Revenue Efficiency Interval Based On Data Envelopment Analysis

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Abstract

In this paper, we introduce the revenue efficiency interval of a DMU. The revenue interval DEA model has been formulated to obtain an efficiency interval in which the lower bound and upper bound of this interval are respectively, the efficiency obtained from pessimistic and optimistic viewpoints.

Keywords: DEA, Interval Efficiency, Revenue Efficiency

1 Introduction

Data Envelopment Analysis (DEA) developed by Charnes, Cooper and Rhodes (1978) usually evaluate decision making units (DMUs) from the angle of the best possible relative efficiency. The technical efficiency of a DMU in DEA is measured as the maximum ratio of weighted sum of outputs to the weighted sum of inputs, and hence, this efficiency measure is calculated from optimistic viewpoint. Revenue efficiency of a DMU, measures the DMUs success in choosing an optimal set of outputs with a given set of output price and so, this measure is calculated from the optimistic viewpoint. In the current paper, we propose a DEA model to derive a revenue efficiency measure from the pessimistic viewpoint. The paper is organized as follows: the next section of the paper addresses an introduction to basic DEA model. The revenue efficiency interval is introduced in section three. An application on Iranian bank branches appears in section four. Conclusions appear in section five.

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2 Background

Suppose we have a set of units, DMU_j, (j=1,…,n). Each DMU uses m inputs x_{ij} (i=1,…, m) to produce s outputs y_{rj} (r=1,…, s). Then the relative efficiency of DMU_j can be expressed as

$$e_j = \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \tag{1}$$

Where u_r and v_i are output and input multipliers, respectively. In DEA, e_j is obtained by solving the following CCR ratio model

$$\begin{aligned} e_j^* &= Max \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\ s.t.& \\ &\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \qquad j = 1,...,n, \\ &u_r, v_i \ge 0. \end{aligned} \tag{2}$$

Where o represents one of the $DMUs$, DMU_o.
Another type of relative efficiency in DEA is revenue efficiency. For testing revenue efficiency of DMU_o, we solve the following linear program

$$\begin{aligned} r_o^* &= Max \sum \rho_r \bar{y}_r \\ s.t& \\ &\sum_{j=1}^{n} \lambda_j x_{ij} \le x_{io}, \qquad i = 1,...,m \\ &\sum_{j=1}^{n} \lambda_j y_{rj} \ge \bar{y}_r \qquad r = 1,...,s \\ &\lambda_j, \bar{y}_r \ge 0 \end{aligned} \tag{3}$$

Here, ρ_r s are output prices and $\bar{y} = (\bar{y}_1,...,\bar{y}_s)$ is output vector optimally decided by DMU_o. If $(\lambda^*, \bar{y}^*)$ be an optimal solution to (3), then the maximum revenue is given by

$r_o^* = \sum_{r=1}^{s} \bar{\rho}_r \bar{y}_r^*$, where as the observed revenue is $r_o = \sum_{r=1}^{s} \rho_r y_{ro}$. Hence the revenue efficiency of DMU_o is determined as

$$re_o = \frac{r_o}{r_o^*} \leq 1.$$

It is easy to show that $re_o = \frac{1}{\phi_o^*}$, in which ϕ_o^* is the optimal value of the following linear programming problem:

$$\begin{aligned} \phi_o^* = Max\ & \frac{\sum_{r=1}^{s} \rho_r \bar{y}_r}{\sum_{r=1}^{s} \rho_r y_{ro}} \\ s.t\ & \\ & \sum_{j=1}^{n} \lambda_j x_{ij} \leq x_{io}, \quad i = 1,...,m \\ & \sum_{r=1}^{s} \lambda_j y_{rj} \geq \bar{y}_r, \quad r = 1,...,s \\ & \lambda_j \geq 0, \quad j = 1,...,n \\ & \bar{y}_r \geq 0, \quad r = 1,...,s. \end{aligned} \tag{4}$$

Definition 1 DMU_o is revenue efficient if and only if $re_o = 1$.
The dual formulation of (4) is as

$$\begin{aligned} Min\ & \sum_{i=1}^{m} v_i x_{io} \\ s.t\ & \\ & \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} \geq 0, \quad j = 1,...,n \\ & u_r \geq \frac{\rho_r}{\sum_{r=1}^{s} \rho_r y_{ro}}, \quad r = 1,...,s \\ & u_r, v_i \geq 0, \quad for\ all\ i, r. \end{aligned} \tag{5}$$

Here, the dual variables v_i, i=1,…,m are related to the first group of constraints in (4) and u_r, r=1,…,s are those related to the second group of constraints. In(5) the objective is to minimize the weighted sum of inputs of DMU_o, subject to the conditions that the weighted sum of outputs to the weighted sum of inputs for each DMU_j can not exceed unity and the dual variables u_r are greater than or equal to $\dfrac{\rho_r}{\sum_{r=1}^{s}\rho_r y_{ro}}$.

3 Revenue Efficiency Interval

In this section, we introduce the revenue efficiency interval. Let x_{ij} and y_{rj} be the amount of input i and output r, respectively, corresponding to DMU_j for i=1,…,m and r =1,…,s and j=1,…,n. Consider the following mathematical problem:

$$\varphi_o^* = \underset{u,v}{Min} \ \frac{\dfrac{v^t x_o}{\rho^t y_o}}{\underset{1\le j\le n}{Min} \ \dfrac{v^t x_j}{u^t y_j}}, \tag{6}$$

$$\begin{aligned} s.t \quad & \\ & u \ge \rho, \\ & v \ge 0, \\ & u \ge 0. \end{aligned}$$

In Which u = (u_1 ,…u_s)^t and ν = (ν$_1$,…ν$_n$)^t. Here ρ_i are the output prices. In this formulation we minimize the ratio of weighted sum of input to the weighted sum of output to that of minimum. This fractional problem can be converted to a linear format. Make the transformation

$\left[\underset{1\le j\le n}{Min} \left\{ {v^t x_j}\big/{u^t y_j} \right\}\right]^{-1} = \pi$ and $\pi v = \bar{v}$, then we have

$$\varphi_o^+ = Min \ \left\{\frac{\bar{v}^t x_o}{\rho^t y_o}\right\} \tag{7}$$

$$\begin{aligned} s.t \quad & \\ & \bar{v}_t x_j - u^t y_j \ge 0, \quad j = 1,...,n, \\ & u \ge \rho, \\ & v \ge 0, \\ & u \ge 0. \end{aligned}$$

Obviously $re_o^+ = \dfrac{1}{\varphi_o^+}$. Letting ${\bar{v}}\big/{\rho^t y_o} = \tilde{v}$ and ${u}\big/{\rho^t y_o} = \tilde{u}$, (7) reduces to the following from

$$
\begin{aligned}
\varphi_o^* = & Min \quad \tilde{v}^t x_o \\
& s.t \\
& \quad \tilde{v}^t x_j - \tilde{u}^t \geq 0, \qquad j = 1,\ldots,n, \\
& \quad \tilde{u} \geq \rho \Big/ \rho^t y_o , \\
& \quad \tilde{v} \geq 0, \\
& \quad \tilde{u} \geq 0.
\end{aligned} \tag{8}
$$

Which is equivalent to (5). re_o^+ is derived from the optimistic viewpoint and hence it is the upper bound of revenue efficiency interval. Considering the maximum problem of (8), the lower bound of revenue efficiency interval for DMU_o can defined as follows:

$$
\begin{aligned}
\varphi_o^- = & Max \quad \left\{ \frac{\dfrac{v^t x_o}{\rho^t y_o}}{\underset{1\leq j\leq n}{Min} \left\{ \dfrac{v^t x_j}{u^t y_j} \right\}} \right\} \\
& s.t \\
& \quad u \geq \rho, \\
& \quad u, v \geq 0.
\end{aligned} \tag{9}
$$

Letting $Min\left\{\dfrac{\bar{v}^t x_o}{u^t y_j}\right\} = \dfrac{1}{t}, 1 \leq j \leq n, tv = \bar{v}$, (9) reduces to the following format:

$$
\begin{aligned}
\varphi_o^- = & Max \left\{ \frac{\bar{v}^t x_o}{\rho^t y_o} \right\} \\
& s.t \\
& \quad \underset{1\leq j\leq n}{Min} \left\{ \frac{\bar{v}_t x_j}{\rho^t y_j} \right\} = 1, \\
& \quad u \geq \rho, \\
& \quad u, v \geq 0.
\end{aligned} \tag{10}
$$

Problem (10) cannot be replaced with an equivalent linear programming problem, but we can transform it to a mixed integer binary linear program. Toward this end, we use the slack variables s_j^+ and rewrite the constraint $Min\left\{\dfrac{\bar{v}^t x_j}{u^t y_j}\right\} = 1, 1 \leq j \leq n$ as follows

$$\begin{cases} \bar{v}^t x_j - u^t y_j - s_j^+ = 0, & j = 1,...,n, \\ s_j^+ \le (1-\delta_j)M, & j = 1,...,n, \\ \sum_{j=1}^{n} \delta_j \ge 1, & \\ \delta_j \in \{0,1\}, & j = 1,...,n, \\ s_j^+ \ge 0, & \\ u, \bar{v} \ge 0. & \end{cases}$$

Constraint $s_j^+ \le (1-\delta_j)M$, $s_j^+ \ge 0$, $\sum_{j=1}^{n} \delta_j \ge 1$ and $\delta_j \in \{0,1\}$ force some of the s_j^+ at zero level (M is a large positive constant). Then we have following mixed binary problem:

$$\varphi_o^- = Max \quad \frac{\bar{v}^t x_o}{\rho^t y_o}$$

$$\begin{aligned} s.t \quad & \bar{v}^t x_j - u^t y_j - s_j^- = 0, \quad j = 1,...,n, \\ & s_j^+ \le (1-\delta_j)M, \quad j = 1,...,n, \\ & \sum_{j=1}^{n} \delta_j \ge 1, \\ & \delta_j \in \{0,1\}, \\ & s_j^+ \ge 0, \\ & u \ge \rho, \\ & u, \bar{v} \ge 0. \end{aligned} \tag{11}$$

In this program $\frac{1}{\varphi_o^-}$ is the lower bound of efficiency interval. So, the revenue efficiency interval is $\left[\frac{1}{\varphi_o^-}, \frac{1}{\varphi_o^+}\right]$.

4 Example

To illustrate the application of the interval efficiency model, we have utilized the data set for 20 Iranian bank branches. The data for this application are derived from operations during 2006. We use six variables from the data set as inputs and outputs. Inputs are staff, budget and space, and outputs are sales; loan and charge. The normal data are summarized in table 1. The revenue efficiency intervals are reported in last column of the table. As the table indicates, five branches (#1, #2, #11, and #16) are technically efficiency.

Table 1
The normalized data and revenue efficiency interval

	staff	Budget	Space	Sales	Loan	Charge	$[L_j, U_j]$
1	0.6500	0.14257	0.5042	0.9263	1.0000	0.4621	[0.0012,1.0000]
2	1.0000	0.1657	0.4344	1.0000	0.8215	0.6064	[0.0009,1.0000]
3	0.8400	0.1535	0.4841	0.3724	0.2224	0.1355	[0.0003,0.2589]
4	0.6800	0.2607	0.5153	0.5388	0.4473	0.4213	[0.0007,0.5917]
5	0.7600	0.4814	0.9004	0.4818	0.7130	0.3078	[0.0005,0.5598]
6	0.7200	0.2600	0.4410	0.5323	0.5102	1.0000	[0.0011,0.9899]
7	0.9200	0.1935	0.6297	0.5200	0.5844	0.2457	[0.0005,0.4131]
8	0.8800	0.1957	0.4812	0.4409	0.4709	0.5391	[0.0007,0.6035]
9	0.6700	0.1414	0.5312	0.3877	0.5600	0.3629	[0.0007,0.5821]
10	0.7200	0.1728	0.3708	0.5690	0.1608	0.2376	[0.0004,0.4266]
11	0.6800	0.0535	0.4907	0.4463	0.5400	0.8721	[0.0011,1.0000]
12	0.7200	0.4614	1.0000	0.6700	0.7515	0.4564	[0.0006,0.7354]
13	0.6400	0.3814	0.4585	0.6524	0.5454	0.6755	[0.0019,0.8833]
14	0.7200	0.1485	0.5005	0.5988	0.6073	0.3913	[0.0008,0.6433]
15	0.6800	0.3507	0.7735	0.5518	0.2530	0.7658	[0.0007,0.7134]
16	0.8400	0.1807	0.5308	0.9823	0.7602	0.9780	[0.0012,1.0000]
17	0.9200	0.1328	0.4405	0.2900	0.8744	0.5647	[0.0009,0.8568]
18	0.8800	0.0871	0.4864	0.8709	0.4989	0.7651	[0.0019,0.9618]
19	0.7200	1.0000	0.4040	0.6777	0.6300	0.6229	[0.0007,0.9195]
20	0.6000	0.2657	0.6834	0.4890	0.8708	0.2589	[0.0008,0.7582]

5 Conclusions

Model of DEA have been generalized here to determine a revenue efficiency interval. In this paper, DMUs have been evaluated from two viewpoints so that the efficiency from optimistic viewpoint is the upper bound and the efficiency from pessimistic viewpoint is the lower bound of revenue efficiency interval.

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