# High-Order Balanced Multiwavelets: Theory, Factorization, and Design

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***Abstract*—This paper deals with multiwavelets and the different properties of approximation and smoothness associated with them. In particular, we focus on the important issue of the preservation of discrete-time polynomial signals by multifilterbanks. We introduce and detail the property of *balancing* for higher degree discrete-time polynomial signals and link it to a very natural factorization of the refinement mask of the lowpass synthesis multifilter. This factorization turns out to be the counterpart for multiwavelets of the well-known *zeros at* $\pi$ condition in the usual (scalar) wavelet framework. The property of balancing also proves to be central to the different issues of the preservation of smooth signals by multifilterbanks, the approximation power of finitely generated multiresolution analyses, and the smoothness of the multiscaling functions and multiwavelets. Using these new results, we describe the construction of a family of orthogonal multiwavelets with symmetries and compact support that is indexed by increasing order of balancing. In addition, we also detail, for any given balancing order, the orthogonal multiwavelets with minimum-length multifilters.**

***Index Terms*—Balancing, Gröbner basis, multicoiflets, multifilterbank, multiwavelets, time-varying filterbank.**

## I. Introduction

Wavelet constructions from iterated filterbanks, as pioneered by Daubechies [5], have become a standard way to derive orthogonal and biorthogonal wavelet bases. The underlying filterbanks are well studied, and thus, the design procedure is well understood. By the structure of the problem, certain issues are ruled out: the impossibility of constructing orthogonal, FIR, linear phase filterbanks implies that there is no orthogonal wavelet with compact support and symmetry. Nevertheless, by relaxing the requirement of time invariance and allowing periodically time-varying filterbanks, it is easily seen that new solutions are possible. As mentioned in [35], such filterbanks are closely related to some matrix two-scale equations leading to multiwavelets.

In the usual framework of wavelets (scalar case), the two important issues of the reproduction of continuous-time polynomials by the associated multiresolution analysis (approximation theory issue) and the preservation/cancellation of discrete-time polynomial signals by the associated filterbank (subband coding and compression issue) are tightly connected since they have been proved to be equivalent to the same condition: the number of zeros at $\pi$ in the factorization of the lowpass synthesis filter $m_0(e^{j\omega})$ of the filterbank. In the orthogonal case, we then say that the lowpass filter $m_0(e^{j\omega})$ has *regularity* $p$ iff any of the following equivalent conditions [5] hold.

- The lowpass filter $m_0(e^{j\omega})$ has a zero of order $p$ at $\omega = \pi$.
- The corresponding highpass filter $m_1(e^{j\omega})$ has a zero of order $p$ at $\omega = 0$ (discrete-time polynomial signals of degree $n < p$ are thus canceled by the highpass branch).
- Discrete-time polynomial signals of degree $n < p$ are preserved by the lowpass branch of the filterbank.
- The associated wavelet $\psi(t)$ has $p$ vanishing moments.
- The multiresolution analysis has approximation power ($p$; continuous-time polynomials of degree $n < p$ are perfectly reproduced from integer shifts of the scaling function $\phi(t)$).

Furthermore, the smoothness of the scaling function $\phi(t)$ (and thus of the wavelet $\psi(t)$ if the filters are FIR) is closely related [7] to the regularity of the lowpass filter. Similar relations are easily obtained for the biorthogonal scalar case. Without much loss of generality, we will, in this paper, look at the orthogonal case for multiwavelets (the interest of biorthogonality in the multiwavelet framework being not as obvious since there is no negative result [6] preventing us from constructing orthogonal, FIR, linear-phase multifilters).

The regularity issue is indeed different for multiwavelets. Interested in the subband coding issue in general and the problem of processing one-dimensional (1-D) signals with multiwavelets in particular, we showed in [20] that the approximation power property did not assure the preservation of discrete-time polynomial signals by the lowpass branch of the filterbank. Consequently, we introduced the concept of *balanced* multiwavelets, which is now also further investigated by several other authors [17], [27], [28]. One of the goals of this concept is to avoid the intricate steps of pre/post filtering [11], [37] that are required with systems based on multiwavelets that do not satisfy the interpolation/approximation properties of balancing. Inspired by some of the results from [4], [23], and [24], we will clarify the relations between balancing order (discrete-time property) and approximation power (continuous-time property) and prove that the notion of balancing order is truly central to the whole issue of *regularity* for multiwavelets. Balanced multiwavelets of order $p$ behave as bona-fide wavelets up to the order $p$ of interpolation and approximation.

Furthermore, the introduction of the balancing property will enable us to construct a family of orthogonal compactly sup-

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ported multiwavelets with symmetries that are naturally indexed by increasing order of balancing. We will also clarify the relations between scalar wavelets, balanced multiwavelets, and nonbalanced multiwavelets with a surprising theorem giving the shortest length orthogonal multiwavelets for any given balancing order.

The outline of the paper is as follows. In Section II, the fundamentals of multiwavelet theory are reviewed with a special highlight on the connection to time-varying filterbanks. Section III reviews balancing and describes equivalence results between balancing order and a special case of Plonka's factorization of the refinement mask. These results are the key in the construction of balanced multiwavelets families. Section IV relates balancing order, approximation power, and smoothness. This leads, in Section V, to the construction of a balanced and smooth family of orthonormal multiwavelets with symmetries. In that section, we also detail the result on minimal-length orthogonal balanced multiwavelets.

*Notations:* In this text, regular symbols will refer to scalar values, whereas bold symbols will imply vector/matrix values.

## II. MULTIWAVELETS

Generalizing the wavelet case, one can allow a multiresolution analysis $\{V_n\}_{n\in\mathbb{Z}}$ of $L^2(\mathbb{R})$ to be generated by a finite number of scaling functions $\phi_0(t), \phi_1(t), \ldots, \phi_{r-1}(t)$ and their integer translates (the multiresolution analysis is then said to be of *multiplicity* $r$). Then, the multiscaling function $\boldsymbol{\phi}(t) := [\phi_0(t), \ldots, \phi_{r-1}(t)]^\top$ satisfies a two-scale equation

$$\boldsymbol{\phi}(t) = \sum_k \mathbf{M}[k]\boldsymbol{\phi}(2t-k) \tag{1}$$

where $\{\mathbf{M}[k]\}_k$ is a sequence of $r \times r$ matrices of real coefficients. By the multiresolution analysis structure, $V_1 = V_0 \oplus W_0$, where $W_0$ is the orthogonal complement of $V_0$ in $V_1$, and we can construct an orthonormal basis of $W_0$ generated by the multiwavelets $\psi_0(t), \psi_1(t), \ldots, \psi_{r-1}(t)$ and their integer translates by introducing $\boldsymbol{\psi}(t) := [\psi_0(t), \ldots, \psi_{r-1}(t)]^\top$ by

$$\boldsymbol{\psi}(t) := \sum_k \mathbf{N}[k]\boldsymbol{\phi}(2t-k) \tag{2}$$

where $\{\mathbf{N}[k]\}_k$ is a sequence of $r \times r$ matrices of real coefficients obtained by *completion* of $\{\mathbf{M}[k]\}_k$ (a detailed exposition of the completion scheme is given in [18]). Introducing in the $z$-domain the refinement masks $\mathbf{M}(z) := 1/2\sum_n \mathbf{M}[n]z^{-n}$ and $\mathbf{N}(z) := 1/2\sum_n \mathbf{N}[n]z^{-n}$, (1) and (2) translate in the Fourier domain into

$$\Phi(2\omega) = \mathbf{M}(e^{j\omega})\Phi(\omega) \quad \text{and} \quad \Psi(2\omega) = \mathbf{N}(e^{j\omega})\Phi(\omega). \tag{3}$$

We can then derive the behavior of the multiscaling function by iterating the first product above. If this iterated matrix product converges, we get, in the limit

$$\Phi(\omega) = \mathbf{M}_\infty(\omega)\Phi(0) = \prod_{i=1}^{\infty} \mathbf{M}\left(e^{j\omega/2^i}\right)\Phi(0). \tag{4}$$

In the sequel, we will assume that the sequences $\{\mathbf{M}[k]\}_k$ and $\{\mathbf{N}[k]\}_k$ are finite and, thus, that $\boldsymbol{\phi}(t)$ and $\boldsymbol{\psi}(t)$ have compact support [22]. Many people worked on the convergence conditions. For more details about these results, see [4], [16], [25], and [35]. Here, we will assume that $\mathbf{M}(z)$ satisfies the following two *basic conditions* (following Strang's notations [32]).

*Condition E (Existence and Uniqueness):* The transition operator [4] associated with $\mathbf{M}(z)$ has all its eigenvalues $|\lambda| < 1$ except for a simple eigenvalue $\lambda = 1$.

*Condition A1 (Approximation of Order 1):* There exists $\mathbf{r}_0$, which is a left eigenvector of $\mathbf{M}(1)$ for the eigenvalue 1 such that $\mathbf{r}_0^\top\mathbf{M}(-1) = \mathbf{0}^\top$.

These two conditions, with $\Phi(0)$ being a right eigenvector of $\mathbf{M}(1)$ for the eigenvalue 1, assure convergence in the weak sense of the infinite matrix product (4) to a compactly supported distributional solution of (1). Now, if $\mathbf{M}(z)$ also verifies the matrix Smith–Barnwell orthogonality condition (also called *Condition O*)

$$\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{M}^\top(-z^{-1}) = \mathbf{I} \tag{5}$$

for all $z$ on the unit circle, then the convergence is also in the $L^2$ sense to a bona-fide $L^2$ solution.

Now, assuming all of these conditions, the scaling functions and their integer translates form an orthonormal basis of $V_0$. If we also impose orthogonality conditions on $\mathbf{N}(z)$, i.e.,

$$\begin{aligned}\mathbf{N}(z)\mathbf{N}^\top(z^{-1}) + \mathbf{N}(-z)\mathbf{N}^\top(-z^{-1}) &= \mathbf{I}\\ \mathbf{M}(z)\mathbf{N}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{N}^\top(-z^{-1}) &= \mathbf{0}\end{aligned} \tag{6}$$

then we get a fully orthonormal multiresolution analysis. For $s(t) \in V_0$, we have

$$s(t) = \sum_n \mathbf{s}_0^\top[n]\boldsymbol{\phi}(t-n). \tag{7}$$

Then, from $V_0 = V_{-1} \oplus W_{-1}$, we get

$$s(t) = \sum_n \mathbf{s}_{-1}^\top[n]\boldsymbol{\phi}\left(\frac{t}{2}-n\right) + \mathbf{d}_{-1}^\top[n]\boldsymbol{\psi}\left(\frac{t}{2}-n\right) \tag{8}$$

and we have the well-known relations between the coefficients at the analysis step

$$\mathbf{s}_{-1}[n] = \sum_k \mathbf{M}[k-2n]\mathbf{s}_0[k] \tag{9}$$

$$\mathbf{d}_{-1}[n] = \sum_k \mathbf{N}[k-2n]\mathbf{s}_0[k] \tag{10}$$

and for the synthesis, we get

$$\mathbf{s}_0[n] = \sum_k \mathbf{M}^\top[n-2k]\mathbf{s}_{-1}[k] + \mathbf{N}^\top[n-2k]\mathbf{d}_{-1}[k]. \tag{11}$$

These relations enable us to construct a multi-input multi-output filterbank (multifilterbank), as seen in Fig. 1. In case of a 1-D signal, it requires vectorization of the input signal to produce an input signal that is $r$-dimensional. The natural way to do that is

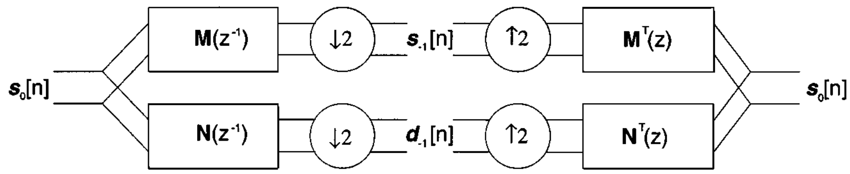


Fig. 1. Orthogonal multifilterbank.

to split a 1-D signal into its polyphase components. Introducing $m_0(z), m_1(z), \ldots, m_{r-1}(z)$, the associated scalar polyphase filters given by

$$\begin{bmatrix} m_0(z) \\ m_1(z) \\ \vdots \\ m_{r-1}(z) \end{bmatrix} := 2\mathbf{M}(z^r) \begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix} \tag{12}$$

and $n_0(z), n_1(z), \ldots, n_{r-1}(z)$ in the same way from $\mathbf{N}(z)$, the system can be rewritten as a $2r$ channel time-varying filterbank (see Fig. 2 for the case $r = 2$).

## III. High-Order Balancing

In [20], we showed that if the associated scalar polyphase filters have different spectral behavior, e.g., lowpass behavior for one and highpass for another, it then leads to unbalanced channels that mix the coarse resolution and detail coefficients and creates strong oscillations (see Fig. 3) if the signal is reconstructed from the lowpass subband coefficients only (compression issue). The idea is thus to impose some class of smooth signals to be preserved by the lowpass branch and canceled by the highpass branch. The natural choice is to take the class of polynomial signals since in a wavelet-based filterbank, the polynomial signals are preserved by the lowpass branch up to the order of regularity.

### A. Balancing

Let $\mathbf{L}$: $\ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$ be the block Toeplitz operator corresponding to the lowpass analysis. We can write $\mathbf{L}$ as an infinite-size matrix, shown at the bottom of the next page, and in the same way, let $\mathbf{H}$ be the block Toeplitz operator corresponding to the highpass analysis. We want constant signals to be preserved by the lowpass branch. Introducing $\mathbf{u}_0 := [\ldots, 1, 1, 1, 1, \ldots]^\top$, we get

*Definition 1:* An orthonormal multiwavelet system is said to be balanced (of order 1) iff the lowpass synthesis operator $\mathbf{L}^\top$ preserves the constant signals, i.e., $\mathbf{L}^\top \mathbf{u}_0 = \mathbf{u}_0$.

By the orthonormality relations

$$[\mathbf{L}^\top \quad \mathbf{H}^\top] \begin{bmatrix} \mathbf{L} \\ \mathbf{H} \end{bmatrix} = \mathbf{I} \quad \text{and} \quad \begin{bmatrix} \mathbf{L} \\ \mathbf{H} \end{bmatrix} [\mathbf{L}^\top \quad \mathbf{H}^\top] = \mathbf{I}$$

we get $\mathbf{L}^\top\mathbf{L} + \mathbf{H}^\top\mathbf{H} = \mathbf{I}$, $\mathbf{L}\mathbf{L}^\top = \mathbf{I}$, $\mathbf{L}\mathbf{H}^\top = \mathbf{0}$ and $\mathbf{H}\mathbf{H}^\top = \mathbf{I}$. Therefore, $\mathbf{L}^\top\mathbf{u}_0 = \mathbf{u}_0$ implies $\mathbf{L}^\top\mathbf{L}\mathbf{u}_0 = \mathbf{u}_0$ and $\mathbf{H}\mathbf{u}_0 = \mathbf{0}$, i.e., $\mathbf{u}_0$ is preserved by the lowpass branch and canceled by the highpass branch.

Now, we can state the following result giving equivalent conditions for balancing and especially linking balancing to a simple factorization of the refinement mask $\mathbf{M}(z)$ (which

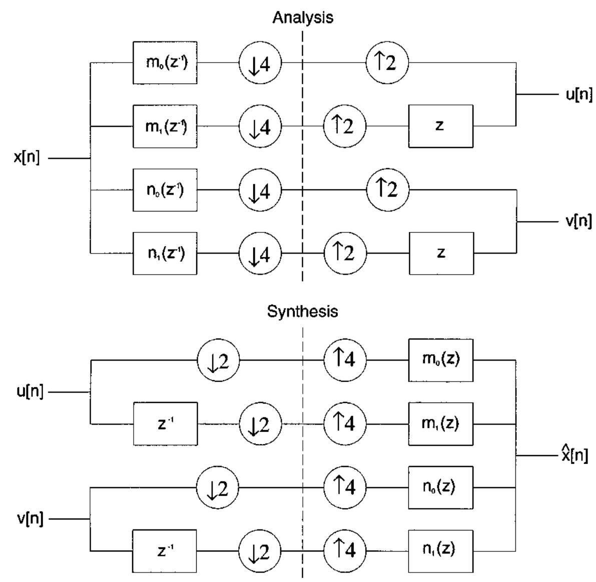


Fig. 2. Multifilterbank seen as a time-varying filterbank.

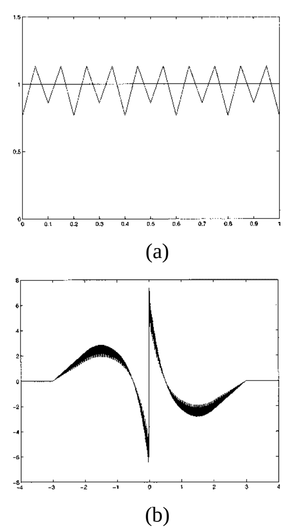

Fig. 3. Reproduction of two input signals [(a) Constant signal. (b) Piecewise polynomial] by the lowpass branch of a DGHM multiwavelet based filterbank; it shows the poor robustness of a system based on the DGHM multiwavelet without prefiltering.

is a special case of the factorizations (the so-called two-scale similarity transforms) introduced by Plonka and Strela in [23] and [24]).

*Theorem 2:* Balancing of order 1 is equivalent (in the case of orthogonal multiwavelet systems) to any of the following conditions.

**B0**) $\mathbf{L}^\top \mathbf{u}_0 = \mathbf{u}_0$.
**B1**) $[1 \cdots 1]\mathbf{M}(1) = [1 \cdots 1]$ and $[1 \cdots 1]\mathbf{M}(-1) = \mathbf{0}^\top$.
**B2**) $\Phi(0) = [1 \cdots 1]^\top$.

**B3**) $\mu(z) := \sum_{i=0}^{r-1} m_i(z)$ has zeros[1] at $z = e^{jk\pi/r}$ for $k = 1, \ldots, 2r-1$ and $\mu(1) = 2r$.

**B4**) One can factorize $\mathbf{M}(z) = 1/2\,\mathbf{\Delta}(z^2)\mathbf{M}_0(z)\mathbf{\Delta}^{-1}(z)$ with

$$\mathbf{\Delta}(z) := \begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-1} & 0 & \ldots & 0 & 1 \end{bmatrix}$$

and $\mathbf{M}_0(1)[1 \cdots 1]^\top = [1 \cdots 1]^\top$.

*Proof:* This rather easy proof is given in the Appendix. ■

### B. High-Order Balancing

A natural generalization of the concept of balancing is then to impose higher degree discrete-time polynomial signals $[\ldots, x(-2), x(-1), x(0), x(1), x(2), \ldots]^\top$ (where $x(t)$ is any polynomial of degree smaller than $p$) to also be preserved by the lowpass branch. Introducing $\mathcal{C}_p$, the vector space of polynomial sequences generated by polynomials of degree up to $p$ (included), we define the following.

*Definition 3:* An orthonormal multiwavelet system is said to be balanced of order $p$ iff the lowpass synthesis operator $\mathbf{L}^\top$ preserves discrete-time polynomial signals of degree less than $p$, i.e., $\mathcal{C}_{p-1}$ is invariant by $\mathbf{L}^\top$.

This condition does not imply that $\mathbf{L}^\top$ exactly preserves polynomial signals. It just says that any polynomial input is transformed into another polynomial signal of a lesser or equal degree (Fig. 4). However, since $\mathcal{C}_{p-1}$ has finite dimension, by the orthonormality condition $\mathbf{L}\mathbf{L}^\top = \mathbf{I}$, we have that $\mathbf{L}^\top \mathcal{C}_{p-1} = \mathcal{C}_{p-1}$. Therefore, $\mathcal{C}_{p-1}$ is globally preserved. From the other orthonormality conditions, $\mathbf{H}\mathcal{C}_{p-1} = \mathbf{H}\mathbf{L}^\top \mathcal{C}_{p-1} = \{\mathbf{0}\}$. This gives that $\mathbf{L}^\top \mathbf{L}\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in \mathcal{C}_{p-1}$, i.e., the polynomial structure (up to degree $p-1$) of the input signal is exactly preserved by the lowpass branch and canceled by the highpass branch.

To deal easily with high order balancing, an important issue is the interpolation of all the polyphase components of a discrete-time polynomial signal from one of the phases. On this subject, the following lemma will prove to be the cornerstone of the further developments. With this lemma, we will get that on discrete-time polynomial signals of degree smaller than the order of balancing, the lowpass synthesis operator (with its intricate time-varying structure) is, in fact, equivalent to a scalar subdivision scheme (on which the classical results from the scalar wavelet theory apply).

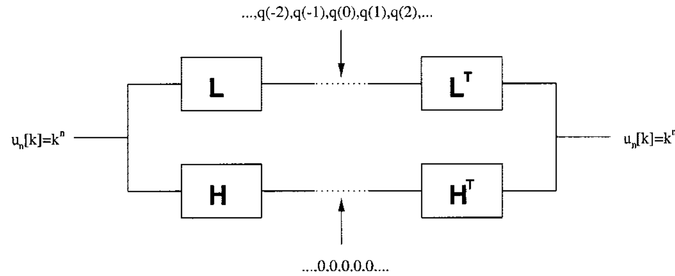


Fig. 4. Fundamental condition of high-order balancing.

*Lemma 4:* Let $u_{0,r}^{(n)}(z), u_{1,r}^{(n)}(z), \ldots, u_{r-1,r}^{(n)}(z)$ be the formal series associated with the $r$ phases of the monomials

$$u_{i,r}^{(n)}(z) := \sum_{k \in \mathbb{Z}} (kr+i)^n z^{-k}.$$

Then, for $i = 1, \ldots, r-1$, there exists a unique polynomial $\alpha_{i,r}^{(n)}(z)$ of degree $n$ such that $u_{i,r}^{(n)}(z) = \alpha_{i,r}^{(n)}(z) u_{0,r}^{(n)}(z)$ and $\alpha_{i,r}^{(n)}(1) = 1$.

*Proof:* Using Padé approximants [1], we can construct the Hörner scheme of interpolation $\alpha_{i,r}^{(n)}(z)$ of the sequence $\mathbf{u}_{i,r}^{(n)}[k] := (kr+i)^n$ from the sequence $\mathbf{u}_{0,r}^{(n)}[k] := (kr)^n$ by

$$\begin{aligned} \alpha_{i,r}^{(n)}(z) = 1 + \frac{i}{r}(1-z^{-1}) \Big[_1\, 1 + &\frac{i+r}{2r}(1-z^{-1}) \Big[_2\, 1 + \\ &\frac{i+2r}{3r}(1-z^{-1}) \Big[_3\, 1 + \ldots \\ &\ldots) \Big[_{n-2}\, 1 + \\ &\frac{i+(n-2)r}{(n-1)r}(1-z^{-1}) \Big[_{n-1}\, 1 + \\ &\frac{i+(n-1)r}{nr}(1-z^{-1}) \Big]_{n-1} \Big]_{n-2} \ldots \Big]_1 . \end{aligned}$$

Thus, we have

$$\alpha_{i,r}^{(n)}(z) = 1 + \sum_{k=1}^{n} \frac{\Gamma\left(k + \frac{i}{r}\right)}{\Gamma(k+1)\Gamma\left(\frac{i}{r}\right)} (1-z^{-1})^k. \qquad (13)$$

[1]Condition B3) and its generalization to higher order balancing were first given by [28].

$$\mathbf{L} := \begin{bmatrix} \cdots & & & & & & & & & \\ & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \ldots & & & & \\ & & & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \ldots & & \\ & & & & & \mathbf{M}[0] & \mathbf{M}[1] & \mathbf{M}[2] & \mathbf{M}[3] & \ldots \\ & & & & & & & & & \ldots \end{bmatrix}$$

Furthermore, it is easily seen that $u_{i,r}^{(n)}(z)$ cannot be formally canceled by multiplication with a polynomial, hence, the uniqueness. ■

*Remark 5:* For the usual case $r = 2$, we give $\alpha_{1,2}^{(n)}(z)$ for $n = 0, 1, 2, 3$.

$$\begin{aligned}\alpha_{1,2}^{(0)}(z) &= 1\\ \alpha_{1,2}^{(1)}(z) &= \tfrac{1}{2}(3 - z^{-1})\\ \alpha_{1,2}^{(2)}(z) &= \tfrac{1}{8}(15 - 10z^{-1} + 3z^{-2})\\ \alpha_{1,2}^{(3)}(z) &= \tfrac{1}{16}(35 - 35z^{-1} + 21z^{-2} - 5z^{-3}).\end{aligned}$$

By natural extension, we write $\alpha_{0,r}^{(n)}(z) := 1$. We also introduce the vectors

$$\boldsymbol{\alpha}_n^\top(z) := \left[\alpha_{0,r}^{(n)}(z), \quad \alpha_{1,r}^{(n)}(z), \quad \ldots, \quad \alpha_{r-1,r}^{(n)}(z)\right].$$

Now, generalizing the wavelet case, we get the following.

*Definition 6:* We say that the refinement mask $\mathbf{M}(z)$ associated with an orthonormal multiwavelet system has balanced vanishing moments of order $p$ iff there exist $r_0, r_1, \ldots, r_{p-1}$, with $r_0 = 1$, such that if we define for $n = 0, \ldots, p-1$, $\mathbf{y}_n^\top := [\rho_n(0/r), \quad \rho_n(1/r), \quad \ldots, \quad \rho_n(r-1/r)]$, where

$$\rho_n(t) := \sum_{k=0}^{n} \binom{n}{k} r_{n-k} t^k$$

we have the vanishing moments

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} \mathbf{y}_k^\top (2j)^{k-n} \frac{d^{n-k}}{d\omega^{n-k}} [\mathbf{M}(e^{j\omega})]_{|\omega=0} &= 2^{-n}\mathbf{y}_n^\top\\ \sum_{k=0}^{n} \binom{n}{k} \mathbf{y}_k^\top (2j)^{k-n} \frac{d^{n-k}}{d\omega^{n-k}} [\mathbf{M}(e^{j\omega})]_{|\omega=\pi} &= \mathbf{0}^\top. \qquad (14)\end{aligned}$$

*Remark 7:* The sequence of polynomials defined by $\rho_n(t) := \sum_{k=0}^{n} \binom{n}{k} r_{n-k} t^k$ with $r_0 = 1$ is called an Appell sequence [2]. It verifies $(d/dt)\rho_n(t) = n\rho_{n-1}(t)$ and $\deg \rho_n(t) = n$. It also satisfies the Appell identity $\rho_n(t+h) := \sum_{k=0}^{n} \binom{n}{k} \rho_{n-k}(t) h^k$ (a generalization of the binomial formula). $\rho_n(t) = t^n$ defines an Appell sequence.

*Theorem 8:* Balancing of order $p$ is equivalent (in the case of orthogonal multiwavelet systems) to any of the following conditions.

- **B0$_\mathrm{p}$)** There exists an Appell sequence $\{\rho_n(t)\}_{n=0}^{p-1}$ such that the discrete-time polynomial signals $\mathbf{v}_n[l] := \rho_n(l/r)$ verify $\mathbf{L}^\top \mathbf{v}_n = 2^{-n}\mathbf{v}_n$ for $n = 0, \ldots, p-1$.
- **B1$_\mathrm{p}$)** $\mathbf{M}(z)$ has balanced vanishing moments of order $p$.
- **B3$_\mathrm{p}$)** $\mu_{p-1}(z) := \sum_{k=0}^{r-1} \alpha_{k,r}^{(p-1)}(z^{2r}) m_k(z)$ has zeros of order $p$ at $z = e^{jk\pi/r}$ for $k = 1, \ldots, 2r-1$ and $\mu_{p-1}(1) = 2r$.
- **B4$_\mathrm{p}$)** For $n = 1, \ldots, p$, $\mathbf{M}(z)$ can be factored as

$$\mathbf{M}(z) = \frac{1}{2^n} \boldsymbol{\Delta}^n(z^2) \mathbf{M}_{n-1}(z) \boldsymbol{\Delta}^{-n}(z) \qquad (15)$$

with $\mathbf{M}_{n-1}(1)[1 \cdots 1]^\top = [1 \cdots 1]^\top$ and $\boldsymbol{\Delta}(z)$ as above.

*Remark 9:* Using the equivalence between conditions $\mathbf{B1}_\mathrm{p}$) and $\mathbf{B3}_\mathrm{p}$), condition $\mathbf{B1}_\mathrm{p}$) (balanced vanishing moments of order $p$) can be weakened in the following more elegant form.

- **B1$^*_\mathrm{p}$)** $\mathbf{M}(z)$ verifies B1) and for $n = 0, \ldots, p-1$ $(d^n/d\omega^n)[\boldsymbol{\alpha}_{p-1}^\top(e^{j2\omega})\mathbf{M}(e^{j\omega})]_{|\omega=\pi} = \mathbf{0}^\top$.

## IV. Balancing Order, Approximation Power, and Smoothness

Here, we will clarify how the results obtained above relate to the classical notions of *regularity*, i.e., approximation power and smoothness.

### A. Approximation Power and Balancing Order

First, let us recall that a multiscaling function $\boldsymbol{\phi}(t)$ has approximation power $m$ if one can exactly decompose polynomials $1, t, t^2, \ldots, t^{m-1}$ using only $\phi_0(t), \phi_1(t), \ldots, \phi_{r-1}(t)$ and their integer translates, i.e., for $n = 0, \ldots, p-1$, there exists a sequence $\theta[n]$ such that

$$t^n = \sum_k \theta_n^\top[k] \boldsymbol{\phi}(t-k). \qquad (16)$$

Now, assuming that $\boldsymbol{\phi}(t)$ is balanced of order $p$, we get that $\mathbf{M}(z)$ factorizes as in (15) so that applying $p$ times Theorem 2.6. from [24], we get that $\boldsymbol{\phi}(t)$ has at least an approximation power of $p$. Therefore, if an orthonormal multiwavelet system is balanced of order $p$, then the associated multiscaling function $\boldsymbol{\phi}(t)$ has an approximation power of at least $p$. We can notice that the converse is false: the DGHM [6] multiscaling function has an approximation power of 2 but is not even balanced [20]. However, we have the following theorem.

*Theorem 10:* Balancing of order $p$ is equivalent (in the case of orthogonal multiwavelet systems) to any of the following conditions:

- **B2$_\mathrm{p}$)** $\boldsymbol{\phi}(t)$ has an approximation power of $p$ and for $i = 0, \ldots, r-1$, the shifted scaling functions $\phi_i(t + (i/r))$ have identical $p$ first moments, i.e., $\int \phi_i(t + (i/r)) t^n \, dt = \int \phi_0(t) t^n \, dt$ for $i = 0, \ldots, r-1$ and $n = 0, \ldots, p-1$.
- **B5$_\mathrm{p}$)** $\varphi_0$ defined by $\hat{\varphi}_0(\omega) := (1/\sqrt{r})\boldsymbol{\alpha}_{p-1}^\top(e^{j\omega}) \Phi(\omega)$ verifies the Strang–Fix conditions of order $p$: $\hat{\varphi}_0(0) = 1$ and $(d^n/d\omega^n)\hat{\varphi}_0(k2\pi) = 0$, for $n = 0, \ldots, p-1$ and $k \neq 0$.

*Remark 11:* By B2$_\mathrm{p}$), balanced multiwavelets of order $p$ behave as bona-fide wavelets up to the order $p$ of interpolation and approximation. $\varphi_0(t)$ is the *superfunction* [26] associated with $\phi_0(t)$. $\{\varphi_0(t-k)\}_{k\in\mathbb{Z}}$ generates a closed linear subspace $V(\varphi_0) \subset V_0$ having the same approximation power as $\boldsymbol{\phi}(t)$.

*Proof:*

- [B1$_\mathrm{p}$) $\Leftrightarrow$ B2$_\mathrm{p}$)]: The ($\Rightarrow$) part is derived from Lemma 2.1 [23] and orthonormality gives us $r_n = \int \phi_0(t) t^n \, dt$. The converse ($\Leftarrow$) is obtained by using Theorem 3.2 [23] and verifying that the $\mathbf{y}_k$ can be written in the proper form if we take $r_n := \int \phi_0(t) t^n \, dt$.
- [B1$_\mathrm{p}$) $\Leftrightarrow$ B5$_\mathrm{p}$)]: The ($\Rightarrow$) part is derived from Theorem 2.2 [23] for the special case of balanced vanishing moments for $\mathbf{M}(z)$. The ($\Leftarrow$) part is an adaptation of

the proof of Corollary 2.3 [15] to this special form of superfunction. ■

*Remark 12:* If the scaling function $\phi_0(t)$ has, furthermore, $p-1$ vanishing moments (i.e., $r_n = \delta_n$ for $n = 0, \ldots, p-1$), we get a multiwavelet generalization of Coiflets [5]. We have then the following properties:

- $\rho_n(t) := t^n$ and $\mathbf{y}_{0,n}^\top := [(0/r)^n, (1/r)^n, \ldots, (r-1/r)^n]$.
- $\int \phi_i(t) t^n \, dt = (i/r)^n$ for $n = 0, \ldots, p-1$.
- $\varphi_0(t)$ is now the *canonical* [26] superfunction, i.e., it verifies the *extended* Strang–Fix conditions $(d^n/d\omega^n)\hat{\varphi}_0(k2\pi) = \delta_n \delta_k$ for $n = 0, \ldots, p-1$.

*Multicoiflets* are then constructed as balanced multiwavelets with more stringent conditions on the moments of $\phi_0(t)$.

$$\frac{d^n}{d\omega^n}\left[\boldsymbol{\alpha}_{p-1}^\top(e^{j2\omega})\mathbf{M}(e^{j\omega})\right]_{|\omega=0} = j^n \mathbf{y}_{0,n}^\top$$
$$\frac{d^n}{d\omega^n}\left[\boldsymbol{\alpha}_{p-1}^\top(e^{j2\omega})\mathbf{M}(e^{j\omega})\right]_{|\omega=\pi} = \mathbf{0}^\top.$$

A family of orthonormal symmetric multicoiflets with compact support is detailed in [21].

### B. Smoothness and Balancing Order

We introduce the classical Sobolev smoothness

$$s(\boldsymbol{\phi}) := \sup\left\{ s \middle| \int \|\hat{\boldsymbol{\phi}}(\omega)\|^2 (1+|\omega|^2)^s \, d\omega < \infty \right\}.$$

Characterizations of the Sobolev smoothness can be done by analyzing the decay of $\Phi(\omega)$ as $|\omega| \to \infty$. For example, we get Sobolev smoothness $s$ by proving that for $\epsilon > 0$ arbitrarily small, we have

$$|\Phi(\omega)| \le C(1+|\omega|)^{-s+\epsilon}.$$

Now, in the special case the multifilterbanks has balancing order $p$, we have the factorization for $n = 1, \ldots, p$

$$\mathbf{M}(z) = \frac{1}{2^n} \boldsymbol{\Delta}^n(z^2)\mathbf{M}_{n-1}(z)\boldsymbol{\Delta}^{-n}(z)$$

with $\mathbf{M}_{n-1}(1)[1, \ldots, 1]^\top = [1, \ldots, 1]^\top$. Assuming furthermore that $\rho(\mathbf{M}_{p-1}(1)) < 2$ and introducing

$$\gamma_k := \frac{1}{k} \log_2 \sup_{\omega \in ]-\pi, \pi]} \left\| \mathbf{M}_{p-1}\left(e^{j(\omega/2)}\right) \cdots \mathbf{M}_{p-1}\left(e^{j(\omega/2^k)}\right) \right\| \quad (17)$$

we get by Theorem 4.1 [4] that there exists a constant $C > 0$, such that $\forall \omega \in \mathbb{R}$

$$|\Phi(\omega)| \le C(1+|\omega|)^{-p+\gamma_k}. \quad (18)$$

However, the computation of this supremum is highly impractical. Here, we introduce the heuristic of the invariant cycles that have been proved to be optimal in many cases [2]. Intuitively, to characterize the smoothness, we are interested in the decay as $n \to \infty$ of $\Phi(2^{kn}\omega_0)$ for $\omega_0 \in ]-\pi, \pi]$. From the convergence (4), we form the truncated products $\mathbf{M}_{p-1}^{(n)}(\omega) := \prod_{i=1}^{n} \mathbf{M}(e^{-j(\omega/2^i)})$. Evaluating these on the invariant cycle $\{\omega_0, \ldots, \omega_{k-1}\}$ of $\omega \mapsto 2\omega (\mathrm{mod}\, 2\pi)$, we get

$$\mathbf{M}_{p-1}^{(kn)}(2^{kn}\omega_0) = \prod_{i=1}^{kn} \mathbf{M}\left(e^{-j2^{-i}2^{kN}\omega_0}\right)$$
$$= \left(\mathbf{M}(e^{-j\omega_{k-1}}) \cdots \mathbf{M}(e^{-j\omega_0})\right)^n. \quad (19)$$

Then, we study the asymptotic behavior of this product by looking at the eigenvalues of

$$\mathbf{M}(e^{-j\omega_{k-1}}) \cdots \mathbf{M}(e^{-j\omega_0}) = \mathbf{U}_k \boldsymbol{\Lambda}_k \mathbf{U}_k^\top \quad (20)$$

where $\boldsymbol{\Lambda}_k = \mathrm{diag}(\lambda_0^{(k)}, \lambda_1^{(k)}, \ldots, \lambda_{r-1}^{(k)})$. If $\rho(\boldsymbol{\Lambda}_k) = \max\{|\lambda_0^{(k)}|, |\lambda_1^{(k)}|, \ldots, |\lambda_{r-1}^{(k)}|\} \ge 2^{-ki}$, then the scaling functions cannot have a Sobolev exponent of more than $i$ and cannot be more than $\lfloor i - 1/2 \rfloor$ times continuously differentiable [7], [14]. Thus, we get an upper bound on the smoothness.

*Proposition 13:* If an orthonormal multiwavelet system has balancing order $p$ and the spectral radius of $\mathbf{M}_{p-1}(z)$ in the factorization (15) verifies $\rho(\mathbf{M}_{p-1}(1)) < 2$, then defining

$$\gamma_k := \frac{1}{k} \log_2 \rho(\mathbf{M}_{p-1}(e^{-j\omega_{k-1}}) \cdots \mathbf{M}_{p-1}(e^{-j\omega_0})) \quad (21)$$

with $\{\omega_0, \ldots, \omega_{k-1}\}$ invariant cycles of $\omega \mapsto 2\omega (\mathrm{mod}\, 2\pi)$, and $\gamma := \inf_k \gamma_k$, we get that $\boldsymbol{\phi}(t)$ is at most $\lfloor p - \gamma - 1/2 \rfloor$ Hölder continuous (and has at most Sobolev exponent $s = p - \gamma$).

As proved in some simple cases [2], [14], the supremum

$$\sup_{\omega \in ]-\pi, \pi]} \left\| \mathbf{M}_{p-1}(e^{j(\omega/2)}) \cdots \mathbf{M}_{p-1}(e^{j(\omega/2^k)}) \right\|$$

is usually attained on invariant cycles. Furthermore, it is often achieved on the smallest length invariant cycle. One can then take $s = p - \gamma$ for the smallest invariant cycle as a good estimate of the Sobolev exponents of $\boldsymbol{\phi}(t)$, and therefore, $\boldsymbol{\psi}(t)$.

For example, in the case of the Haar multiwavelet (multiplexed scalar Haar filter [35]), with $\omega_0 = 2\pi/3$, $\lambda_0 = 0$, $\lambda_1 = 1/4$, it is then proven that the scaling functions cannot be continuous. In the case of the DGHM multiwavelet $\lambda_0 = 1/100$, $\lambda_1 = 1/4^2$, it is proven that the scaling functions can be at most $C^1$. DGHM scaling functions and wavelets are in fact Lipschitz.

In [4], another method was developed using the transition operator. This method gives the exact Sobolev smoothness of $\boldsymbol{\phi}(t)$ and $\boldsymbol{\psi}(t)$. An approach giving a good lower bound of the Sobolev smoothness for each scaling function $\phi_i(t)$ is detailed in [26].

## V. Construction of High-Order Balanced Multiwavelets

### A. Bat Family

Using the results above, we are now able to investigate the construction of orthonormal multiwavelets of arbitrary balancing order in a similar way to what Daubechies did for her well-known wavelet family. The scheme of construction is the following.

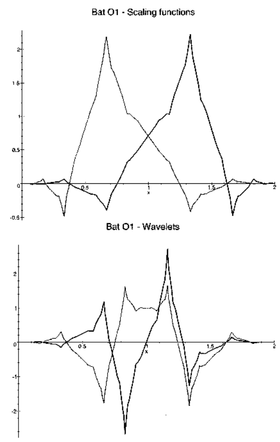


Fig. 5. First-order balanced orthogonal multiwavelet. Scaling functions are flipped around 1, the wavelets are symmetric/antisymmetric, the length is three taps, and an estimate of the smoothness using Proposition 13 gives the Sobolev exponent $s = 0.64$.

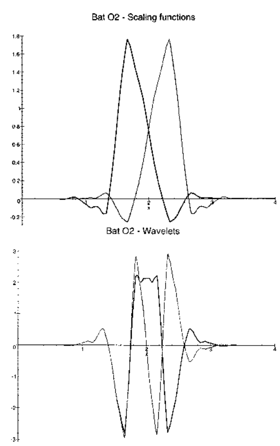


Fig. 6. Order 2 balanced orthogonal multiwavelet. Scaling functions are flipped around 2, the wavelets are symmetric/antisymmetric, the length is five taps ($2 \times 2$), and an estimate of the smoothness using Proposition 13 gives the Sobolev exponent $s = 1.15$.

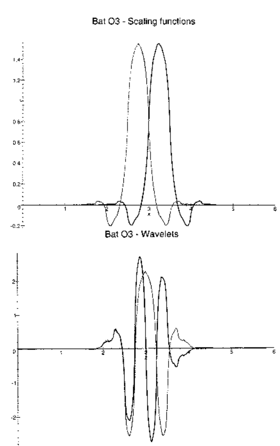


Fig. 7. Order–3 balanced orthogonal multiwavelet. Scaling functions are flipped around 3, the wavelets are symmetric/antisymmetric, the length is seven taps ($2 \times 2$), and an estimate of the smoothness using Proposition 13 gives the Sobolev exponent $s = 1.71$.

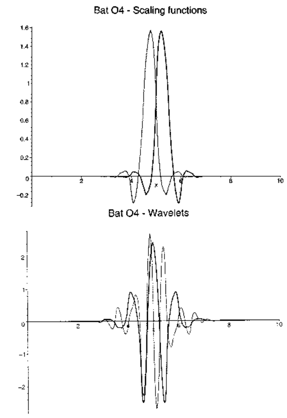


Fig. 8. Order 4 balanced orthogonal multiwavelet. The scaling functions are flipped around 5, the wavelets are symmetric/antisymmetric, the length is 11 taps ($2 \times 2$), and an estimate of the smoothness using Proposition 13 gives the Sobolev exponent $s = 2.07$.

TABLE I
COEFFICIENTS OF BAT O1: FIRST ORDER BALANCED ORTHOGONAL MULTIWAVELET

| | | | | | | |
|---|---|---|---|---|---|---|
| $m_0[k]$ | 0 | $\frac{1}{2}+\frac{1}{4}\sqrt{7}$ | $\frac{3}{4}$ | $\frac{1}{4}$ | $\frac{1}{2}-\frac{1}{4}\sqrt{7}$ | 0 |
| $m_1[k]$ | 0 | $\frac{1}{2}-\frac{1}{4}\sqrt{7}$ | $\frac{1}{4}$ | $\frac{3}{4}$ | $\frac{1}{2}+\frac{1}{4}\sqrt{7}$ | 0 |
| $n_0[k]$ | 0 | $\frac{-1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{-1}{2}$ | 0 |
| $n_1[k]$ | 0 | $\frac{1}{4}$ | $-\frac{1}{4}\sqrt{7}$ | $\frac{1}{4}\sqrt{7}$ | $\frac{-1}{4}$ | 0 |

TABLE II
COEFFICIENTS OF BAT O2: ORDER 2 BALANCED ORTHOGONAL MULTIWAVELET

| | | | | | |
|---|---|---|---|---|---|
| $m_0[k]$ | 0 | $-\frac{31}{640}+\frac{1}{640}\sqrt{31}$ | $\frac{93}{640}-\frac{13}{640}\sqrt{31}$ | $\frac{217}{640}+\frac{23}{640}\sqrt{31}$ | $\frac{341}{640}-\frac{11}{640}\sqrt{31}$ |
| $m_1[k]$ | 0 | $-\frac{13}{640}+\frac{3}{640}\sqrt{31}$ | $-\frac{1}{640}+\frac{1}{640}\sqrt{31}$ | $\frac{11}{640}-\frac{11}{640}\sqrt{31}$ | $\frac{23}{640}+\frac{7}{640}\sqrt{31}$ |
| $n_0[k]$ | 0 | $\frac{23}{160}-\frac{3}{160}\sqrt{31}$ | $\frac{11}{160}-\frac{1}{160}\sqrt{31}$ | $-\frac{91}{160}+\frac{1}{160}\sqrt{31}$ | $\frac{57}{160}+\frac{3}{160}\sqrt{31}$ |
| $n_1[k]$ | 0 | $\frac{47}{320}-\frac{7}{320}\sqrt{31}$ | $\frac{9}{320}+\frac{1}{320}\sqrt{31}$ | $-\frac{159}{320}+\frac{9}{320}\sqrt{31}$ | $\frac{103}{320}+\frac{17}{320}\sqrt{31}$ |

| | | | | |
|---|---|---|---|---|
| $\frac{23}{640}+\frac{7}{640}\sqrt{31}$ | $\frac{11}{640}-\frac{11}{640}\sqrt{31}$ | $-\frac{1}{640}+\frac{1}{640}\sqrt{31}$ | $-\frac{13}{640}+\frac{3}{640}\sqrt{31}$ | 0 |
| $\frac{341}{640}-\frac{11}{640}\sqrt{31}$ | $\frac{217}{640}+\frac{23}{640}\sqrt{31}$ | $\frac{93}{640}-\frac{13}{640}\sqrt{31}$ | $-\frac{31}{640}+\frac{1}{640}\sqrt{31}$ | 0 |
| $\frac{57}{160}+\frac{3}{160}\sqrt{31}$ | $-\frac{91}{160}+\frac{1}{160}\sqrt{31}$ | $\frac{11}{160}-\frac{1}{160}\sqrt{31}$ | $\frac{23}{160}-\frac{3}{160}\sqrt{31}$ | 0 |
| $-\frac{103}{320}-\frac{17}{320}\sqrt{31}$ | $\frac{159}{320}-\frac{9}{320}\sqrt{31}$ | $-\frac{9}{320}-\frac{1}{320}\sqrt{31}$ | $-\frac{47}{320}+\frac{7}{320}\sqrt{31}$ | 0 |

1) Impose the order of balancing to be $p$ by writing for $n = 1, \ldots, p$

$$\mathbf{M}(z) = \frac{1}{2^n}\,\boldsymbol{\Delta}^n(z^2)\mathbf{M}_{n-1}(z)\boldsymbol{\Delta}^{-n}(z)$$

with $\mathbf{M}_{n-1}(1)[1\cdots 1]^\top = [1\cdots 1]^\top$. This way, we reduce the number of degrees of freedom in the design.

2) Impose the condition $\mathbf{O}$ (orthonormality) (5) on $\mathbf{M}(z)$, which gives quadratic equations on the free variables of $\mathbf{M}_{p-1}(z)$ (the idea is to introduce the Laurent polynomial matrix $\mathbf{V}_{p-1}(z) := 2^{-p}(1-z^{-r})^p\mathbf{M}_{p-1}(z)\boldsymbol{\Delta}^{-p}(z)$ and to translate the orthonormality condition on this matrix; for more details, see the proof of Lemma 15 given in the Appendix).

3) Impose a flipping property on $m_0(z), m_1(z)$ [i.e., $m_1(z) = z^{-2L+1}m_0(z^{-1})$]. The flipping property enables an easy lossless symmetrization (as seen in [36]) of finite length input signals both for the lowpass filters and the highpass.

TABLE III
COEFFICIENTS OF BAT O3: ORDER 3 BALANCED ORTHOGONAL MULTIWAVELET

| | | | | |
|---|---|---|---|---|
| $m_0[k]$ | 0 | $-\frac{2989}{2232320}+\frac{97}{2232320}\sqrt{15199}$ | $\frac{537}{2232320}+\frac{69}{2232320}\sqrt{15199}$ | $-\frac{75969}{2232320}-\frac{105}{446464}\sqrt{15199}$ |
| $m_1[k]$ | 0 | $\frac{2481}{446464}-\frac{21}{446464}\sqrt{15199}$ | $-\frac{4701}{446464}+\frac{39}{446464}\sqrt{15199}$ | $\frac{13769}{2232320}-\frac{43}{2232320}\sqrt{15199}$ |
| $n_0[k]$ | 0 | $-\frac{2871}{279040}+\frac{33}{279040}\sqrt{15199}$ | $-\frac{1177}{279040}+\frac{1}{279040}\sqrt{15199}$ | $-\frac{12001}{279040}-\frac{11}{55808}\sqrt{15199}$ |
| $n_1[k]$ | 0 | $\frac{12021}{1116160}-\frac{63}{1116160}\sqrt{15199}$ | $\frac{7697}{1116160}-\frac{101}{1116160}\sqrt{15199}$ | $-\frac{25921}{1116160}-\frac{331}{1116160}\sqrt{15199}$ |

| | | | |
|---|---|---|---|
| $m_0[k]$ | $\frac{14785}{446464}+\frac{551}{2232320}\sqrt{15199}$ | $\frac{601461}{1116160}-\frac{441}{1116160}\sqrt{15199}$ | $\frac{450639}{1116160}+\frac{819}{1116160}\sqrt{15199}$ |
| $m_1[k]$ | $\frac{22083}{2232320}-\frac{111}{2232320}\sqrt{15199}$ | $-\frac{96093}{1116160}+\frac{45}{223232}\sqrt{15199}$ | $\frac{30005}{223232}-\frac{667}{1116160}\sqrt{15199}$ |
| $n_0[k]$ | $-\frac{1555}{55808}-\frac{71}{279040}\sqrt{15199}$ | $\frac{75083}{139520}+\frac{19}{139520}\sqrt{15199}$ | $-\frac{63171}{139520}+\frac{27}{139520}\sqrt{15199}$ |
| $n_1[k]$ | $-\frac{44869}{1116160}-\frac{241}{1116160}\sqrt{15199}$ | $\frac{150307}{558080}+\frac{743}{558080}\sqrt{15199}$ | $-\frac{348777}{558080}+\frac{333}{558080}\sqrt{15199}$ |

| | | | |
|---|---|---|---|
| $m_0[k]$ | $\frac{30005}{223232}-\frac{667}{1116160}\sqrt{15199}$ | $-\frac{96093}{1116160}+\frac{45}{223232}\sqrt{15199}$ | $\frac{22083}{2232320}-\frac{111}{2232320}\sqrt{15199}$ |
| $m_1[k]$ | $\frac{450639}{1116160}+\frac{819}{1116160}\sqrt{15199}$ | $\frac{601461}{1116160}-\frac{441}{1116160}\sqrt{15199}$ | $\frac{14785}{446464}+\frac{551}{2232320}\sqrt{15199}$ |
| $n_0[k]$ | $-\frac{63171}{139520}+\frac{27}{139520}\sqrt{15199}$ | $\frac{75083}{139520}+\frac{19}{139520}\sqrt{15199}$ | $-\frac{1555}{55808}-\frac{71}{279040}\sqrt{15199}$ |
| $n_1[k]$ | $\frac{348777}{558080}-\frac{333}{558080}\sqrt{15199}$ | $-\frac{150307}{558080}-\frac{743}{558080}\sqrt{15199}$ | $\frac{44869}{1116160}+\frac{241}{1116160}\sqrt{15199}$ |

| | | | | |
|---|---|---|---|---|
| $m_0[k]$ | $\frac{13769}{2232320}-\frac{43}{2232320}\sqrt{15199}$ | $-\frac{4701}{446464}+\frac{39}{446464}\sqrt{15199}$ | $\frac{2481}{446464}-\frac{21}{446464}\sqrt{15199}$ | 0 |
| $m_1[k]$ | $-\frac{75969}{2232320}-\frac{105}{446464}\sqrt{15199}$ | $\frac{537}{2232320}+\frac{69}{2232320}\sqrt{15199}$ | $-\frac{2989}{2232320}+\frac{97}{2232320}\sqrt{15199}$ | 0 |
| $n_0[k]$ | $-\frac{12001}{279040}-\frac{11}{55808}\sqrt{15199}$ | $-\frac{1177}{279040}+\frac{1}{279040}\sqrt{15199}$ | $-\frac{2871}{279040}+\frac{33}{279040}\sqrt{15199}$ | 0 |
| $n_1[k]$ | $\frac{25921}{1116160}+\frac{331}{1116160}\sqrt{15199}$ | $-\frac{7697}{1116160}+\frac{101}{1116160}\sqrt{15199}$ | $-\frac{12021}{1116160}+\frac{63}{1116160}\sqrt{15199}$ | 0 |

4) Solve the system of equations using a Gröbner bases approach (here, we used the programs Singular [10] and the web version of FGb [8]).
5) The highpass filters are easily derived from the lowpass by imposing $n_0(z)$ to be symmetric and $n_1(z)$ to be antisymmetric. The orthonormality conditions (6) give unique solutions up to a change of sign.

Using this approach, we have been able to construct all the shortest length (as defined below) orthonormal multiwavelets with flipped scaling functions and symmetric/antisymmetric wavelets for balancing order up to 4. Figs. 5–8 show the smoothest high-order balanced multiwavelets with these properties. In Tables I–III,[2] we detail closed-form expressions of the coefficients. For order 4 of balancing, because of the degree of the leading polynomial in the Gröbner basis, only numerical solutions have been obtained (the coefficients can be downloaded from http: \\lcavwww.epfl.ch\~lebrun).

*B. Minimal-Length BMW*

In this section, we will prove for $r = 2$ the following surprising result.

*Theorem 14:* The multiwavelets of multiplicity $r = 2$ and balancing order $p$ with the shortest length refinement mask are the Daubechies wavelets of length $2p$.

First, let us define the length of a matrix Laurent polynomial $\mathbf{M}(z) = \sum_{k=-N_1}^{N_2} \mathbf{M}[k]z^{-k}$ with $\mathbf{M}[-N_1] \neq \mathbf{0}$ and $\mathbf{M}[N_2] \neq \mathbf{0}$ to be $\ell(\mathbf{M}(z)) = N_2 - N_1 + 1 = \deg(z^{N_2}\mathbf{M}(z)) + 1$. One verifies easily that $\ell(\mathbf{M}(z)\mathbf{N}(z)) \leq \ell(\mathbf{M}(z)) + \ell(\mathbf{N}(z)) - 1$. Now, to prove the theorem, we will first prove that the minimal length condition with balancing and orthogonality implies that the refinement mask has a multiplexed filter structure.

[2]The coefficients of BAT O1 already appeared in [3] and [34].

*Lemma 15:* Let $\mathbf{M}(z)$ be the refinement mask associated with an orthonormal multiwavelet system of multiplicity $r = 2$ and balancing order $p$. If $\mathbf{M}(z)$ is of minimal length, then $m_1(z) = z^{-2}m_0(z)$.

*Proof:* This rather lengthy and technical proof is given in the Appendix. ■

*Proof of Theorem 14:* Using Lemma 15 and the balancing order condition **B3**), we get that

$$m_0(z) + \alpha_{1,2}^{(p-1)}(z^4)m_1(z) = (1 + z^{-2}\alpha_{1,2}^{(p-1)}(z^4))m_0(z)$$

must have zeros of order $p$ at $j$, $-1$, $-j$, and for $z = -1$

$$\begin{aligned} 1 + z^{-2}\alpha_{1,2}^{(p-1)}(z^4) &= 1 + (-1)^{-2}\alpha_{1,2}^{(p-1)}((-1)^4) \\ &= 1 + \alpha_{1,2}^{(p-1)}(1) = 2 \end{aligned}$$

which implies that $m_0(z)$ must have $p$ zeros at $z = -1$. Since

$$\mathbf{M}(z) = \begin{bmatrix} m_{00}(z) & m_{01}(z) \\ z^{-1}m_{00}(z) & z^{-1}m_{01}(z) \end{bmatrix}$$

where $m_{00}(z)$ and $m_{01}(z)$ are the polyphase components of $m_0(z)$, then the orthonormality condition (5) gives that $m_0(z)$ is a real conjugate mirror filter. Then, from the well-known theorem of Daubechies [5], this implies that $m_0(z)$ has at least $2p$ nonzero coefficients and that the minimal length filters are the Daubechies filters (i.e., the classical $D_{2p}$ and Symlets of order $p$). ■

This also implies the following.

*Corollary 16:* An orthonormal multiwavelet system of multiplicity $r = 2$ and balancing order $p$ has a refinement mask $\mathbf{M}(z)$ with at least $p + 1$ nonzero ($2 \times 2$) taps.

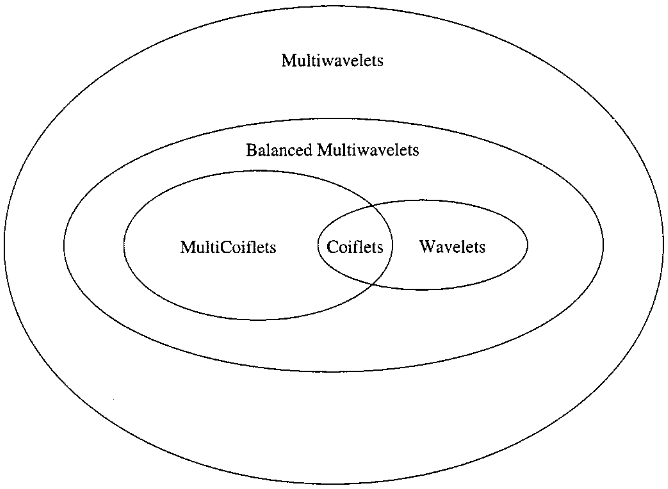


Fig. 9. Relations between the different multiwavelets.

This result can be generalized to any multiplicity and to multicoiflets [21]. Fig. 9 gives an overview of the relations between the different multiwavelets.

## VI. CONCLUSION

By introducing the concept of high-order balancing, we have clarified an important issue in the design of multiwavelets. We have proved that this concept is the natural counterpart of the *zeros at* $\pi$ condition in the standard wavelet theory. With these results, we made it possible to design general families of high-order balanced multiwavelets with the required properties for practical signal processing (preservation/cancellation of discrete-time polynomial signals in the lowpass/highpass subbands, FIR, linear phase, and orthogonality). The proposed scheme of construction is making use of computationally heavy methods (Gröbner basis decomposition), and it is not clear at this point that closed-form designs will be feasible for multiwavelets of balancing order $p \geq 4$. Matrix spectral factorization could be a way to overcome this limitation. Another subject of interest would also be to use the invariant cycles scheme developed for estimating the smoothness to link the smoothness of the scaling functions to a particular factorization of the refinement mask (the counterpart of the *zeros at pre-periodic points* condition [14] in the standard wavelet theory).

## APPENDIX

*Proof of Theorem 2:*

- [B0) $\Rightarrow$ B1)]: Assuming B0), we have by transposition $\mathbf{u}_0^\top \mathbf{L} = \mathbf{u}_0^\top$. Writing the equations explicitly, we get $[1\cdots 1]\sum_k \mathbf{M}[2k+1] = [1\cdots 1]\sum_k \mathbf{M}[2k] = [1\cdots 1]$. Therefore, $[1\cdots 1]\sum_k \mathbf{M}[k] = 2[1\cdots 1]$, and since $\mathbf{M}(1) = (1/2)\sum_k \mathbf{M}[k]$, we have condition B1).
- [B1) $\Rightarrow$ B2)]: Conditions E and A1 imply that $\mathbf{r}_0^\top = [1\cdots 1]$, and from condition O, we get $[1\cdots 1]\mathbf{M}^\top(1) = [1\cdots 1]$. From (3), we derive that $\Phi(0)$ is also a right eigenvector associated with the eigenvalue 1 of $\mathbf{M}(1)$, and using again condition E, we get the result.
- [B2) $\Rightarrow$ B3)]: From (3) and condition O, we have $\mathbf{r}_0 = \Phi(0) = [1\cdots 1]^\top$. $\mu(z) = \sum_{i=0}^{r-1} m_i(z) = [1, 1, \ldots, 1]2\mathbf{M}(z^r)[1, z^{-1}, \ldots, z^{-(r-1)}]^\top$. Therefore, $\mu(1) = [1, 1, \ldots, 1]2\mathbf{M}(1)[1, 1, \ldots, 1]^\top = 2r$. Now, if $z = e^{jk\pi/r}$ with $k = 2l+1$ and $l = 0, \ldots, r$, then $z^r = -1$, and therefore, $\mu(z) = 0$. If $z = e^{jk\pi/r}$ with $k = 2l$ and $l = 1, \ldots, r-1$, then $z^r = 1$ and $\mu(z) = 2\sum_{k=0}^{r-1} z^k = 2((1-z^{-r})/(1-z^{-1})) = 0$. Therefore, $\mu(z)$ has roots at $z = e^{jk\pi/r}$ for $k = 1, \ldots, 2r-1$.
- [B3) $\Rightarrow$ B0)]: Taking $u[n] := 1$ from the time-varying filterbank representation (see Fig. 2), we get that the $2r$ possible outputs are $\mathbf{y}[2rn+l] = \sum_k \sum_{i=0}^{r-1} m_i[2rk+l]$ for $l = 0, \ldots, 2r-1$. Denoting $\mu^{(l)}(z)$ as the lth polyphase component of $\mu(z)$, we get that

$$\mathbf{y}[2rn+l] = \mu^{(l)}(1) = \frac{1}{2r}\sum_{k=0}^{2r-1} \mu(e^{jk\pi/r}) = \frac{1}{2r}(\mu(1)) = 1.$$

Hence, $\mathbf{u}_0$ is an eigenvector of the operator $\mathbf{L}^\top$.
- [B1) $\Rightarrow$ B4)]: This is a direct consequence of Theorem 4.1 in [23].
- [B4) $\Rightarrow$ B3)]: Assuming B4)

$$\sum_{i=0}^{r-1} m_i(z) = [1, \quad \ldots, \quad 1]\,\mathbf{M}(z^r)\begin{bmatrix} 1 \\ \vdots \\ z^{-(r-1)} \end{bmatrix}$$

$$= \tfrac{1}{2}[1\cdots 1]\begin{bmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-2r} & 0 & \ldots & 0 & 1 \end{bmatrix}$$

$$\cdots \mathbf{M}_0(z^r)\frac{1}{1-z^{-r}}\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ z^{-r} & 1 & 1 & \ddots & \vdots \\ \vdots & z^{-r} & \ddots & \ddots & 1 \\ \vdots & \vdots & \ddots & 1 & 1 \\ z^{-r} & z^{-r} & \ldots & z^{-r} & 1 \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 \\ z^{-1} \\ z^{-2} \\ \vdots \\ z^{-(r-1)} \end{bmatrix}$$

so that

$$\sum_{i=0}^{r-1} m_i(z) = \frac{1}{2}\left(\frac{1-z^{-2r}}{1-z^{-1}}\right)[1, \quad 0\cdots 0]\mathbf{M}_0(z^r)\begin{bmatrix} 1 \\ z^{-1} \\ \vdots \\ z^{-(r-1)} \end{bmatrix}$$

and this is condition B3).

■

*Proof of Theorem 10:*

- [Balancing $\Rightarrow$ B3$_\text{p}$)]: From Lemma 4, for $k = 1, \ldots, r$, we have that $\alpha_{k,r}^{(p-1)}(z)$ interpolates the kth polyphase component of any polynomial sequence of degree smaller than $p$ from the zeroth polyphase component. That means that on $\mathcal{C}_{p-1}$, the lowpass synthesis operator $\mathbf{L}^\top$ is equivalent to the scalar subdivision operator $\mathbf{S}_{\mu_{p-1}}x[n] := \sum_k \mu_{p-1}[n-2rk]x[rk]$, where $\mu_{p-1}(z) := \sum_{k=0}^{r-1} \alpha_{k,r}^{(p-1)}(z^{2r})m_k(z)$, i.e., on $\mathcal{C}_{p-1}$, the operator $\mathbf{L}^\top$ is equivalent to the multirate system

$$-(\downarrow r) - (\uparrow 2r) - \boxed{\sum_{k=0}^{r-1} \alpha_{k,r}^{(p-1)}(z^{2r})m_k(z)} - .$$

Now, using the discrete-time version of the Strang–Fix theorem (cf. [2], [13]), $\mathcal{C}_{p-1}$ being invariant by $\mathbf{L}^\top$ and then by $\mathbf{S}_{\mu_{p-1}}$ (since $(\downarrow r)$ preserves the polynomial signals), we get that $\mu_{p-1}(z) = \sum_{k=0}^{r-1} \alpha_{k,r}^{(p-1)}(z^{2r})m_k(z)$ must have zeros of order $p$ at $z = e^{jk\pi/r}$ for $k = 1, \ldots, 2r-1$.

- $[\mathrm{B3_p}) \Rightarrow \mathrm{B0_p})]$: From [2], this condition implies that the existence of an Appell sequence of polynomials $\rho_n(t) := \sum_{k=0}^{n} \binom{n}{k} r_{n-k}t^k$ with $r_0 = 1$ such that $\mathbf{v}_n[l] := \rho_n(l/r)$ are eigenvectors of the subdivision operator $\mathbf{S}_{\mu_{p-1}}$ for the eigenvalues $2^{-n}$, i.e., $\mathbf{S}_{\mu_{p-1}}\mathbf{v}_n = 2^{-n}\mathbf{v}_n$. From the equivalence of $\mathbf{L}^\top$ and $\mathbf{S}_{\mu_{p-1}}$ on $\mathcal{C}_{p-1}$, we get the result.
- $[\mathrm{B0_p}) \Rightarrow \mathrm{B1_p})]$: $\mathbf{M}(z)$ satisfies the conditions of Theorem 2.1 in [24] with

$$\mathbf{y}_n^\top := \left[\rho_n\left(\frac{0}{r}\right), \quad \rho_n\left(\frac{1}{r}\right), \ldots, \rho_n\left(\frac{r-1}{r}\right)\right]$$

for $n = 0, \ldots, p-1$. The balanced vanishing moments of order $p$ are just a rewriting of these conditions.
- $[\mathrm{B1_p}) \Rightarrow \mathrm{B4_p})]$: Applying Corollary 4.3. from [23], we get the factorization

$$\mathbf{M}(z) = \frac{1}{2^p}\mathbf{C}_0(z^2)\cdots\mathbf{C}_{p-1}(z^2)\mathbf{M}_{p-1}(z)\mathbf{C}_{p-1}^{-1}(z)\cdots\mathbf{C}_0^{-1}(z)$$

with

$$\mathbf{C}_n(z) := \begin{bmatrix} c_{0,n}^{-1} & -c_{0,n}^{-1} & 0 & \cdots & 0 \\ 0 & c_{1,n}^{-1} & -c_{1,n}^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & c_{r-2,n}^{-1} & -c_{r-2,n}^{-1} \\ -z^{-1}c_{r-1,n}^{-1} & 0 & \cdots & 0 & c_{r-1,n}^{-1} \end{bmatrix}$$

and the polynomial matrix $\mathbf{M}_{p-1}(z)$ verifying

$$\mathbf{M}_{p-1}(1)\mathbf{c}_{p-1} = \mathbf{c}_{p-1}$$

where

$$\mathbf{c}_n^\top := [c_{0,n}, \quad \ldots, \quad c_{r-1,n}] = 2^{-n}[1, \quad \ldots, \quad 1]$$

obtained recursively from $\mathbf{y}_0[0], \ldots, \mathbf{y}_{p-1}[0]$. Therefore, for $n = 0, \ldots, p-1$, we get $\mathbf{C}_n(z) = 2^n\mathbf{\Delta}(z)$ and $\mathbf{M}_{n-1}(1)[1, \ldots, 1]^\top = [1, \ldots, 1]^\top$.

- $[\mathrm{B4_p}) \Rightarrow \mathrm{B3_p})]$: First, we give a digest of the proof in the case $r = 2$ for $p = 2, 3$ (case $p = 1$ is a consequence of Theorem 2).

For $p = 2$, we have

$$\begin{aligned} 2(m_0(z) &+ \alpha_{1,2}^{(1)}(z^4)m_1(z)) \\ &= 2m_0(z) + (3 - z^{-4})m_1(z) \\ &= [2, \quad 3 - z^{-4}]\mathbf{M}(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\ &= \frac{1}{4}\left(\frac{1-z^{-4}}{1-z^{-1}}\right)^2 [2, \quad -1]\mathbf{M}_1(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}. \end{aligned}$$

For $p = 3$

$$\begin{aligned} 8(m_0(z) &+ \alpha_{1,2}^{(2)}(z^4)m_1(z)) \\ &= 8m_0(z) + (15 - 10z^{-4} + 3z^{-8})m_1(z) \\ &= \frac{1}{8}\left(\frac{1-z^{-4}}{1-z^{-1}}\right)^3 [8 + 3z^{-4}, \quad -9]\mathbf{M}_2(z^2)\begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}. \end{aligned}$$

For the general case, writing $\boldsymbol{\pi}(z) = [1, z^{-1}, \ldots, z^{-(r-1)}]^\top$, we will first prove that $\forall n \geq 0$, and we can factorize

$$\boldsymbol{\alpha}_n^\top(z)\mathbf{\Delta}^{n+1}(z) = (1 - z^{-1})^{n+1}\boldsymbol{\gamma}_n^\top(z) \tag{22}$$

where $\boldsymbol{\gamma}_n(z) \in \mathbb{Q}^r[z^{-1}]$, i.e., vector polynomial in $z^{-1}$. Namely, by induction on $n$, $n = 0$

$$\begin{aligned} \boldsymbol{\alpha}_0^\top(z)\mathbf{\Delta}^1(z) &= \\ &= [1, \quad \ldots, \quad 1]\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & -1 \\ -z^{-1} & 0 & \cdots & 0 & 1 \end{bmatrix} \\ &= (1 - z^{-1})[1, \quad 0, \quad \ldots, \quad 0]. \end{aligned}$$

Now, assume for $k = 0, \ldots, n-1$ that

$$\boldsymbol{\alpha}_k^\top(z)\mathbf{\Delta}^{k+1}(z) = (1 - z^{-1})^{k+1}\boldsymbol{\gamma}_k^\top(z)$$

with $\boldsymbol{\gamma}_k(z)$ polynomial. Then, introducing for $k \geq 1$

$$\mathbf{\Gamma}_k^\top := \left[0, \frac{\Gamma\left(k + \frac{1}{r}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{r}\right)}, \ldots, \frac{\Gamma\left(k + \frac{r-1}{r}\right)}{\Gamma(k+1)\Gamma\left(\frac{r-1}{r}\right)}\right]$$

and $\mathbf{\Gamma}_0 := [1, \ldots, 1]^\top$, we get $\boldsymbol{\alpha}_k(z) = \boldsymbol{\alpha}_{k-1}(z) + (1 - z^{-1})^k\mathbf{\Gamma}_k$; therefore

$$\begin{aligned} \boldsymbol{\alpha}_n^\top(z)\mathbf{\Delta}^{n+1}(z) &= (\boldsymbol{\alpha}_{n-1}^\top(z) + (1 - z^{-1})^n\mathbf{\Gamma}_n^\top)\mathbf{\Delta}^{n+1}(z) \\ &= (1 - z^{-1})^n(\boldsymbol{\gamma}_{n-1}^\top(z) + \mathbf{\Gamma}_n^\top\mathbf{\Delta}^n(z))\mathbf{\Delta}(z). \end{aligned}$$

Now, it is easily computed that $\boldsymbol{\gamma}_{n-1}^\top(1) + \mathbf{\Gamma}_n^\top\mathbf{\Delta}^n(1) = 2^{-n}[1, \ldots, 1]^\top$. Therefore, there exists $\boldsymbol{\gamma}_n(z) \in \mathbb{Q}^r[z^{-1}]$ such that

$$(\boldsymbol{\gamma}_{n-1}^\top(z) + \mathbf{\Gamma}_n^\top\mathbf{\Delta}^n(z))\mathbf{\Delta}(z) = (1 - z^{-1})\boldsymbol{\gamma}_n^\top(z). \tag{23}$$

Thus $\boldsymbol{\alpha}_n^\top(z)\boldsymbol{\Delta}^{n+1}(z) = (1-z^{-1})^{n+1}\boldsymbol{\gamma}_n^\top(z)$ with $\boldsymbol{\gamma}_n(z)$ polynomial; hence, we have the result. Now, by the hypothesis

$$\begin{aligned}\mu_{p-1}(z) &= \boldsymbol{\alpha}_{p-1}^\top(z^{2r})2\mathbf{M}(z^r)\boldsymbol{\pi}(z)\\ &= \frac{1}{2^{p-1}}\boldsymbol{\alpha}_{p-1}^\top(z^{2r})\boldsymbol{\Delta}^p(z^{2r})\mathbf{M}_{p-1}(z^r)\boldsymbol{\Delta}^{-p}(z^r)\boldsymbol{\pi}(z)\\ &= \frac{1}{2^{p-1}}(1-z^{-2r})^p\boldsymbol{\gamma}_{p-1}^\top(z^{2r})\mathbf{M}_{p-1}(z^r)\frac{1}{(1-z^{-1})^p}\boldsymbol{\pi}(z)\\ &= \frac{1}{2^{p-1}}\left(\frac{1-z^{-2r}}{1-z^{-1}}\right)^p\boldsymbol{\gamma}_{p-1}^\top(z^{2r})\mathbf{M}_{p-1}(z^r)\boldsymbol{\pi}(z).\end{aligned}$$

Now, we get the result since $\boldsymbol{\gamma}_{p-1}^\top(z^{2r})\mathbf{M}_{p-1}(z^r)\boldsymbol{\pi}(z)$ is polynomial.

- [$\mathrm{B3_p}$) $\Rightarrow$ Balancing)]: As mentioned in [28], condition $\mathrm{B3_p}$) says that the multirate system

$$-(\downarrow r)-(\uparrow 2r)-\boxed{\sum_{k=0}^{r-1}\alpha_{k,r}^{(p-1)}(z^{2r})m_k(z)}-$$

has zeros of order $p$ at the roots of the unity $e^{jk\pi/r}$ with $k=1,\ldots,2r-1$. Therefore, from the rank $M$ wavelet theory [13, Th. 2.1], we get that this system preserves discrete polynomial sequences of degree $n=0,\ldots,p-1$, and since this multirate system is equivalent to the lowpass synthesis branch for polynomial sequences of degree up to $p-1$ (cf. [Balancing $\Rightarrow$ $\mathrm{B3_p}$)]), this translates in time domain into the definition of balancing. ■

*Proof of Lemma 15:* Assuming $\mathbf{M}(z)$ is the refinement mask associated with an orthonormal multiscaling function of balancing order $p$, we then have by the orthonormality condition

$$\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) + \mathbf{M}(-z)\mathbf{M}^\top(-z^{-1}) = \mathbf{I}.$$

Besides, balancing of order $p$ gives us

$$\mathbf{M}(z) = \frac{1}{2^p}\boldsymbol{\Delta}^p(z^2)\mathbf{M}_{p-1}(z)\boldsymbol{\Delta}^{-p}(z)$$

with

$$\mathbf{M}_{p-1}(1)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \quad\text{and}\quad \boldsymbol{\Delta}(z) = \begin{bmatrix}1 & -1\\ -z^{-1} & 1\end{bmatrix}.$$

Introducing

$$\mathbf{V}_{p-1}(z) := 2^{-p}(1-z^{-2})^p\mathbf{M}_{p-1}(z)\boldsymbol{\Delta}^{-p}(z)$$

one gets

$$\begin{aligned}&\mathbf{V}_{p-1}(z)\mathbf{V}_{p-1}^\top(z^{-1}) + \mathbf{V}_{p-1}(-z)\mathbf{V}_{p-1}^\top(-z^{-1})\\ &\quad= \begin{bmatrix}2 & 1+z^2\\ 1+z^{-2} & 2\end{bmatrix}^p.\end{aligned}$$

Furthermore, one can write

$$\begin{aligned}\mathbf{U}_{p-1}(z) &:= \begin{bmatrix}2 & 1+z^2\\ 1+z^{-2} & 2\end{bmatrix}^p\\ &= \sum_{k=-\lceil p/2\rceil}^{k=\lceil p/2\rceil}\mathbf{U}_{p-1}[k]z^{-2k}\end{aligned}$$

with $\mathbf{U}_{p-1}[-k] = \mathbf{U}_{p-1}^\top[k]$. Thus, for $\mathbf{W}(z) = \sum_{k=-N_1}^{k=N_2}\mathbf{W}[k]z^{-k}$ to verify

$$\mathbf{W}(z) + \mathbf{W}(-z) = \mathbf{U}_{p-1}(z) \tag{24}$$

one needs $N_1, N_2 \geq 2\lceil p/2\rceil$. Introducing $\mathbf{u}(z) := \left[\begin{smallmatrix}1\\ z^{-1}\end{smallmatrix}\right]$, then

$$\begin{aligned}\mathbf{W}_{p-1}(z) &:= (1+z^{-1})^p(1+z)^p\mathbf{u}(z)\mathbf{u}^\top(z)\\ &= (1+z^{-1})^p(1+z)^p\begin{bmatrix}1 & z\\ z^{-1} & 1\end{bmatrix}\end{aligned}$$

is an obvious minimal length solution of (24). Therefore, one has to prove now that there is no other minimal length solution. Since all even degrees of $\mathbf{W}(z)$ are uniquely determined by (24), all the other minimal-length solutions will be of the form $\mathbf{W}(z) = \mathbf{W}_{p-1}(z) + z^{-1}\mathbf{R}(z^2)$ and should factorize as $\mathbf{W}(z) = \mathbf{V}(z)\mathbf{V}^\top(z^{-1})$. We have

$$\begin{aligned}z^{-1}\mathbf{R}(z^2) &= \mathbf{V}(z)\mathbf{V}^\top(z^{-1})\\ &\quad - (1+z^{-1})^p(1+z)^p\begin{bmatrix}1\\ z^{-1}\end{bmatrix}[1\quad z].\end{aligned} \tag{25}$$

Since $\mathbf{u}^\top(z)\mathbf{u}(-z^{-1}) = \mathbf{u}^\top(-z^{-1})\mathbf{u}(z) = 0$, multiplying (25) by $\mathbf{u}^\top(-z^{-1})$ on the left and by $\mathbf{u}(-z)$ on the right, we have

$$\begin{aligned}&z^{-1}\mathbf{u}^\top(-z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z)\\ &\quad= \mathbf{u}^\top(-z^{-1})\mathbf{V}(z)\mathbf{V}^\top(z^{-1})\mathbf{u}(-z).\end{aligned}$$

For $|z|=1$, we obtain

$$z^{-1}\mathbf{u}^\top(-z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = |\mathbf{V}^\top(z^{-1})\mathbf{u}(-z)|^2.$$

Changing $z \to -z$, we also get

$$-z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z) = |\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)|^2. \tag{26}$$

Now again, multiplying (25) by $\mathbf{u}^\top(z^{-1})$ on the left and by $\mathbf{u}(z)$ on the right, we get

$$\begin{aligned}&z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z) = \mathbf{u}^\top(z^{-1})\mathbf{V}(z)\mathbf{V}^\top(z^{-1})\mathbf{u}(z)\\ &\quad - (1+z^{-1})^p(1+z)^p[1\quad z]\begin{bmatrix}1\\ z^{-1}\end{bmatrix}[1\quad z]\begin{bmatrix}1\\ z^{-1}\end{bmatrix}\end{aligned}$$

i.e., for $|z|=1$

$$\begin{aligned}&z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(z)\\ &\quad= |\mathbf{V}^\top(z^{-1})\mathbf{u}(z)|^2 - 4(1+z^{-1})^p(1+z)^p.\end{aligned} \tag{27}$$

Therefore, adding equations (26) and (27), one gets

$$|\mathbf{V}^\top(z^{-1})\mathbf{u}(z)|^2 + |\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)|^2 = 4(1+z^{-1})^p(1+z)^p$$

then

$$\begin{aligned}|\mathbf{V}^\top(z^{-1})\mathbf{u}(z)| &= O((1+z^{-1})^p\\ |\mathbf{V}^\top(-z^{-1})\mathbf{u}(z)| &= O((1+z^{-1})^p)\end{aligned}$$

and then

$$|\mathbf{V}^\top(z^{-1})\mathbf{u}(-z)| = O((1-z^{-1})^p).$$

Besides, multiplying (25) by $\mathbf{u}^\top(z^{-1})$ on the left and by $\mathbf{u}(-z)$ on the right, we get

$$z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = \mathbf{u}^\top(z^{-1})\mathbf{V}(z)\mathbf{V}^\top(z^{-1})\mathbf{u}(-z)$$

and then

$$z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = O((1-z^2)^p)$$

i.e., there exits a unique Laurent polynomial $q(z)$ such that

$$k(z) := z^{-1}\mathbf{u}^\top(z^{-1})\mathbf{R}(z^2)\mathbf{u}(-z) = (1-z^2)^p q(z). \quad (28)$$

Furthermore, from (25), $z\mathbf{R}^\top(z^{-2}) = z^{-1}\mathbf{R}(z^2)$, then $k(-z^{-1}) = -k(z)$, and therefore, $q(-z^{-1}) = -q(z)$. Hence, $q(z)$ is at least of length 3. Thus, $z^{-1}\mathbf{R}(z^2)$ is at least of length $2p$, and because it is of odd length by structure, it is at least of length $2p+1$. Hence, we have that $\mathbf{W}_{p-1}(z)$ is the unique minimal length solution.

Furthermore, since $\mathbf{W}_{p-1}(z) = \mathbf{V}_{p-1}(z)\mathbf{V}_{p-1}^\top(z^{-1})$ for $\mathbf{V}_{p-1}(z) = 2^{-p}(1-z^{-2})^p\mathbf{M}_{p-1}(z)\mathbf{\Delta}^{-p}(z)$, then

$$\begin{aligned}\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) \\ = 2^{-2p}(1-z^{-2})^{-p}(1-z^2)^{-p}\mathbf{\Delta}^p(z^2)\mathbf{W}_{p-1}(z)\mathbf{\Delta}^{p\top}(z^{-2}).\end{aligned}$$

Now, since $\mathbf{\Delta}(z^2)\mathbf{u}(z) = (1-z^{-1})\mathbf{u}(z)$, we have

$$\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) = \tfrac{1}{2}\begin{bmatrix} 1 & z \\ z^{-1} & 1\end{bmatrix}.$$

On the other side, since $\det \mathbf{W}_{p-1}(z) = 0$, one can write

$$\mathbf{M}(z) = \begin{bmatrix} 1 \\ \lambda(z)\end{bmatrix}[m_{00}(z) \quad m_{01}(z)]$$

so that

$$\begin{aligned}\mathbf{M}(z)\mathbf{M}^\top(z^{-1}) = &(m_{00}(z)m_{00}(z^{-1}) \\ &+ m_{01}(z)m_{01}(z^{-1}))\begin{bmatrix} 1 & \lambda(z^{-1}) \\ \lambda(z) & \lambda(z)\lambda(z^{-1})\end{bmatrix}.\end{aligned}$$

Hence, $\lambda(z) = z^{-1}$, and therefore, $m_1(z) = z^{-2}m_0(z)$. ■

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