Intersecting families of permutations and partial permutations

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The first intersection theorem was discovered by Erdős, Ko and Rado. According to reference [4], they had already proved the following theorem by the year 1938, although it was only published in 1961 in their famous paper [5]. Since then it has turned out to be one of the most celebrated results in combinatorial extremal set theory.

Theorem 1 (EKR, [5]) *If $\mathcal{A}$ is an intersecting family of k-element subsets of $[n] = \{1, \ldots, n\}$ (that is every two members of $\mathcal{A}$ have at least one element in common) and $n > 2k$, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$ and this bound is attained. Moreover, equality holds if and only if $\mathcal{A}$ has the form $\{A : A \subseteq [n], |A| = k, x \in A\}$ for some fixed element x.*

1 Intersecting families in S_n

The above result motivated the study of intersecting families of permutations which was initiated by Deza and Frankl, see [2]. Let S_n be the symmetric group on $[n]$, that is the group of all permutations of $[n]$. For a positive integer t, a subset $\mathcal{A}$ of S_n is said to be *t-intersecting* if, for any g, $h \in \mathcal{A}$ with $g \neq h$, we have $|\{x : g(x) = h(x)\}| \geq t$. By an *intersecting family*, we mean an 1-intersecting set of permutations.

Denote by $\alpha(n, t)$ the maximum cardinality of an t-intersecting family of S_n. Comparatively little is known about the number $\alpha(n, t)$ even though the analogous problem for subsets of a finite set has been widely studied: see [3] for a survey. The following result is the starting point of my research.

Theorem 2 (Deza–Frankl, [2]) *Let $\mathcal{A}$ be an intersecting family of permutations of n elements. Then $|\mathcal{A}| \leq (n-1)!$.*

It is easy to see that the above bound is attained by taking $\mathcal{A}$ to be the stabiliser of a point. In the joint work with my supervisor, Peter J. Cameron, we are able to determine completely the structure of such families of maximal size, namely we prove the following:

Theorem 3 (Cameron–Ku, [1]) *The above bound is attained if and only if $\mathcal{A}$ is a coset of the stabiliser of a point, that is $\mathcal{A} = \{g \in S_n : g(x) = y\}$ for some x, $y \in [n]$.*

In connection with Theorem 2, Deza and Frankl [2] made the following conjecture:

Conjecture 1 *For $n \geq n_0(t)$, we have*

$$\alpha(n,t) = (n-t)!.$$

Note that we do require n to be sufficiently large in the above conjecture for otherwise it is possible to construct an intersecting family $\mathcal{A}$ of size greater that $(n-t)!$. For example, let $\mathcal{A} = \{(xy) : x, y \in [8], x \neq y\} \subseteq S_8$ where (xy) is the transposition interchanging x and y. It is easy to check that $\mathcal{A}$ is 4-intersecting and $|\mathcal{A}| = \binom{8}{2} = 28 > (8-4)!$.

Despite much effort in my research, this problem remains open. However, I can show that the following holds:

Theorem 4 *Let $t \geq 2$ and $\mathcal{A} \subseteq S_n$ be a family of permutations such that any* **three** *of them agree in at least t points. Then, for $n \geq n_0(t) = \max\{5t, t+40\}$, we have $|\mathcal{A}| \leq (n-t)!$ such that equality holds if and only if $\mathcal{A}$ is a coset of the stabiliser of t points.*

2 Intersecting families in $\Gamma(b,d,n) \times \cdots \times \Gamma(b,d,n)$

The study of intersecting families can be translated into that of stable sets of graphs. (Recall that a stable set of a graph is a set of pairwise non-adjacent vertices.) To see this, form the graph $\Gamma(S_n)$ as follows: the vertex-set $V(\Gamma(S_n))$ is S_n and that any two vertices are adjacent if and only if they do not agree in any points. Thus a stable set of $\Gamma(S_n)$ is just an intersecting family of permutations. In view of this, we now introduce the graph $\Gamma(b,d,n)$ as an extension of our preceding notion: the vertex-set $V(\Gamma(b,d,n))$ consists of all n-tuples $(a_1,\ldots,a_n)$ for which there is a permutation $\pi \in S_n$ such that $(a_1,\ldots,a_n) = (x_{\pi(1)},\ldots,x_{\pi(n)})$ where $(x_1,\ldots,x_n) = (1,\ldots,1,2,\ldots,2,\ldots,b,\ldots,b)$ with each symbol appearing exactly d times, and that any two vertices $(a_1,\ldots,a_n)$, $(b_1,\ldots,b_n)$ are adjacent if and only if $a_i \neq b_i$ for all i. In particular, $\Gamma(n,1,n)$ is isomorphic to $\Gamma(S_n)$.

Recently, Larose and Malvenuto [8] determined the structure of all stable sets of maximal size in $\Gamma(b,d,n)$ (for a more general result, see Theorem 5.7 in [8]): given $n \geq 2d$ and $b \geq 5$, the only stable sets of maximal size in $\Gamma(b,d,n)$ are the families $\mathcal{A}_p^q = \{(a_1,\ldots,a_n) : a_p = q\}$ for some fixed p, q. Clearly this generalizes Theorem 3. Armed with this result as the base case of an induction argument and a mixture of techniques from [1] and [8], I prove that, under mild restrictions, similar results still hold for the weak (categorical) product of any m copies of $\Gamma(b,d,n)$, denoted by $\Gamma^m = \underbrace{\Gamma(b,d,n) \times \cdots \times \Gamma(b,d,n)}_{m \ \ times}$: (Recall that for any two graphs Γ_1 and Γ_2, the weak product $\Gamma_1 \times \Gamma_2$ is defined as follows: the vertex-set $V(\Gamma_1 \times \Gamma_2) = \{(u,v) : u \in V(\Gamma_1), v \in V(\Gamma_2)\}$ and any two vertices (u_1,v_1) and (u_2,v_2) are adjacent if and only if both u_1u_2 and v_1v_2 are edges in Γ_1 and Γ_2 respectively.)

Theorem 5 *Let $b > \max\{2(d+1), 3/d+3, 6-4/d, 4\}$, $n \geq 2d$ and m be a positive integer. Then $\mathcal{A}$ is a stable set of maximal size in Γ^m if and only if $\mathcal{A} = \mathcal{A}_p^q(\Gamma^m; u) = \{(g_1,\ldots,g_m) \in V(\Gamma^m) : g_u(p) = q\}$ for some fixed u, p, q where $g_u(p)$ denotes the p-th coordinate entry of the n-tuple g_u.*

3 Intersecting families of partial permutations

The work on partial permutations is another extension of my study on intersecting families of permutations. An *r-partial permutation* of $[n]$ is a pair (A,f) where $A \subseteq [n]$, $|A| = r$ and $f : A \rightarrow$

$[n]$ is an injective map. A family $\mathcal{A}$ of r-partial permutations is *t-intersecting* if for any (A,f), $(B,g) \in \mathcal{A}$, we have $|\{x : x \in A \cap B, f(x) = g(x)\}| \geq t$. Imre Leader expressed his interest in this work during my visit to Cambridge. As a result of our collaboration [7], we prove the following:

Theorem 6 *Fix r, n with $r \leq n-1$. Let $\mathcal{A}$ be an intersecting family of r-partial permutations. Then $|\mathcal{A}| \leq \binom{n-1}{r-1}\frac{(n-1)!}{(n-r)!}$. For $8 \leq r \leq n-3$, equality holds if and only if $\mathcal{A}$ consists of all r-partial permutations (A,f) such that $x \in A$ and $f(x) = y$ for some fixed x, $y \in [n]$.*

We conjecture that the above restriction $8 \leq r \leq n-3$ is unnecessary.

Having proved Theorem 6 led me to consider intersecting families which are **not** of the form $\mathcal{A}(x,y) = \{(A,f) : x \in A, f(x) = y\}$. First, we need some definitions. The *size* of a partial permutation (A,f), denoted by $|(A,f)|$, is the cardinality of A. For two partial permutations (A,f) and (B,g), we say that (B,g) *contains* (A,f), denoted by $(A,f) \sqsubseteq (B,g)$, if $A \subseteq B$ and $g|_A = f$. Also we let $(A,f) \sqcap (B,g)$ denote the largest partial permutation (C,h) such that (C,h) is contained in both (A,f) and (B,g).

An t-intersecting family $\mathcal{A}$ of r-partial permutations of $[n]$ is *centred* if there exists t-subsets $X = \{x_1,\ldots,x_t\}$, $Y = \{y_1,\ldots,y_t\}$ of $[n]$ such that $\mathcal{A} = \{(A,f) : X \subseteq A,\ |A| = r,\ f(x_i) = y_i \text{ for all } i\}$. Here I am interested in intersecting families of maximal size which are **non-centred**, in particular, I determine the structure of such families when n is sufficiently large. The analogous problem for finite sets has been studied by Frankl [6].

Before stating the main theorem, let us construct two large non-centred interseting families $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows:

Let t, r and n be positive integers such that $t < r \leq n$. Let (D_1,d_1) and (D_2,d_2) be t- and $(r-t+1)$-partial permutations of $[n]$ respectively satisfying the following:

(i) $D_1 \cap D_2 = \emptyset$; and

(ii) $d_1(D_1) \cap d_2(D_2) = \emptyset$.

We define the family $\mathcal{F}_1$ of r-partial permutations to be the union of the following two families:

$$\begin{aligned} \mathcal{G}_1 &= \{(A,f) : (D_1,d_1) \sqsubseteq (A,f),\ |(A,f) \sqcap (D_2,d_2)| \neq 0\}; \\ \mathcal{G}_2 &= \{(A,f) : (D_2,d_2) \sqsubseteq (A,f),\ |(A,f) \sqcap (D_1,d_1)| \geq t-1\}. \end{aligned}$$

It is readily checked that $\mathcal{G}_1$ and $\mathcal{G}_2$ are disjoint and that $\mathcal{F}_1$ is t-intersecting.

We define another family $\mathcal{F}_2$ of r-partial permutations as follows:

$$\mathcal{F}_2 = \{(A,f) : |(A,f) \sqcap (D,d)| \geq t+1\},$$

where (D,d) is a fixed $(t+2)$-partial permutation. Again, it is easy to verify that $\mathcal{F}_2$ is t-intersecting.

The following can be regarded as an analogue of Frankl's theorem:

Theorem 7 *Let t, r be positive integers, $t < r$, and $\mathcal{A}$ be a non-centred t-intersecting family of r-partial permutations of $[n]$ of maximal size. Then for $n \geq n_0(r,t)$,*

(i) if $r > 2t+1$, then $\mathcal{A} = \mathcal{F}_1$;

(ii) if $r \leq 2t+1$, $r \neq 3$, $t \neq 1$, then $\mathcal{A} = \mathcal{F}_2$;

(iii) if $r = 3$, $t = 1$, then $\mathcal{A} = \mathcal{F}_1$ or $\mathcal{F}_2$.

4 My current research

Being motivated by several insights and tools which have been developed so far in my research, I am currently investigating intersecting families of partitions. Recall that a *partition* of $[n]$ is a collection of non-empty, pairwise disjoint subsets of $[n]$ (which are called *blocks*) whose union is $[n]$. The total number of partitions is the Bell number $B(n)$. A family $\mathcal{A}$ of partitions is said to be *intersecting* if any two members of $\mathcal{A}$ have at least one common block. We conjecture that an intersecting family has size at most $B(n-1)$ and that equality holds if and only if it consists of all partitions containing a fixed block of size 1.

5 Conclusion

Finite sets, permutations and set partitions are basic combinatorial objects. While the study of extremal properties of finite sets has attracted the attention of many researchers over the past four decades, the analogous problems for permutations and partitions have not been extensively studied. It seems to me that these problems are appealing to somewhat similar intuition, if not common techniques and ideas. This is definitely an exciting time in exploring and discovering further and deeper connections between finite sets, permutations and partitions, at least from the viewpoint of intersecting families.

References

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