Multiclass Markovian Fluid Queues

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Abstract This paper considers a multiclass Markovian fluid queue with a buffer of infinite capacity. Input rates of fluid flows in respective classes and the drain rate from the buffer are modulated by a continuous-time Markov chain with finite states. We derive the joint Laplace-Stieltjes transform for the stationary buffer contents in respective classes, assuming the FIFO service discipline. Further we develop a numerically feasible procedure to compute the joint and marginal moments of the stationary buffer contents in respective classes. Some numerical examples are then provided.

Keywords Markovian fluid queue, multiclass, FIFO, buffer contents, stationary joint distribution

AMS subject classification: 60K25, 60K37

1 Introduction

Fluid queues have been studied extensively for a few decades and surveys of early developments can be found in [4, 6]. In recent years, fluid queues have been attracting considerable attention again, with a great advance in matrix-analytic methods. For example, recent studies on Markovian fluid queues are found in [1, 2, 5, 11, 14]. More general models are also studied in [3, 9, 10, 15].

Almost of all researches of fluid queues focus on the stationary distribution of the total amount of fluid in the buffer. In particular, Asmussen [3] showed that in Markovian fluid queues, the stationary distribution of the amount of fluid in the buffer is of phase-type (see [12] also). Further, considering the time-reversed process, Ramaswami [11] showed that the kernel of the phase-type distribution is given in terms of the G-matrix of a discrete-time quasi birth-and-death process (QBD). da Silva Soares and Latouche [5] provided a physical interpretation of the discrete-time QBD associated with the Markovian fluid queue. See [1] also. These results enable us to compute the stationary distribution of the total amount of fluid in the buffer efficiently.

This paper considers a multiclass Markovian fluid queue with a buffer of infinite capacity and discusses the stationary *joint* distribution of the buffer contents (i.e., the amount of fluid in the buffer) in respective classes under the FIFO service discipline. The multiclass FIFO fluid queue arises naturally when we construct a macroscopic model of the ordinary FIFO queueing system with multiple types of customers. Kulkarni and Glazebrook [7] studied the output process of a multiclass FIFO fluid queue fed by independent Markovian on-off processes.

Our model is described as follows. There are K classes of fluid flows, labeled 1 to K. Let $\mathcal{K}$ denote $\{1, 2, \ldots, K\}$. Input rates of these K fluid flows are modulated by a continuous-time Markov chain $\{S(t); t \geq 0\}$ with finite state space $\mathcal{M} = \{1, 2, \ldots, M\}$, whose sample paths are right-continuous. We call $\{S(t); t \geq 0\}$ the underlying Markov chain hereafter. We denote the infinitesimal generator of the underlying Markov chain by $\boldsymbol{T}$, which is assumed to be irreducible. While the underlying Markov chain is being in state $j \in \mathcal{M}$, class k ($k \in \mathcal{K}$) fluid arrives to the buffer at rate $f_{k,j} \geq 0$. The drain rate of fluid from the buffer is also modulated by the underlying Markov chain. Namely, while the underlying Markov chain is being in state $j \in \mathcal{M}$, the drain rate of fluid from the buffer is equal to r_j, where we assume $r_j > 0$ for all $j \in \mathcal{M}$. For convenience, the state space $\mathcal{M}$ is divided into three disjoint subsets $\mathcal{M}_+$, $\mathcal{M}_-$, and $\mathcal{M}_0$ in the following way:

$$\begin{aligned}\mathcal{M}_+ &= \{j \in \mathcal{M};\, f_j - r_j > 0\},\\ \mathcal{M}_- &= \{j \in \mathcal{M};\, f_j - r_j < 0\},\\ \mathcal{M}_0 &= \{j \in \mathcal{M};\, f_j - r_j = 0\},\end{aligned}$$

where $f_j = f_{1,j} + f_{2,j} + \cdots + f_{K,j}$. To avoid trivialities, we assume $\mathcal{M}_+ \neq \emptyset$.

Let $\boldsymbol{f}_k$ ($k \in \mathcal{K}$) and $\boldsymbol{r}$ denote $M \times 1$ vectors whose jth ($j \in \mathcal{M}$) elements are given by $f_{k,j}$ and r_j, respectively. Thus the multiclass fluid queue can be characterized by $(\boldsymbol{T}, \boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_K, \boldsymbol{r})$. Because input rates of respective classes and the drain rate are modulated by a single Markov chain, those rates are correlated in general and the independent case [7] is included as a special case. For example, we consider a fluid queue fed by two independent fluid flows $(\boldsymbol{T}^{[1]}, \boldsymbol{f}^{[1]})$ and $(\boldsymbol{T}^{[2]}, \boldsymbol{f}^{[2]})$, where $\boldsymbol{T}^{[i]}$ ($i = 1, 2$) denotes an infinitesimal generator of an irreducible Markov chain and $\boldsymbol{f}^{[i]}$ ($i = 1, 2$) denotes an input rate vector of fluid. These flows arrive to the buffer whose drain rate is modulated by an independent Markov chain with an infinitesimal generator $\boldsymbol{T}^{[3]}$. Let $\boldsymbol{r}^{[3]}$ denote the drain rate vector. This fluid queue can be characterized by $(\boldsymbol{T}^{[1]} \oplus \boldsymbol{T}^{[2]} \oplus \boldsymbol{T}^{[3]}, \boldsymbol{f}^{[1]} \otimes \boldsymbol{e}^{[2]} \otimes \boldsymbol{e}^{[3]}, \boldsymbol{e}^{[1]} \otimes \boldsymbol{f}^{[2]} \otimes \boldsymbol{e}^{[3]}, \boldsymbol{e}^{[1]} \otimes \boldsymbol{e}^{[2]} \otimes \boldsymbol{r}^{[3]})$, where $\boldsymbol{e}^{[i]}$ ($i = 1, 2$) denotes a column vector of ones with the same dimension as $\boldsymbol{f}^{[i]}$ and $\boldsymbol{e}^{[3]}$ denotes a column vector of ones with the same dimension as $\boldsymbol{r}^{[3]}$.

We define $\boldsymbol{\pi}$ as a $1 \times M$ vector whose jth ($j \in \mathcal{M}$)

element π_j is given by

$$\pi_j = \lim_{t\to\infty} \Pr[S(t) = j].$$

Note that $\boldsymbol{\pi}$ satisfies $\boldsymbol{\pi}\boldsymbol{T} = \boldsymbol{0}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ denotes a column vector with an appropriate dimension, whose elements are all equal to one. Owing to the irreducibility of the underlying Markov chain, $\boldsymbol{\pi}$ is uniquely determined. In what follows, we assume

$$\rho = \sum_{k\in\mathcal{K}} \boldsymbol{\pi}\boldsymbol{f}_k / \boldsymbol{\pi}\boldsymbol{r} < 1, \tag{1}$$

which ensures that the queue is ergodic, e.g., see [6]. We denote the input rate of class k ($k \in \mathcal{K}$) fluid at time t by $f_k(t)$, i.e.,

$$f_k(t) = f_{k,S(t)}.$$

Further let $f(t)$ and $r(t)$ denote the total input rate and the drain rate, respectively, at time t, i.e.,

$$f(t) = \sum_{k\in\mathcal{K}} f_k(t), \qquad r(t) = r_{S(t)}.$$

This paper studies the stationary joint distribution of the buffer contents in respective classes under the FIFO service discipline that is formally defined as follows.

Definition 1 (FIFO discipline [7]). For any time $t \in [0,\infty)$, fluid arriving in interval $[t,\infty)$ is drained from the buffer only after all fluid arriving in interval $[0,t)$ is removed completely.

The FIFO service discipline can be achieved in the following way. Suppose that the initial buffer content of class k ($k \in \mathcal{K}$) at time zero is equal to $\phi_k \geq 0$. While the initial buffer contents in respective classes are being drained, the drain rate $r(t)$ at time t is shared among classes, proportional to their initial buffer contents ϕ_k's. After those initial buffer contents are completely drained from the buffer, the drain rate $r(t)$ is shared among classes, proportional to their input rates at the arrival epoch of fluid being drained at time t. For example, suppose $S(t) = j$ for all $t \in [t_1, t_2)$ and $f_j > 0$. Further we assume that fluid arriving in interval $[t_1, t_2)$ starts to be drained at time t_3. The drain rate of class k ($k \in \mathcal{K}$) fluid at time t_3 is then given by $r(t_3)f_{k,j}/f_j$, and the sharing ratio $f_{k,j}/f_j$ remains constant while fluid arriving in interval $[t_1, t_2)$ is being drained.

Let $V_k(t)$ ($k \in \mathcal{K}$) denote the amount of class k fluid in the buffer at time t under the FIFO service discipline. We are interested in the stationary distribution of the multivariate process $\{(V_1(t), V_2(t), \ldots, V_K(t), S(t));\ t \geq 0\}$, so that without loss of generality, we assume $V_k(0) = 0$ for all $k \in \mathcal{K}$. Let $\boldsymbol{x}$ denote $(x_1, x_2, \ldots, x_K)$, where $x_k \geq 0$ for all $k \in \mathcal{K}$. We then define $\boldsymbol{\pi}(\boldsymbol{x})$ as a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $\pi_j(\boldsymbol{x})$ represents

$$\pi_j(\boldsymbol{x}) = \lim_{t\to\infty} \Pr[V_k(t) \leq x_k\ (k \in \mathcal{K}), S(t) = j].$$

To analyze the stationary joint distribution $\boldsymbol{\pi}(\boldsymbol{x})$, we first convert the original fluid queue to a new fluid queue with constant drain rate of one. Next we introduce *attained waiting time* in fluid queues, which is an analog of that for ordinary FIFO single-server queues [13, 16]. We then characterize the joint Laplace-Stieltjes transform (LST) of $\boldsymbol{\pi}(\boldsymbol{x})$, by examining the joint distribution of the buffer contents in respective classes, which arrive during the stationary attained waiting time. Further we provide recursions to compute the joint and marginal moments of the stationary buffer contents in respective classes.

The remainder of this paper is organized as follows. Section 2 provides some preliminary results. Section 3 considers the attained waiting time distribution in steady state and with this, we derive the joint LST of the stationary joint distribution of the buffer contents in respective classes. Section 4 discusses the computation of the joint and marginal moments of the stationary buffer contents in respective classes. Finally, section 5 provides some numerical examples.

2 Preliminary results

In general, any fluid queue with randomly changing (positive) drain rate can be converted to that with constant drain rate by change of time-scale [9, 17]. Associated with a multiclass FIFO fluid queue characterized by $(\boldsymbol{T}, \boldsymbol{f}_1, \boldsymbol{f}_2, \ldots, \boldsymbol{f}_K, \boldsymbol{r})$, we consider a multiclass FIFO fluid queue with constant drain rate of one, which is characterized by $(\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{f}}_1, \widetilde{\boldsymbol{f}}_2, \ldots, \widetilde{\boldsymbol{f}}_K, \boldsymbol{e})$, where

$$\widetilde{\boldsymbol{T}} = \mathrm{diag}\,(\boldsymbol{r})^{-1}\boldsymbol{T}, \qquad \widetilde{\boldsymbol{f}}_k = \mathrm{diag}\,(\boldsymbol{r})^{-1}\boldsymbol{f}_k \quad (k \in \mathcal{K}).$$

Let $\widetilde{V}_k(t)$ ($k \in \mathcal{K}$) and $\widetilde{S}(t)$ denote the amount of class k fluid in the buffer and the state of the underlying Markov chain, respectively, at time t in the FIFO fluid queue characterized by $(\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{f}}_1, \widetilde{\boldsymbol{f}}_2, \ldots, \widetilde{\boldsymbol{f}}_K, \boldsymbol{e})$. We then define $\widetilde{\boldsymbol{\pi}}(\boldsymbol{x})$ as a $1\times M$ vector whose jth ($j \in \mathcal{M}$) element $\widetilde{\pi}_j(\boldsymbol{x})$ ($\boldsymbol{x} \geq \boldsymbol{0}$) represents

$$\widetilde{\pi}_j(\boldsymbol{x}) = \lim_{t\to\infty} \Pr[\widetilde{V}_k(t) \leq x_k\ (k \in \mathcal{K}), \widetilde{S}(t) = j].$$

The following theorem can be proved in exactly the same way as Theorems 1 and 2 in [17].

Theorem 1. *The multiclass FIFO fluid queue characterized by $(\widetilde{\boldsymbol{T}}, \widetilde{\boldsymbol{f}}_1, \widetilde{\boldsymbol{f}}_2, \ldots, \widetilde{\boldsymbol{f}}_K, \boldsymbol{e})$ is stable if and only if (1) holds, i.e.,*

$$\rho = \sum_{k\in\mathcal{K}} \widetilde{\boldsymbol{\pi}}\widetilde{\boldsymbol{f}}_k < 1,$$

where $\widetilde{\boldsymbol{\pi}}$ denotes the stationary probability vector of $\widetilde{\boldsymbol{T}}$. Further $\boldsymbol{\pi}(\boldsymbol{x})$ is given in terms of $\widetilde{\boldsymbol{\pi}}(\boldsymbol{x})$:

$$\boldsymbol{\pi}(\boldsymbol{x}) = \boldsymbol{\pi}\boldsymbol{r} \cdot \widetilde{\boldsymbol{\pi}}(\boldsymbol{x})\mathrm{diag}\,(\boldsymbol{r})^{-1}.$$

Thus, without loss of generality, we assume $\boldsymbol{r} = \boldsymbol{e}$ unless otherwise stated.

We now summarize some known results on the stationary distribution of the total amount of fluid in the

buffer [5, 11], assuming constant drain rate of one. Let $V(t)$ denote the total amount of fluid in the buffer at time t, i.e., $V(t) = V_1(t) + V_2(t) + \cdots + V_K(t)$. We define $\boldsymbol{\varpi}$ as a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element ϖ_j is given by

$$\varpi_j = \lim_{t\to\infty} \Pr[V(t) = 0, S(t) = j].$$

Further let $\boldsymbol{v}(x)$ ($x > 0$) denote a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $v_j(x)$ represents the stationary probability density function of $V(t)$ when $S(t) = j$, i.e.,

$$\lim_{t\to\infty} \Pr[x < V(t) \le x+\Delta x, S(t) = j] = v_j(x)\Delta x + o(\Delta x).$$

For convenience, let $v_j(0) = \lim_{x\to 0+} v_j(x)$ ($j \in \mathcal{M}$) by continuity. As a result, $\boldsymbol{v}(x)$ is defined for all $x \ge 0$ and $v_j(x)$ ($j \in \mathcal{M}$) satisfies

$$\int_0^x v_j(y)dy = \lim_{t\to\infty} \Pr[0 < V(t) \le x, S(t) = j], \quad x \ge 0.$$

From the definition of $\boldsymbol{\varpi}$ and $\boldsymbol{v}(x)$, we have

$$\int_0^\infty \boldsymbol{v}(x)dx = \boldsymbol{\pi} - \boldsymbol{\varpi}. \tag{2}$$

We define $\boldsymbol{F}$ as

$$\boldsymbol{F} = \sum_{k\in\mathcal{K}} \mathrm{diag}\,(\boldsymbol{f}_k),$$

and we decompose $\boldsymbol{\pi}$, $\boldsymbol{T}$, and $\boldsymbol{F}$ to conform with disjoint subsets $\mathcal{M}_+$, $\mathcal{M}_-$, and $\mathcal{M}_0$ of $\mathcal{M}$:

$$\boldsymbol{\pi} = \begin{array}{c} \\ \end{array} \begin{array}{ccc} \mathcal{M}_+ & \mathcal{M}_- & \mathcal{M}_0 \end{array} \\ \begin{pmatrix} \boldsymbol{\pi}_{[+]} & \boldsymbol{\pi}_{[-]} & \boldsymbol{\pi}_{[0]} \end{pmatrix},$$

$$\boldsymbol{T} = \begin{array}{c} \\ \mathcal{M}_+ \\ \mathcal{M}_- \\ \mathcal{M}_0 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{M}_+ & \mathcal{M}_- & \mathcal{M}_0 \end{array} \\ \begin{pmatrix} \boldsymbol{T}_{[++]} & \boldsymbol{T}_{[+-]} & \boldsymbol{T}_{[+0]} \\ \boldsymbol{T}_{[-+]} & \boldsymbol{T}_{[--]} & \boldsymbol{T}_{[-0]} \\ \boldsymbol{T}_{[0+]} & \boldsymbol{T}_{[0-]} & \boldsymbol{T}_{[00]} \end{pmatrix} \end{array},$$

$$\boldsymbol{F} = \begin{array}{c} \\ \mathcal{M}_+ \\ \mathcal{M}_- \\ \mathcal{M}_0 \end{array} \begin{array}{c} \begin{array}{ccc} \mathcal{M}_+ & \mathcal{M}_- & \mathcal{M}_0 \end{array} \\ \begin{pmatrix} \boldsymbol{F}_{[+]} & \boldsymbol{O} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{F}_{[-]} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{F}_{[0]} \end{pmatrix} \end{array}.$$

We also define $\overline{\boldsymbol{T}}_{[ij]}$ ($i, j = +, -$) as

$$\overline{\boldsymbol{T}}_{[ij]} = \boldsymbol{T}_{[ij]} + \boldsymbol{T}_{[i0]}(-\boldsymbol{T}_{[00]})^{-1}\boldsymbol{T}_{[0j]}. \tag{3}$$

Let $\boldsymbol{\Psi}$ denote the minimal nonnegative solution of

$$\boldsymbol{\Psi} = \int_0^\infty \exp\left(\boldsymbol{\Lambda}_{[+]}^{-1}\overline{\boldsymbol{T}}_{[++]}x\right)\boldsymbol{\Lambda}_{[+]}^{-1}\overline{\boldsymbol{T}}_{[+-]} \cdot \exp\left[\boldsymbol{\Lambda}_{[-]}^{-1}(\overline{\boldsymbol{T}}_{[--]} + \overline{\boldsymbol{T}}_{[-+]}\boldsymbol{\Psi})x\right]dx,$$

where

$$\boldsymbol{\Lambda}_{[+]} = \boldsymbol{F}_{[+]} - \boldsymbol{I}, \qquad \boldsymbol{\Lambda}_{[-]} = \boldsymbol{I} - \boldsymbol{F}_{[-]}. \tag{4}$$

An efficient way to compute $\boldsymbol{\Psi}$ can be found in [5, 11], which is summarized in Appendix A.

We decompose $\boldsymbol{\varpi}$ and $\boldsymbol{v}(x)$ to conform with disjoint subsets $\mathcal{M}_+$, $\mathcal{M}_-$, and $\mathcal{M}_0$ of $\mathcal{M}$:

$$\boldsymbol{\varpi} = \begin{array}{c} \\ \end{array} \begin{array}{ccc} \mathcal{M}_+ & \mathcal{M}_- & \mathcal{M}_0 \end{array} \\ \begin{pmatrix} \boldsymbol{\varpi}_{[+]} & \boldsymbol{\varpi}_{[-]} & \boldsymbol{\varpi}_{[0]} \end{pmatrix},$$

$$\boldsymbol{v}(x) = \begin{array}{c} \\ \end{array} \begin{array}{ccc} \mathcal{M}_+ & \mathcal{M}_- & \mathcal{M}_0 \end{array} \\ \begin{pmatrix} \boldsymbol{v}_{[+]}(x) & \boldsymbol{v}_{[-]}(x) & \boldsymbol{v}_{[0]}(x) \end{pmatrix}.$$

Proposition 1 ([5, 11]). *$\boldsymbol{\varpi} = (\boldsymbol{\varpi}_{[+]}\ \boldsymbol{\varpi}_{[-]}\ \boldsymbol{\varpi}_{[0]})$ is given by*

$$\begin{aligned} \boldsymbol{\varpi}_{[+]} &= \boldsymbol{0}, \\ \boldsymbol{\varpi}_{[-]} &= \boldsymbol{\pi}_{[-]} - \boldsymbol{\pi}_{[+]}\boldsymbol{\Lambda}_{[+]}\boldsymbol{\Psi}\boldsymbol{\Lambda}_{[-]}^{-1}, \\ \boldsymbol{\varpi}_{[0]} &= \boldsymbol{\varpi}_{[-]}\boldsymbol{T}_{[-0]}(-\boldsymbol{T}_{[00]})^{-1}. \end{aligned}$$

Further $\boldsymbol{v}(x) = (\boldsymbol{v}_{[+]}(x)\ \boldsymbol{v}_{[-]}(x)\ \boldsymbol{v}_{[0]}(x))$ is given by

$$\begin{aligned} \boldsymbol{v}_{[+]}(x) &= \boldsymbol{\varpi}_{[-]}\overline{\boldsymbol{T}}_{[-+]}\exp[\boldsymbol{\Omega}x]\boldsymbol{\Lambda}_{[+]}^{-1}, \\ \boldsymbol{v}_{[-]}(x) &= \boldsymbol{\varpi}_{[-]}\overline{\boldsymbol{T}}_{[-+]}\exp[\boldsymbol{\Omega}x]\boldsymbol{\Psi}\boldsymbol{\Lambda}_{[-]}^{-1}, \\ \boldsymbol{v}_{[0]}(x) &= \boldsymbol{\varpi}_{[-]}\overline{\boldsymbol{T}}_{[-+]}\exp[\boldsymbol{\Omega}x]\boldsymbol{\Theta}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{\Omega} &= \boldsymbol{\Lambda}_{[+]}^{-1}\overline{\boldsymbol{T}}_{[++]} + \boldsymbol{\Psi}\boldsymbol{\Lambda}_{[-]}^{-1}\overline{\boldsymbol{T}}_{[-+]}, \\ \boldsymbol{\Theta} &= \left(\boldsymbol{\Lambda}_{[+]}^{-1}\boldsymbol{T}_{[+0]} + \boldsymbol{\Psi}\boldsymbol{\Lambda}_{[-]}^{-1}\boldsymbol{T}_{[-0]}\right)(-\boldsymbol{T}_{[00]})^{-1}. \end{aligned}$$

3 Joint distribution

This section analyzes the stationary joint distribution $\boldsymbol{\pi}(\boldsymbol{x})$ of the buffer contents in respective classes. For this purpose, we first introduce attained waiting time, and utilizing it, we derive the vector joint LST of $\boldsymbol{\pi}(\boldsymbol{x})$.

3.1 Attained waiting time

We first define the attained waiting time $\{W(t);\ t \ge 0\}$ in FIFO fluid queues with constant drain rate of one. Note that $W(t)$ is an analog of the attained waiting time for ordinary FIFO queues [13, 16].

Definition 2. The attained waiting time $W(t)$ ($t \ge 0$) in FIFO fluid queues with constant drain rate of one is defined as

$$W(t) = \inf\{x \ge 0;\ V(t-x) = x\}. \tag{5}$$

Remark 1. The value of $W(t)$ at $t = u$ is equal to the value of $V(t)$ at the closest point to time u among intersection points between a sample path of $V(t)$ and a line passing through $(u, 0)$ with slope -1. See Figure 1 also.

Remark 2. The attained waiting time in (5) has the following properties. In FIFO fluid queues with constant drain rate of one, all fluid in the buffer at time $t - W(t)$ has just been drained at time t and the amount of fluid arriving in interval $(t - W(t), t]$ is equal to $V(t)$.

We now consider the bivariate process $\{(W(t), J(t)); t \geq 0\}$, where $J(t)$ is defined as

$$J(t) = S(t - W(t)). \tag{6}$$

Let $\boldsymbol{w}(x)$ denote a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $w_j(x)$ is given by

$$\begin{aligned} w_j(x) &= \lim_{t\to\infty} \Pr[W(t) \leq x, J(t) = j] \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{I}(W(t) \leq x, J(t) = j)dt, \quad \text{w.p.1,} \end{aligned}$$

where $\mathbb{I}(\chi)$ denotes the indicator function of event χ, and the second equality comes from the ergodicity of $\{(W(t), J(t)); t \geq 0\}$.

Note here that

$$W(t) = V(t - W(t)), \tag{7}$$

which is immediate from (5). Note also that $W(t) = 0$ when $V(t) = 0$ because $V(t - 0) = 0$. These two facts show that $W(t) = 0$ if and only if $V(t) = 0$. Further, because $J(t) = S(t)$ when $W(t) = 0$, $\boldsymbol{w}(0)$ is given by

$$\boldsymbol{w}(0) = \boldsymbol{\varpi}.$$

Therefore we have

$$\boldsymbol{w}(x) = \boldsymbol{\varpi} + \boldsymbol{w}^+(x), \qquad x \geq 0, \tag{8}$$

where $\boldsymbol{w}^+(x)$ denotes a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $w_j^+(x)$ is given by

$$\begin{aligned} w_j^+(x) &= \lim_{t\to\infty} \Pr[0 < W(t) \leq x, J(t) = j] \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{I}(0 < W(t) \leq x, J(t) = j)dt, \end{aligned}$$

where the second equality holds w.p.1. It is easy to see that

$$\lim_{x\to\infty} \boldsymbol{w}^+(x)\boldsymbol{e} = 1 - \boldsymbol{\varpi}\boldsymbol{e}. \tag{9}$$

Figure 1 describes typical sample paths of the total amount $V(t)$ of fluid in the buffer and the corresponding attained waiting time $W(t)$. We first observe that $V(t)$ is continuous and piece-wise linear in time. Note also that $W(t)$ is piece-wise linear in time, and each linear segment in $W(t)$ has the corresponding linear segment in $V(t)$. More precisely, linear segments in $W(t)$ have one-to-one correspondence to linear segments in $V(t) = 0$ or $V(t) > 0$ with $f(t) > 0$, and downward jumps in $W(t)$ are caused by linear segments in $V(t) > 0$ with $f(t) = 0$.

Thus $\{W(t); t \geq 0\}$ is regarded as a weighted censored process of $\{V(t); t \geq 0\}$, constructed by observing $\{V(t); t \geq 0\}$ only when $V(t) = 0$ or $V(t) > 0$ with $f(t) > 0$, where the weight for linear segments in $V(t) > 0$ is equal to $f(t)$, because a linear segment in $W(t) > 0$ is obtained by extending the corresponding linear segment in $V(t) > 0$ $f(t)$ times (see Figure 2).

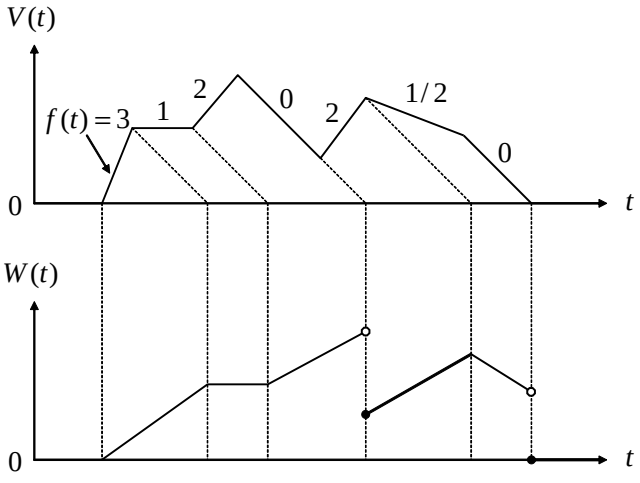

Figure 1: Sample paths of $V(t)$ and $W(t)$.

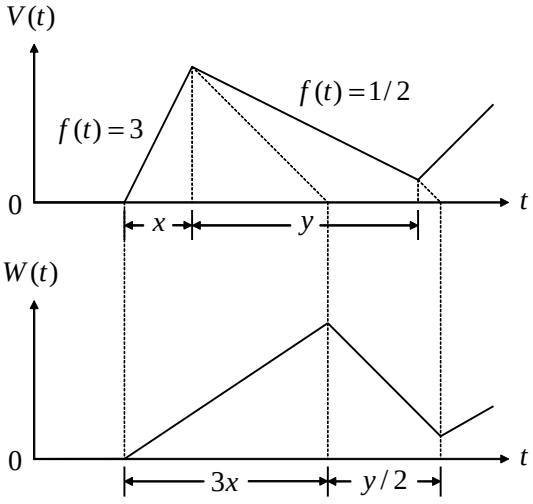

Figure 2: Linear segments in $V(t)$ and $W(t)$.

Therefore, noting (2) and (9), we obtain $\boldsymbol{w}^+(x)$ in terms of $\boldsymbol{v}(x)$:

$$\boldsymbol{w}^+(x) = \frac{1 - \boldsymbol{\varpi}\boldsymbol{e}}{(\boldsymbol{\pi} - \boldsymbol{\varpi})\boldsymbol{F}\boldsymbol{e}} \int_0^x \boldsymbol{v}(y)dy\boldsymbol{F}, \qquad x \geq 0. \tag{10}$$

The following theorem now follows from (8) and (10).

Theorem 2. *$\boldsymbol{w}(x)$ is given by*

$$\boldsymbol{w}(x) = \boldsymbol{\varpi} + \frac{1 - \boldsymbol{\varpi}\boldsymbol{e}}{(\boldsymbol{\pi} - \boldsymbol{\varpi})\boldsymbol{F}\boldsymbol{e}} \int_0^x \boldsymbol{v}(y)dy\boldsymbol{F}, \qquad x \geq 0. \tag{11}$$

The formal proof of Theorem 2 is given in Appendix B.

3.2 Vector joint LST for buffer contents

This subsection derives the vector joint LST of $\boldsymbol{\pi}(\boldsymbol{x})$. For notational convenience, we shift the origin of time from zero to $-\infty$, and we assume that the two processes $\{(V_1(t), V_2(t), \ldots, V_K(t), S(t))\}$ and $\{(W(t), J(t))\}$ are jointly stationary at time 0. Let $\boldsymbol{s}$

denote $(s_1, s_2, \ldots, s_K)$, where $\mathrm{Re}(s_k) > 0$ $(k \in \mathcal{K})$. We then define $\boldsymbol{\pi}^*(\boldsymbol{s})$ as a $1 \times M$ vector whose jth $(j \in \mathcal{M})$ element $\pi_j^*(\boldsymbol{s})$ is given by

$$\pi_j^*(\boldsymbol{s}) = \mathrm{E}\left[\prod_{k\in\mathcal{K}} e^{-s_k V_k(0)} \cdot \mathbb{I}(S(0) = j)\right]. \tag{12}$$

Let $A_k(t,u]$ $(k \in \mathcal{K};\ -\infty < t \le u)$ denote the amount of class k fluid arriving in interval $(t,u]$, where $A_k(u,u] = 0$. Further we define $A(t,u]$ $(-\infty < t \le u)$ as the total amount of fluid arriving in interval $(t,u]$, i.e.,

$$A(t,u] = A_1(t,u] + A_2(t,u] + \cdots + A_K(t,u].$$

Lemma 1. *The vector joint LST $\boldsymbol{\pi}^*(\boldsymbol{s})$ is given by*

$$\boldsymbol{\pi}^*(\boldsymbol{s}) = \int_0^\infty d\boldsymbol{w}(x)\boldsymbol{A}^*(\boldsymbol{s} \mid x), \tag{13}$$

where $\boldsymbol{A}^*(\boldsymbol{s} \mid x)$ $(x \ge 0)$ denotes an $M \times M$ matrix whose (i,j)th $(i,j \in \mathcal{M})$ element $A_{i,j}^*(\boldsymbol{s} \mid x)$ is given by

$$A_{i,j}^*(\boldsymbol{s} \mid x) = \mathrm{E}\left[\prod_{k\in\mathcal{K}} e^{-s_k A_k(-x,0]} \cdot \mathbb{I}(S(0) = j) \;\middle|\; S(-x) = i\right].$$

Proof. Note first that on each sample path,

$$V_k(t) = A_k(t - W(t), t], \qquad k \in \mathcal{K},$$

which is immediate from the definition of $W(t)$ (see Remark 2). We then rewrite (12) as

$$\pi_j^*(\boldsymbol{s}) = \mathrm{E}\left[\prod_{k\in\mathcal{K}} e^{-s_k A_k(-W(0),0]} \cdot \mathbb{I}(S(0) = j)\right]. \tag{14}$$

Because $W(0) = V(-W(0))$ (see (7)), $W(0)$ is fully determined by fluid arriving in interval $(-\infty, -W(0)]$. Further fluid arriving in interval $(-W(0), 0]$ does not affect $W(0)$. Thus, given $W(0) = V(-W(0))$ and $J(0) = S(-W(0))$, the joint distribution of $A_k(-W(0),0]$ $(k \in \mathcal{K})$ is conditionally independent of the past history of the queue in interval $(-\infty, -W(0))$. Noting $w_i(x) = \Pr[W(0) \le x, J(0) = i]$, we obtain from (14)

$$\begin{aligned}
\pi_j^*(\boldsymbol{s}) &= \int_0^\infty \sum_{i\in\mathcal{M}} dw_i(x)\mathrm{E}\left[\prod_{k\in\mathcal{K}} e^{-s_k A_k(-W(0),0]} \right.\\
&\qquad \left. \cdot\, \mathbb{I}(S(0) = j) \;\middle|\; J(0) = i, W(0) = x\right]\\
&= \int_0^\infty \sum_{i\in\mathcal{M}} dw_i(x)\mathrm{E}\left[\prod_{k\in\mathcal{K}} e^{-s_k A_k(-x,0]} \right.\\
&\qquad \left. \cdot\, \mathbb{I}(S(0) = j) \;\middle|\; S(-x) = i\right]\\
&= \int_0^\infty \sum_{i\in\mathcal{M}} dw_i(x) A_{i,j}(\boldsymbol{s} \mid x),
\end{aligned}$$

from which (13) follows. □

Lemma 2. *$\boldsymbol{A}^*(\boldsymbol{s} \mid x)$ is given by*

$$\boldsymbol{A}^*(\boldsymbol{s} \mid x) = \exp\left[\left(\boldsymbol{T} - \sum_{k\in\mathcal{K}} s_k \boldsymbol{F}_k\right) x\right], \qquad x \ge 0, \tag{15}$$

where $\boldsymbol{F}_k = \mathrm{diag}\,(\boldsymbol{f}_k)$ $(k \in \mathcal{K})$.

The proof of Lemma 2 is given in Appendix C. Substituting (11) and (15) into (13), we obtain the following theorem.

Theorem 3. *$\boldsymbol{\pi}^*(\boldsymbol{s})$ is given by*

$$\boldsymbol{\pi}^*(\boldsymbol{s}) = \boldsymbol{\varpi} + \frac{1 - \boldsymbol{\varpi}\boldsymbol{e}}{(\boldsymbol{\pi} - \boldsymbol{\varpi})\boldsymbol{F}\boldsymbol{e}} \int_0^\infty \boldsymbol{v}(x)\boldsymbol{F} \cdot \exp\left[\left(\boldsymbol{T} - \sum_{k\in\mathcal{K}} s_k \boldsymbol{F}_k\right) x\right] dx,$$

where $\boldsymbol{v}(x)$ is given in Proposition 1.

4 Joint and marginal moments

This section provides recursions to compute the joint and marginal moments of the buffer contents in respective classes, assuming that the system is stationary at time 0. We start with the marginal moment because it is simpler than the joint moment.

4.1 Marginal moments

Note that matrix $\boldsymbol{\Omega}$ appeared in $\boldsymbol{v}(x)$ has no probabilistic meanings. Thus, according to [11], we first convert it to a defective generator of a certain continuous-time Markov chain. Let $\widehat{\boldsymbol{T}}$ denote a proper generator of a Markov chain, which is given by

$$\widehat{\boldsymbol{T}} = \begin{pmatrix} \boldsymbol{\Lambda}_{[+]}^{-1} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_{[-]}^{-1} \end{pmatrix} \begin{pmatrix} \overline{\boldsymbol{T}}_{[++]} & \overline{\boldsymbol{T}}_{[+-]} \\ \overline{\boldsymbol{T}}_{[-+]} & \overline{\boldsymbol{T}}_{[--]} \end{pmatrix},$$

where $\overline{\boldsymbol{T}}_{[ij]}$ $(i,j = +,-)$ and $\boldsymbol{\Lambda}_{[i]}$ $(i = +,-)$ are given in (3) and (4), respectively. We then define $\widehat{\boldsymbol{\pi}}$ as the stationary probability vector of $\widehat{\boldsymbol{T}}$, which is decomposed to conform with subsets $\mathcal{M}_+$ and $\mathcal{M}_-$, i.e.,

$$\widehat{\boldsymbol{\pi}} = \begin{array}{c} \begin{array}{cc} \mathcal{M}_+ & \mathcal{M}_- \end{array} \\ \begin{pmatrix} \widehat{\boldsymbol{\pi}}_{[+]} & \widehat{\boldsymbol{\pi}}_{[-]} \end{pmatrix} \end{array}.$$

With $\widehat{\boldsymbol{\pi}}_{[+]}$, we define $\widehat{\boldsymbol{\Omega}}$ as

$$\widehat{\boldsymbol{\Omega}} = \mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right)^{-1} \boldsymbol{\Omega}^T \mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right),$$

where T stands for the transpose operator. It is known that $\widehat{\boldsymbol{\Omega}}$ is a defective generator of a certain continuous-time Markov chain [11]. Note here that

$$\exp(\boldsymbol{\Omega}x) = \mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right)^{-1} \exp\left(\widehat{\boldsymbol{\Omega}}^T x\right) \mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right). \tag{16}$$

We now define $\boldsymbol{\pi}_k^*(s)$ $(k \in \mathcal{K})$ as a $1 \times M$ vector whose jth $(j \in \mathcal{M})$ element $\pi_{k,j}^*(s)$ is given by

$$\pi_{k,j}^*(s) = \mathrm{E}\left[\exp(-sV_k(0)) \cdot \mathbb{I}(S(0) = j)\right].$$

It follows from Proposition 1, Theorem 3, and (16) that for $k \in \mathcal{K}$,

$$\boldsymbol{\pi}_k^*(s) = \boldsymbol{\varpi} + \boldsymbol{\alpha} \int_0^\infty \exp\left(\widehat{\boldsymbol{\Omega}}^T x\right) \boldsymbol{L} \cdot \exp\left[(\boldsymbol{T} - s\boldsymbol{F}_k)\, x\right] dx, \quad (17)$$

where

$$\boldsymbol{\alpha} = \frac{1 - \boldsymbol{\varpi}\boldsymbol{e}}{(\boldsymbol{\pi} - \boldsymbol{\varpi})\boldsymbol{F}\boldsymbol{e}} \cdot \boldsymbol{\varpi}_{[-]}\overline{\boldsymbol{T}}_{[-+]}\mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right)^{-1},$$
$$\boldsymbol{L} = \mathrm{diag}\left(\widehat{\boldsymbol{\pi}}_{[+]}\right) \begin{pmatrix} \boldsymbol{\Lambda}_{[+]}^{-1} & \boldsymbol{\Psi}\boldsymbol{\Lambda}_{[-]}^{-1} \\ \boldsymbol{\Theta} & \end{pmatrix} \boldsymbol{F}.$$

Recall that $\widehat{\boldsymbol{\Omega}}$ is a defective generator of a Markov chain. Also $\boldsymbol{T} - \boldsymbol{F}_k$ $(k \in \mathcal{K})$ is considered as a defective generator. Thus we shall apply the uniformization technique to expanding matrix exponentials in (17). Let η and θ_k $(k \in \mathcal{K})$ denote maximum absolute values of diagonal elements of $\widehat{\boldsymbol{\Omega}}$ and $\boldsymbol{T} - \boldsymbol{F}_k$, respectively. Applying the uniformization technique to (17), we obtain

$$\begin{aligned}
\boldsymbol{\pi}_k^*(s) &= \boldsymbol{\varpi} + \boldsymbol{\alpha} \sum_{n=0}^\infty \sum_{l=0}^\infty \int_0^\infty e^{-\eta x} \frac{(\eta x)^n}{n!} e^{-\theta_k x} \frac{(\theta_k x)^l}{l!} dx \\
&\qquad \cdot \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right]^n \boldsymbol{L} \\
&\qquad \cdot \left[\boldsymbol{I} + \theta_k^{-1}(\boldsymbol{T} - s\boldsymbol{F}_k)\right]^l \\
&= \boldsymbol{\varpi} + \boldsymbol{\alpha} \sum_{m=0}^\infty \int_0^\infty e^{-(\eta+\theta_k)x} \frac{[(\eta+\theta_k)x]^m}{m!} dx \\
&\qquad \cdot \sum_{l=0}^m \binom{m}{l} \left(\frac{\eta}{\eta+\theta_k}\right)^{m-l} \left(\frac{\theta_k}{\eta+\theta_k}\right)^l \\
&\qquad \cdot \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right]^{m-l} \boldsymbol{L} \\
&\qquad \cdot \left[\boldsymbol{I} + \theta_k^{-1}(\boldsymbol{T} - s\boldsymbol{F}_k)\right]^l.
\end{aligned}$$

Because for all $m = 0, 1, \ldots$,

$$\int_0^\infty e^{-(\eta+\theta_k)x} \frac{[(\eta+\theta_k)x]^m}{m!} dx = \frac{1}{\eta+\theta_k},$$

we have

$$\boldsymbol{\pi}_k^*(s) = \boldsymbol{\varpi} + \frac{1}{\eta+\theta_k} \boldsymbol{\alpha} \sum_{m=0}^\infty \boldsymbol{H}_{k;m}^*(s), \qquad k \in \mathcal{K}, \quad (18)$$

where for $m = 0, 1, \ldots$,

$$\boldsymbol{H}_{k;m}^*(s) = \sum_{l=0}^m \binom{m}{l} \left(\frac{\eta}{\eta+\theta_k}\right)^{m-l} \left(\frac{\theta_k}{\eta+\theta_k}\right)^l \cdot \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right]^{m-l} \boldsymbol{L} \left[\boldsymbol{I} + \theta_k^{-1}(\boldsymbol{T} - s\boldsymbol{F}_k)\right]^l.$$

We now define $\boldsymbol{\pi}_k^{(n)}$ $(k \in \mathcal{K};\ n = 0, 1, \ldots)$ as

$$\boldsymbol{\pi}_k^{(n)} = \lim_{s \to 0+} \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \boldsymbol{\pi}_k^*(s).$$

Note that the jth $(j \in \mathcal{M})$ element of $\boldsymbol{\pi}_k^{(n)}$ represents

$$\frac{1}{n!}\mathrm{E}\left[\{V_k(0)\}^n \cdot \mathbb{I}(S(0) = j)\right].$$

It then follows from (18) that for $k \in \mathcal{K}$,

$$\boldsymbol{\pi}_k^{(n)} = \frac{1}{\eta+\theta_k} \boldsymbol{\alpha} \sum_{m=n}^\infty \boldsymbol{H}_{k;m}^{(n)}, \qquad n = 1, 2, \ldots, \quad (19)$$

where

$$\boldsymbol{H}_{k;m}^{(n)} = \lim_{s \to 0+} \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \boldsymbol{H}_{k;m}^*(s), \qquad m, n = 0, 1, \ldots.$$

In deriving (19), we use $\boldsymbol{H}_{k;m}^{(n)} = \boldsymbol{0}$ for $n > m$. Thus the computation of the marginal moment $\boldsymbol{\pi}_k^{(n)}$ is reduced to that of $\boldsymbol{H}_{k;m}^{(n)}$.

Theorem 4. *The sequence of matrices $\boldsymbol{H}_{k;m}^{(n)}$ $(n = 0, 1, \ldots;\ m = n, n+1, \ldots)$ can be computed by the following recursion: for $n = 0$,*

$$\begin{aligned}
\boldsymbol{H}_{k;0}^{(0)} &= \boldsymbol{L}, \\
\boldsymbol{H}_{k;m}^{(0)} &= \left(\frac{\eta}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right] \boldsymbol{H}_{k;m-1}^{(0)} \\
&\quad + \boldsymbol{H}_{k;m-1}^{(0)} \left(\frac{\theta_k}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \theta_k^{-1}\boldsymbol{T}\right], \quad m = 1, 2, \ldots.
\end{aligned}$$

Further for $n = 1, 2, \ldots$,

$$\begin{aligned}
\boldsymbol{H}_{k;n}^{(n)} &= \boldsymbol{H}_{k;n-1}^{(n-1)} \left(\frac{1}{\eta+\theta_k}\right) \boldsymbol{F}_k, \\
\boldsymbol{H}_{k;m}^{(n)} &= \left(\frac{\eta}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right] \boldsymbol{H}_{k;m-1}^{(n)} \\
&\quad + \boldsymbol{H}_{k;m-1}^{(n)} \left(\frac{\theta_k}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \theta_k^{-1}\boldsymbol{T}\right] \\
&\quad + \boldsymbol{H}_{k;m-1}^{(n-1)} \left(\frac{1}{\eta+\theta_k}\right) \boldsymbol{F}_k, \quad m = n+1, n+2, \ldots.
\end{aligned}$$

Proof. It is easy to verify that for $m = 1, 2, \ldots$,

$$\begin{aligned}
\boldsymbol{H}_{k;m}^*(s) &= \left(\frac{\eta}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right] \boldsymbol{H}_{k;m-1}^*(s) \\
&\quad + \boldsymbol{H}_{k;m-1}^*(s) \left(\frac{\theta_k}{\eta+\theta_k}\right) \left[\boldsymbol{I} + \theta_k^{-1}(\boldsymbol{T} - s\boldsymbol{F}_k)\right],
\end{aligned} \quad (20)$$

where $\boldsymbol{H}_{k;0}^*(s) = \boldsymbol{L}$. Differentiating both sides of (20) n times, letting $s \to 0+$ on both sides, and multiplying them by $(-1)^n/n!$, we can prove this theorem. Because those calculations are routine, we omit the details. □

4.2 Joint moments

This subsection considers the joint moments of the buffer contents in two specific classes k and ν ($k, \nu \in \mathcal{K}$). Let $\boldsymbol{\pi}^*_{k,\nu}(s_k, s_\nu)$ ($k, \nu \in \mathcal{K}$) denote a $1 \times M$ vector whose jth ($j \in \mathcal{M}$) element $\pi^*_{k,\nu,j}(s_k, s_\nu)$ represents

$$\pi^*_{k,\nu,j}(s_k, s_\nu) = \mathrm{E}\left[\exp(-s_k V_k(0)) \cdot \exp(-s_\nu V_\nu(0)) \cdot \mathbb{I}(S(0) = j)\right].$$

It follows from Proposition 1, Theorem 3, and (16) that

$$\boldsymbol{\pi}^*_{k,\nu}(s_k, s_\nu) = \boldsymbol{\varpi} + \boldsymbol{\alpha} \int_0^\infty \exp\left(\widehat{\boldsymbol{\Omega}}^T x\right) \boldsymbol{L} \cdot \exp\left[(\boldsymbol{T} - s_k \boldsymbol{F}_k - s_\nu \boldsymbol{F}_\nu)\, x\right] dx. \quad (21)$$

Let $\theta_{k,\nu}$ ($k, \nu \in \mathcal{K}$) denote the maximum absolute value of the diagonal elements of $\boldsymbol{T} - \boldsymbol{F}_k - \boldsymbol{F}_\nu$. Applying the uniformization technique to (21) and manipulating it in a way very similar to $\boldsymbol{\pi}^*_k(s)$, we obtain for $k, \nu \in \mathcal{K}$,

$$\boldsymbol{\pi}^*_{k,\nu}(s_k, s_\nu) = \boldsymbol{\varpi} + \frac{1}{\eta + \theta_{k,\nu}} \boldsymbol{\alpha} \sum_{m=0}^\infty \boldsymbol{H}^*_{k,\nu;m}(s_k, s_\nu), \quad (22)$$

where

$$\boldsymbol{H}^*_{k,\nu;m}(s_k, s_\nu) = \sum_{l=0}^m \binom{m}{l} \left(\frac{\eta}{\eta + \theta_{k,\nu}}\right)^{m-l} \left(\frac{\theta_{k,\nu}}{\eta + \theta_{k,\nu}}\right)^l \cdot \left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right]^{m-l} \boldsymbol{L} \cdot \left[\boldsymbol{I} + \theta^{-1}_{k,\nu}\left(\boldsymbol{T} - s_k \boldsymbol{F}_k - s_\nu \boldsymbol{F}_\nu\right)\right]^l.$$

We define $\boldsymbol{\pi}^{(n,l)}_{k,\nu}$ ($k, \nu \in \mathcal{K}$; $n, l = 0, 1, \ldots$) as

$$\boldsymbol{\pi}^{(n,l)}_{k,\nu} = \lim_{s_k \to 0+} \lim_{s_\nu \to 0+} \frac{(-1)^n}{n!} \frac{(-1)^l}{l!} \cdot \frac{d^n}{ds_k^n} \frac{d^l}{ds_\nu^l} \boldsymbol{\pi}^*_{k,\nu}(s_k, s_\nu).$$

Note that the jth ($j \in \mathcal{M}$) element of $\boldsymbol{\pi}^{(n,l)}_{k,\nu}$ represents

$$\frac{1}{n!}\frac{1}{l!}\mathrm{E}\left[\{V_k(0)\}^n \{V_\nu(0)\}^l \cdot \mathbb{I}(S(0) = j)\right].$$

It then follows from (22) that for $k, \nu \in \mathcal{K}$,

$$\boldsymbol{\pi}^{(n,l)}_{k,\nu} = \frac{1}{\eta + \theta_{k,\nu}} \boldsymbol{\alpha} \sum_{m=n+l}^\infty \boldsymbol{H}^{(n,l)}_{k,\nu;m}, \qquad (n,l) \neq (0,0),$$

where

$$\boldsymbol{H}^{(n,l)}_{k,\nu;m} = \lim_{s_k \to 0+} \lim_{s_\nu \to 0+} \frac{(-1)^n}{n!} \frac{(-1)^l}{l!} \cdot \frac{d^n}{ds_k^n} \frac{d^l}{ds_\nu^l} \boldsymbol{H}^*_{k,\nu;m}(s_k, s_\nu).$$

Note here that

$$\boldsymbol{\pi}^{(n,0)}_{k,\nu} = \boldsymbol{\pi}^{(n)}_k, \qquad \boldsymbol{\pi}^{(0,l)}_{k,\nu} = \boldsymbol{\pi}^{(l)}_\nu.$$

Theorem 5. *The sequence of matrices $\boldsymbol{H}^{(n,0)}_{k,\nu;m}$ (resp. $\boldsymbol{H}^{(0,l)}_{k,\nu;m}$) can be computed by the recursion for $\boldsymbol{H}^{(n)}_{k;m}$ (resp. $\boldsymbol{H}^{(l)}_{\nu;m}$) in Theorem 4, where θ_k (resp. θ_ν) is replaced with $\theta_{k,\nu}$. Further the sequence of matrices $\boldsymbol{H}^{(n,l)}_{k,\nu;m}$ ($n, l = 1, 2, \ldots$; $m = n + l, n + l + 1, \ldots$) can be computed by the following recursion:*

$$\boldsymbol{H}^{(n,l)}_{k,\nu;m} = \left(\frac{\eta}{\eta + \theta_{k,\nu}}\right)\left[\boldsymbol{I} + \eta^{-1}\widehat{\boldsymbol{\Omega}}^T\right] \boldsymbol{H}^{(n,l)}_{k,\nu;m-1} + \boldsymbol{H}^{(n,l)}_{k,\nu;m-1}\left(\frac{\theta_{k,\nu}}{\eta + \theta_{k,\nu}}\right)\left[\boldsymbol{I} + \theta^{-1}_{k,\nu}\boldsymbol{T}\right] + \boldsymbol{H}^{(n-1,l)}_{k,\nu;m-1}\left(\frac{1}{\eta + \theta_{k,\nu}}\right)\boldsymbol{F}_k + \boldsymbol{H}^{(n,l-1)}_{k,\nu;m-1}\left(\frac{1}{\eta + \theta_{k,\nu}}\right)\boldsymbol{F}_\nu,$$

where $\boldsymbol{H}^{(n,l)}_{k,\nu;n+l-1} = \boldsymbol{O}$.

Theorem 5 can be shown in the same way as Theorem 4, so that we omit the proof.

5 Numerical examples

We provide some numerical examples for fluid queues with two classes. In all of these examples, the marginal behavior of class k ($k = 1, 2$) fluid flow is characterized by $(\boldsymbol{T}^{[k]}, \boldsymbol{f}^{[k]})$, where

$$\boldsymbol{T}^{[1]} = \boldsymbol{T}^{[2]} = \boldsymbol{T}_{\mathrm{c}} = \begin{pmatrix} -0.1 & 0.1 & 0.0 & 0.0 \\ 0.1 & -0.2 & 0.1 & 0.0 \\ 0.0 & 0.1 & -0.2 & 0.1 \\ 0.0 & 0.0 & 0.1 & -0.1 \end{pmatrix}.$$

In this setting, we examine various $\boldsymbol{f}^{[k]}$ ($k = 1, 2$).

In the first example, we assume that

$$\boldsymbol{f}^{[1]} = \boldsymbol{f}^{[2]} = \begin{pmatrix} 1.2 & 0.6 & 0.0 & 0.0 \end{pmatrix}^T.$$

Further, in constructing the input process $(\boldsymbol{T}, \boldsymbol{f}_1, \boldsymbol{f}_2)$ of the queue, we consider three types of superpositions of two fluid flows $(\boldsymbol{T}^{[1]}, \boldsymbol{f}^{[1]})$ and $(\boldsymbol{T}^{[2]}, \boldsymbol{f}^{[2]})$:

Case P: $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{c}}$, $\boldsymbol{f}_1 = \boldsymbol{f}^{[1]}$, and $\boldsymbol{f}_2 = \boldsymbol{f}^{[2]}$.

Case I: $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{c}} \oplus \boldsymbol{T}_{\mathrm{c}}$, $\boldsymbol{f}_1 = \boldsymbol{f}^{[1]} \otimes \boldsymbol{e}_4$, and $\boldsymbol{f}_2 = \boldsymbol{e}_4 \otimes \boldsymbol{f}^{[2]}$, where $\boldsymbol{e}_4 = (1 \quad 1 \quad 1 \quad 1)^T$.

Case N: $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{c}}$, $\boldsymbol{f}_1 = \boldsymbol{f}^{[1]}$, and $\boldsymbol{f}_2 = (0.0 \quad 0.0 \quad 0.6 \quad 1.2)^T$.

Because of the descending arrangement of elements in $\boldsymbol{f}^{[k]}$ ($k = 1, 2$), two fluid flows are positively correlated in Case P, while they have negative correlation in Case N. Note also that they are independent in Case I. Further we assume constant drain rate of one (i.e., $\boldsymbol{r} = \boldsymbol{e}$), unless otherwise stated. Thus the offered load ρ is equal to

Table 1: Numerical results for homogeneous fluid flows.

	$\mathrm{E}[V_1]$	$\mathrm{E}[V_2]$	$\mathrm{C_V}[V_1]$	$\mathrm{C_V}[V_2]$	$\mathrm{C_C}[V_1, V_2]$
Case P	70.946	70.946	1.129	1.129	1.000
Case I	32.756	32.756	1.289	1.289	0.754
Case N	1.500	1.500	2.003	2.003	−0.169

Table 2: Comparison of Case P, Case P-1, and Case P-2.

	$\mathrm{E}[V_1]$	$\mathrm{E}[V_2]$	$\mathrm{C_V}[V_1]$	$\mathrm{C_V}[V_2]$	$\mathrm{C_C}[V_1, V_2]$
Case P	70.946	70.946	1.129	1.129	1.000
Case P-1	94.594	47.297	1.129	1.129	1.000
Case P-2	97.394	44.497	1.176	1.169	0.824

0.9 in all three cases. Let V_1 and V_2 denote generic random variables for $V_1(t)$ and $V_2(t)$, respectively, in steady state.

Table 1 shows the mean $\mathrm{E}[V_k]$, the coefficient of variation $\mathrm{C_V}[V_k] = \sqrt{\mathrm{Var}[V_k]}/\mathrm{E}[V_k]$, and the correlation coefficient $\mathrm{C_C}[V_1, V_2] = \mathrm{Cov}[V_1, V_2] \ /(\sqrt{\mathrm{Var}[V_1]}\sqrt{\mathrm{Var}[V_2]})$ of the amount of class k $(k = 1, 2)$ fluid in the buffer. We observe that owing to the homogeneity and *symmetry*, the two classes have the same mean and coefficient of variation. However, the correlation coefficients are different among the three cases. In Case P, $f_{1,j} = f_{2,j}$ for all $j \in \mathcal{M}$, so that the drain rate is always shared equally between them. As a result, when $V(0) = 0$, $V_1(t) = V_2(t) = V(t)/2$ for all $t \geq 0$, which leads to $\mathrm{C_C}[V_1, V_2] = 1.0$. In Case I, V_1 and V_2 are positively correlated due to interaction in the buffer, even though the two fluid flows are independent. Finally, in Case N, we observe that V_1 and V_2 are negatively correlated.

Next we compare Case P with the following two cases with $\boldsymbol{T} = \boldsymbol{T}_{\mathrm{c}}$.

Case P-1:

$\boldsymbol{f}_1 = (1.6 \quad 0.8 \quad 0.0 \quad 0.0)^T,$
$\boldsymbol{f}_2 = (0.8 \quad 0.4 \quad 0.0 \quad 0.0)^T.$

Case P-2:

$\boldsymbol{f}_1 = (2.4 \quad 0.0 \quad 0.0 \quad 0.0)^T,$
$\boldsymbol{f}_2 = (0.0 \quad 1.2 \quad 0.0 \quad 0.0)^T.$

In the three cases, Case P, Case P-1, and Case P-2, $\boldsymbol{f} = \boldsymbol{f}_1 + \boldsymbol{f}_2$ is identical and therefore the total buffer content $V(t)$ has the same stationary distribution. Table 2 shows the numerical results.

In Case P-1, $\boldsymbol{f}_1 = 2\boldsymbol{f}_2$, and therefore the drain rate of class 1 fluid is always the double of that of class 2 fluid. As a result, when $V(0) = 0$, $V_1(t) = 2V_2(t)$ for all $t \geq 0$. Thus the correlation coefficient in Case P-1 is the same as in Case P, i.e., $\mathrm{C_C}(V_1, V_2) = 1.0$. In Case P-2, the correlation between V_1 and V_2 decreases to 0.824, so that the coefficient of variation slightly increases, compared with Case P and Case P-1.

Finally, we consider the fluid queue $(\boldsymbol{T}, \boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{r})$, where

$$\boldsymbol{T} = \boldsymbol{T}_{\mathrm{c}},$$

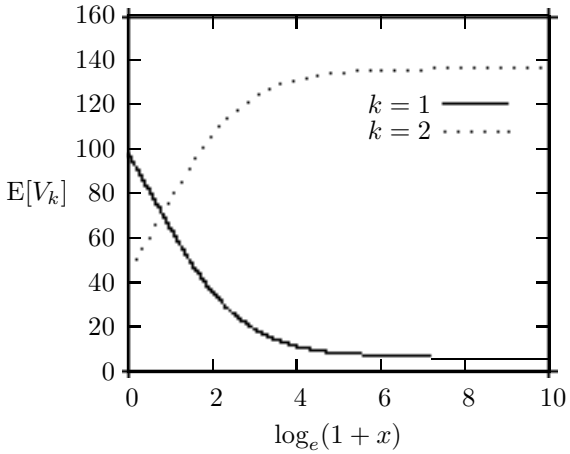

Figure 3: Mean buffer content.

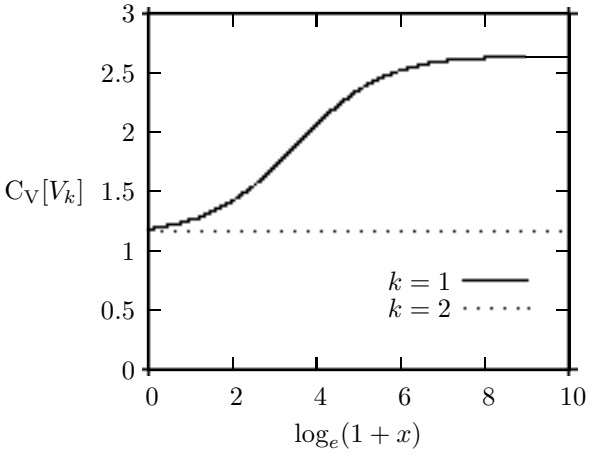

Figure 4: Coefficient of variation of the buffer content.

$$\boldsymbol{f}_1 = (2.4 \quad 0.0 \quad 0.0 \quad 0.0)^T,$$
$$\boldsymbol{f}_2 = (0.0 \quad 1.2(1+x) \quad 0.0 \quad 0.0)^T, \qquad x \geq 0,$$
$$\boldsymbol{r} = \boldsymbol{e} + (0.0 \quad 1.2x \quad 0.0 \quad 0.0)^T, \qquad x \geq 0.$$

Note here that Case P-2 in Table 2 corresponds to $x = 0$. Because $\boldsymbol{f}_1 + \boldsymbol{f}_2 - \boldsymbol{r}$ remains fixed for all $x \geq 0$, the stationary distribution of the total amount $V(t)$ of fluid in the buffer is irrelevant to parameter x.

Figures 3 and 4 show the mean and the coefficient of variation, respectively, of the amount of class k $(k = 1, 2)$ fluid in the buffer as a function of parameter x. From Figure 3, we observe that $\mathrm{E}[V_2]$ increases monotonously with parameter x. This phenomenon can be explained as follows. Suppose x is large enough. In such a case, as soon as the underlying Markov chain changes its state from state 1 to state 2, class 1 fluid in the buffer is drained very rapidly as if it would be replaced by class 2 fluid. This is also the reason why in Figure 4, the variation in V_1 increases with x. We also observe that the variation in V_2 is almost irrelevant to parameter x, even though it increases very slightly. This phenomenon seems to be interesting, but we cannot provide any intu-

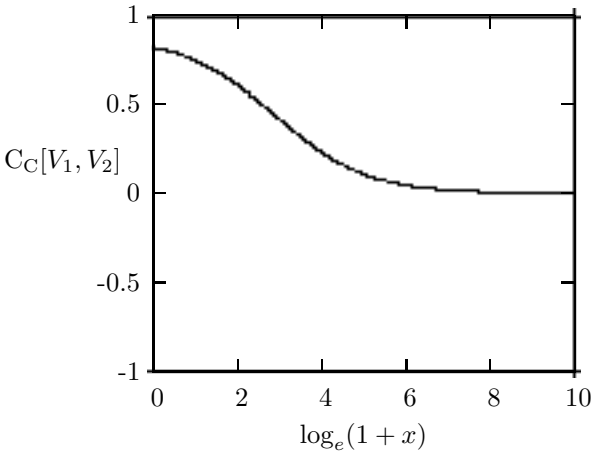

Figure 5: Correlation coefficient of the buffer contents of two classes.

Table 3: Conditional mean and coefficient of variation ($x = e^{10}$).

S	1	2	3	4
$\mathrm{E}[V_1 \mid S]$	22.7399	0.0139	0.0124	0.0117
$\mathrm{C_V}[V_1 \mid S]$	0.9949	57.0779	60.4161	62.2609
$\mathrm{E}[V_2 \mid S]$	139.5840	148.3101	132.3116	124.5827
$\mathrm{C_V}[V_2 \mid S]$	1.1450	1.0835	1.1992	1.2608

itive explanations of it unfortunately.

Figure 5 shows the correlation coefficient $\mathrm{C_C}[V_1, V_2]$ as a function of parameter x. We observe that the correlation between V_1 and V_2 becomes very weak with the increase of x, even though they are positively correlated. We examine this phenomenon more closely. For $x = e^{10}$, Table 3 shows the conditional mean and coefficient of variation of V_k $(k = 1, 2)$ given the state S of the underlying Markov chain in steady state. We observe that V_1 is nearly equal to zero unless $S = 1$. Thus

$$\begin{aligned}\mathrm{Cov}[V_1, V_2] &= \mathrm{E}[(V_1 - \mathrm{E}[V_1])(V_2 - \mathrm{E}[V_2])] \\ &= \mathrm{E}[V_1(V_2 - \mathrm{E}[V_2])] \\ &\approx \mathrm{E}[V_1(V_2 - \mathrm{E}[V_2]) \cdot \mathbb{I}(S = 1)].\end{aligned}$$

We now consider a typical sojourn in state 1 of the underlying Markov chain and assume that the underlying Markov chain changes its state to state 1 at time t_1. Let T_1 denotes a generic random variable for sojourn times in state 1 of the underlying Markov chain (i.e., T_1 is an exponential random variable with mean 10). From Table 3, we observe that $\mathrm{E}[V_2 \mid S = 1]$ is slightly greater than $\mathrm{E}[V_2] = 136.1971$. Because $V_2(t)$ decreases during the sojourn $[t_1, t_1 + T_1)$ in state 1, it is likely to be $V_2(t_1) > \mathrm{E}[V_2] > V_2(t_1 + T_1)$. If so, $V_1(t)(V_2(t) - \mathrm{E}[V_2]) > 0$ in the former part of $[t_1, t_1 + T_1)$, and then it turns to be negative in the latter part. As a result, the values of $V_1(t)(V_2(t) - \mathrm{E}[V_2])$ are canceled out and therefore $\mathrm{E}[V_1(V_2 - \mathrm{E}[V_2])\mathbb{I}(S = 1)]$ takes a value close to zero in this specific case.

A Computation of $\boldsymbol{\Psi}$

This section summarizes an efficient way to compute $\boldsymbol{\Psi}$. For the details, readers are referred to [5, 11]. We define $\boldsymbol{P}_{[ij]}$ $(i, j = +, -)$ as

$$\boldsymbol{P}_{[ij]} = \begin{cases} \mu^{-1}\boldsymbol{\Lambda}_{[i]}^{-1}\overline{\boldsymbol{T}}_{[ij]}, & j \neq i, \\ \boldsymbol{I} + \mu^{-1}\boldsymbol{\Lambda}_{[i]}^{-1}\overline{\boldsymbol{T}}_{[ii]}, & j = i, \end{cases}$$

where μ denotes the maximum absolute value of diagonal elements of $\boldsymbol{\Lambda}_{[+]}^{-1}\overline{\boldsymbol{T}}_{[++]}$ and $\boldsymbol{\Lambda}_{[-]}^{-1}\overline{\boldsymbol{T}}_{[--]}$. We then introduce an irreducible and positive-recurrent discrete-time QBD with state space $\{0, 1, \ldots\} \times (\mathcal{M}_+ \cup \mathcal{M}_-)$, whose transition probability matrix $\boldsymbol{B}$ takes a form:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{B}_0 & \boldsymbol{O} & \boldsymbol{O} & \cdots \\ \boldsymbol{B}_2 & \boldsymbol{B}_1 & \boldsymbol{B}_0 & \boldsymbol{O} & \cdots \\ \boldsymbol{O} & \boldsymbol{B}_2 & \boldsymbol{B}_1 & \boldsymbol{B}_0 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_2 & \boldsymbol{B}_1 & \cdots \\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{B}_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $\boldsymbol{C}_1$ is a substochastic matrix such that $\boldsymbol{C}_1\boldsymbol{e} + \boldsymbol{B}_0\boldsymbol{e} = \boldsymbol{e}$, and

$$\boldsymbol{B}_0 = \begin{pmatrix} \frac{1}{2}\boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{pmatrix}, \quad \boldsymbol{B}_1 = \begin{pmatrix} \frac{1}{2}\boldsymbol{P}_{[++]} & \boldsymbol{O} \\ \boldsymbol{P}_{[-+]} & \boldsymbol{O} \end{pmatrix},$$
$$\boldsymbol{B}_2 = \begin{pmatrix} \boldsymbol{O} & \frac{1}{2}\boldsymbol{P}_{[+-]} \\ \boldsymbol{O} & \boldsymbol{P}_{[--]} \end{pmatrix}.$$

In [5, 11], it is shown that the G-matrix $\boldsymbol{G}$ of this QBD, satisfying $\boldsymbol{G} = \boldsymbol{B}_2 + \boldsymbol{B}_1\boldsymbol{G} + \boldsymbol{B}_0\boldsymbol{G}^2$ and $\boldsymbol{G}\boldsymbol{e} = 1$, takes the following from:

$$\boldsymbol{G} = \begin{array}{c} \\ \mathcal{M}_+ \\ \mathcal{M}_- \end{array} \begin{array}{c} \begin{array}{cc} \mathcal{M}_+ & \mathcal{M}_- \end{array} \\ \begin{pmatrix} \boldsymbol{O} & \boldsymbol{\Psi} \\ \boldsymbol{O} & \boldsymbol{I} + \mu^{-1}\boldsymbol{U} \end{pmatrix} \end{array},$$

where

$$\boldsymbol{U} = \boldsymbol{\Lambda}_{[-]}^{-1}(\overline{\boldsymbol{T}}_{[--]} + \overline{\boldsymbol{T}}_{[-+]}\boldsymbol{\Psi}).$$

Thus we can obtain $\boldsymbol{\Psi}$ through $\boldsymbol{G}$, where the latter can be computed efficiently by the logarithmic reduction algorithm [8].

B Proof of Theorem 2

Let $\mathcal{M}^{(+)} = \{j \in \mathcal{M}; f_j > 0\}$. We start with the following lemma.

Lemma 3. *For an arbitrarily fixed $t \geq 0$ such that $V(t) = 0$ or $V(t) > 0$ with $S(t) \in \mathcal{M}^{(+)}$, the following equations hold:*

$$V(t) = W(c(t)), \tag{23}$$
$$S(t) = J(c(t)), \tag{24}$$

where

$$c(t) = t + V(t). \tag{25}$$

The proof of the lemma is given in the end of this section.

Because $V(t) \geq 0$ is continuous and the drain rate is equal to one, $c(t)$ is a continuous, non-decreasing, piecewise linear function of t and goes to infinity w.p.1 as $t \to \infty$. Furthermore, when $V(0) = 0$, $0 = c(0) \leq u \leq c(u)$ for all $u \geq 0$. Therefore for all $u \geq 0$, there exists $d(u)$ such that

$$d(u) = \max(t;\ 0 \leq t \leq u, c(t) = u). \tag{26}$$

Note that $d(u)$ is uniquely determined on each sample path and $W(u) = V(d(u))$ (see Remark 1). Let $\mathcal{D}(T)$ $(T > 0)$ denote the set of such subintervals in $(0, d(T)]$ as $V(t) = 0$ or $V(t) > 0$ with $S(t) \in \mathcal{M}^{(+)}$. It then follows from Lemma 3 that for each $j \in \mathcal{M}$,

$$\begin{aligned} &w_j(x) \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{I}(W(u) \leq x, J(u) = j)du \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(W(c(t)) \leq x, J(c(t)) = j)c_R'(t)dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(V(t) \leq x, S(t) = j)c_R'(t)dt, \end{aligned}$$

where $c_R'(t)$ $(t \geq 0)$ denotes the right hand side derivative of $c(t)$:

$$c_R'(t) = \lim_{\Delta t\to 0+} \frac{c(t + \Delta t) - c(t)}{\Delta t}.$$

Note here that

$$c_R'(t) = \begin{cases} 1, & \text{if} \quad V(t) = 0, \\ f_j, & \text{if} \quad V(t) > 0 \text{ and } S(t) = j \in \mathcal{M}. \end{cases}$$

We then have

$$\begin{aligned} &w_j(x) \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(V(t) = 0, S(t) = j)dt \\ &\quad + \lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(0 < V(t) \leq x, S(t) = j)f_j dt. \end{aligned} \tag{27}$$

It follows from (25) and (26) that

$$\begin{aligned} c(d(T)) &= d(T) + V(d(T)), \\ c(d(T)) &= T, \end{aligned}$$

so that

$$d(T) + V(d(T)) = T.$$

Thus $d(T) = T$ if $V(d(T)) = 0$. On the other hand, if $V(d(T)) = T - d(T) > 0$, $V(t) > 0$ for all $t \in [d(T), T)$. Therefore the first term on the right hand side of (27) is rewritten to be

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(V(t) = 0, S(t) = j)dt \\ &\qquad = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathbb{I}(V(t) = 0, S(t) = j)dt \\ &\qquad = \varpi_j, \quad \text{w.p.1.} \end{aligned} \tag{28}$$

We also rewrite the second term on the right hand side of (27) to be

$$\begin{aligned} &\lim_{T\to\infty} \frac{1}{T} \int_{t\in\mathcal{D}(T)} \mathbb{I}(0 < V(t) \leq x, S(t) = j)f_j dt \\ &\quad = \left(\lim_{T\to\infty} \frac{L(T)}{T}\right) \\ &\quad \cdot \left(\lim_{T\to\infty} \frac{\displaystyle\int_{t\in\mathcal{D}(T)} \mathbb{I}(0 < V(t) \leq x, S(t) = j)dt}{L(T)}\right) f_j, \end{aligned} \tag{29}$$

where $L(T)$ denotes the total length of subintervals in $\mathcal{D}(T)$, i.e.,

$$\begin{aligned} L(T) = &\int_0^{d(T)} \mathbb{I}(V(t) = 0)dt \\ &+ \int_0^{d(T)} \mathbb{I}(V(t) > 0 \text{ and } S(t) \in \mathcal{M}^{(+)})dt. \end{aligned}$$

Note here that $L(T)$ goes to infinity w.p.1 as $T \to \infty$. Thus the factor in the second parentheses on the right hand side of (29) represents the conditional stationary probability that $0 < V(t) \leq x$ and $S(t) = j$ given that $V(t) = 0$ or $V(t) > 0$ with $S(t) \in \mathcal{M}^{(+)}$. Let ζ denote $\lim_{t\to\infty} \Pr[V(t) = 0 \text{ or } V(t) > 0 \text{ with } S(t) \in \mathcal{M}^{(+)}]$. We then have

$$\zeta = \boldsymbol{\varpi}\boldsymbol{e} + \sum_{j\in\mathcal{M}^{(+)}} (\pi_j - \varpi_j) = 1 - \sum_{j\in\mathcal{M}^{(0)}} (\pi_j - \varpi_j),$$

where $\mathcal{M}^{(0)} = \{j \in \mathcal{M}; f_j = 0\}$. Thus, from (27), (28), and (29), we obtain

$$w_j(x) = \varpi_j + \left(\lim_{T\to\infty} \frac{L(T)}{T}\right) \zeta^{-1} \int_0^x v_j(y)dy f_j, \quad j \in \mathcal{M},$$

or equivalently,

$$\boldsymbol{w}(x) = \boldsymbol{\varpi} + \left(\lim_{T\to\infty} \frac{L(T)}{T}\right) \zeta^{-1} \int_0^x \boldsymbol{v}(y)dy\boldsymbol{F}. \tag{30}$$

By ergodicity, $L(T)/T$ converges to some constant w.p.1 as $T \to \infty$. Using $\boldsymbol{w}(\infty)\boldsymbol{e} = 1$ and (2), we obtain

$$\lim_{T\to\infty} \frac{L(T)}{T} \zeta^{-1} = \frac{1 - \boldsymbol{\varpi}\boldsymbol{e}}{(\boldsymbol{\pi} - \boldsymbol{\varpi})\boldsymbol{F}\boldsymbol{e}}. \tag{31}$$

Theorem 2 now follows from (30) and (31). □

Proof of Lemma 3. From (5), we have

$$W(t + V(t)) = \inf\{x \geq 0;\ V(t + V(t) - x) = x\}. \tag{32}$$

Note here that the equality $V(t + V(t) - x) = x$ holds for $x = V(t)$. Thus from (32), we have

$$W(t + V(t)) \leq V(t), \quad t \geq 0. \tag{33}$$

In what follows, we first prove (23) by considering the contraposition: If $W(t+V(t)) < V(t)$, then $V(t) > 0$ and $S(t) \in \mathcal{M}^{(0)} = \{j \in \mathcal{M}; f_j = 0\}$.

Suppose for an arbitrarily fixed $t \geq 0$, there exists a positive a $(0 < a \leq V(t))$ such that $W(t+V(t)) = V(t) - a$. It then follows from (32) that

$$V((t+V(t)) - (V(t)-a)) = V(t) - a,$$

or equivalently,

$$V(t+a) = V(t) - a. \tag{34}$$

Because $V(t+a) \geq 0$, we have

$$V(t) \geq a > 0. \tag{35}$$

On the other hand, because of the constant drain rate of one, (34) implies that for all $\tau \in (t, t+a)$, $f(\tau) = 0$, i.e., $S(\tau) \in \mathcal{M}^{(0)}$. Further, because $\{S(t); t \geq 0\}$ is right-continuous, we obtain

$$S(t) = \lim_{\varepsilon \to 0+} S(t+\varepsilon) \in \mathcal{M}^{(0)},$$

from which and (35), (23) follows.

Next we consider (24). Suppose $W(t+V(t)) = V(t)$. It then follows from (6) that

$$\begin{aligned} J(t+V(t)) &= S(t+V(t) - W(t+V(t))) \\ &= S(t+V(t) - V(t)) \\ &= S(t), \end{aligned}$$

which completes the proof. □

C Proof of Lemma 2

We denote $(y_1, y_2, \ldots, y_K)$ by $\boldsymbol{y}$, where $y_k \geq 0$ for all $k \in \mathcal{K}$. Let $\boldsymbol{A}(\boldsymbol{y} \mid x)$ $(x \geq 0)$ denote an $M \times M$ matrix whose (i,j)th $(i,j \in \mathcal{M})$ element $A_{i,j}(\boldsymbol{y} \mid x)$ represents

$$\begin{aligned} &A_{i,j}(\boldsymbol{y} \mid x) \\ &\quad = \Pr[A_k(-x,0] \leq y_k \ (k \in \mathcal{K}), S(0) = j \mid S(-x) = i]. \end{aligned}$$

Considering the probability of events occurring in an interval of length Δx, we obtain

$$\begin{aligned} &A_{i,j}(\boldsymbol{y} \mid x + \Delta x) \\ &\quad = (1 + T_{i,i}\Delta x) A_{i,j}(y_1 - f_{1,i}\Delta x, \ldots, y_K - f_{K,i}\Delta x \mid x) \\ &\qquad + \sum_{\substack{\nu \in \mathcal{M} \\ \nu \neq i}} (T_{i,\nu}\Delta x) A_{\nu,j}(\boldsymbol{y} \mid x) + o(\Delta x), \end{aligned} \tag{36}$$

where $T_{i,j}$ denotes the (i,j)th element of $\boldsymbol{T}$. Dividing both sides of (36) by Δx, rearranging terms, and taking the limit $\Delta x \to 0$ on both sides yield

$$\begin{aligned} \frac{\partial}{\partial x} A_{i,j}(\boldsymbol{y} \mid x) = &\sum_{\nu \in \mathcal{M}} T_{i,\nu} A_{\nu,j}(\boldsymbol{y} \mid x) \\ &- \sum_{k \in \mathcal{K}} f_{k,i} \frac{\partial}{\partial y_k} A_{i,j}(\boldsymbol{y} \mid x), \end{aligned}$$

or equivalently,

$$\frac{\partial}{\partial x} \boldsymbol{A}(\boldsymbol{y} \mid x) = \boldsymbol{T} \boldsymbol{A}(\boldsymbol{y} \mid x) - \sum_{k \in \mathcal{K}} \boldsymbol{F}_k \frac{\partial}{\partial y_k} \boldsymbol{A}(\boldsymbol{y} \mid x). \tag{37}$$

Further taking the LSTs of both sides of (37), we obtain

$$\frac{\partial}{\partial x} \boldsymbol{A}^*(\boldsymbol{s} \mid x) = \left[\boldsymbol{T} - \sum_{k \in \mathcal{K}} s_k \boldsymbol{F}_k \right] \boldsymbol{A}^*(\boldsymbol{s} \mid x).$$

Note here that $\boldsymbol{A}^*(\boldsymbol{s} \mid 0) = \boldsymbol{I}$, from which (15) follows. □

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