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PORTFOLIO INSURANCE: THE EXTREME VALUE OF THE CPPI METHOD

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Abstract This paper applies the extreme value theory to the Constant Proportion Portfolio Insurance (CPPI) . In particular, the choice of the standard multiple is detailed according to the statistical estimation of the behaviour of extreme variations in rates of assets returns. Moreover, we introduce the distributions of interarrival times of these extreme movements and show their impact on the portfolio insurance. We illustrate these results on S&P 500 data.

Keywords: insurance portfolio; CPPI; extreme value theory.

JEL classi…cation: G11.

Résumé Ce papier applique la théorie des valeurs extrêmes à la méthode CPPI d'assurance de portefeuille. En particulier, le choix du multiple standard est analysé en fonction de l'estimation statistique du comportement des variations extrêmes des taux de rendements des actifs …nanciers. De plus, nous introduisons les instants d'arrivée entre deux variations extrêmes et montrons leur in‡uence sur l'assurance de portefeuille. Ceci est illustré sur des données du S&P 500.

Mots-clés: assurance de portefeuille; méthode CPPI; théorie des valeurs extrêmes.

1 Introduction

Portfolio insurance is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. This return pattern has seemed attractive to many investors who have poured up to billions of dollars into various portfolio insurance products.

There exist many methods of insurance portfolio : OBPI (Option Based Portfolio Insurance), CPPI (Constant Proportion Portfolio Insurance), Stop-loss, ...(See for example Poncet and Portait (1997) for a comparison of theses methods). Here, we're interested in a widely used one : the CPPI introduced by Black and Jones (1987) for equity instruments and Perold (1986), (1988) for …xed-income instruments (see also Black and Rouhani (1987), Roman , Kopprash and Hakanoglu (1989), Black and Perold (1992)).

Usually, banks don't bear directly market risks on the asset portfolios they manage for their customers. This is not necessarily true when we considermanagement of insured portfolios. In that case, consequences of sudden large market's decrease may be totally borne by the banks, depending on the method of management they use[2]¢ For example, in the case of the CPPI method, the bank must, at least, provision the di¤erence on its own capital if the value of the portfolio drops below the ‡oor.

The CPPI method uses a simpli…ed strategy to allocate assets dynamically over time. The investor starts by setting a ‡oor equal to the lowest acceptable value of the portfolio, computes the cushion as the excess of the portfolio value over the ‡oor, and then determines the amount allocated to the active asset by multiplying the cushion by a predetermined multiple. Both the ‡oor and the multiple are functions of the investor's risk tolerance. The total amount allocated to the riskier asset is known as the exposure. The remaining funds are invested in the reserve asset, usually Treasury bills or other liquid money market instruments.

The higher the multiple, the more the investor will participate in a sustained increase in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the ‡oor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero, too. Normally, this keeps portfolio value from falling below the ‡oor. Portfolio value will fall below the ‡oor only when there is a very sharp drop in the market before the investor has a chance to trade.

Advantages of this strategy over other approaches to portfolio insurance are its simplicity and its ‡exibility (see for example De Vitry and Moulin (1994) and Black and Rouhani (1987)). Initial cushion, multiple, ‡oor and tolerance can be chosen according to the own investor's objective.

One crucial question for the bank which promotes such funds is : what exposure to the risky asset should I choose if I don't want that the probability that portfolio's value dropping below the ‡oor exceeds some small …xed level ? The problematic is quite

[2] Customers indirectly bear part of the risk in that they abandon part of the expected return when they invest in an insured portfolio.

comparable to the Value-at-Risk one.

If we take into account the possibility of occurrence of extreme events in the risky asset returns, the answer to this question is non trivial and has practical implications in the management process implementation.

This paper is organized as follows. Section 2 presents the model and basic properties of marked point processes and extreme value theory. Section 3 examines the CPPI method and provide upper bounds on the multiple when using a quantile hedging approach. Extreme value theory allows to approximate these bounds. Finally, empirical data on the S&P500 illustrate our approach .

2 The Model

2.1 The financial market.

Changes in asset prices are supposed to occur at discrete times along a whole period $[0; T_0]$ (for example, thirty years which may be the period of all observations of financial data).

Particular subperiods indexed by l, $[lT; (l+1)T]_{l \leq m}$ with $mT = T_0$ can be introduced: they may correspond to standard portfolio management periods (for example one year). Finally, for any n and for each subperiod l, consider the sequence of deterministic prices variations times $(t^l_{n;k})_k$ that for simplicity for each l, we denote by $(t_k)_k$ (for example, daily variations).

The variations of the stock price S between two times t_k and t_{k+1} are defined by

$$\Delta S_{t_{k+1}} = S_{t_{k+1}} - S_{t_k} = (-X_{k+1}) S_{t_k}$$

Here, we are interested by possible restrictions on the range of the jumps

$$X_{k+1} = \frac{S_{t_k} - S_{t_{k+1}}}{S_{t_k}}$$

In fact, some particular values can be excluded because they are too small to be considered (they have never been observed) or their frequency on the period under consideration is nil. For this purpose, we can introduce the truncated jumps $X_k^{[a;b]}$ defined by :

$$X_k^{[a;b]} = X_k 1_{a \leq X_k \leq b}$$

where $1_{a \leq . \leq b}$ denotes the indicator function of the interval $[a; b]$. That is:

$1_{a \leq X_k \leq b}$ is equal to 1 if $a \leq X_k \leq b$ and equal to 0 otherwise.

Denote $(T_k^{[a;b]})_k$ the corresponding sequence of prices variations times .

Then, the sequence $(T_k^{[a;b]}; X_k^{[a;b]})_k$ is a marked point process. [3].

[3] For the basic definitions and properties about marked point processes, see for example Bremaud (1981), Last and Brandt (1995) for marked point processes on the real line, Jacod (1977) for more general multivariate point processes and Prigent (1997) for financial applications.

2.2 Some properties of Marked Point Process (MPP) $(T_k; X_k)_k$:

Consider the filtration F generated by this marked point process. At time T_k; F_{T_k} is the information given by the observations of all the variables $(T_l; X_l)$ with $l \leq k$: Denote by $B(\mathbb{R})$ the set of all borelian subsets of $\mathbb{R}$.

Define $\mu = \mu(\omega; ds; dx)$ the measure of jumps of $(T_k; X_k)$. That is :

$$\forall A \in B(\mathbb{R}); \ \mu(\omega; (0; t] \times A) = \sum_{T_k \leq t} 1_{(X_k \in A)}$$

$\mu(\omega; (0; t] \times A)$ counts the number of jumps whose values are in the set A , during the period $(0; t]$:

In fact, the MPP $(T_k; X_k)_k$ is characterized by another measure, called the compensator of μ that we denote by ν:

Recall that this measure $\nu = \nu(\omega; ds; dx)$ is a predictable measure, such that $\mu - \nu$ is a local martingale measure (in particular, for each $A \in B(\mathbb{R} - \{0\})$, the process $(\mu(\omega; (0; t] \times A) - \nu(\omega; (0; t] \times A)_t$ is a local martingale).

It implies for example that the expectations of the number of jumps during the period $[0; t]$ is given by

$$E[\int_{\mathbb{R}} \mu(\omega; (0; t] \times dx] = E[\sum_{T_k \leq t} 1_{(X_k \in \mathbb{R})}] = E[\int_{\mathbb{R}} \nu(\omega; (0; t] \times dx]$$

A quasi-explicit formula of the compensator measure ν can be given (see Jacod (1977) for all details) :

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Denote $G_k(dt; dx)$ a regular version of the conditional distribution of $(T_{k+1}; X_{k+1})$ with respect to F_{T_k}. Then, one deduces:

$$\nu(dt; dx) = \sum_{n \geq 0} 1_{\{T_k < t \leq T_{k+1}\}} \frac{G_k(dt; dx)}{G_k([t; \infty[\times \mathbb{R})}$$

Denote by F_k a regular version of the conditional distribution of T_{k+1} with respect to F_{T_k}:

$$F_k(dt) = G_k(dt \times \mathbb{R})$$

Finally, introduce the kernel K such that $K(T_{k+1}; dx)$ is the conditional distribution of X_{k+1} with respect to F_{T_k} and T_{k+1}.

As previously mentioned, the above distributions characterize the MPP $(T_k; X_k)_k$ and allow to calculate all useful quantities like expectations, variances, options prices... (see for example Prigent (1997))

Consider some basic examples of such processes which can be used in financial modelling:

Example 1: Independent marking.

This kind of model introduces more general interarrival times distributions than for example usual Levy processes (for this kind of processes whose increments are independent, the interarrival times are independent and exponentially distributed). Nevertheless, the marks X_k do not depend on times of variations T_k:

In our financial model, it means that the amplitudes of the relative stock variations X_k are independent from each other and of the times T_k However, the distributions of these times T_k are quite general and can be estimated from market data without any preliminary assumption.

Let $\bar{\Phi} = (T_k)_k$ be an arbitrary point process and $(X_k)_k$ a sequence of i.i.d. random variables with distribution G independent of $\bar{\Phi}$.
Then the marked point process $\Phi = ((T_k; X_k))_k$ is called an independent G-marking of $\bar{\Phi}$.
It is easily verified that Φ has the intensity measure :

$$\nu(dt; dx) = \sum_{n \geq 0} 1_{\{T_k < t \leq T_{k+1}\}} \frac{F_k(dt)}{F_k([t; \infty[)} G(dx)$$

Introduce a special case: the independent marked point process.
Recall the definition of an homogeneous Poisson process $\bar{\Phi} = (T_n)_n$. It is defined by the two following properties :

-i) Denote by $N(I_j)$ the number of jumps during the space of time I_j.The random numbers $N(I_1); \ldots; N(I_n)$ are independent for pairwise disjoint measurable subsets $I_1; \ldots; I_n$ (property of independent increments).
-ii) For all bounded $I \in B^+$, $P[N(I) = k] = \frac{\Lambda(I)^k}{k!} e^{-\Lambda(I)}$ where $\Lambda(I) = \lambda|I|$ and $|I|$ is the Lebesgue measure of I.

The Independent marked Poisson process is an independent marked process such that the times component is an homogeneous Poisson process.

Consider now models that allow to take account of more general distributions of interarrival times : they need no longer to be deterministic or exponential with a constant parameter. Moreover, it is possible to correlate stock prices variations with the times of transactions.

Example 2: Dependent marking.

Example 2.1. Position Dependent Marking

This kind of model can be used to introduce for example exogeneous factors like macroeconomic variables (inflation or exchange rates, indexes of economic performances like economic growth indicators....) to explain stocks prices variations. For this purpose, an exogeneous process $(Y_t)_t$ can be involved to describe in particular the distributions of interarrival times.

More generally, an MPP $\Phi = ((T_k, X_k))_k$ is called a position-dependent marking of $\bar{\Phi} = (T_k)_k$ if $X_1, ..., X_k ...$ are conditionally independent given $\bar{\Phi}$ and $P[X_k|\bar{\Phi}] = G(T_k, dx)$. In this case, the compensator of Φ is given by $\nu(dt, dx) = G(t, dx)\Psi(dt)$.

The following is an important special case: the Marked Poisson process.

$\bar{\Phi}$ is a Poisson process with a continuous and locally bounded intensity measure $\bar{\Lambda}$ and $\nu(dt, dx) = G(t, dx)\bar{\Lambda}(dt)$. The process $Z_t = \sum_{T_n \leq t} X_k$ is a compound Poisson process. In particular, consider the doubly stochastic or conditional Poisson processes (Cox Processes) (DSPP).

Denote by $N_t(B)$ the number of jumps with sizes in the subset B of $\mathbb{R}$ during $[0, t]$. Φ is called an $\mathcal{F}_t$-doubly stochastic marked Poisson process (denoted DSPP) if there exists an $\mathcal{F}_0$-measurable random measure ν on $[0, T] \times \mathbb{R}$ such that

$$P[(N_t - N_s)(B) = k|\mathcal{F}_s] = \frac{(\nu(]s, t] \times B))^k}{k!} e^{-(\nu(]s,t] \times B))}$$

One particular case of doubly stochastic marked Poisson process is the Markov modulated marked Poisson process: The intensity of jumps are directed by a Markov process Y. It is defined in the following manner: There exists a markov chain $Y = (Y_t)_t$ with a finite number of jumps in finite intervals. Let $\{\lambda_i; i \in Y\}$ be a family of nonnegative numbers. The point process $(T_k)_k$ is called a Markov modulated Poisson process if the conditional distributions $P[N \in \cdot|Y]$ is P-almost surely equal to the distribution of a Poisson process with the intensity function $t \to \lambda_{Y_t}$.
In order to have a doubly stochastic marked Poisson process, one must suppose that $\mathcal{F}_0$ contains $\mathcal{F}^Y_\infty$. Then :

$$P[(N_t - N_s) = k|Y] = \frac{(\int_s^t \lambda_{Y_u} du)^k}{k!} e^{-\int_s^t \lambda_{Y_u} du}$$

Thus the compensator of $\bar{\Phi}$ is given by

$$\Psi(dt) = \lambda_{Y_t} dt$$

This process can be interpreted as a Poisson process in a random environment. It is driven by the process Y.

Consider a position-dependent marked process with the above structure to model the times of stock variations. Then:

$$\nu(dt;dx) = \lambda_{Y_t} dt G(t;dx)$$

Example 2.2 Semi-Markov process

Introduce now a model where the times of variations and the amplitudes of these variations are linked by a markovian dependence. Thus, for instance, stock prices variations can be "explained" by both times and amplitudes of past variations, taking much more account of financial market activity.

Let $((U_k;X_k))_k$ denote a sequence of $]0;\infty] \times E$ valued random variables where E is at most countable set. For notational convenience, E is a subset of $\mathbb{N}$.
Assume that :

$$P[U_0 \in dt; X_1 = j | X_0] = \mathbb{Q}^0_{X_0 j}(dt)$$

and

$$P[U_k \in dt; X_{k+1} = i | X_0; U_0; \dots; U_{k-1}; X_k] = \mathbb{Q}_{X_k i}(dt)$$

where $\mathbb{Q}^0_{ij}$ and $\mathbb{Q}_{ij}$ are finite measures on $]0;\infty]$ for all $(i;j) \in E \times E$ such that :

$$\sum_j \mathbb{Q}^0_{ij}(]0;\infty[) = 1 : \text{and} : \sum_j \mathbb{Q}_{ij}(]0;\infty[) = 1$$

Clearly, the conditional distribution of $(T_{k+1};X_{k+1})$ under F_{T_n} is only dependent on the values of X_n.

Then, the sequence $((T_k;X_k))_k$ with $T_k = \sum_{i=0}^{k-1} U_i$ is called a semi-markov process (SMP) with state space E, the family $\mathbb{Q}^0 = \{\mathbb{Q}^0_{ij} / (i;j) \in E \times E\}$ is called the delay kernel and the family $\mathbb{Q} = \{\mathbb{Q}_{ij} / (i;j) \in E \times E\}$ is called the semi-Markov kernel of the SMP.

Recall that if $\sum_{j \in R} \mathbb{Q}^0_{ij} = A^0$ and $\sum_{j \in R} \mathbb{Q}_{ij} = A$ independent of i then the SMP is a renewal process (the interarrival times U_i are iid with interarrival distribution A and the delay distribution A^0).

2.3 Extreme Value Theory

We measure stock markets price movements by the daily rates of returns $Y_k = \frac{S_{t_{k+1}} - S_{t_k}}{S_{t_k}}$ which determine the conditions to impose on insurance strategies. If these variables are statistically independent, and drawn from the same distribution, then the exact distribution of the maximum of $Y_1, \ldots, Y_n$ is equal to the power F_Y^n of the common distribution of the Y_i. But, in most cases, this distribution is not exactly known. Nevertheless, like for the well-known central limit theorem, there exists a normalization procedure to get non-degenerate distributions at the limit.

Recall the Fisher-Tippett theorem (1928) concerning the limit laws for maxima: Let $(Y_k)_k$ be a sequence of iid random variables. If there exist norming constants $c_k > 0$, d_k and some non-degenerate distribution function H such that

$$\frac{Max(Y_1, \ldots, Y_n) - d_n}{c_n} \longrightarrow H$$

then H belongs to the type of one of the following three distribution functions:

Fréchet :

$$\Phi_\alpha(x) = \exp(-x^{\alpha}), x > 0, \alpha > 0$$

Weibull :

$$\Psi_\alpha(x) = \exp(-(-x)^{\alpha}), x < 0, \alpha > 0$$

Gumbel :

$$\Lambda(x) = \exp(-e^{(-x)}), x \in \mathbb{R}$$

These three kinds of distributions are particular cases of the generalised extreme distibutions (GEV) defined by:

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{\frac{-1}{\xi}}) \text{ if } \xi \neq 0 \\ \exp(-\exp(-x)) \text{ if } \xi = 0 \end{cases} \quad \text{where } 1 + \xi x > 0. \text{ [4]}$$

When examining the sequence $X_{k+1} = \frac{S_{t_k} - S_{t_{k+1}}}{S_{t_k}}$, it is obvious that its distribution has a right end limit equal to 1. So, we can verify that in general, the normalized maxima $\frac{Max(Y_1, \ldots, Y_n) - d_n}{c_n}$ converges to the Gumbel distribution Λ, (for example, if the normalized sequence of the logreturns converge to a Frechet distribution)[5] .

[4] For the basic properties about GEV distributions, see for example Embrechts and alii (1997). For applications of extreme distributions in finance, see Longin (1996).

[5] Proof available on request.

Recall the characterizations of the result of Resnik (1987) for the maximum domain of attraction of Λ; based on generalizations of the Von Mises functions (see also for example Embrechts and alii (1997)):

The distribution F with right end point limit $x_F \le \infty$ belongs to the maximum domain of attraction of Λ if and only if there exists some $z < x_F$ such that F has representation

$$\bar{F}(x) = c(x)\exp\left(-\int_z^x \frac{g(t)}{\alpha(t)}dt\right); z < x < x_F$$

where c and g are functions satisfying $c(x) \to c > 0$; $g(x) \to 1$ as $x \uparrow x_F$ and $\alpha(x)$ is a positive, absolutely continuous function with respect to Lebesgue measure with density $\alpha'(x)$ having $\lim_{x \uparrow x_F} \alpha'(x) = 0$.

Denote $F^{\leftarrow}(q)$ the q quantile of the distribution F, more precisely:

$$F^{\leftarrow}(q) = \inf\{x \in \mathbb{R}; F(x) \ge q\}$$

For F with the previous representation, we can choose:

$$d_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right) \text{ and } c_n = \alpha(d_n)$$

A possible choice of the function $\alpha(.)$ is :

$$\alpha(x) = \int_z^{x_F} \frac{\bar{F}(t)}{\bar{F}(x)}dt; x < x_F$$

Besides, if we truncate the jumps X_k on the interval $[a;b]$ with $b < \infty$ and examine the sequence $(X_k^{[a;b]})_k$ then we can check that there exist norming constants $c_k^{[a;b]}$ and $d_k^{[a;b]}$ such that

$$\frac{Max(X_1^{[a;b]}; ...; X_n^{[a;b]}) - d_n^{[a;b]}}{c_n^{[a;b]}} \to \Lambda$$

3 CPPI and Quantile Hedging :

3.1 Upper Bounds on the Multiple

Recall that, for fixed n, the sequence of deterministic prices variations times is denoted by $(t_k)_k$.

The variations of the stock price S between two times t_k and t_{k+1} are defined by

$$\Delta S_{t_{k+1}} = S_{t_{k+1}} - S_{t_k} = (-X_{k+1}) S_{t_k}$$

$X^{[a;b]}_{k+1}$ denotes the truncated jumps $X^{[a;b]}_{k+1} = X_{k+1} 1_{a \leq X_{k+1} \leq b}$ and $(T^{[a;b]}_{k+1})_k$ is the corresponding sequence of prices variations times .

The sequence $(T^{[a;b]}_{k+1}; X^{[a;b]}_{k+1})_k$ is a marked point process, associated to the measure of jumps $\mu^{[a;b]} = \mu^{[a;b]}(\omega; ds; dx)$. $\nu^{[a;b]} = \nu^{[a;b]}(\omega; ds; dx)$ is its compensator. Denote $G^{[a;b]}_k(dt; dx)$ a regular version of the conditional distribution of $(T^{[a;b]}_k; X^{[a;b]}_k)$ with respect to $F^{[a;b]}$

Denote by $F^{[a;b]}_{k+1}$ a regular version of the conditional distribution of $T^{[a;b]}_{k+1}$. Finally, introduce the kernel $K^{[a;b]}$ such that $K^{[a;b]}(T^{a;b}_{k+1}; dx)$ is the conditional distribution of $X^{[a;b]}_{k+1}$ with respect to $F^{[a;b]}$ and $T^{[a;b]}_{k+1}$.

Standard condition on the multiple:

Recall that $X_k = -\frac{\Delta S_k}{S_{k-1}}$. Define $M_n = Max(X_1; ...; X_n)$. Denote by V_t the value of the portfolio at time t . The condition of portfolio insurance is the following: the investor chooses a deterministic floor P_t. At any time t, the value V_t must be above the floor. Denote by e_t the total amount invested on the underlying asset S (usually called the exposure). Introduce the difference $V_t - P_t$ between the portfolio value and the floor P_t. This quantity is called the cushion and is denoted by C_t. The usual CPPI method consists in letting $e_t = mC_t$ where m is a constant, called the multiple.
Then, for all times t_k; the cushion must be positive. This leads to :

$$(1 - m) r_n - m X_k \geq -1$$

and, in fact, since r_n is relatively small, it is reduced to the relation:

Proposition 1:

$$\forall k; X_k \leq \frac{1}{m} \text{ or equivalently } Max(X_k) \leq \frac{1}{m}$$

Since the right end point d of the distribution F is positive, we deduce that the insurance is perfect along any period T' if and only if m is smaller than $\frac{1}{d}$.

Nevertheless, this strong condition can be modified if a quantile hedging approach is adopted, like for the Value-at-Risk (see Föllmer and Leukert (1999) for recent application of this notion in financial modelling): for example, an auxiliary floor can be chosen above the initial floor and the new condition is to guarantee that the portfolio value will be always above this new floor at a given probability $1-\varepsilon$. This gives the following relation for a period T' :

$$P[C_t \geq 0, \forall t \leq T'] \geq 1-\varepsilon$$

or, equivalently, if $[\frac{nT'}{T}]$ denotes the integer part of $\frac{nT'}{T}$:

$$P[\forall k \leq [\frac{nT'}{T}], X_k \leq \frac{1}{m}] = P[M_{[\frac{nT'}{T}]} \leq \frac{1}{m}] \geq 1-\varepsilon$$

If we reduce the range of the distribution of the variables X_k to the interval $[a,b]$, then only the variables $X_k^{[a;b]}$ are involved.

Define as above $M_{[\frac{nT'}{T}]}^{[a;b]} = Max(X_1^{[a;b]}, ..., X_{[\frac{nT'}{T}]}^{[a;b]})$.

This leads to :

$$P[\forall k \leq [\frac{nT'}{T}], X_k^{[a;b]} \leq \frac{1}{m}] = P[M_{[\frac{nT'}{T}]}^{[a;b]} \leq \frac{1}{m}] \geq 1-\varepsilon$$

As it has been previously mentioned, the times $T_k^{[a;b]}$ are random variables and the range of their distributions is the set of times $(t_k)_k$.

Suppose for example that the number n of times t_k is sufficiently great so that we can consider that the set of t_k seems like an interval (continuous time at the limit). Assume also that the sequence of interarrival times $(T_{k+1}^{[a;b]} - T_k^{[a;b]})_k$ is an iid sequence and so is exponentially distributed with a parameter $\lambda^{[a;b]}$ (which implies that the expectation of the number of variations with values in $[a,b]$, during the period t, is equal to $t\lambda^{[a;b]}$). Then we can detail the quantile hedging condition. For this, introduce the function $F^{[a;b]-1}$ defined as the inverse of the distribution function $F^{[a;b]}$ which is assumed to be strictly increasing. Then, we get (see Prigent (1999)):

Proposition 2:

$$m \leq \frac{1}{F^{[a;b]-1}\left(1+\frac{\ln(1-\varepsilon)}{\lambda^{[a;b]}T'}\right)}$$

This condition gives an upper limit on the multiple m which is obviously greater than the standard limit $\frac{1}{b}$ (which is given in proposition 1 with $b = d$ if the distribution is not truncated). It takes account of both the distribution of the variations and of the interarrival times. Note that if the intensity $\lambda^{[a;b]}$ increases, then this upper limit is decreasing. So, like intuitively, if the frequency $\lambda^{[a;b]}$ increases, then the multiple has to be reduced and if $\lambda^{[a;b]}$ goes to infinity, then the previous upper limit converges to the standard limit $\frac{1}{b}$.

Suppose now that the true distribution $F^{[a;b]}$ is not known.

1) First case: assume that the times of variations $X_k^{[a;b]}$ are the t_k themselves.

Denote $N = [\frac{nT'}{T}]$. Then the insurance condition on the multiple m along the period T' can be still analysed by applying the extreme value theory. Indeed we know that there exist scale and location parameters $c_N^{[a;b]}$ and $d_N^{[a;b]}$ such that $\frac{M_N^{[a;b]} - d_N^{[a;b]}}{c_N^{[a;b]}}$ converge in distribution to one of the extreme value distributions $H_\xi^{[a;b]}$. Consequently, we can get the following approximation of the upper limit on the multiple m :

Proposition 3:

$$m \le \frac{1}{c_N^{[a;b]} H_\xi^{[a;b]\,-1}(1-\varepsilon) + d_N^{[a;b]}}$$

Proof: from the condition $P[M^{[a;b]}_{[\frac{nT'}{T}]} \le \frac{1}{m}] \ge 1-\epsilon$, and by applying the extreme value theory to $M^{[a;b]}_{[\frac{nT'}{T}]}$, we get $H_\xi^{[a;b]}\left(\frac{\frac{1}{m}-d_N^{[a;b]}}{c_N^{[a;b]}}\right) \ge 1-\epsilon$, which leads to the result.

2) For the general case, we can get also an upper bound. Now, the times of prices variations are random variables, defined on the set of $(t_k)_k$.

Denote by $N^{[a;b]}_{T'}$ the random number of variations of the underlying asset with values in the interval $[a;b]$ during the period $[0;T']$. The quantile hedging condition becomes:

$$P[\forall k \le [\frac{nT'}{T}]; C_k \ge 0] = P[\forall T_k^{[a;b]} \le T'; X_k \le \frac{1}{m}]$$

but, by conditioning with respect to the number $N^{[a;b]}_{[\frac{nT'}{T}]}$

$$\begin{aligned} P[\forall T_k^{[a;b]} \le T'; X_k \le \frac{1}{m}] &= \sum_{l \le [\frac{nT'}{T}]} P[M_l^{[a;b]} \le \frac{1}{m}; N_{T'}^{[a;b]} = l] \\ &= \sum_{l \le [\frac{nT'}{T}]} P[M_l^{[a;b]} \le \frac{1}{m} \Big| N_{T'}^{[a;b]} = l\,]P[N_{T'}^{[a;b]} = l] \\ &= \sum_{l \le [\frac{nT'}{T}]} P[M_l^{[a;b]} \le \frac{1}{m} \Big| T_l^{[a;b]} \le T' < T_{l+1}^{[a;b]}\,]P[T_l^{[a;b]} \le T' < T_{l+1}^{[a;b]}] \end{aligned}$$

Introduce the function $L^{[a;b]}_{T'}$ defined by:

$$L^{[a;b]}_{T'}(x) = \sum_{l} P[M_l^{[a;b]} \le x \Big| T_l^{[a;b]} \le T' < T_{l+1}^{[a;b]}\,]P[T_l^{[a;b]} \le T' < T_{l+1}^{[a;b]}]$$

Since $F^{[a;b]}$ is assumed to be strictly increasing then $L^{[a;b]}_{T'}$ has also the same property. Consider its inverse $L^{[a;b]\,-1}_{T'}$. We get the following result:

Proposition 4:
The general quantile hedging condition $P[C_t \ge 0; \forall t \le T'] \ge 1-\epsilon$ is equivalent to

$$m \le \frac{1}{L^{[a;b]\,-1}_{T'}(1-\epsilon)}$$

Remark 1: As it can be seen, this condition involves the joint conditional distributions of the marked point process $(T^{[a;b]}_{k+1}; X^{[a;b]}_{k+1})_k$.
We can illustrate it for basic examples:

i) Independent marking: since $(T_k^{[a,b]})_k$ and $(X_k^{[a,b]})_k$ are independent, we get:

$$L_{T'}^{[a,b]}(x) = \sum_l P[M_l^{[a,b]} \le x]\,P[N_{T'}^{[a,b]} = l]$$

And, for the special case of Independent marked Poisson process:

$$L_{T'}^{[a,b]}(x) = \sum_l P[M_l^{[a,b]} \le x]\frac{(\lambda^{[a,b]}T')^l}{l!}e^{-\lambda^{[a,b]}T'}$$

ii) Position-Dependent marking : if the times of variations depend on the realization of a stochastic process Y (for example, an exogeneous economic variable, then :

$$L_{T'}^{[a,b]}(x) = \sum_l P[M_l^{[a,b]} \le x \mid T_l^{[a,b]} \le T' < T_{l+1}^{[a,b]}, Y] \int \left(\frac{(\int_0^{T'} \lambda_{Y_u} du)^l}{l!} e^{-\int_0^{T'} \lambda_{Y_u} du}\right) dP_Y$$

iii) Semi-Markov process: consider for example the case of a renewal process with interarrival distribution A: E is for example the set of ticks of the underlying asset.

Denote $f(l, T') = \iint_{\{(u,v): u \le T', u+v \ge T'\}} A^{*l}(du)A(dv)$: $f(l, T')$ is equal to $P[T_l^{[a,b]} \le T' < T_{l+1}^{[a,b]}]$ where A^{*l} denotes the l-convolution product: Then, the quantile hedging condition is:

$$L_{T'}^{[a,b]}(x) = \sum_l P[M_l^{[a,b]} \le x \mid T_l^{[a,b]} \le T' < T_{l+1}^{[a,b]}]\,f(l, T')$$

Remark 2: Estimation of the function $L_{T'}^{[a,b]}(x)$:

If T' is sufficiently great, the probabilities $P[N_{T'}^{[a,b]} = l]$ are small, for small values of l. So, by applying the extreme value theory, we can get an approximation of each $P[M_l^{[a,b]} \le x \mid T_l^{[a,b]} \le T' < T_{l+1}^{[a,b]}]$ if the variations $X_k^{[a,b]}$ are for example conditionally independent with respect to the sequence $(T_k^{[a,b]})_k$ (case (i) and (ii) for example).

Thus, we get :

$$L_{T'}^{[a,b]}(x) \simeq \sum_l H_\xi^{[a,b]}\left(\frac{x - d_l^{[a,b]}}{c_l^{[a,b]}}\right) P[T_l^{[a,b]} \le T' < T_{l+1}^{[a,b]}]$$

3.2 Empirical estimations.

A. Estimations of the variations $X_k^{[a,b]}$:

We examine the variations $X_k^{[a,b]}$ of the S&P 500 during the period 01/1969-09/1997. We refer to Longin (1996) for details about estimation procedures when dealing with extreme value theory (especially, for the estimation of the Frechet extreme distribution when dealing with log returns). Here, we are mainly interested by the extreme values of the sequence of rates of returns.

First, we can estimate the distribution function without limit (i.e.a=-1 and b=1). The results indicate that the daily variations have a range which is mostly included in the interval [-3%,+4%].

Using this , we can give approximations of the norming sequences $c_N^{[a;b]}$ and $d_N^{[a;b]}$ according to N. Recall that here $\frac{M_N^{[a;b]} - d_N^{[a;b]}}{c_N^{[a;b]}}$ converges to the Gumbel distribution (see paragraph 2.3). For example, the following table indicates the values of these sequences for $[a;b] = [0;10\%]$:

Table1

N	d	c	N	d	c
12	0.0143	0.0167	35	0.0191	0.0117
13	0.0148	0.0160	40	0.0197	0.0109
15	0.0154	0.0153	45	0.0203	0.0096
16	0.0155	0.0150	50	0.0210	0.0081
17	0.0157	0.0147	100	0.0338	0.0051
18	0.0160	0.0145	200	0.0367	0.0042
19	0.0163	0.0140	300	0.0572	0.0033
20	0.0165	0.0138	400	0.0687	0.0025
25	0.0175	0.0127	500	0.0788	0.0015

B. Estimations of the interarrival distributions $T_{k+1}^{[a;b]} - T_k^{[a;b]}$:

When dealing with the sequence of interarrival times $T_{k+1}^{[a;b]} - T_k^{[a;b]}$ along the whole period of observations, it is possible to examine the influence of the interval $[a;b]$ on the frequency λ of the daily variations in these intervals (the estimation of λ is deduced from the average of the $T_{k+1}^{[a;b]} - T_k^{[a;b]}$): This is illustrated by the following table:

Table 2

$[a;b]$	λ	$[a;b]$	λ	$[a;b]$	λ
$[-1\%;0]$	0;40045	$[-2\%;0]$	0;48044	$[-3\%;0]$	0;49232
$[-1;25\%;0]$	0;43424	$[-2;25\%;0]$	0;48458	$[-3;25\%;0]$	0;49326
$[-1;5\%;0]$	0;45614	$[-2;5\%;0]$	0;48805	$[-3;5\%;0]$	0;49326
$[-1;75\%;0]$	0;47256	$[-2;75\%;0]$	0;49072	$[-3;75\%;0]$	0;49366

We observe that the frequency converges to approximatively 0.5 as λ increases: this is due to the fact that daily returns have almost the same probability to be positive than to be negative.

For each period T^0, the distribution of the number of variations in the interval [a,b] is estimated. This gives the values of the probabilities $P[N_{T^0}^{[a;b]} = l]$: The next table indicates the kind of distribution that we can get. (we choose $[a;b] = [0;10\%]$; $T^0 = 1$ year: Nv is the number of variations in this interval. Nd is the number of days. f is the corresponding frequency. The period of observation is 01/1971-09/1997).

Table 3

Nv	Nd	f	Nv	Nd	f	Nv	Nd	f
124	261	47.51%	117	262	44.66%	109	260	41.92%
123	260	47.31%	144	261	55.17%	125	261	47.89%
144	261	57.17%	146	261	55.94%	136	261	52.11%
152	261	58.24%	123	260	47.31%	132	262	50.38%
124	261	47.51%	147	261	56.32%	131	261	50.19%
126	262	48.09%	126	262	48.09%	127	260	48.85%
142	260	54.62%	122	261	46.74%	104	260	40.00%
130	260	50.00%	111	261	42.53%	123	262	46.95%
121	261	46.36%	121	261	46.36%	83	184	45.11%

The following …gure illustrates this distribution (the …rst curve is the density and the second the cumulative distribution).

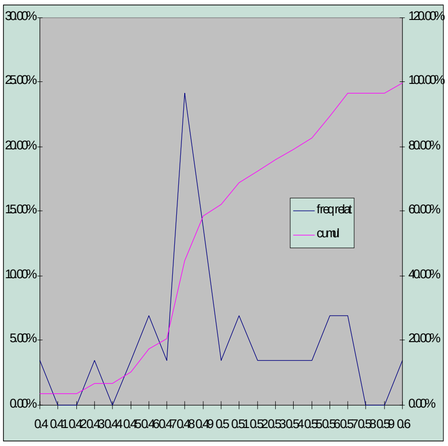

Figure 1:

Applying the previous results, we can provide an approximation of the upper bound for the multiple if we assume that the variations are conditionally independent from the times of variations. Consider a period T' which corresponds to one year. From table 3, we deduce that the range of the distribution of $N_{T'}^{[0;10\%]}$ is [100;150]. Then, from table 1, we take the values of the norming constants $c_N^{[0;10\%]}$ and $d_N^{[0;10\%]}$ for N in [100;150]. Finally, using the approximation given by the extreme value theory, we deduce:

$$L_{T'}^{[0;10\%]}(x) \simeq \sum_{100 \leq l \leq 150} \Lambda\left(\frac{x - d_l^{[0;10\%]}}{c_l^{[0;10\%]}}\right) P[N_{T'}^{[0;10\%]]} = l]$$

Now, if we want to hegde at the level $1 - \varepsilon = 0{:}95$, by applying the relation of proposition 4, we deduce that the multiple m must be chosen smaller than approximatively 16 whereas the standard upper bound is equal to 10. This result is partially explained by the empirical distribution of the variations $X_k^{[0;10\%]}$ that in fact has most of its range in [0,5%]. Note also that the frequency of $X_k^{[0;10\%]}$ is less than 0.5.

4 Conclusion

As it can be seen, it is possible to use higher multiples with the CPPI method if quantile hedging is used. The upper bounds can be calculated for each level of probability, according to the distributions of the marked point process which indicates the variations of the underlying asset and its times of variations. The extreme value theory allows to approximate theses distributions to provide a numerical upper bound. The difference with the standard multiple is significant. Moreover, these variations can be related to exogeneous processes, like economic variables by using for example semi-Markov processes. The study of general dependent marking is the purpose of further research.

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