We also show that, complementary to results in [14], a renewal process in which the waiting time between jumps has a slowly varying probability tail can be analyzed in much more detail. These results may be of independent interest. Finally we note that the general stochastic solutions to distributed-order time-fractional diffusion equations that we develop here may be useful in other contexts [44].

This paper is organized as follows: In Section 2 we define a generalization of the classical continuous time random walk (CTRW) model, using a triangular array of waiting times. In Section 3 a special triangular array with slowly varying tails is considered and the limiting Lévy process together with its hitting time process is analyzed. These results are then used in Section 4 to derive a limit theorem for generalized CTRWs with slowly varying waiting times and jumps in some generalized domain of attraction. We then derive various properties of the limiting process and we show that the density of this limiting process solves a variant of the distributed order time fractional diffusion equation considered in [11, 12].

2. Generalized CTRW

Given any scale $c \geq 1$, let $J_1^{(c)}, J_2^{(c)}, \ldots$ be nonnegative and independent and identically distributed (i.i.d.) random variables, modelling the waiting times between particle jumps at scale c. Let

$$T^{(c)}(0) = 0 \quad \text{and} \quad T^{(c)}(t) = \sum_{i=1}^{[t]} J_i^{(c)}, \tag{2.1}$$

so that $T^{(c)}(n)$ is the time of the nth jump at scale c. Let

$$N_t^{(c)} = \max\{n \geq 0 : T^{(c)}(n) \leq t\} \tag{2.2}$$

be the number of jumps by time $t \geq 0$ at scale c.

To model the particle jumps let $Y_1, Y_2, \ldots$ be i.i.d. $\mathbb{R}^d$-valued random vectors. Let $S(0) = 0$ and $S(t) = \sum_{i=1}^{[t]} Y_i$, so that $S(n)$ is the position of the particle after n jumps at scale $c = 1$. We assume that Y_1 belongs to the generalized domain of attraction of some full operator stable law with exponent E. Then there exists a norming function $B \in \mathrm{RV}(-E)$, meaning that $B(c) \in \mathrm{GL}(\mathbb{R}^d)$ for all $c > 0$ and $B(\lambda c)B(c)^{-1} \to \lambda^{-E}$ as $c \to \infty$ for any $\lambda > 0$, such that

$$\{B(c)S(ct)\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \{A(t)\}_{t\geq 0} \quad \text{as } c \to \infty \tag{2.3}$$

where $\{A(t)\}_{t\geq 0}$ is an operator Lévy motion with $A(t) \overset{d}{=} t^E A(1)$. Here $\overset{f.d.}{\Longrightarrow}$ denotes convergence in distribution of all finite dimensional marginals. See [27], Example 11.2.18 for details.

At scale $c \geq 1$ the jumps are given by $B(c)Y_i$ and hence $B(c)S(n)$ is the position of a particle after n jumps at scale c. Therefore

$$X^{(c)}(t) = B(c)S(N_t^{(c)}) \tag{2.4}$$

describes the position of a particle a time $t \geq 0$ and scale c. We call $\{X^{(c)}(t)\}_{t\geq 0}$ a *generalized continuous time random walk*.

Remark 2.1. The classical continuous time random walk model considered in [29] is a special case of our construction above. In fact, assume that $J_1, J_2, \ldots$ are nonnegative and i.i.d. belonging to the domain of attraction of some β-stable law with $0 < \beta < 1$. Then, for some norming function $b \in \mathrm{RV}(-1/\beta)$ we have

$$b(c) \sum_{i=1}^{[ct]} J_i \Rightarrow D(t) \quad \text{as } c \to \infty$$

where $\{D(t)\}_{t \geq 0}$ is a β-stable subordinator. If we set $J_i^{(c)} = b(c) J_i$, then $T^{(c)}(n) = b(c) \sum_{i=1}^{n} J_i$ is the time of the nth jump at scale $c \geq 1$. In this case the generalized CTRW converges as $c \to \infty$ to a limit process $M(t)$ whose density $h(t, x)$ solve the fractional partial differential equation

$$\frac{\partial^\beta h(x,t)}{\partial t^\beta} = L h(x,t) + \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{2.5}$$

Here $\delta(x)$ is the Dirac delta function, the fractional derivative $\partial^\beta h(x,t)/\partial t^\beta$ is defined as the inverse Laplace transform of $s^\beta \tilde{h}(x,s)$, where $\tilde{h}(x,s) = \int_0^\infty e^{-st} h(x,t)\, dt$ is the usual Laplace transform, and L is the generator of the continuous convolution semigroup associated with the Lévy process $\{A(t)\}_{t \geq 0}$. For example, if $\{A(t)\}_{t \geq 0}$ is a one-dimensional Brownian motion then $L_x = \partial^2/\partial x^2$. See [29] for more details. A different norming scheme is used in [5] for $J_i > 0$ belonging to the domain of attraction of some β-stable law with $1 < \beta < 2$. Now for some norming function $b \in \mathrm{RV}(-1/\beta)$ we have

$$c^{-1}\mu[ct] + b(c) \sum_{i=1}^{[ct]} (J_i - \mu) \Rightarrow \bar{D}(t) \quad \text{as } c \to \infty$$

where $\mu = \mathbb{E} J_i$ and $\{\bar{D}(t)\}_{t \geq 0}$ is a totally positively skewed β-stable Lévy motion with drift such that $\mathbb{E} D(t) = \mu t$. Letting $J_i^{(c)} = b(c)(J_i - \mu) + c^{-1}\mu$, the resulting generalized CTRW limit process has a density that solves a fractional partial differential equation similar to (2.5) but with both a first order and a β-order time derivative on the left-hand side.

3. The time process

In this section we construct and analyze a class of specific triangular arrays $\{J_i^{(c)} : i \geq 1, c \geq 1\}$ which corresponds to waiting times with slowly varying tails. It is shown that the corresponding partial sum processes $\{T^{(c)}(t)\}_{t \geq 0}$ defined by (2.1) converge to a class of Lévy processes complementing β-stable subordinators to the limiting case $\beta = 0$.

Our approach gives a much larger class of possible limiting processes than the nonlinear scaling of a random walk with slowly varying tails considered in [14, 22, 47]. There only one process, the so-called extremal process, can appear. See [16] for details on extremal processes. Our approach decomposes the case $\beta = 0$ of slowly varying tails into a family of different processes described by an additional parameter $\alpha > 0$, where any positive α is possible.

Stimulated by [11], our approach is based on the following idea: Given a measurable nonnegative function $p :]0,1[\to \mathbb{R}_+$ with $0 < \int_0^1 p(\beta) d\beta < \infty$ and some constant $C > 0$ let

$$L(t) = C \int_0^1 t^{-\beta} p(\beta)\, d\beta \tag{3.1}$$

for $t > 0$. In the following we always assume that the function p is defined on $\mathbb{R}_+$ but vanishes outside $]0,1[$. Observe that L is decreasing and continuous. Moreover L is a mixture of the tail functions $Ct^{-\beta}$ with respect to $p(\beta)$. The following lemma describes the behavior of L near infinity in terms of regular variation of p. Recall that a function R is regularly varying at infinity with exponent $\gamma \in \mathbb{R}$, if R is measurable, eventually positive and $R(\lambda t)/R(t) \to \lambda^\gamma$ as $t \to \infty$ for any $\lambda > 0$. We write $R \in \mathrm{RV}_\infty(\gamma)$ in this case. Similarly, R is called regularly varying at zero with exponent $\gamma \in \mathbb{R}$, if R is measurable, positive in some neighborhood $(0, t_0)$ of the origin and $R(\lambda t)/R(t) \to \lambda^\gamma$ as $t \to 0$. We write $R \in \mathrm{RV}_0(\gamma)$ in this case. Note that $R(t) \in \mathrm{RV}_0(\gamma)$ if and only if $R(1/t) \in \mathrm{RV}_\infty(-\gamma)$.

Lemma 3.1. *For $\alpha > 0$ let $p \in \mathrm{RV}_0(\alpha - 1)$ and define $L(t)$ by (3.1). Then there exists a function $L^* \in \mathrm{RV}_\infty(0)$ such that*

$$L(t) = (\log t)^{-\alpha} L^*(\log t). \tag{3.2}$$

Especially $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and $L \in \mathrm{RV}_\infty(0)$, so L is slowly varying at infinity. Conversely, if for L defined by (3.1) we have $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and $\alpha > 0$, then $p \in \mathrm{RV}_0(\alpha - 1)$.

Proof. First note that since $p \in \mathrm{RV}_0(\alpha - 1)$ with $\alpha > 0$, we have for any $\delta > 0$ there exists a $\beta_0 > 0$ and some constant K such that $p(\beta) \le K\beta^{\alpha-1-\delta}$ for all $0 < \beta \le \beta_0$ (see, e.g., [42] p.18). Hence $\int_0^1 p(\beta) d\beta$ is finite and positive. Moreover

$$L(t) = C \int_0^1 e^{-\beta \log t} p(\beta) d\beta = C\tilde{p}(\log t)$$

where $\tilde{p}(s) = \int_0^1 e^{-s\beta} p(\beta) d\beta$ denotes the Laplace transform of a function p with $\mathrm{supp}(p) \subset [0,1]$. Since p vanishes outside the interval $[0,1]$, it is ultimately monotone in the sense of Feller [17], p.446. Then, since $p \in \mathrm{RV}_0(\alpha-1)$ by Theorem 4 on p. 446 of [17] we know $\tilde{p} \in \mathrm{RV}_\infty(-\alpha)$, so $\tilde{p}(s) = s^{-\alpha} L^*(s)$ for some $L^* \in \mathrm{RV}_\infty(0)$. Hence (3.2) holds.

Conversely, if $L(t) = R(\log t)$ for some $R \in \mathrm{RV}_\infty(-\alpha)$ and some $\alpha > 0$, since $L(t) = C\tilde{p}(\log t)$, we have $\tilde{p}(u) = C^{-1} R(u)$. Using Theorem 4 on p. 446 of [17] again, we conclude $p \in \mathrm{RV}_0(\alpha - 1)$ and the proof is complete. □

We now construct a triangular array $\{J_i^{(c)} : i \geq 1, c \geq 1\}$ with i.i.d. rows $J_1^{(c)}, J_2^{(c)}, \ldots$ of nonnegative random variables. In the following we assume that $p \in \mathrm{RV}_0(\alpha - 1)$ for some $\alpha > 0$ is supported in $[0,1]$. Then we can take $C^{-1} = C^{-1}(p) = \int_0^1 p(\beta) d\beta$ is finite and positive, so Cp is a probability density. We will assume without loss of generality, that $C = 1$ so p is a probability density on $[0,1]$. Note that by Lemma 3.1 the function $L(t) = \int_0^1 t^{-\beta} p(\beta)\, d\beta$ is in $\mathrm{RV}_\infty(0)$ with $L(t) = (\log t)^{-\alpha} L^*(\log t)$ for $t > 1$. We do need an additional integrability condition on $p(\beta)$ for $\beta \to 1$. This condition does not change the asymptotic behavior of $L(t)$ near infinity, but is necessary for our analysis. We assume that p also fulfills

$$\int_0^1 \frac{p(\beta)}{1-\beta}\, d\beta < \infty. \tag{3.3}$$

Note that (3.3) trivially holds true, if p vanishes in some open neighborhood of one.

Now let $B_1, B_2, \ldots$ be i.i.d. with density p. Given any scale $c \geq 1$ let $J_1^{(c)}, J_2^{(c)}, \ldots$ be nonnegative i.i.d. random variables such that for any $0 < \beta < 1$ we have

$$P\{J_i^{(c)} > u | B_i = \beta\} = \begin{cases} 1 & 0 \leq u < c^{-1/\beta} \\ c^{-1} u^{-\beta} & u \geq c^{-1/\beta} \end{cases}. \tag{3.4}$$

Then the density $\psi_c(u|\beta)$ of $J_i^{(c)}$ given $B_i = \beta$ is

$$\psi_c(u|\beta) = \begin{cases} 0 & 0 \leq u < c^{-1/\beta} \\ c^{-1} \beta u^{-\beta-1} & u \geq c^{-1/\beta} \end{cases}. \tag{3.5}$$

Remark 3.2. If we define for $0 < \beta < 1$

$$P\{J_1 > t | B_1 = \beta\} = \begin{cases} 1 & 0 \leq t < 1 \\ t^{-\beta} & t \geq 1 \end{cases}$$

we get by letting $u = c^{-1/\beta} t$ that

$$P\{c^{-1/\beta} J_1 > u | B_1 = \beta\} = \begin{cases} 1 & 0 \le u < c^{-1/\beta} \\ c^{-1} u^{-\beta} & u \ge c^{-1/\beta} \end{cases}$$

so conditionally on $B_1 = \beta$ we have $J_1^{(c)} \stackrel{d}{=} c^{-1/\beta} J_1$. Moreover, for $t \ge 1$

$$P\{J_1 > t\} = \int_0^1 t^{-\beta} p(\beta) d\beta$$

so by Lemma 3.1 J_1 has a slowly varying tail.

Remark 3.3. An application of ultraslow diffusion to disorded systems in [13] illustrates the physical meaning of the generalized CTRW model described here. The parameter $\beta = B_i$ relates to the shallowness of a potential well from which a particle must escape, and the waiting time J_i until escape from the well has a probability tail that falls off like a power law with exponent β. The probability density $p(\beta)$ governs the depth distribution for potential wells, and the index α indicates the scarcity of very deep wells.

Theorem 3.4. *Given $p \in \mathrm{RV}_0(\alpha - 1)$ for some $\alpha > 0$ as above and define the triangular array $\{J_i^{(c)} : 1 \le i \le [ct], c \ge 1\}$ by (3.4). Assume that (3.3) holds. Then for the partial sum process $\{T^{(c)}(t)\}_{t \ge 0}$ defined by (2.1) we have*

$$\{T^{(c)}(ct)\}_{t\ge 0} \stackrel{f.d.}{\Longrightarrow} \{D(t)\}_{t \ge 0} \quad \text{as } c \to \infty, \tag{3.6}$$

where $\{D(t)\}_{t\ge0}$ is a subordinator such that $D(1)$ has Lévy-Khinchin representation $[0, 0, \phi]$ with Lévy measure ϕ given by

$$\phi(u, \infty) = \int_0^1 u^{-\beta} p(\beta)\, d\beta \tag{3.7}$$

for any $u > 0$, and furthermore $\phi(u, \infty) = (\log u)^{-\alpha} L^*(\log u)$ for any $u > 1$ where $\alpha > 0$ and $L^* \in \mathrm{RV}_\infty(0)$.

Proof. Since $T^{(c)}(ct) = \sum_{i=1}^{[ct]} J_i^{(c)}$ is a sum of i.i.d. random variables, the convergence of all finite dimensional marginals follows from the convergence for one fixed $t > 0$ by considering increments. See [27], Example 11.2.18 for details. Fix any $t > 0$ and observe that $\{J_i^{(c)} : 1 \le i \le [ct], c \ge 1\}$ is an infinitesimal triangular array. By standard convergence criteria for triangular arrays, see e.g. [27], Theorem 3.2.2, we know that

$$T^{(c)}(ct) - a_{[ct]} \Rightarrow D(t) \quad \text{as } c \to \infty \tag{3.8}$$

where

$$a_{[ct]} = [ct] \int_0^R x \, dP_{J_1^{(c)}}(x) \tag{3.9}$$

for some $R > 0$ and $D(t)$ has Lévy-Khinchin representation $[0, 0, t \cdot \phi]$ if

$$[ct] \cdot P_{J_1^{(c)}} \to t \cdot \phi \quad \text{as } c \to \infty \tag{3.10}$$

and

$$\lim_{\varepsilon \downarrow 0} \limsup_{c \to \infty} \, [ct] \int_0^\varepsilon u^2 \, dP_{J_1^{(c)}}(u) = 0. \tag{3.11}$$

Fix any $u > 0$. Then, for all large c we obtain from (3.4) that

$$\begin{aligned} [ct] P\{J_1^{(c)} > u\} &= [ct] \int_0^1 P\{J_1^{(c)} > u | B_1 = \beta\} p(\beta) d\beta \\ &= \frac{[ct]}{c} \int_0^1 u^{-\beta} p(\beta) d\beta \\ &\to t \cdot L(u) = t \cdot \phi(u, \infty) \end{aligned}$$

as $c \to \infty$. Hence (3.10) holds and by Lemma 3.1 the Lévy measure ϕ has the form (3.7). Moreover, for the Gaussian part we compute using (3.5) that

$$\begin{aligned} [ct] \int_0^\varepsilon u^2 \, dP_{J_1^{(c)}}(u) &= [ct] \int_0^\varepsilon u^2 \int_0^1 \psi_c(u|\beta) p(\beta) \, d\beta \, du \\ &= [ct] \int_0^1 \int_0^\varepsilon u^2 \psi_c(u|\beta) \, du \, p(\beta) \, d\beta \\ &= [ct] \int_0^1 \int_{c^{-1/\beta}}^\varepsilon u^2 \beta c^{-1} u^{-\beta-1} \, du \; p(\beta) \, d\beta \\ &= \frac{[ct]}{c} \int_0^1 \frac{\beta}{2-\beta} \big(\varepsilon^{2-\beta} - c^{1-2/\beta}\big) p(\beta) \, d\beta \\ &= \frac{[ct]}{c} \int_0^1 \varepsilon^{2-\beta} \frac{\beta}{2-\beta} p(\beta) \, d\beta - \frac{[ct]}{c} \int_0^1 c^{1-2/\beta} \frac{\beta}{2-\beta} p(\beta) \, d\beta \end{aligned}$$

Observe that $\beta/(2-\beta) \le 1$ and $1 - 2/\beta \le -1$. Then dominated convergence yields

$$\limsup_{c \to \infty} [ct] \int_0^\varepsilon u^2 dP_{J_1^{(c)}}(u) = t \int_0^1 \varepsilon^{2-\beta} \frac{\beta}{2-\beta} p(\beta) d\beta \to 0 \quad \text{as } \varepsilon \to 0.$$

Hence (3.11) holds and therefore (3.8) holds true. Note that since ϕ has a Lebesgue density any $R > 0$ in (3.9) is possible. We show now that the shifts $a_{[ct]}$ can be made arbitrary small for all large c, by choosing $R > 0$ small enough. This implies that we can choose $a_{[ct]} = 0$ for all $c \geq 1$ and then (3.8) holds without $a_{[ct]}$. For $R > 0$ we get from (3.5) that

$$
\begin{aligned}
[ct] \int_0^R x \, dP_{J_1^{(c)}}(x) &= [ct] \int_0^R x \int_0^1 \psi_c(x|\beta) p(\beta) \, d\beta \, dx \\
&= \frac{[ct]}{c} \int_0^1 \int_{c^{-1/\beta}}^R x^{-\beta} \, dx \, \beta p(\beta) \, d\beta \\
&= \frac{[ct]}{c} \int_0^1 \frac{\beta}{1-\beta} R^{1-\beta} p(\beta) \, d\beta - \frac{[ct]}{c} \int_0^1 c^{1-1/\beta} \frac{\beta}{1-\beta} p(\beta) \, d\beta \\
&= I(c,R) - J(c).
\end{aligned}
$$

Now, since $1 - 1/\beta < 0$ we get from (3.3) and dominated convergence that $J(c) \to 0$ as $c \to \infty$. Moreover, by the same argument we see that $I(c,R) \to 0$ as $R \to 0$ uniformly in $c \geq 1$. Hence $a_{[ct]}$ can be made arbitrary small for all large c by choosing $R > 0$ small enough. This concludes the proof. □

As an immediate corollary we get convergence in the Skorohod space $D([0,\infty),[0,\infty))$ in the J_1-topology.

Corollary 3.5. *Under the assumptions of Theorem 3.4 we also have*

$$\{T^{(c)}(ct)\}_{t \geq 0} \Rightarrow \{D(t)\}_{t \geq 0} \quad \text{as } c \to \infty$$

in the J_1–topology on $D([0,\infty),[0,\infty))$.

Proof. Note that the sample paths of $\{T^{(c)}(ct)\}_{t \geq 0}$ and $\{D(t)\}_{t \geq 0}$ are nondecreasing. Moreover, as a Lévy-process, $\{D(t)\}_{t \geq 0}$ is stochastically continuous. Then Theorem 3.4 together with Theorem 3 of [8] yields the assertion. □

Corollary 3.6. *Assume that $\{D(t)\}_{t \geq 0}$ is the limit process obtained in* (3.6) *with Lévy measure of the form* (3.7) *for some $p \in \mathrm{RV}_0(\alpha - 1)$ and some $\alpha > 0$. Let* $\log_+(x) = \max(\log x, 0)$. *Then for $\rho \geq 0$ and any $t > 0$ we have*

$$\mathbb{E}((\log_+ D(t))^\rho) \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha \end{cases}.$$

Proof. Let $g(x) = (\log(\max(x,e)))^\rho$. Then it is easy to see that the assertion follows if we can show that $\mathbb{E}(g(D(t))) < \infty$ if $\rho < \alpha$ and $\mathbb{E}(g(D(t))) = \infty$ if $\rho > \alpha$. Note that by Proposition 25.4 of [37] the function g is submultiplicative. Hence, by Theorem 25.3 of [37] the assertion follows if $\int_1^\infty g(x)d\phi(x) < \infty$ for $\rho < \alpha$ and $\int_1^\infty g(x)d\phi(x) = \infty$ for $\rho > \alpha$. By definition of g this is equivalent to

$$\int_e^\infty (\log x)^\rho \, d\phi(x) \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha \end{cases}. \tag{3.12}$$

Note that by (3.7) the Lévy measure ϕ has density $x \mapsto \int_0^1 x^{-\beta-1}\beta p(\beta)\, d\beta$. Then, by Tonelli's theorem and a change of variable we obtain

$$\begin{aligned}\int_e^\infty (\log x)^\rho \, d\phi(x) &= \int_0^1 \int_e^\infty (\log x)^\rho x^{-\beta-1} \, dx \, \beta p(\beta)\, d\beta \\ &= \int_0^1 \int_1^\infty y^\rho e^{-\beta y} \, dy \, \beta p(\beta)\, d\beta \\ &= \int_0^1 \Big(\int_\beta^\infty s^\rho e^{-s} \, ds\Big) \beta^{-\rho} p(\beta)\, d\beta.\end{aligned}$$

Since $\int_\beta^\infty s^\rho e^{-s} \, ds \to \Gamma(\rho+1)$ as $\beta \to 0$, it is easy to see that (3.12) follows from

$$\int_0^1 \beta^{-\rho} p(\beta)\, d\beta \begin{cases} < \infty & \rho < \alpha \\ = \infty & \rho > \alpha \end{cases}. \tag{3.13}$$

Since $p \in \mathrm{RV}(\alpha - 1)$, for any $\delta > 0$ there exist constants $C_1, C_2 > 0$ such that $C_1\beta^{\alpha-1+\delta} \le p(\beta) \le C_2\beta^{\alpha-1-\delta}$ for all $0 < \beta < 1$, a simple calculation shows that (3.13) holds true and the proof is complete. □

Corollary 3.7. *Assume that* $\{D(t)\}_{t\ge0}$ *is the limit process obtained in* (3.6) *with Lévy measure of the form* (3.7) *for some* $p \in \mathrm{RV}_0(\alpha - 1)$ *and some* $\alpha > 0$. *Then every* $D(t)$ *has a* C^∞*-density* $g(t, y)$ *and all derivatives of the density with respect to* y *vanish at infinity.*

Proof. We use the following sufficient condition due to Orey, see [37], Proposition 28.3. It says that, if there exists any $0 < \rho < 2$ such that

$$\liminf_{r\downarrow 0} r^{\rho-2} \int_{|x|\le r} x^2 d\phi(x) > 0 \tag{3.14}$$

then $D(t)$ has a C^∞ density with the desired property. Since the Lévy measure of $D(t)$ is $t \cdot \phi$ is suffices to show the assertion for $D(1)$.

Note that $u \mapsto \int_0^1 u^{-\beta-1}\beta p(\beta)d\beta$ is the density of ϕ. Note that since $p \in \mathrm{RV}_0(\alpha-1)$ with $\mathrm{supp}(p) \subset [0, 1]$ we know that for some $0 < \rho_0 < 1/2$ we have $p(\beta) > 0$ for all

$0 < \beta < 2\rho_0$. By Tonelli's theorem we have

$$\int_{|x|\le r} x^2\, d\phi(x) = \int_0^1 \int_0^r x^{1-\beta}\, dx\, \beta p(\beta)\, d\beta$$
$$= \int_0^1 r^{2-\beta} \frac{\beta}{2-\beta} p(\beta)\, d\beta.$$

Now, for $\rho = \rho_0$, we obtain

$$r^{\rho_0 - 2} \int_{|x|\le r} x^2\, d\phi(x) = \int_0^1 r^{\rho_0-\beta} \frac{\beta}{2-\beta} p(\beta)\, d\beta \ge \int_{\rho_0}^{2\rho_0} r^{\rho_0-\beta} \frac{\beta}{2-\beta} p(\beta)\, d\beta$$

and hence, by Fatou's lemma

$$\liminf_{r\downarrow 0} r^{\rho_0-2} \int_{|x|\le r} x^2\, d\phi(x) \ge \liminf_{r\downarrow 0} \int_{\rho_0}^{2\rho_0} r^{\rho_0-\beta} \frac{\beta}{2-\beta} p(\beta)\, d\beta$$
$$\ge \int_{\rho_0}^{2\rho_0} \big(\liminf_{r\downarrow 0} r^{\rho_0-\beta}\big) \frac{\beta}{2-\beta} p(\beta)\, d\beta$$
$$= \infty.$$

Therefore (3.14) holds with $\rho = \rho_0$ and the proof is complete. □

In view of the form of the Lévy measure ϕ of $\{D(t)\}_{t\ge 0}$ in (3.7), this process is not a stable process. However, our next result shows that $\{D(t)\}_{t\ge 0}$ is a *selfdecomposable* process in the sense of Definition 15.6 of [37]. For an introduction to selfdecomposable laws see Section 3.15 in [37] or Chapter 2 in [21]. We do not need this property in our analysis of CTRWs but we include it for sake of completeness.

Corollary 3.8. *The limiting process $\{D(t)\}_{t\ge 0}$ obtained in Theorem 3.4 above is selfdecomposable. That is, the distribution of any $D(t)$ is selfdecomposable.*

Proof. It suffices to show the assertion for $D(1)$. Since the Lévy measure ϕ of $D(1)$ has the density $\tilde{k}(x) = \int_0^1 x^{-\beta-1} \beta p(\beta)\, d\beta$ it follows from the Lévy-Khinchin representation (see, e.g., Theorem 8.1 in [37]) that the log-characteristic function ψ of the distribution of $D(1)$ has the form

$$\psi(\xi) = ia\xi + \int_0^\infty \big(e^{i\xi x} - 1 - i\xi x I(|x|\le 1)\big)\tilde{k}(x)\, dx$$
$$= ia\xi + \int_0^\infty \big(e^{i\xi x} - 1 - i\xi x I(|x|\le 1)\big)\frac{x\tilde{k}(x)}{x}\, dx$$

Since $k(x) = x\tilde{k}(x) = \int_0^1 x^{-\beta}\beta p(\beta)\, d\beta$ is decreasing on $(0,\infty)$, it follows from Corollary 15.11 of [37] that $D(1)$ has a selfdecomposable distribution. □

Let $\{D(u)\}_{u\geq 0}$ be the Lévy process obtained in Theorem 3.4. Note that, by Theorem 21.3 of [37] and the fact that the integral in (3.7) tends to infinity as $u \to 0$, the sample paths are strictly increasing. Note also that, by Theorem 48.1 in [37] and the fact that the Lévy measure (3.7) is concentrated on the positive reals, $D(u) \to \infty$ as $u \to \infty$ almost surely. Define the *hitting time* process by

$$E(t) = \inf\{x \geq 0 : D(x) > t\}. \tag{3.15}$$

Then it is easy to see that for $t, x \geq 0$

$$\{E(t) \leq x\} = \{D(x) \geq t\}. \tag{3.16}$$

Theorem 3.9. *Let $\{D(u)\}_{u\geq 0}$ be the Lévy process with Lévy measure given by* (3.7). *Then, for any $t > 0$ the random variable $E(t)$ defined by* (3.15) *has the density*

$$f(t,x) = \int_0^1 \int_0^t (t-y)^{-\beta} g(x,y)\, dy\, p(\beta)\, d\beta \tag{3.17}$$

where $g(x,\cdot)$ is the density of $D(x)$. We can also write

$$f(t,x) = \int_0^t L(t-y) g(x,y)\, dy \tag{3.18}$$

where L is given by (3.1).

Proof. First note that by Corollary 3.7 for $x > 0$ the density $g(x,\cdot)$ of $D(x)$ is a bounded C^∞–function. Let us compute the Laplace transform of $f(t,x)$ given by (3.17) with respect to t for any fixed $x > 0$. Observe that for $s > 0$ and $0 < \beta < 1$, by changing the order of integration, we get

$$\begin{aligned}
\int_0^\infty e^{-st} \int_0^t (t-y)^{-\beta} g(x,y)\, dy\, dt &= \int_0^\infty \Big(\int_y^\infty e^{-st}(t-y)^{-\beta}\, dt\Big) g(x,y)\, dy \\
&= s^{\beta-1}\Gamma(1-\beta) \int_0^\infty e^{-sy} g(x,y)\, dy \\
&= s^{\beta-1}\Gamma(1-\beta) e^{x\psi(s)}
\end{aligned}$$

and then the well-known formula for the Laplace transform of a subordinator (see, e.g., Theorem 30.1 of [37]) and (3.7) yield

$$\begin{aligned}\psi(s) &= \int_0^\infty (e^{-su}-1)\,d\phi(u)\\ &= \int_0^1 \Big(\int_0^\infty (e^{-su}-1)\beta u^{-\beta-1}\,du\Big)p(\beta)\,d\beta \qquad (3.19)\\ &= -\int_0^1 \Gamma(1-\beta)s^\beta p(\beta)\,d\beta.\end{aligned}$$

Note that in view of (3.3) the last integral exists since $\Gamma(x)\sim 1/x$ as $x\to 0$. Then, by (3.17) we obtain

$$\begin{aligned}\int_0^\infty e^{-st}f(t,x)\,dt &= e^{x\psi(s)}\int_0^1 s^{\beta-1}\Gamma(1-\beta)p(\beta)\,d\beta \qquad (3.20)\\ &= -\frac{1}{s}\psi(s)e^{x\psi(s)}.\end{aligned}$$

Now, for $t>0$ let $F_t(z)=P\{E(t)\le z\}$ denote the distribution function of $E(t)$. Note that in view of (3.16) we know $F_t(z)=P\{D(z)\ge t\}=\int_t^\infty g(z,y)\,dy=R_z(t)$. On the other hand, if $E(t)$ has density $f(t,x)$ given by (3.17) we should also have $L_z(t)=F_t(z)=\int_0^z f(t,x)\,dx$. Hence, the assertion follows if we can show that for any fixed $z>0$ we have $L_z(t)=R_z(t)$ for all $t>0$. By the uniqueness theorem for Laplace transforms (see, e.g., the Corollary on [17] p. 433), this follows if $\tilde L_z(s)=\tilde R_z(s)$ for $s>0$. In view of (3.20) we obtain

$$\begin{aligned}\tilde L_z(s) &= \int_0^\infty e^{-st}L_z(t)\,dt\\ &= \int_0^z\int_0^\infty e^{-st}f(t,x)\,dt\,dx\\ &= -\frac{1}{s}\psi(s)\int_0^z e^{x\psi(s)}\,dx\\ &= \frac{1}{s}\big(1-e^{z\psi(s)}\big).\end{aligned}$$

Moreover

$$\begin{aligned}\tilde{R}_z(s) &= \int_0^\infty e^{-st} R_z(t)\,dt \\ &= \int_0^\infty e^{-st} \int_t^\infty g(z,y)\,dy\,dt \\ &= \int_0^\infty \Big(\int_0^y e^{-st}\,dt\Big) g(z,y)\,dy \\ &= \frac{1}{s}\int_0^\infty \big(1-e^{-sy}\big) g(z,y)\,dy \\ &= \frac{1}{s}\big(1-e^{z\psi(s)}\big)\end{aligned}$$

and the proof is complete. □

As an immediate corollary to the proof of Theorem 3.9 we derive the governing equation of the density $f(t,x)$ of the hitting time $E(t)$. For suitable functions $h : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}$ we define the *Fourier-Laplace transform* (FLT) by

$$\bar{\hat{h}}(s,k) = \int_{\mathbb{R}^d} \int_0^\infty e^{i\langle k,x\rangle} e^{-st} h(t,x)\,dt\,dx \tag{3.21}$$

where $(s,k) \in (0,\infty) \times \mathbb{R}^d$. It follows from a general theory of FLTs on semigroups, that this transform has properties similar to the usual Fourier or Laplace transform. See [7] and Theorem 1 in [35] for details. Recall from [9, 33] that for $0 < \beta < 1$ and suitable functions g the Caputo derivative $(\frac{\partial}{\partial t})^\beta g(t)$ has Laplace transform $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ where $\tilde{g}(s) = \int_0^\infty e^{-st} g(t)\,dt$ denotes the usual Laplace transform. As in [3] we say that a function $h(t,x)$ is a *mild solution* to a fractional partial differential equation, if the FLT $\bar{\hat{h}}(s,k)$ solves the equivalent algebraic equation in Fourier-Laplace space.

Corollary 3.10. *Under the assumptions of Theorem 3.9 the density $f(t,x)$ of $E(t)$ is the mild solution of the distributed-order time-fractional partial differential equation*

$$\int_0^1 \Big(\frac{\partial}{\partial t}\Big)^\beta f(t,x)\Gamma(1-\beta)p(\beta)\,d\beta = -\frac{\partial}{\partial x} f(t,x) \quad , \quad f(0,x) = \delta(x). \tag{3.22}$$

Proof. For $s > 0$ and $k \in \mathbb{R}$ let $\bar{\hat{f}}(s,k)$ be the FLT of $f(t,x)$. Then it follows from (3.20) that

$$\bar{\hat{f}}(s,k) = -\frac{1}{s}\psi(s) \int_0^\infty e^{ikx} e^{x\psi(s)}\,dx = \frac{1}{s}\frac{\psi(s)}{ik+\psi(s)}.$$

Take Laplace transforms in (3.22) to get

$$\int_0^1 \big(s^\beta \tilde{f}(s,x) - s^{\beta-1}\delta(x)\big)\Gamma(1-\beta)p(\beta)\,d\beta = -\frac{\partial}{\partial x}\tilde{f}(s,x)$$

and use (3.19) to obtain

$$-\psi(s)\tilde{f}(s,x) + \frac{1}{s}\psi(s)\delta(x) = -\frac{\partial}{\partial x}\tilde{f}(s,x).$$

Then take Fourier transforms, using the fact that if $g(x)$ has Fourier transform $\mathcal{F}(g)(k)$ then $\mathcal{F}(g')(k) = (-ik)\mathcal{F}(g)(k)$, to get

$$-\psi(s)\hat{\tilde{f}}(s,k) + \frac{1}{s}\psi(s) = ik\hat{\tilde{f}}(s,k).$$

Then it follows easily that $f(t,x)$ is the mild solution of (3.22). □

Remark 3.11. Note that the proof of Theorem 3.9 and Corollary 3.10 also hold true, if we replace integration with respect to $p(\beta)\,d\beta$ for some probability density p supported in $[0,1]$ and satisfying (3.3) by integration with respect to a probability measure $\rho(d\beta)$ with support in $[0,1]$ and $\int_0^1 \frac{\rho(d\beta)}{1-\beta} < \infty$. Hence, if $\{D(u)\}_{u\geq 0}$ is a subordinator with Lévy measure ϕ of the form $\phi(u,\infty) = \int_0^1 u^{-\beta}\rho(d\beta)$ and having a bounded C^∞ density $g(u,\cdot)$ for $D(u)$, then the hitting time $E(t)$ has the density

$$f(t,x) = \int_0^1 \int_0^t (t-y)^{-\beta} g(x,y)\,dy\,\rho(d\beta). \tag{3.23}$$

Moreover, by Corollary (3.10) we know that $f(t,x)$ is the mild solution of

$$\int_0^1 \Big(\frac{\partial}{\partial t}\Big)^\beta f(t,x)\Gamma(1-\beta)\,\rho(d\beta) = -\frac{\partial}{\partial x}f(t,x)\ ,\ f(0,x) = \delta(x).$$

Especially, if $\rho = \varepsilon_\gamma$ is the point mass in some $0 < \gamma < 1$, then $\{D(u)\}_{u\geq 0}$ is a γ-stable subordinator and its density is given by $g(u,y) = u^{-1/\gamma}g_0(u^{-1/\gamma}y)$ where g_0 is the bounded C^∞-density of $D(1)$. Then the density $f(t,x)$ of the corresponding hitting time $E(t)$ is given by

$$f(t,x) = x^{-1/\gamma}\int_0^t (t-y)^{-\gamma} g_0(x^{-1/\gamma}y)\,dy.$$

On the other hand, in view of Corollary 3.2 of [29]

$$f(t,x) = \frac{t}{\gamma}x^{-1-1/\gamma}g_0(tx^{-1/\gamma}).$$

Hence, the density g_0 of a γ-stable random variable D solves the integral equation

$$g_0(z) = \frac{\gamma}{z}\int_0^z (z-y)^{-\gamma} g_0(y)\,dy. \tag{3.24}$$

To our knowledge this property of the density g_0 of a γ-stable random variable is new and may be of independent interest. Moreover, the density $f(t,x)$ of $E(t)$ in this case is the mild solution of

$$\Gamma(1-\gamma)\Big(\frac{\partial}{\partial t}\Big)^{\gamma} f(t,x) = -\frac{\partial}{\partial x} f(t,x)\ ,\ f(0,x)=\delta(x)$$

which agrees with (3.8) in [5].

Later we do need the asymptotic behavior of $\mathbb{E}(E(t))$ as $t\to\infty$. We present a more general result on the asymptotic behavior of all moments of $E(t)$. We write $f(x)\sim g(x)$ if $f(x)/g(x)\to 1$.

Theorem 3.12. *Let $E(t)$ be the hitting time of the subordinator $\{D(u)\}_{u\geq 0}$ obtained in Theorem 3.4 for $p\in \mathrm{RV}_0(\alpha-1)$ and some $\alpha>0$. Then there exists a function $\bar L\in \mathrm{RV}_\infty(0)$ such that for any $\gamma>0$*

$$\mathbb{E}(E(t)^\gamma) \sim (\log t)^{\alpha\gamma} \bar L(\log t)^{-\gamma} \quad \textit{as } t\to\infty.$$

Proof. Since $p\in\mathrm{RV}_0(\alpha-1)$ and $\Gamma(1)=1$ it follows that $q(\beta)=\Gamma(1-\beta)p(\beta)\in \mathrm{RV}_0(\alpha-1)$ as well. Note that by (3.3) and the fact that $\Gamma(x)\sim 1/x$ as $x\to 0$ we have $\int_0^1 q(\beta)\,d\beta<\infty$. Then, by Lemma 3.1, there exists a function $\bar L\in\mathrm{RV}_\infty(0)$ such that

$$\int_0^1 t^{-\beta} q(\beta)\,d\beta = (\log t)^{-\alpha}\bar L(\log t) \quad \text{as } t\to\infty.$$

Hence

$$I(s) = \int_0^1 s^\beta q(\beta)\,d\beta = \big(\log(1/s)\big)^{-\alpha}\bar L(\log(1/s)) \quad \text{as } s\to 0. \tag{3.25}$$

Note that by (3.19) we have $I(s)=-\psi(s)$. Fix any $\gamma>0$. Then, for $t>0$ we have by (3.16) and a well-known formula for fractional moments (see, e.g., Lemma 1 on p.150 of [17]) that

$$\begin{aligned} h_\gamma(t) = \mathbb{E}(E(t)^\gamma) &= \gamma\int_0^\infty x^{\gamma-1} P\{E(t)>x\}\,dx \\ &= \gamma\int_0^\infty x^{\gamma-1}P\{D(x)<t\}\,dx. \end{aligned} \tag{3.26}$$

Now let $F(t) = P\{D(x) < t\}$ denote the distribution function of $D(x)$ for some fixed $x > 0$. Then, by Theorem 30.1 of [37] together with (3.19), we get $\int_0^\infty e^{-st}\, dF(t) = e^{-xI(s)}$. Moreover, by integration by parts $\int_0^\infty e^{-st}\, dF(t) = s\int_0^\infty e^{-st} P\{D(x) < t\}\, dt$ and hence

$$\int_0^\infty e^{-st} P\{D(x) < t\}\, dt = \frac{1}{s} e^{-xI(s)}. \tag{3.27}$$

Using (3.27) and Tonelli we therefore compute

$$\tilde{h}_\gamma(s) = \int_0^\infty e^{-st} h_\gamma(t)\, dt = \frac{\gamma}{s}\int_0^\infty x^{\gamma-1} e^{-xI(s)}\, dx = \Gamma(\gamma+1)s^{-1}I(s)^{-\gamma}.$$

In view of (3.25) this implies

$$\tilde{h}_\gamma(s) = \Gamma(\gamma+1)s^{-1}\big(\log(1/s)\big)^{\alpha\gamma}\bar{L}(\log(1/s))^{-\gamma} \quad \text{as } s \to 0.$$

By a Tauberian theorem (see [17], Theorem 4 on p.446) we conclude

$$h_\gamma(t) \sim \frac{\Gamma(\gamma+1)}{\Gamma(\gamma)}\big(\log t\big)^{\alpha\gamma}\bar{L}(\log t)^{-\gamma} \quad \text{as } t \to \infty.$$

Note that in view of (3.26) the function $h_\gamma(t)$ is ultimately monotone. □

After investigating the hitting time process $\{E(t)\}_{t\geq 0}$ we now show that the rescaled counting process $\{N_t^{(c)}\}_{t\geq 0}$ defined by (2.2) converges to $\{E(t)\}_{t\geq 0}$.

Theorem 3.13. *Suppose that we are given a probability density $p \in \mathrm{RV}_0(\alpha-1)$ for some $\alpha > 0$ such that (3.3) holds as in Theorem 3.4. Define the triangular array $\Delta = \{J_i^{(c)} : 1 \leq i \leq [ct], c \geq 1\}$ by (3.4) and the counting process $\{N_t^{(c)}\}_{t\geq 0}$ by (2.2). Then*

$$\Big\{\frac{1}{c}N_t^{(c)}\Big\}_{t\geq 0} \overset{f.d.}{\Longrightarrow} \{E(t)\}_{t\geq 0} \quad \text{as } c \to \infty, \tag{3.28}$$

where $\{E(t)\}_{t\geq 0}$ is the hitting time process defined by (3.15) of the subordinator $\{D(u)\}_{u\geq 0}$ corresponding to the triangular array Δ obtained in Theorem 3.4.

Proof. Observe that for $t \geq 0$ and $c \geq 1$ we have $\{N_t^{(c)} \geq x\} = \{T^{(c)}(\lceil x\rceil) \leq t\}$ where $\lceil x\rceil$ denotes the smallest integer greater than or equal to $x \geq 0$. Note that by Corollary 3.7 and Theorem 3.9 both $D(u)$ and $E(t)$ have a density with respect to Lebesgue measure. Fix any $0 \leq t_1 < \cdots < t_m$ and $x_1, \ldots, x_m \geq 0$ and let $\forall i$ mean for

$i = 1, \ldots, m$. Since $T^{(c)}(x)$ has nondecreasing sample paths, Theorem 3.4 together with (3.16) imply

$$\begin{aligned} P\{c^{-1} N_{t_i}^{(c)} < x_i \ \forall i\} &= P\{N_{t_i}^{(c)} < cx_i \ \forall i\} \\ &= P\{T^{(c)}(\lceil cx_i \rceil) > t_i \ \forall i\} \\ &\geq P\{T^{(c)}(cx_i) > t_i \ \forall i\} \\ &\to P\{D(x_i) > t_i \ \forall i\} \\ &= P\{E(t_i) < x_i \ \forall i\} \end{aligned}$$

as $c \to \infty$. Also for any $\varepsilon > 0$ for all $c > 0$ sufficiently large we have

$$\begin{aligned} P\{c^{-1} N_{t_i}^{(c)} < x_i \ \forall i\} &= P\{N_{t_i}^{(c)} < cx_i \ \forall i\} \\ &= P\{T^{(c)}(\lceil cx_i \rceil) > t_i \ \forall i\} \\ &\leq P\{T^{(c)}(c(1+\varepsilon)x_i) > t_i \ \forall i\} \\ &\to P\{D((1+\varepsilon)x_i) > t_i \ \forall i\} \\ &= P\{E(t_i) < (1+\varepsilon)x_i \ \forall i\} \end{aligned}$$

as $c \to \infty$. Now let $\varepsilon \to 0$ and use Theorem 3.9 to complete the proof. □

Corollary 3.14. *Under the assumptions of Theorem 3.13 we have*

$$\{\frac{1}{c} N_t^{(c)}\}_{t \geq 0} \Rightarrow \{E(t)\}_{t \geq 0} \quad \text{as } c \to \infty$$

in the J_1-topology on $D([0,\infty),[0,\infty))$.

Proof. Note that the sample paths of $\{N_t^{(c)}\}_{t \geq 0}$ and $\{E(t)\}_{t \geq 0}$ are nondecreasing. Moreover, since the sample path of $\{E(t)\}_{t \geq 0}$ are continuous, the process $\{E(t)\}_{t \geq 0}$ is stochastically continuous. Then Theorem 3.13 together with Theorem 3 in [8] yields the assertion. □

4. CTRW limit theorem

Assume that (Y_i) are i.i.d. $\mathbb{R}^d$-valued random vectors independent of the triangular array $\{J_i^{(c)} : 1 \leq i \leq [ct], c \geq 1\}$ of waiting times defined by (3.4). We assume that (3.3) holds. Moreover it is assumed that Y_1 belongs to the strict generalized domain of attraction of some full operator stable law with exponent E and (2.3) holds.

Theorem 4.1. *Under the assumptions of the beginning of this section we have for the generalized CTRW process* $\{X^{(c)}(t)\}_{t\geq 0}$ *defined in* (2.4) *that*

$$\{X^{(c)}(t)\}_{t\geq 0} \stackrel{f.d.}{\Longrightarrow} \{A(E(t))\}_{t\geq 0} \quad \text{as } c \to \infty.$$

Here $\{A(t)\}_{t\geq 0}$ *is the operator Lévy motion corresponding to the jumps* (Y_i) *and* $\{E(t)\}_{t\geq 0}$ *is the hitting time process corresponding to the subordinator* $\{D(u)\}_{u\geq 0}$ *obtained in Theorem 3.4.*

Proof. The proof is similar to the proof of Theorem 4.2 in [5], so we only sketch the argument. Fix any $0 < t_1 < \cdots < t_m$ and let $\forall i$ mean for $i = 1, \ldots, m$. Note that by Theorem 3.13

$$\Big(\frac{1}{c}N_{t_i}^{(c)}\ \forall i\Big) \Rightarrow \big(E(t_i)\ \forall i\big) \quad \text{as } c \to \infty.$$

Moreover, for any $x_1, \ldots, x_m \geq 0$ we know that

$$\big(B(c)S(cx_i)\ \forall i\big) \Rightarrow \big(A(x_i)\ \forall i\big) \quad \text{as } c \to \infty$$

uniformly on compact sets of $\mathbb{R}_+^m$ as was established in the proof of Theorem 4.2 in [5]. Independence of (Y_i) and $\{N_t^{(c)}\}$ yields

$$\begin{aligned}
P_{(X^{(c)}(t_i)\ \forall i)} &= P_{(B(c)S(N_{t_i}^{(c)})\ \forall i)} \\
&= \int_{\mathbb{R}_+^m} P_{(B(c)S(cx_i)\ \forall i)} dP_{(c^{-1}N_{t_i}^{(c)}\ \forall i)}(x_1, \ldots, x_m) \\
&\Rightarrow \int_{\mathbb{R}_+^m} P_{(A(x_i)\ \forall i)} dP_{(E(t_i)\ \forall i)}(x_1, \ldots, x_m) \\
&= P_{(A(E(t_i))\ \forall i)}
\end{aligned}$$

as $c \to \infty$, by a transfer theorem, Proposition 4.1 in [5]. □

Corollary 4.2. *Under the assumptions of Theorem 4.1, if* $A(1)$ *has no normal component, for every* $t > 0$ *the distribution* λ_t *of* $M(t) = A(E(t))$ *belongs to the domain of normal attraction of* $A(1)$*. That is, if* $m(t) = \mathbb{E}(E(t))$*, then for some sequence* (b_n) *of shifts*

$$\big(m(t)n\big)^{-E}\lambda_t^{*n} * \varepsilon_{b_n} \Rightarrow \nu \quad \text{as } n \to \infty,$$

where ν *is the distribution of* $A(1)$ *and* E *is an exponent of* ν*.*

Proof. Since by Theorem 3.12 we know that $m(t) = \mathbb{E}(E(t))$ is finite and ν is assumed to be a strictly operator stable law with exponent E having no normal component, the assertion follows from Corollary 4.2 of [24]. □

Remark 4.3. It follows from Theorem 4.1 of [24] that, under the additional condition that $A(1)$ has no normal component, the distribution λ_t of $M(t) = A(E(t))$ varies regularly with exponent E. See [27] for a comprehensive introduction to regularly varying measures on $\mathbb{R}^d$. Therefore various results on the tail and moment behavior of λ_t can be obtained from [27]. Let $a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of E. Then Theorem 8.2.14 in [27] implies that there exists a function $\rho : \Gamma \to \{a_p^{-1}, \ldots, a_1^{-1}\}$ such that for all $\theta \in \Gamma = \mathbb{R}^d \setminus \{0\}$ the radial moments

$$\int |\langle y, \theta\rangle|^\gamma \lambda_t(dy) = \mathbb{E}(|\langle M(t), \theta\rangle|^\gamma) \tag{4.1}$$

exist for $0 \le \gamma < \rho(\theta)$ and diverge for $\gamma > \rho(\theta)$. Corollary 8.2.15 in [27] implies that

$$\int \|y\|^\gamma \lambda_t(dy) = \mathbb{E}(\|M(t)\|^\gamma) \tag{4.2}$$

exists if $\gamma < 1/a_p$ and is infinite if $\gamma > 1/a_p$. Also, Theorem 6.4.15 in [27] gives the power law tail behavior of the truncated moments and tail moments

$$\int_{|\langle y,\theta\rangle| \le r} |\langle y, \theta\rangle|^\zeta \lambda_t(dy) \quad \text{and} \quad \int_{|\langle y,\theta\rangle| > r} |\langle y, \theta\rangle|^\eta \lambda_t(dy) \tag{4.3}$$

in terms of multivariable R-O variation. Roughly speaking, this result says that the tail $P(|\langle M(t), \theta\rangle| > r)$ falls off like $r^{-\rho(\theta)}$ as $r \to \infty$.

Recall from Theorem 7.2.7 of [27] that the full operator stable random vector $A(t)$ has a density $p(t, x)$ for any $t > 0$. As an immediate corollary to Theorem 4.1 we get:

Corollary 4.4. *Under the assumptions of Theorem 4.1, for every $t > 0$ the random vector $M(t) = A(E(t))$ has the density*

$$h(t, x) = \int_0^\infty p(u, x) f(t, u)\, du \tag{4.4}$$

where $f(t, u)$ is the density of $E(t)$ given by (3.17).

Proof. This is a simple conditioning argument using the fact that $\{A(t)\}_{t\ge0}$ and $\{E(t)\}_{t\ge0}$ are independent. □

Recall that as an infinitely divisible law, the operator stable random vector $A(t)$ has log-characteristic function $t \cdot \psi_A(k)$ so that $\mathbb{E}(e^{i\langle k,A(t)\rangle}) = e^{t\psi_A(k)}$. It is well known that, under some regularity conditions, the log-characteristic function of an infinitely divisible distribution is the symbol of the pseudo-differential operator defined by the generator

$$Lf(x) = \lim_{t\downarrow 0} \frac{T(t)f(x) - f(x)}{t}$$

of the corresponding C_0-semigroup $T(t)f(x) = \mathbb{E}[f(x - A(t))]$. In particular, for a C^∞ function $u : \mathbb{R}^d \to \mathbb{R}$ with compact support we define the *pseudo-differential operator* $L = \psi_A(iD_x)$ with *symbol* $\psi_A(k)$ by requiring $Lu(x)$ to have Fourier transform $\psi_A(k)\hat{u}(k)$. Since $\hat{u}(k)$ is rapidly decreasing it follows that, since ψ_A grows at a polynomial rate at infinity, the function $\psi_A(iD_x)u(x)$ is pointwise defined. Furthermore, it usually can be extended to larger spaces of functions (or even distributions), where the extension is also denoted by $\psi_A(iD_x)$. For example, a one-dimensional Brownian motion $A(t)$ with variance $2t$ has symbol $\psi_A(k) = -k^2$ and $L = \partial^2/\partial x^2$. For a one-dimensional α-stable Lévy motion, L is a fractional space derivative of order α, and for a d-dimensional operator stable Lévy motion, L is a multivariable fractional space derivative. For more details see [1, 2, 20, 25, 26, 28, 38]. Recall the definition of the Fourier-Laplace transform (FLT) from (3.21).

Theorem 4.5. *Let $h(t,x)$ be the density of $A(E(t))$ obtained in Corollary 4.4 above. Then h has FLT*

$$\bar{\hat{h}}(s,k) = \frac{1}{s}\frac{I(s)}{I(s) - \psi_A(k)}\ ,\ (s,k) \in (0,\infty) \times \mathbb{R}^d, \tag{4.5}$$

where

$$I(s) = -\psi(s) = \int_0^1 s^\beta \Gamma(1-\beta)p(\beta)\,d\beta. \tag{4.6}$$

Moreover, h is the mild solution of the distributed-order time-fractional partial differential equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x)\Gamma(1-\beta)p(\beta)\,d\beta = \psi_A(iD_x)h(t,x)\ ,\ h(0,x) = \delta(x). \tag{4.7}$$

Proof. Recall from Corollary 4.4 that $h(t,x)$ is given by (4.4). Moreover, since $|e^{\psi_A(k)}| \le 1$ we know $\operatorname{Re}\psi_A(k) \le 0$. In view of (3.20) we get

$$\begin{aligned}\hat{\tilde{h}}(s,k) &= \int_0^\infty \hat{p}(u,k)\tilde{f}(s,u)\,du \\ &= \frac{1}{s}I(s)\int_0^\infty e^{-u(I(s)-\psi_A(k))}\,du \\ &= \frac{1}{s}\frac{I(s)}{I(s)-\psi_A(k)},\end{aligned}$$

so (4.5) holds true. Equivalently $I(s)\hat{\tilde{h}}(s,k) - s^{-1}I(s) = \psi_A(k)\hat{\tilde{h}}(s,k)$ and in view of (3.19) this is equivalent to

$$\int_0^1 \left(s^\beta\hat{\tilde{h}}(s,k) - s^{\beta-1}\right)\Gamma(1-\beta)p(\beta)\,d\beta = \psi_A(k)\hat{\tilde{h}}(s,k).$$

Taking Laplace and then Fourier transforms in (4.7) as in the proof of Corollary 3.10 yields the same equation. Hence $h(t,x)$ is the mild solution of (4.7) and the proof is complete. □

Remark 4.6. In the degenerate case $A(t) = t$ the process $A(E(t)) = E(t)$ has density $f(t,x)$ given by Theorem 3.9. Note that this density solves (3.22) which is formally equivalent to (4.7) if we take $\psi_A(k) = ik$, which is the symbol of the semigroup generator $-\partial/\partial x$ for the associated semigroup $T(t)f(x) = f(x-t)$.

Example 4.7. Let $\{A(t)\}_{t\ge 0}$ be a one-dimensional Brownian motion with $\operatorname{Var}(A(t)) = 2t$. Then $\psi_A(k) = -k^2$ and the pseudodifferential operator $\psi_A(iD_x) = -(iD_x)^2 = D_x^2$, and by Theorem 4.5 the density $h(t,x)$ of $M(t) = A(E(t))$ has FLT

$$\hat{\tilde{h}}(s,k) = \frac{1}{s}\frac{I(s)}{I(s)+k^2}, \tag{4.8}$$

where $I(s)$ is given by (4.6). Furthermore $h(t,x)$ is the mild solution of the distributed-order time-fractional partial differential equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x)\Gamma(1-\beta)p(\beta)\,d\beta = \frac{\partial^2}{\partial x^2}h(t,x)\ ,\ h(0,x) = \delta(x). \tag{4.9}$$

Corollary 4.4 shows that in this case

$$h(t,x) = \int_0^\infty \frac{1}{\sqrt{4\pi u}}e^{-x^2/4u}f(t,u)\,du \tag{4.10}$$

where $f(t,u)$ is the density of $E(t)$ given by (3.17). Equation (4.9) first appeared in [13] together with (4.8). They show that $h(x,t)$ is a probability density for every $t > 0$ by using (4.10) along with the fact that (3.20) is completely monotone. The present paper improves on those results by extending (4.9) to the case of a more general pseudodifferential operator, and identifying the stochastic process for which $h(t,x)$ is a density. A simple conditioning argument along with Theorem 3.12 shows that the mean square displacement of a particle governed by (4.9) is

$$\mathbb{E}(M(t)^2) = 2\mathbb{E}(E(t)) \sim (\log t)^\alpha \bar{L}(\log t)^{-1}$$

as $t \to \infty$ for some $\bar{L} \in \mathrm{RV}_\infty(0)$. This agrees with [11, 12] and shows that (4.9) describes an ultraslow diffusion, where a cloud of diffusing particles spreads at the rate $(\log t)^{\alpha/2}$.

Remark 4.8. Following [11] we note that, in the case where $A(t)$ is a one-dimensional Brownian motion, the Fourier transform formula $\mathcal{F}[(c/2)e^{-c|x|}] = c^2/(c^2+k^2)$ along with (4.8) implies that

$$\tilde{h}(s,x) = \frac{I(s)^{1/2}}{2s} e^{-I(s)^{1/2}|x|}.$$

Under the additional assumption that $h(t,x)$ is ultimately monotone, a Tauberian theorem (Theorem 4 on p. 446 of Feller [17]) yields that

$$h(t,x) \sim \frac{I(1/t)^{1/2}}{2} e^{-I(1/t)^{1/2}|x|} \quad \text{as } t \to \infty.$$

If $p \in \mathrm{RV}_0(\alpha - 1)$ then it follows from Lemma 3.1 as in the proof of Theorem 3.12 that $I(1/t) = (\log t)^{-\alpha} L_1(\log t)$ for some $L_1 \in \mathrm{RV}_\infty(0)$. Hence $h(t,x)$ is asymptotically equivalent to a Laplace density whose variance grows like $(\log t)^\alpha$. A different stochastic model for ultraslow diffusion presented in [30], using nonlinear rescaling for the waiting time process, leads exactly to a Laplace limit with density

$$h_1(t,x) = \frac{(\log t)^{-\alpha/2}}{2} e^{-(\log t)^{-\alpha/2}|x|}.$$

Using the converse of the same Tauberian theorem yields

$$\tilde{h}_1(s,x) = \frac{(\log(1/s))^{-\alpha/2}}{2s} e^{-(\log(1/s))^{-\alpha/2}|x|}$$

and then the same Fourier tranform formula leads to (4.8) with $I(1/t) = (\log t)^{-\alpha}$. Now suppose that

$$I(s) = \int_0^\infty s^\beta q(\beta)\, d\beta = (\log(1/s))^{-\alpha}$$

for some function $q(\beta)$, which is equivalent to $\tilde{q}(s) = s^{-\alpha}$. In view of the Laplace transform pair $\mathcal{L}[t^{\alpha-1}/\Gamma(\alpha)] = s^{-\alpha}$ for $\alpha > 0$ this implies that $q(\beta) = \beta^{\alpha-1}/\Gamma(\alpha)$ supported on the positive real line $\beta > 0$. Then the uniqueness theorem for Laplace transforms implies that we cannot write $q(\beta) = \Gamma(1-\beta)p(\beta)$ for any $p(\beta)$ supported on $0 < \beta < 1$. Hence the family of Laplace densities $h_1(t,x)$ cannot be the mild solution of (4.9) for any choice of $p(\beta)$, so the two process densities are only asymptotically equivalent. This resolves an open question in [30].

Remark 4.9. As in Remark 3.11 we can also consider the more general equation

$$\int_0^1 \left(\frac{\partial}{\partial t}\right)^\beta h(t,x)\Gamma(1-\beta)\,\rho(d\beta) = \psi_A(iD_x)h(t,x)\ ,\ h(0,x) = \delta(x)$$

whose mild solution is given by (4.4) and (3.23). An application where $\psi_A(iD_x) = \partial^2/\partial x^2$ and $\rho(d\beta)$ consists of two atoms at $0 < \beta_1 < \beta_2 < 1$ is considered in [45], Section 4.2 and [12]. The results of this paper give a different and perhaps simpler proof that the solutions in these papers are probability distributions, and also illuminate the nature of the stochastic limit.

References

[1] W. Arendt, C. Batty, M. Hieber, and F. Neubrander (2001) *Vector-valued Laplace transforms and Cauchy problems.* Monographs in Mathematics, Birkhaeuser-Verlag, Berlin.

[2] B. Baeumer and M. Meerschaert (2001) Stochastic solutions for fractional Cauchy problems. *Frac. Calc. Appl. Anal.* **4**, 481–500.

[3] B. Baeumer, D.A. Benson and M. Meerschaert (2003) Advection and dispersion in time and space. Preprint available at http://unr.edu/homepage/mcubed/hit.pdf

[4] E. Barkai, R. Metzler and J. Klafter (2000) From continuous time random walks to the fractional Fokker-Planck equation. *Phys. Rev. E* **61** 132-138.

[5] P. Becker-Kern, M.M. Meerschaert and H.P. Scheffler (2003) Limit theorem for continuous time random walks with two time scales. *J. Applied Probab.*, to appear.

[6] D.A. Benson, S.W. Wheatcraft and M.M. Meerschaert, (2000) The fractional-order governing equation of Lévy motion. *Water Resour. Res.* **36**, 1413–1424.

[7] C. Berg, J.P. Reus Christensen and P. Ressel (1984) *Harmonic analysis on Semigroups.* Springer-Verlag, New York

[8] N. H. Bingham (1971) Limit theorems for occupation times of Markov processes. *Z. Wharsch. verw. Geb.* **17**, 1–22.

[9] M. Caputo (1967) Linear models of dissipation whose Q is almost frequency independent, Part II. *Geophys. J. R. Astr. Soc.* **13**, 529–539.

[10] A. Chaves (1998) A fractional diffusion equation to describe Lévy flights. *Phys. Lett. A* **239** , pp. 13–16.

[11] A. V. Chechkin, J. Klafter and I.M. Sokolov (2003) Fractional Fokker-Plank equation for ultraslow kinetics. *Europhys. Lett.* **63**(3), 326–332.

[12] A. V. Chechkin, R. Gorenflo, I. M. Sokolov and V. Yu. Gonchar (2003) Distributed order time fractional diffusion equation. *Frac. Calc. Appl. Anal.* **6**(3), 259–279.

[13] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov (2002) Retarding subdiffusion and accelerating superdiffusion governed by distributed-order fractional diffusion equations. *Phys Rev. E* **66**, 046129–046135.

[14] D. A. Darling (1952) The influence of the maximum term in the addition of independent random variable. *Trans. Amer. Math. Soc.***73**, 95–107.

[15] J. Dräger and J. Klafter (2000) Strong anomaly in diffusion generated by iterated maps. *Phys. Rev. Lett.* **84**, 59986001.

[16] M. Dwass (1964) Extermal processes. *Ann. Math. Statist.* **35**, no. 4, 1718–1725.

[17] Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, Vol. II, 2nd Ed., Wiley, New York.

[18] R. Gorenflo and F. Mainardi (2003) Fractional diffusion processes: Probability distribution and continuous time random walk. Preprint.

[19] F. Iglói (1999) Anomalous diffusion in aperiodic environments. *Phys. Rev. E* **59**, 1465-1474.

[20] N. Jacob (1996) *Pseudo-Differential Operators and Markov Processes.* Berlin : Akad. Verl.

[21] Z. Jurek and J.D. Mason (1993) *Operator-Limit Distributions in Probability Theory.* Wiley, New York.

[22] Y. Kasahara (1986) A limit theorem for sums of i.i.d. random variables with slowly varying tail probability. *J. Math. Kyoto Univ.* **26**, no. 3, 437–443.

[23] J. Klafter, A. Blumen and M.F. Shlesinger (1987) Stochastic pathways to anomalous diffusion. *Phys. Rev. A* **35**, 3081–3085.

[24] T.J. Kozubowski, M.M. Meerschaert and H.P. Scheffler (2003) ν-operator stable laws. *Publ. Math. Debrecen.* **63**(4), 569–585.

[25] M.M. Meerschaert, D.A. Benson and B. Baeumer (1999) Multidimensional advection and fractional dispersion. *Phys. Rev. E* **59**, 5026–5028.

[26] M.M. Meerschaert, D.A. Benson and B. Baeumer (2001) Operator Lévy motion and multiscaling anomalous diffusion. *Phys. Rev. E* **63**, 1112–1117.

[27] M.M. Meerschaert and H.P. Scheffler (2001) *Limit Distributions for Sums of Independent Random Vectors: Heavy Tails in Theory and Practice.* Wiley Interscience, New York.

[28] M.M. Meerschaert, D.A. Benson, H.P. Scheffler and B. Baeumer (2002) Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E* **65**, 1103–1106.

[29] M.M. Meerschaert and H.P. Scheffler (2003) Limit theorems for continuous time random walks with infinite mean waiting times. *J. Applied Probab.*, to appear. Available at http://unr.edu/homepage/mcubed/LimitCTRW.pdf

[30] M.M. Meerschaert and H.P. Scheffler (2004) Limit theorems for continuous time random walks with slowly varying waiting times. Preprint available at http://unr.edu/homepage/mcubed/ultraslow.pdf

[31] Metzler R. and J. Klafter (2000) The random walk's guide to anomalous diffusion: A fractional dynamics approach. *Phys. Rep.* **339**, 1–77.

[32] E.W. Montroll and G.H. Weiss (1965) Random walks on lattices. II. *J. Mathematical Phys.* **6**, 167–181.

[33] I. Podlubny (1999) *Fractional differential equations.* Academic Press, San Diego.

[34] T. Prosen and M. Znidaric (2001) Anomalous diffusion and dynamical localization in polygonal billiards. *Phys. Rev. Lett.* **87**, 114101–114104.

[35] P. Ressel (1991) Semigroups in probability theory. In *Probability measures on groups* X. Proceedings of the Tenth Conference held in Oberwolfach, November 4–10, 1990. Edited by Herbert Heyer. Plenum Press, New York.

[36] A.I. Saichev and G.M. Zaslavsky (1997) Fractional kinetic equations: Solutions and applications. *Chaos* **7**, 753–764.

[37] K.I. Sato (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[38] S. Samko, A. Kilbas and O. Marichev (1993) *Fractional Integrals and derivatives: Theory and Applications*. Gordon and Breach, London.

[39] H. Scher and M. Lax (1973) Stochastic transport in a disordered solid. I. Theory. *Phys. Rev. B* **7**, 4491–4502.

[40] H. Schiessel, I. M. Sokolov, and A. Blumen (1997) Dynamics of a polyampholyte hooked around an obstacle. *Phys. Rev. E* **56**, R2390-R2393.

[41] M. Shlesinger, J. Klafter and Y.M. Wong (1982) Random walks with infinite spatial and temporal moments. *J. Statist. Phys.* **27**, 499–512.

[42] E. Seneta (1976) *Regularly Varying Functions*. Lecture Notes Mathematic **508**, Springer-Verlag, Berlin.

[43] Y.G. Sinai (1982) The limiting behavior of a one-dimensional random walk in a random medium. *Theor. Probab. Appl.* **27**, 256–268.

[44] I.M. Sokolov, A.V. Chechkin and J. Klafter (2003) Fractional diffusion equation for power-law-truncated Lévy processes. Preprint.

[45] I.M. Sokolov, A.V. Chechkin and J. Klafter (2004) Distributed-order fractional kinetics. *Acta Physica Polonica B* **35**, 1323.

[46] V.V. Uchaikin and V.M. Zolotarev (1999) *Chance and stability. Stable distributions and their applications.* VSP, Utrecht.

[47] S. Watanabe (1980) A limit theorem for sums of non-negative i.i.d. random variables with slowly varying tail probabilities. *Proc. 5th International Conf. Multivariate Anal.* 249–261, North-Holland, Amsterdam.

[48] Zaslavsky, G. (1994) Fractional kinetic equation for Hamiltonian chaos. Chaotic advection, tracer dynamics and turbulent dispersion. *Phys. D* **76**, 110–122.

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