Title	On the structure of parabolic Humphreys-Verma modules (New Trends in Combinatorial Representation Theory)
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Citation	数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2009), B11: 117-123
Issue Date	2009-04
URL	http://hdl.handle.net/2433/176774
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

RIMS Kôkyûroku Bessatsu
B11 (2009), 117–123

On the structure of parabolic Humphreys-Verma modules

By

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Through an investigation into the bounded derived category $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$ of the coherent sheaves on a projective homogeneous variety $\mathcal{P}$ we have been led to study the parabolic Humphreys-Verma modules. Although these modules are defined only in positive characteristic, our geometric application appears effective characteristic-free.

Write $\mathcal{P} = G/P$ with G a reductive algebraic group over an algebraically closed field of positive characteristic and P a parabolic subgroup of G. Let G_1 be the Frobenius kernel of G and let $\hat{\nabla}_P(\varepsilon)$ be the G_1P-module induced from 1-dimensional trivial P-module ε, a parabolic Humphreys-Verma module. We have recently found a way, though verified in only few limited cases yet, to parametrize certain components of the G_1T-socle series of $\hat{\nabla}_P(\varepsilon)$ by the set W^P of distinguished coset representatives of the Weyl group of G by the Weyl group of P such that the associated coherent sheaves $\mathcal{E}_w$, $w \in W^P$, on $\mathcal{P}$ form a Karoubian complete strongly exceptional poset for $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$. Those sheaves are defined over $\mathbb{Z}$ to verify Catanese's conjecture [Bö] transferring over to $\mathbb{C}$; in some cases our constructions offer a new evidence to the conjecture in complex algebraic geometry. In this note, however, we will focus on the structure of Humphreys-Verma modules.

In order to be precice in which category the morphisms are taken, we will write $\mathcal{C}(X, Y)$ for the set of morphisms in catergory $\mathcal{C}$ from object X to Y. For an algebraic group H we let $H\mathbf{Mod}$ denote the category of rational H-modules, and for a variety X the category of modules over the structure sheaf of X will be denoted by $\mathbf{Mod}_X$.

This is an expanded version of the author's talk at a RIMS meeting under the title of the present volume. Subsequent to the talk I have come up with a description of the structure of Humphreys-Verma modules for projective spaces, which is included in §3; in the talk I could merely exhibit the computations in the cases of GL_2 and GL_3.

Received January 26, 2008. Accepted May 23, 2008.
2000 Mathematics Subject Classification(s): 20G05
Key Words: parabolic Humphreys-Verma modules, Frobenius direct image
Supported in part by JSPS Grant in Aid for Scientific Research.
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§ 1. Humphreys-Verma modules

Let $\Bbbk$ be an algebraically closed field of positive characteristic p. We will assume p is sufficiently large. Let G be a reductive algebraic group over $\Bbbk$, P a parabolic subgroup of G, and G_1 the Frobenius kernel of G. We call the functor

$$\mathrm{ind}_P^{G_1P} = \mathbf{Sch}_{\Bbbk}(G_1P, ?)^P : P\mathbf{Mod} \to G_1P\mathbf{Mod}$$

parabolic Humphreys-Verma induction and write $\hat{\nabla}_P$ for short.

In case $P = B$ a Borel subgroup of G put $\hat{\nabla} = \hat{\nabla}_B$. Let Λ be the character group of B, T a maximal torus of B, $R \subset \Lambda$ the root system of G relative to T. We choose a positive system R^+ of R such that the roots of B are $-R^+$, and let R^s be the set of simple roots. Let $W = \mathrm{N}_G(T)/T$ the Weyl group of G, $W_p = W \ltimes p\mathbb{Z}R$, $S_p = \{s_i, s_0 \mid \alpha_i \in R^s\}$ with s_i the reflexion associated to α_i and s_0 the reflexion in the wall $\{v \in \Lambda \otimes_{\mathbb{Z}} \mathbb{R} \mid \langle v, \alpha_0^\vee \rangle = -p\}$, $\alpha_0^\vee$ the highest coroot. Thus (W_p, S_p) forms a Coxeter system. We will consider the dot action of W_p on Λ such that $x \bullet \lambda = x(\lambda + \rho) - \rho$, $x \in W_p$, $\lambda \in \Lambda$, $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. One has $\hat{\nabla}(\lambda + p\mu) \simeq \hat{\nabla}(\lambda) \otimes p\mu$ $\forall \mu \in \Lambda$, and each $\hat{\nabla}(\lambda)$ has a G_1T-simple socle $\hat{L}(\lambda)$ of highest weight λ with all other composition factors having their highest weights $< \lambda$. If $\Lambda_1 = \{\lambda \in \Lambda \mid \langle \lambda, \alpha^\vee \rangle \in [0, p[\ \forall \alpha \in R^s\}$ the set of restricted dominant weights, a simple G_1T-module of highest weight $\lambda \in \Lambda_1$ admits a structure of simple G-module $L(\lambda)$. Thus, the determination of the composition factors of all $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda_1$, will yield all irreducible characters for G by Steinberg's tensor product theorem. Moreover, let $\mathcal{A}$ be the set of alcoves on Λ with respect to the dot action of W_p. $\forall A \in \mathcal{A}$, $\forall w \in W$, $\forall \gamma \in \mathbb{Z}R$, we will write $Aw^{-1}t_{p\gamma}$ for the alcove $(w \bullet A) + p\gamma$. Let 0_A be the image of $0 \in \Lambda$ in A under the W_p-action. By the translation principle [J, II.7] the structure of $\hat{\nabla}(0_A)$ describes that of all other $\hat{\nabla}(\lambda)$, $\lambda \in \Lambda \cap A$, and the determination of irreducible characters of all $L(0_A)$, $0_A \in \Lambda_1$, will obtain all irreducible characters for G, which has now been achieved for indefinitely large p by Andersen, Jantzen and Soergel [AJS] to verify Lusztig's conjecture. We will thus write $\hat{\nabla}(A)$ for $\hat{\nabla}(0_A)$.

The Lusztig conjecture on the irreducible G-characters, in turn, determines the G_1T-socle series of each $\hat{\nabla}(A)$; let $\mathrm{soc}_i\hat{\nabla}(A)$ be the i-th G_1T-socle layer of $\hat{\nabla}(A)$ and let $\hat{L}(C)$ be the simple G_1T-module of highest weight 0_C; by the linkage principle [J, II.9.15] the composition factors of $\hat{\nabla}(A)$ are of the form $\hat{L}(C)$. Let $w_0 \in W$ be the longest element of W with respect to $\{s_i \mid \alpha_i \in R^s\}$. Then

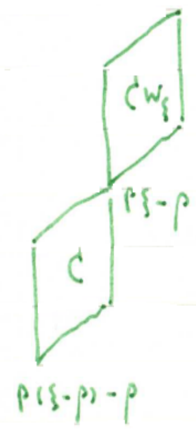
Theorem 1.1 ([AK]/[RIMS]). *Let $A \in \mathcal{A}$.*

(i) *The Loewy length, that is the length of the socle series, of each $\hat{\nabla}(A)$ is $\ell(w_0)+1$.*

(ii) *If* $0_C \in \Lambda_1 + p(\xi - \rho)$, $\xi \in \Lambda$, *then with* $w_\xi = w_0 t_{p(\xi - w_0 \xi)}$

$$Q_{A, C w_\xi} = \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A, C w_\xi) + i - \ell(w_0) - 1\}} [\mathrm{soc}_i \hat{\nabla}(A) : \hat{L}(C)],$$

where $Q_{A,C}$ *is the periodic inverse Kazhdan-Lusztig polynomial [L] associated to* $A, C \in \mathcal{A}$ *and* $d(A, C w_\xi)$ *is the distance from* A *to* $C w_\xi$.



§ 2. Parabolic Humphreys-Verma modules

We wish to obtain a formula to describe the socle series for general parabolic P. Let $\Lambda_P = \mathbf{Grp}_{\Bbbk}(P, \mathrm{GL}_1)$, L the standard Levi subgroup of P, U_L the unipotent radical of $B \cap L$, $\mathrm{Dist}(U_L)$ the algebra of distributions on U_L, $R_L \subseteq R$ the root system of L, and $W_P = \langle s_\alpha \mid \alpha \in R_L \rangle \leq W$ the Weyl group of P and also of L. $\forall \nu \in \Lambda_P$, regarded as G_1T-modules

$$\hat{\nabla}_P(\nu) = \mathrm{ind}_P^{G_1 P}(\nu) \simeq \mathrm{ind}_{P_1 T}^{G_1 T}(\nu),$$

where P_1 is the Frobenius kernel of P. We call $\hat{\nabla}_P(\nu)$ the parabolic Humphreys-Verma module of highest weight ν; if $\rho_P = \frac{1}{2} \sum_{\alpha \in R^+ \setminus R_L} \alpha$,

$$\begin{aligned} \hat{\nabla}_P(\nu) &\simeq \mathrm{ind}_{P_1}^{G_1}(\nu) \\ &\simeq \mathrm{Dist}(G_1) \otimes_{\mathrm{Dist}(P_1)} (\nu - 2(p-1)\rho_P) \quad \text{by [J, II.9.2]} . \end{aligned}$$

If U_P^+ is the subgroup of G generated by the root subgroups U_α, $\alpha \in R^+ \setminus R_L$, and if $U_{P,1}^+$ is the Frobenius kernel of U_P^+, then $\mathrm{Dist}(G_1) \simeq \mathrm{Dist}(U_{P,1}^+) \otimes_{\Bbbk} \mathrm{Dist}(P_1)$, and hence

$$\hat{\nabla}(\nu) \simeq \mathrm{Dist}(U_{P,1}^+) \otimes_{\Bbbk} (\nu - 2(p-1)\rho_P)$$

with $\mathrm{Dist}(U_{P,1}^+) \simeq \otimes_{\alpha \in R^+ \setminus R_L} \mathrm{Dist}(U_{\alpha,1})$. To relate a parabolic Humphreys-Verma module to ordinary Humphreys-Verma modules, one has at the charactrer level

Proposition 2.1 ([KY]). $\forall \nu \in \Lambda$,

$$\mathrm{ch}\hat{\nabla}_P(\nu) = e^\nu \prod_{\alpha \in R^+ \setminus R_L} \frac{1 - e^{-p\alpha}}{1 - e^{-\alpha}} = \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\mathrm{Dist}(U_L)_\gamma) \mathrm{ch}\hat{\nabla}(w \bullet \nu + p\gamma).$$

Now let $A \in \mathcal{A}$ with $0_A \in \Lambda_P$. As $\hat{\nabla}_P(A) \leq \hat{\nabla}(A)$ as G_1B-modules, a naïve speculation on the G_1T-socle series of $\hat{\nabla}_P(A)$ would be that $\forall C \in \mathcal{A}$ with $0_C \in \Lambda_1 + p(\xi - \rho)$, $\xi \in \Lambda$,

$$\text{(1)} \quad \sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(A, Cw_\xi) + i - \ell(w_0) - 1\}} [\mathrm{soc}_i \hat{\nabla}_P(A) : \hat{L}(C)] = \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\mathrm{Dist}(U_L)_\gamma)$$
$$\sum_{i \in \mathbb{N}} q^{\frac{1}{2}\{d(Aw^{-1}t_{p\gamma}, Cw_\xi) + i - \ell(w_0) - 1\}} [\mathrm{soc}_i \hat{\nabla}(Aw^{-1}t_{p\gamma}) : \hat{L}(C)]$$
$$= \sum_{\substack{w \in W_P \\ \gamma \in \mathbb{Z}R_L}} (-1)^{\ell(w)} \dim(\mathrm{Dist}(U_L)_\gamma) Q_{Aw^{-1}t_{p\gamma}, Cw_\xi}.$$

By design (1) holds under the specialization $q \rightsquigarrow 1$, and also in case $P = B$.

Proposition 2.2 ([KY]). *(1) holds for G of rank ≤ 2.*

If $W^P = \{w \in W \mid wR_L^+ \subseteq R^+\}$, $W = \bigsqcup_{w \in W^P} wW_P$. Let $w_{0,P} \in W_P$ with $w_{0,P}R_L^+ = -R_L^+$, and set $w_0^P = w_0 w_{0,P} \in W^P$. $\forall x \in W_p$, $\exists! w \in W$: $x \bullet 0 \equiv w \bullet 0 \mod p\Lambda$. $\forall w \in W$, choose $0_{P,w} \in \Lambda_1$ such that $0_{P,w} \equiv w_0 w w_{0,P} \bullet 0 \mod p\Lambda$. Then $W_p \bullet 0 + p\Lambda = \bigsqcup_{w \in W} (0_{P,w} + p\Lambda)$.

Theorem 2.3 ([KY]). *Assume* $\mathrm{rk}\, G \leq 2$. *Let $A \in \mathcal{A}$ with $0_A \in \Lambda_P$.*
(i) *The Loewy length of $\hat{\nabla}_P(A)$ is $\ell(w_0^P) + 1$.*
(ii) $\forall i \in [1, \ell(w_0^P) + 1]$, *there is a decomposition as G_1P-modules*

$$\mathrm{soc}_i \hat{\nabla}_P(A) = \coprod_{w \in W^P} L(0_{P,w}) \otimes G_1\mathbf{Mod}(L(0_{P,w}), \mathrm{soc}_i \hat{\nabla}_P(A)).$$

(iii) *In case $A = A^+$ the bottom dominant alcove,*

$$G_1\mathbf{Mod}(L(0_{P,w}), \mathrm{soc}_{\ell(w_0^P)+1-\ell(w)} \hat{\nabla}_P(A^+)) \neq 0 \quad \forall w \in W^P.$$

Remark 2.4. (i) In case $P = B$, each $L(0_{B,w})$, $w \in W$, appears as a G_1-composition factor of any $\hat{\nabla}(A)$.

To see that, $\forall x, y \in W$, $\forall \mu \in \Lambda$,

$$\begin{aligned}[\hat{\nabla}(A^+x) : L(0_{B,y}) \otimes p\mu] &= [\hat{\nabla}(x^{-1}\bullet 0) : \hat{L}(0_{B,y}+p\mu)] = [\hat{\nabla}(x^{-1}\bullet 0) : \hat{L}(0_{B,y}+px^{-1}x\bullet\mu)] \\ &= [\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,y} + px\bullet\mu)] \quad \text{by [J, II.9.16.4] with } \varepsilon \text{ denoting } 0 \in \Lambda.\end{aligned}$$

It is therefore enough to show that $\forall w \in W$, $\exists \mu \in \Lambda$: $[\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p\mu)] \neq 0$. Write $0_{B,w} = w_0 w \bullet 0 + p\eta$ for some $\eta \in \Lambda$. Then

$$[\hat{\nabla}(\varepsilon) : \hat{L}(0_{B,w} + p\mu)] = [\hat{\nabla}(\varepsilon) : \hat{L}(w_0 w \bullet 0 + p(\eta+\mu))] = [\hat{\nabla}(-p(\eta+\mu)) : \hat{L}(w_0 w \bullet 0)].$$

Thus we have only to check $\forall C \in \mathcal{A}$, $\exists \gamma \in \mathbb{Z}R$: $[\hat{\nabla}(A^+t_{p\gamma}) : \hat{L}(C)] \neq 0$. If $C \subseteq \Lambda_1 + p(\xi - \rho)$ for some $\xi \in \Lambda$, then by [Y] $\forall w \in W$, $[\hat{\nabla}(A^+wt_{p\xi}) : \hat{L}(C)] \neq 0$. Write $A^+t_{p\xi} = A^+t_{p\gamma}y$, $\gamma \in \mathbb{Z}R$, $y \in W$. Then

$$A^+y^{-1}t_{p\xi} = A^+t_{p\gamma}yt_{-p\xi}y^{-1}t_{p\xi} = A^+t_{p\gamma}t_{-py\xi}t_{p\xi} = A^+t_{p(\gamma - y\xi + \xi)}.$$

If $y = y_1 y_2$ with $y_1, y_2 \in W$, then $y\xi - \xi = y_1y_2\xi - \xi = y_1(y_2\xi - \xi) + y_1\xi - \xi$, and $\forall \alpha \in R^+$, $s_\alpha\xi - \xi = \xi - \langle \xi, \alpha^\vee\rangle\alpha - \xi = -\langle \xi, \alpha^\vee\rangle\alpha \in \mathbb{Z}R$. Thus $\gamma - y\xi + \xi \in \mathbb{Z}R$, as desired.

(ii) In application to the study of $\mathrm{D}^b(\mathrm{coh}\mathcal{P})$, $\mathcal{P} = G/P$, the P-module structure on each $G_1\mathbf{Mod}(L(0_{P,w}), \mathrm{soc}_{\ell(w_0^P)+1-\ell(w)}\hat{\nabla}_P(A^+))$, $w \in W^P$, appears to play an important role: as G_1 acts trivially on those, untwisting the Frobenius, put $\mathrm{soc}^1_{P,w} = G_1\mathbf{Mod}(L(0_{P,w}), \mathrm{soc}_{\ell(w_0^P)+1-\ell(w)}\hat{\nabla}_P(A^+))^{[-1]}$. It appears from [KY] that each $\mathrm{soc}^1_{P,w}$ admits a direct summand E_w such that, writing $\mathcal{L}_\mathcal{P}(E_w)$ for the locally free sheaf on $\mathcal{P}$ associated to E_w, $\{\mathcal{L}_\mathcal{P}(E_w) \mid w \in W^P\}$ forms a Karoubian complete strongly exceptional poset such that $\mathbf{Mod}_\mathcal{P}(\mathcal{L}_\mathcal{P}(E_x), \mathcal{L}_\mathcal{P}(E_y)) \neq 0$ iff $x \leq y$. Moreover, those E_w are defined over $\mathbb{Z}$ to yield also a Karoubian complete strongly exceptional poset in characteristic 0.

§ 3. Projective spaces

Let E be a $\Bbbk$-linear space of basis $e_1, \ldots, e_{n+1}$, $G = \mathrm{GL}(E)$, and $P = \mathrm{N}_G(\Bbbk e_{n+1})$. Thus $\mathcal{P} = G/P \simeq \mathbb{P}^n_\Bbbk$. If $F_\mathcal{P}$ (resp. $F_\Bbbk$) is the absolute Frobenius morphism on $\mathcal{P}$ (resp. $\mathrm{Spec}(\Bbbk)$) and if $q : G/P \to G/G_1P$ is a natural morphism, one has a commutative diagram of schemes

$$\begin{array}{ccccc} \mathcal{P} & \xrightarrow{F_\mathcal{P}} & \mathcal{P} & \xrightarrow{\text{structure}} & \mathrm{Spec}(\Bbbk) \\ {\scriptstyle q}\downarrow & \overset{F_{\mathcal{P}/\Bbbk}}{\searrow} & \uparrow{\scriptstyle \phi} & \square & \uparrow{\scriptstyle F_\Bbbk} \\ G/G_1P & \xrightarrow[\sim]{} & \mathcal{P}^{(1)} & \longrightarrow & \mathrm{Spec}(\Bbbk). \end{array}$$

If A^+ is the bottom dominant alcove, one has from [Haa]

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \phi_*\mathcal{L}_{G/G_1P}(\hat{\nabla}_P(A^+)).$$

On the other hand, we know from [HKR]

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \coprod_{i=0}^{n} \mathcal{O}_{\mathcal{P}}(-i) \otimes_{\Bbbk} V_i,$$

where $V_i = \coprod_{\substack{j\in[0,p[^{n+1} \\ |j|=pi}} \Bbbk x^j$ in the polynomial algebra $\Bbbk[x_1,\dots,x_{n+1}]$ with $x^j = \prod_{i=1}^{n+1} x_i^{j_i}$ and $|j| = \sum_{i=1}^{n+1} j_i$ if $j = (j_1,\dots,j_{n+1})$. Regarding $x_1,\dots,x_{n+1}$ as the dual basis of $e_1,\dots,e_{n+1}$, let G act on $\Bbbk[x_1,\dots,x_{n+1}]$ and also on $\Bbbk[x_1,\dots,x_{n+1}]/(x_1^p,\dots,x_{n+1}^p)$ contragrediently. Then one can equip V_i with a structure of G-module by identifying it with its image in $\Bbbk[x_1,\dots,x_{n+1}]/(x_1^p,\dots,x_{n+1}^p)$.

Now let B be a Borel subgroup of P consisting of lower triangular matrices and T a maximal torus of B consisting of diagonal matrices. Identify Λ with $\mathbb{Z}^{\oplus n+1}$ via the basis $\varepsilon_i : \mathrm{diag}(a_1,\dots,a_{n+1}) \mapsto a_i$, $i \in [1, n+1]$, and W with the symmetric group $\mathfrak{S}_{n+1}$ permuting the ε_i, $i \in [1, n+1]$. Then $W^P = \{(i\, i{+}1\ \dots\ n{+}1) \mid i \in [1, n{+}1]\}$, and Serre's twisted sheaf on $\mathbb{P}^n$ is given by $\mathcal{O}_{\mathcal{P}}(1) = \mathcal{L}_{\mathcal{P}}(-\varepsilon_{n+1})$. $\forall i \in [1, n{+}1]$, set $\lambda_{(i\ i+1\ \dots\ n+1)} = (i-1)\varepsilon_{n+1}$ and $0_{P,(i\ i+1\ \dots\ n+1)} = -(i-1)\varepsilon_{n+2-i} - (p-1)(\varepsilon_{n+3-i} + \dots + \varepsilon_{n+1}) \in \Lambda_1$, which we agree to be 0 in case $i = 1$. Then $V_{i-1} \simeq L(0_{P,(i\ i+1\ \dots\ n+1)})$ $\forall i \in [1, n+1]$, and hence

$$F_{\mathcal{P}*}\mathcal{O}_{\mathcal{P}} \simeq \coprod_{i=1}^{n+1} \mathcal{L}_{\mathcal{P}}(\lambda_{(i\ i+1\ \dots\ n+1)}) \otimes_{\Bbbk} L(0_{P,(i\ i+1\ \dots\ n+1)}).$$

Confirming the pattern in Theorem 2.3, it holds that

Theorem 3.1 ([K]). *Assume $p \geq n + 1$.*
(i) The Loewy length of $\hat{\nabla}_P(A^+)$ is $n + 1 = \ell(w_0^P) + 1$.
(ii) $\forall i \in [1, n + 1]$, $\mathrm{soc}_i \hat{\nabla}_P(A^+) \simeq L(0_{P,(i\ i+1\ \dots\ n+1)}) \otimes \lambda^{[1]}_{(i\ i+1\ \dots\ n+1)}$.

Remark 3.2. Regardless of characteristic $\{\mathcal{L}_{\mathcal{P}}(\lambda_w) \mid w \in W^P\}$ forms a complete strongly exceptional poset on $\mathbb{P}^n$ such that $\mathbf{Mod}_{\mathbb{P}^n}(\mathcal{L}_{\mathcal{P}}(\lambda_x), \mathcal{L}_{\mathcal{P}}(\lambda_y)) \neq 0$ iff $x \leq y$ [HKR]/[K08].

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