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**INVOLUTIONS ON BANACH SPACES
AND REFLEXIVITY**

S. J. DILWORTH

1. Notation and results. Let E and F be (real or complex) Banach spaces. E is said to be finitely representable in F if, given $\epsilon > 0$ and a finite dimensional subspace E_0 of E , there exists a subspace F_0 of F such that $d(E_0, F_0) \leq 1 + \epsilon$, where

$$d(E_0, F_0) = \inf \{ \|T\| \|T^{-1}\| : T \text{ is an isomorphism from } E_0 \text{ onto } F_0 \}$$

denotes the Banach-Mazur distance coefficient. E is said to be super-reflexive if every Banach space which is finitely representable in E is reflexive. Super-reflexivity has been characterized in terms of the notion of J -convexity: suppose that $n \geq 1$ and that $\epsilon > 0$; E is said to be $J(n, \epsilon)$ -convex if, for all x_1, \dots, x_n in the unit ball of E , we have

$$\inf_{1 \leq k \leq n-1} \|x_1 + \dots + x_k - x_{k+1} - \dots - x_n\| \leq n - \epsilon.$$

The “if” part of the following theorem was proved in [12] and [5], and the “only if” part was proved in [10].

THEOREM A. *E is super-reflexive if and only if E is $J(n, \epsilon)$ -convex for some $n \geq 1$ and $\epsilon > 0$.*

The main purpose of this article is to extend Theorem A to a certain class of operators. To this end we introduce some new definitions: an operator T on E will be said to be $J(n, \epsilon)$ -convexifying ($n \geq 1$ and $\epsilon > 0$) if, for all x_1, \dots, x_n in the unit ball of E , we have

$$\inf_{0 \leq k \leq n} \|x_1 + \dots + x_k + T(x_{k+1} + \dots + x_n)\| \leq n - \epsilon.$$

When no importance is placed on ϵ or n we shall say that T is $J(n)$ -convexifying or simply J -convexifying. T will be said to be an involution (of order $n \geq 1$) if $T^n = I$, where I denotes the identity operator on E . The following main result is proved in Section 4 below.

THEOREM 1.1. *Suppose that E admits a J -convexifying involution. Then either c_0 is finitely representable in E or E is super-reflexive.*

Combining Theorem 1.1 with the “only if” part of Theorem A gives rise to the following characterization of super-reflexive Banach spaces.

THEOREM 1.2. *E is super-reflexive if and only if c_0 is not finitely representable in E and E admits a J -convexifying involution.*

Theorem 1.1 gives the following geometrical characterization of super-reflexive complex Banach spaces.

THEOREM 1.3. *Suppose that E is a complex Banach space, that $|\lambda| = 1$ and that $\lambda \neq 1$. Then E is super-reflexive if and only if there exist $n \geq 2$ and $\epsilon > 0$ such that for all x_1, \dots, x_n in the unit ball of E , we have*

$$\inf_{1 \leq k \leq n} \|x_1 + \dots + x_k + \lambda x_{k+1} + \lambda x_n\| \leq n - \epsilon.$$

PROOF: Necessity is proved in Corollary 2.3 below. Sufficiency follows from Theorem 1.1 when λ is a root of unity (and so multiplication by λ is an involution) by observing that $c_0(C)$ does not satisfy the hypothesis. The case for general $\lambda \neq 1$ is simply a consequence of the density of the roots of unity in the unit circle.

It is not known to me whether the possibility of c_0 being finitely representable in E in the conclusion of Theorem 1.1 may be eliminated, but when E is a complex Banach space this can be done.

THEOREM 1.4. *Suppose that E is a complex Banach space. Then E is super-reflexive if and only if E admits a J -convexifying involution.*

PROOF: We need only prove sufficiency. It follows from the theory of algebraic operators (e.g., [11]) that if T is an involution on E , then E may be written as a direct sum of closed subspaces E_i on which T acts as multiplication by a root of unity. If $c_0(C)$ is finitely representable in E , then $c_0(C)$ is finitely representable in some E_i , but this means that T is not J -convexifying.

We conclude by stating a special case of Theorem 1.3 which may be regarded as a complex version of a theorem of R. C. James on uniformly non-square Banach spaces ([8]).

THEOREM 1.5. *Suppose that E is a complex Banach space, that $\epsilon > 0$, and that for all x, y in the unit ball of E , we have*

$$\min\{\|x + y\|, \|x + iy\|\} \leq 2 - \epsilon.$$

Then E is reflexive.

2. J -convexifying operators. In this section we shall make use of the notion of the numerical range of an operator, which we now define. Suppose that E is a Banach space (either real or complex); the collection Π is defined by

$$\Pi = \{(x, f) : \|f\| = \|x\| = f(x) = 1\} \subset E \times E^*.$$

The numerical range of an operator T on E , denoted $W(T)$, is defined by

$$W(T) = \{f(Tx) : (x, f) \in \Pi\}.$$

PROPOSITION 2.1. *Suppose that T is a J -convexifying operator on E .*

- (a) $\overline{W(T)} \subset \{z : \operatorname{Re}(z) < 1\}$.
- (b) If $\|T\| = 1$ then $\|I + T\| < 2$.

PROOF: (a) Let $\epsilon > 0$ and $n \geq 1$ be given. If $\overline{W(T)} \not\subset \{z : \operatorname{Re}(z) < 1\}$ then there exists $(x, f) \in \Pi$ such that $\operatorname{Re}(f(Tx)) \geq 1 - \epsilon/2n$. It follows that for each $0 \leq k \leq n$, we have

$$\|kx + (n - k)Tx\| \geq |f(kx + (n - k)Tx)| \geq n - \frac{\epsilon}{2}.$$

Hence T is not J -convexifying, and the result follows.

(b) By a theorem of Lumer (e.g., [2, page 82]), we have

$$\sup\{\operatorname{Re}(z) : z \in W(T)\} = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha} (\|I + \alpha T\| - 1);$$

but by (a) there exists $t < 1$ such that $\sup\{\operatorname{Re}(z) : z \in W(T)\} < t$, and so there exists $0 < \alpha < 1$ such that $\|I + \alpha T\| < 1 + \alpha t$. Since $\|T\| = 1$ it follows that $\|I + T\| < 2$.

The next result concerns J -convexifying operators on super-reflexive spaces. It generalizes one of the implications in Theorem A and serves as a partial converse to Proposition 2.1(b).

PROPOSITION 2.2. *Suppose that E is super-reflexive and that T is a norm one operator on E with $\|I + T\| < 2$. Then T is J -convexifying.*

PROOF: Select $\epsilon > 0$ and $\delta > 0$ such that $\|I + T\| + \epsilon + \delta < 2$. If T is not J -convexifying, then for each $n \geq 1$ there exist x_1, \dots, x_n in the unit ball of E such that

$$\inf_{0 \leq k \leq n} \|x_1 + \dots + x_k + T(x_{k+1} + \dots + x_n)\| > n - \epsilon.$$

Now suppose that $1 \leq k \leq n$, that $\alpha_i \geq 0$ and that $\sum_{i=1}^k \alpha_i = \sum_{i=k+1}^n \alpha_i = 1$. Then

$$\begin{aligned} & \|\alpha_1 x_1 + \dots + \alpha_k x_k + T(\alpha_{k+1} x_{k+1} + \dots + \alpha_n x_n)\| \\ & \geq \|x_1 + \dots + x_k + T(x_{k+1} + \dots + x_n)\| - \|(1 - \alpha_1)x_1 \\ & \quad + \dots + (1 - \alpha_k)x_k + T((1 - \alpha_{k+1})x_{k+1} + \dots + (1 - \alpha_n)x_n)\| \\ & \geq n - \epsilon - \sum_{i=1}^k (1 - \alpha_i) - \sum_{i=k+1}^n (1 - \alpha_i) \|T\| \\ & \geq n - \epsilon - (k - 1) - (n - k - 1) \\ & = 2 - \epsilon. \end{aligned}$$

Using the fact that $\|I + T\| + \epsilon + \delta < 2$, we obtain

$$\|\alpha_1 x_1 + \dots + \alpha_k x_k - (\alpha_{k+1} x_{k+1} + \dots + \alpha_n x_n)\| \geq 2 - \epsilon - \|I + T\| \geq \delta.$$

So for each $1 \leq k \leq n$, we have

$$d(\text{conv}(x_1, \dots, x_k), \text{conv}(x_{k+1}, \dots, x_n)) \geq \delta.$$

It follows from a characterization of super-reflexivity in [9] that E is not super-reflexive. This contradiction proves that T is J -convexifying.

The following immediate consequence of Proposition 2.2 completes the proof of Theorem 1.3.

COROLLARY 2.3. *Suppose that E is a super-reflexive complex Banach space, that $|\lambda| = 1$, and that $\lambda \neq 1$. Then there exist $n \geq 2$ and $\epsilon > 0$ such that for all x_1, \dots, x_n in the unit ball of E , we have*

$$\inf_{1 \leq k \leq n} \|x_1 + \dots + x_k + \lambda x_{k+1} + \dots + \lambda x_n\| \leq n - \epsilon.$$

When E is uniformly convex the converse of Proposition 2.1(a) is also true.

THEOREM 2.4. *Suppose that T is an operator on a uniformly convex space E . Then T is J -convexifying if and only if $\overline{W(T)} \subset \{z : \operatorname{Re}(z) < 1\}$.*

PROOF: Necessity is proved in Proposition 2.1 To prove the converse we shall suppose that $\overline{W(T)} \subset \{z : \operatorname{Re}(z) < k\}$, where $k < 1$. Select $\eta > 0$ so that $k + 2\eta\|T\| = m < 1$ and let $\epsilon = \eta^2/8$. Since E is uniformly convex there exists $\delta \in (0, \frac{1}{2}(1 - m))$ such that if $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x + y\| \geq 2 - \delta$, then $\|x - y\| \leq \epsilon$. Now select $n \geq 1$ such that $(\|T\| + \delta - 1)/(n - 1) < \epsilon$. Suppose that x_1, \dots, x_n lie in the unit ball of E and that $\|x_1 + \dots + x_n\| \geq n - \delta$. Then $\|x_i + x_j\| \geq 2 - \delta$ ($1 \leq i < j \leq n$), and so $\|x_i - x_j\| \leq \epsilon$. To obtain a contradiction we shall suppose that $\|x_1 + \dots + x_{n-1} + T(x_n)\| \geq n - \delta$. Select $f \in E^*$ such that $\|f\| = 1$ and $f(x_1 + \dots + x_{n-1} + T(x_n)) = \|x_1 + \dots + x_{n-1} + T(x_n)\|$. It follows that

$$\max_{1 \leq j \leq n} \operatorname{Re}(f(x_j)) \geq \frac{n - \delta - \|T\|}{n - 1} \geq 1 - \epsilon,$$

and so $\operatorname{Re}(f(x_n)) \geq 1 - 2\epsilon \geq 1 - \eta^2/4$. By the Bishop-Phelps-Bollobas Theorem (e.g., [2]) there exists $(x, g) \in \Pi$ such that $\|x - x_n\| < \eta$ and $\|g - f\| < \eta$. So

$$\begin{aligned} \operatorname{Re}(f(Tx_n)) &\leq \operatorname{Re}(g(Tx_n)) + \eta\|T\| \\ &\leq \operatorname{Re}(g(Tx)) + 2\eta\|T\| \\ &\leq k + 2\eta\|T\| \\ &< 1 - 2\delta. \end{aligned}$$

Hence $\|x_1 + \dots + x_{n-1} + Tx_n\| = f(x_1 + \dots + x_{n-1} + Tx_n) \leq n - 2\delta$, which is the desired contradiction. It follows that T is $J(n, \delta)$ -convexifying.

COROLLARY 2.5. *The following are equivalent:*

- (i) E is super-reflexive;
- (ii) if T is an operator on E such that $\overline{W(T)} \subset \{z : \operatorname{Re}(z) < 1\}$ then E can be renormed so that T is a J -convexifying operator with respect to the new norm.

PROOF: (ii) implies (i) follows at once from Theorem A by considering $T = -I$. To prove that (i) implies (ii) we recall that a super-reflexive Banach space admits an equivalent uniformly convex norm ([7]). Moreover,

it is well known (see e.g., [1, page 211]) that if $(E, \|\cdot\|)$ admits an equivalent uniformly convex norm then, given $\epsilon > 0$, there exists a uniformly convex norm $\|\!\|\!\cdot\!\|\!$ on E such that $(1 - \epsilon)\|x\| \leq \|\!\|\!\cdot\!\|\! \leq (1 + \epsilon)\|x\|$ for all $x \in E$. A straightforward perturbation argument involving the Bishop-Phelps-Bollobas theorem proves that provided ϵ is sufficiently small the numerical range of T with respect to $\|\!\|\!\cdot\!\|\!$ still satisfies $\overline{W(T)} \subset \{z : \operatorname{Re}(z) < 1\}$. It now follows from Theorem 2.4 that T is J -convexifying with respect to $\|\!\|\!\cdot\!\|\!$.

We conclude this section by recalling the notion of an ultrapower of a Banach space, which will be needed in Section 4. Let F denote the collection of all bounded sequences $x = (x_n)_{n=1}^\infty$ in E , and let \mathcal{U} be a non-trivial ultrafilter on N . A semi-norm on F is defined by $\|x\| = \lim_{\mathcal{U}} \|x_n\|$. Quotienting F by the kernel of this semi-norm and taking the completion gives rise to the ultrapower E^N/\mathcal{U} . An operator T on E induces an operator \tilde{T} on E^N/\mathcal{U} in the obvious way. The following proposition, whose straightforward proof is omitted, will be needed in Section 4.

LEMMA. *If T is a J -convexifying operator on E , then \tilde{T} is J -convexifying on E^N/\mathcal{U} .*

We shall also need to use the fact that E is super-reflexive if and only if every ultrapower E^N/\mathcal{U} is reflexive.

3. Generalization of the Brunel-Sucheston technique. To prove Theorem 1.1 we need to develop the machinery of the Brunel-Sucheston procedure ([4], [5]) in the more general setting of an algebra of operators acting on a normed space. Once the definitions have been decided upon much of the theory carries across from [4], [5] with only minor modifications; when this is so we shall merely state the corresponding result without proof.

Suppose that A is a real or complex algebra with identity and that E is a normed space. We shall say that E is an A -module if A acts as an algebra of bounded operators on E . Let $N = \{\alpha \in A : \alpha x = 0 \text{ for all } x \in E\}$; then A/N may and shall be regarded as a subalgebra of the algebra of all operators on E . Let S denote the space of all sequences $a = (a_i)_{i=1}^\infty$ of elements of A with only finitely many non-zero terms.

THEOREM 3.1. *Let $(y_n)_{n=1}^\infty$ be a bounded sequence in E and suppose that A/N is separable in the operator norm topology. There exists a semi-norm $\|\cdot\|$ on S and a subsequence $(x_n)_{n=1}^\infty$ such that, for all $a \in S$ and $\epsilon > 0$,*

there exists a positive integer ν such that

$$\left| \left\| \sum_{i=1}^{\infty} a_i(x_{n_i}) \right\| - \|a\| \right| \leq \epsilon$$

for all integers $\nu \leq n_1 < n_2 < \dots$.

We shall assume throughout the remainder of this section that $(x_n)_1^\infty$, $(y_n)_1^\infty$ and the seminorm $\|\cdot\|$ are fixed. If \tilde{K} is the kernel of this semi-norm then S/\tilde{K} is itself a normed A -module with the action defined coordinate-wise.

PROPOSITION 3.2. *S/\tilde{K} is finitely representable in E .*

We now introduce a type of finite representability which appropriately reflects the A -module structure. Suppose that E and F are normed A -modules. Then E will be said to be A -finitely representable in F if, for all positive integers n and N , for all z_1, \dots, z_n in E , and for all n -tuples $(\alpha_1^k, \dots, \alpha_n^k)$ ($1 \leq k \leq N$) of elements of A , there exist w_1, \dots, w_n in F such that

$$\left| \left\| \sum_{i=1}^n \alpha_i^k z_i \right\| - \left\| \sum_{i=1}^n \alpha_i^k w_i \right\| \right| < \epsilon \quad (1 \leq k \leq N).$$

A standard compactness argument shows that the above definition coincides with the usual notion of finite representability when E and F are just normed spaces.

PROPOSITION 3.3. *S/\tilde{K} is A -finitely representable in E .*

PROOF: Suppose that $\hat{z}_1, \dots, \hat{z}_n$ are any vectors in S/\tilde{K} and that $(\alpha_1^k, \dots, \alpha_n^k)$ ($1 \leq k \leq N$) are n -tuples of elements of A . Let z_1, \dots, z_n be representatives from S of $\hat{z}_1, \dots, \hat{z}_n$. For $m \geq 1$, let $R_m : S \rightarrow E$ be the A -module homomorphism uniquely defined by $R_m(e_k) = x_{m+k}$ (here $(e_k)_{k=1}^\infty$ is the canonical basis of S as a free A -module). Given $\epsilon > 0$, there exists $m \geq 1$ such that

$$\left| \left\| \sum_{i=1}^n \alpha_i^k z_i \right\| - \left\| R_m \left(\sum_{i=1}^n \alpha_i^k z_i \right) \right\| \right| < \epsilon$$

for each $1 \leq k \leq N$. Setting $w_i = R_m(z_i)$ and using the fact that R_m is an A -module homomorphism, we obtain

$$\left\| \left\| \sum_{i=1}^n \alpha_i^k \hat{z}_i \right\| - \left\| \sum_{i=1}^n \alpha_i^k w_i \right\| \right\| < \epsilon$$

for each $1 \leq k \leq N$. This completes the proof.

$(S, \|\cdot\|)$ has the property that, for all $k \geq 1$, for all natural numbers $n_1 < n_2 < \dots < n_k$, and for all $a = \sum_{i=1}^k a_i e_i$ in S , we have $\|\sum_{i=1}^k a_i e_i\| = \|\sum_{i=1}^k a_i e_{n_i}\|$. In accordance with [4] such a semi-norm on S will be called “invariant under spreading” (or I.S.). We turn now to define an analogue of the “equal signs additive” norm of [5], [6]. For each $n \geq 1$, the averaging operator $A_n : S \rightarrow S$ is defined by $A_n(e_k) = \frac{1}{n}(e_k + e_{k+1} + \dots + e_{k+n-1})$ with extension to S by A -linearity. Given $a = a_1 e_1 + \dots + a_r e_r$ in S , we consider the vector $a_1 A_{n_1}(e_{s_1}) + \dots + a_r A_{n_r}(e_{s_r})$, where $s_1 > 0$, $s_2 \geq s_1 + n_1, \dots, s_r \geq s_{r-1} + n_{r-1}$. The I.S. property of $\|\cdot\|$ guarantees that the semi-norm of this vector does not depend on the choice of s_1, \dots, s_r ; it shall be denoted by $F(a; n_1, \dots, n_r)$.

PROPOSITION 3.4. *For each $a = a_1 e_1 + \dots + a_r e_r$ in S , the limit of $F(a; n_1, \dots, n_r)$ as $\inf\{n_i : 1 \leq i \leq r\} \rightarrow \infty$ exists. This limit, denoted $\|a\|$, is a semi-norm on S .*

If K denotes the kernel of $\| \cdot \|$, then S/K is a normed A -module; exactly as in Proposition 3.2 we have the following.

PROPOSITION 3.5. *$(S/K, \| \cdot \|)$ is both finitely representable and A -finitely representable in E .*

Let $|\cdot|$ be a semi-norm on S . Then $|\cdot|$ will be said to be “equal terms additive” (E.T.A.) if, for each $a = a_1 e_1 + \dots + a_r e_r$ in S with $a_i = a_{i+1}$ for some $1 \leq i \leq r - 1$, we have

$$|a| = |a_1 e_1 + \dots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \dots + a_r e_r|.$$

It is easily seen that an E.T.A. semi-norm is automatically I.S.

PROPOSITION 3.6: $\| \cdot \|$ is an E.T.A. semi-norm on S .

PROOF: Suppose that $a = a_1 e_1 + \dots + a_r e_r$ with $a_i = a_{i+1}$. Let $b = a_1 e_1 + \dots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \dots + a_r e_r$. The I.S. property implies that $F(a; n_1, \dots, n_{i-1}N, N, n_{i+2}, \dots, n_r) = F(b; n_1, \dots, n_{i-1}, 2N, n_r)$.

$n_{i+2}, \dots, n_r)$ for all $n_1, \dots, n_{i-1}, n_{i+2}, \dots, n_r, N$ and n . The result follows at once.

4. *J*-convexifying involutions. This section is devoted to a proof of Theorem 1.1. We shall use the ideas of the previous section and follow the strategy of the proof of Theorem A given in [5]. Anything from [5] which transfers with only minor alteration will be stated without proof. Using the notation of the previous section we shall show that the sequence $(x_n)_{n=1}^\infty$ contains a subsequence which is convergent in Cesaro mean when the hypotheses of Theorem 1.1 are met. This will show that E has the Banach-Saks property and, in particular, that E is reflexive.

PROPOSITION 4.1 ([5]): If $\|e_1 - e_2\| = 0$ then $(x_n)_{n=1}^\infty$ contains a subsequence which is convergent in Cesaro mean.

LEMMA 4.2. Suppose that T is a *J*-convexifying involution on a Banach space E . Then there exists $k \geq 1$ such that $I + T + T^2 + \dots + T^k = 0$.

PROOF: Suppose that T is an involution of order $k + 1$. Then $\|T^n\|^{1/n} \rightarrow 1$ as $n \rightarrow \infty$, and so $T - \lambda I$ is invertible for all $\lambda > 1$. It follows that either $T - I$ is invertible or that $T - I$ fails to be an isomorphism onto its range: in the complex case this is just the familiar fact that every point in the boundary of the spectrum of T is an approximate eigenvalue. If $T - I$ is not an isomorphism onto its range then, given $\epsilon > 0$, there exists a unit vector x in E with $\|Tx - x\| < \epsilon$. It follows that, for each $n \geq 1$, we have

$$\inf_{0 \leq k \leq n} \|kx + T((n - k)x)\| \geq n - n\epsilon.$$

Since ϵ is arbitrary, this contradicts the fact that T is *J*-convexifying. So $T - I$ is invertible and hence $I + T + T^2 + \dots + T^k = 0$.

Now suppose that the element α of A satisfies $I + \alpha + \dots + \alpha^{k-1} = 0$. We shall say that α is cyclic of order k . A sequence of vectors $(f_n)_{n=1}^\infty$ in S is defined by

$$f_n = e_{(n-1)k+1} + \alpha e_{(n-1)k+2} + \dots + \alpha^{k-1} e_{nk};$$

the real vector subspace spanned by $(f_n)_{n=1}^\infty$ will be denoted F .

PROPOSITION 4.3. *If $\|e_1 - e_2\| \neq 0$ then $(F, \|\cdot\|)$ is a normed space.*

PROPOSITION 4.4. *$(f_n)_{n=1}^\infty$ is an orthogonal sequence in F .*

PROOF: We have to show that, for all $m \geq 1$, for all real $\lambda_1, \dots, \lambda_m$, and for each $1 \leq r \leq m$, we have

$$\|\lambda_1 f_1 + \dots + \lambda_m f_m\| \geq \|\lambda_1 f_1 + \dots + \lambda_{r-1} f_{r-1} + \lambda_{r+1} f_{r+1} + \dots + \lambda_m f_m\|.$$

The I.S. property is used to write the expression on the left hand side as each of the following n expressions:

$$\|y + \lambda_r(e_{(r-1)k+1} + \alpha e_{(r-1)k+2} + \dots + \alpha^{k-1} e_{rk}) + z\|;$$

$$\|y + \lambda_r(e_{(r-1)k+2} + \alpha e_{(r-1)k+3} + \dots + \alpha^{k-1} e_{rk+1}) + z\|;$$

down to

$$\|y + \lambda_r(e_{(r-1)k+n} + \alpha e_{(r-1)k+n+1} + \dots + \alpha^{k-1} e_{rk+n-1}) + z\|,$$

where $y = \sum_{i=1}^{r-1} \lambda_i f_i$ and $z = U_n(\sum_{i=r+1}^m \lambda_i f_i)$ (here $U_n : S \rightarrow S$ is the A -module homomorphism defined by $U_n(e_k) = e_{n+k}$ for all $k \geq 1$). Taking the average of these, and using the triangle inequality and the fact that $I + \alpha \dots + \alpha^{k-1} = 0$, we obtain

$$\|\lambda_1 f_1 + \dots + \lambda_m f_m\| \geq \|y + z\| - \frac{|\lambda_r|}{n} \|a + b\|,$$

where

$$a = e_{(r-1)k+1} + (1 + \alpha)e_{(r-1)k+2} + \dots + (1 + \alpha + \dots + \alpha^{k-2})e_{(r-1)k+k-1}$$

and

$$b = \alpha^{k-1} e_{rk+n-1} + (\alpha^{k-1} + \alpha^{k-2})e_{rk+n-2} + \dots + (\alpha^{k-1} + \dots + \alpha)e_{rk+n-k+1}.$$

Using the I.S. property and taking the limit as n tends to infinity gives the required result.

COROLLARY 4.5. (a) $(f_n)_{n=1}^\infty$ is an unconditional basic sequence in F .

(b) Either c_0 is finitely representable in F or $\|f_1 + \dots + f_n\|$ increases to infinity with n .

PROPOSITION 4.6. Suppose that E is a normed A -module and that α is a J -convexifying cyclic element of A . Then either c_0 is finitely representable in E or E is reflexive.

PROOF: If E is not reflexive then there exists a bounded sequence $(y_n)_{n=1}^\infty$ in E which has no Cesaro mean convergent subsequence. It follows from Propositions 4.1 and 4.3 that the space F constructed above is a normed space. If c_0 is not finitely representable in E , then by Proposition 3.5 c_0 is not finitely representable in S/K , of which F is a subspace; so by Corollary 4.5(b), $\|f_1 + \dots + f_n\|$ increases to infinity with n . Now suppose that α is cyclic of order k and is $J(r, \epsilon)$ -convexifying. For each $1 \leq j \leq r$ and $n \geq 1$, we define

$$v_n^j = (e_{j+r} + \alpha e_{j+2r} + \dots + \alpha^{k-1} e_{j+kr}) \\ + \dots + (e_{j+(n-1)kr+r} + \dots + \alpha^{k-1} e_{j+nkr}).$$

Let $d_n^s = v_n^1 + \dots + v_n^s + \alpha(v_n^{s+1} + \dots + v_n^r)$ ($0 \leq s \leq r$); we write $d_n^s = S_1 + S_2 + S_3$ by grouping the terms as follows:

$$S_1 = -e_{s+1} - \dots - e_r; \\ S_2 = \{(e_{s+1} + \dots + e_{s+r}) + \alpha(e_{r+s+1} + \dots + e_{2r+s}) \\ + \alpha^2(e_{2r+s+1} \\ + \dots + e_{3r+s}) + \dots + \alpha^{k-1}(e_{s+(k-1)r+1} + \dots + e_{s+kr})\} \\ + \dots + \{(e_{s+1+(n-1)kr} + \dots + e_{s+((n-1)k+1)r}) \\ + \dots + \alpha^{k-1}(e_{s+1+(nk-1)r} + \dots + e_{nkr+s})\}; \\ S_3 = e_{nkr+s+1} + \dots + e_{(nk+1)r}.$$

The I.S. property implies that

$$\|v_n^j\| = \|f_1 + \dots + f_n\| \quad (1 \geq j \geq r),$$

and so $\|S_1\|/\|v_n^j\|$ and $\|S_3\|/\|v_n^j\|$ both tend to zero as n tends to infinity. Moreover, the E.T.A. property implies that $\|S_2\| = r\|v_n^j\|$. Let $z_n^j = v_n^j/\|v_n^j\|$, so that $\|z_n^j\| = 1$. Then

$$\inf_{0 \leq j \leq r} \|z_n^1 + \dots + z_n^j + \alpha(z_n^{j+1} + \dots + z_n^r)\| \geq r - \epsilon_n,$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Hence α is not $J(r)$ -convexifying for $(S/K, ||| |||)$. But by Proposition 3.5 S/K is A -finitely representable in E , and so α is not $J(r)$ -convexifying for E . This contradiction completes the proof of the proposition.

PROOF OF THEOREM 1.1: Let T be a J -convexifying involution on a Banach space E and suppose that c_0 is not finitely representable in E . By Lemma 2.6 the induced operator \tilde{T} on the ultrapower E^N/\mathcal{U} is J -convexifying; moreover, \tilde{T} is clearly also an involution. Let A be the subalgebra generated by \tilde{T} of the algebra of all bounded operators on E^N/\mathcal{U} . Then E^N/\mathcal{U} is a normed A -module, and by Lemma 4.2 \tilde{T} is a J -convexifying cyclic element of A . It follows from Proposition 4.6 that E^N/\mathcal{U} is reflexive, and so E is super-reflexive.

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