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ASYMPTOTIC REPRESENTATION OF ZOLOTAREV POLYNOMIALS

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Abstract

In 1868 Zolotarev determined the polynomial which deviates least from zero with respect to the maximum norm on $[-1,1]$ among all polynomials of the form $x^n+\sigma x^{n-1}+a_{n-2}x^{n-2}+\ldots+a_1x+a_0$, where $\sigma\in\mathbb{R}$ is given. The polynomial was given explicitly in terms of elliptic functions by Zolotarev. It is now called the Zolotarev polynomial. Zolotarev also gave an explicit expression for the minimum deviation. In the sequel attempts have been made to replace the elliptic functions and to express the Zolotarev polynomial and the minimum deviation in terms of elementary functions, at least asymptotically. In 1913 Bernstein succeeded in finding an asymptotic formula for the minimum deviation, which has been improved several times since then. Here we give the first asymptotic representation of the Zolotarev polynomials. For the asymptotic representation we use the rational functions introduced by Bernstein. Furthermore, we obtain asymptotic representations of minimal polynomials with interpolation constraints which are of interest in the theory of the iterative solution of inconsistent linear systems of equations.

1. *Introduction*

In 1858 Chebyshev showed that the polynomials $T_n(x)=\cos n\arccos x$, $x\in[-1,1]$, which are now called Chebyshev polynomials, are the minimal polynomials on $[-1,1]$ with respect to the maximum norm. In 1868 in his famous dissertation Zolotarev, a student of Chebyshev, found the polynomial which deviates least from zero on $[-1,1]$ with respect to the maximum norm among all polynomials of the form $x^n+\sigma x^{n-1}+q(x)$, where $q\in\mathbb{P}_{n-2}$ ($\mathbb{P}_n$ denotes the set of all real polynomials of degree less than or equal to n). The fixed parameter σ may be assumed, by symmetry, to be nonnegative. By the Alternation Theorem it follows immediately that for $0<\sigma\leqslant n\tan^2(\pi/2n)$ the shifted Chebyshev polynomials are the minimal polynomials. For $\sigma>n\tan^2(\pi/2n)$ Zolotarev was able to give a representation of the minimal polynomials and of the minimum deviation in terms of elliptic functions (see [**1**, **28**] and for more recent surveys [**3**, **24**]). Though Zolotarev's solution is explicit there was and is still a demand for a description without elliptic functions. In 1913 Bernstein gave an asymptotic formula for the minimum deviation $E_n(x^n+\sigma x^{n-1})=\inf_{q\in\mathbb{P}_{n-2}}\|x^n+\sigma x^{n-1}+q(x)\|$ in terms of elementary functions and also upper and lower bounds. The estimates of Bernstein for the minimum deviation have been improved by Reddy [**18**, **19**] and by Haussmann and Zeller [**8**], where the estimates of the last named authors are rather precise. Though Bernstein has developed a theory in order to deal with the asymptotic representation of minimal polynomials with respect to the L_2-norm, nowadays well known as the so-called Bernstein–Szegő theory on orthogonal polynomials, he did not present an asymptotic description of the Zolotarev polynomials nor of the minimal polynomials

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satisfying certain interpolation restrictions for which he studied the behaviour of the minimum deviation [**2**] intensively. Interesting facts on Zolotarev polynomials can be found also in Erdős and Szegő [**5**]. In the second half of the 1950s Fekete and Walsh [**6**] gave n-roots asymptotics for minimal polynomials with constraints including the Zolotarev polynomials. But, as mentioned later by Widom in [**27**], strong asymptotics for minimal polynomials on $[-1,1]$ with respect to weight functions and moreover with constraints, like Zolotarev-type polynomials, are still lacking. Let us mention that for weighted minimal polynomials asymptotics outside of $[-1,1]$ have been derived by Lubinsky and Saff [**11**].

There have been other attempts to describe the Zolotarev polynomials analytically. We mention Paszkowski [**13**], who for $n = 2, 3$ got explicit expressions by deriving a differential equation for the zeros with respect to σ. See also Voronovskaya [**26**] and [**12**]. Concerning recent attempts to obtain explicit expressions by symbolic computation, using Gröbner bases for instance, see [**10**] and also [**16**, Section 7]. It is interesting that Zolotarev polynomials with respect to the L_1-norm can be given explicitly, even when several leading coefficients are fixed [**14**].

Zolotarev polynomials appear in several fields of mathematics. For instance, they appear as solutions of many extremal problems in approximation theory, in precise Markoff inequalities [**5**, **21**, **22**, **26**], in numerical differentiation [**20**], in signal processing–filter design [**17**, **25**] or in the investigations of modular forms [**9**, Appendix IV, 'Zolotarev polynomials and the modular curve $X_1(N)$']. Recently Zolotarev-type polynomials, that is, minimal polynomials with respect to the maximum norm satisfying a local extrema constraint outside $[-1,1]$, turned out to be of importance in the iterative solution of large sparse inconsistent linear systems of equations [**4**, **7**].

2. *Bernstein's rational functions*

First let us introduce some basic notations used in the following. Let $K \subset \mathbb{R}$ be a compact set. For $f \in C(K)$ let $\|f\|_K = \max_{x\in K} |f(x)|$; and $\|f\|_{[-1,1]}$ is abbreviated by $\|f\|$. A point $y \in K$ is called an extremal point of f on K if $|f(y)| = \|f\|_K$. The points $x_1, \ldots, x_m \in K$, $x_1 < x_2 < \ldots < x_m$, are called alternation points (abbreviated a-points) of f on K if they are extremal points of f and $f(x_j) = -f(x_{j+1})$ for $j = 1, \ldots, m-1$. Let $G \subset \mathbb{P}_{n-1}$ be a linear subspace. Then g^* is called a best approximation from G on K to a given function f if

$$\|f - g^*\|_K = \inf_{g\in G} \|f - g\|_K. \tag{1}$$

We note that, if $G = \mathbb{P}_{n-1}$, then by the Alternation Theorem g^* is a best approximant if and only if $f - g^*$ has $n+1$ a-points on K. Furthermore g^* is uniquely determined. In the case of Zolotarev polynomials we have $f(x) = x^{n+1} - \sigma x^n$, where $\sigma \in \mathbb{R}$ is given. We call a monic polynomial $x^n - q_{n-1}(x)$, $q_{n-1} \in \mathbb{P}_{n-1}$, a minimal polynomial on K if q_{n-1} is a best approximation to x^n.

In the following we use frequently the well-known conformal map from $\bar{\mathbb{C}} \setminus [-1,1]$ onto the open unit disc $\{|z| < 1\}$ and *vice versa*, that is,

$$z = x - \sqrt{x^2 - 1}, \quad x \in \bar{\mathbb{C}} \setminus [-1,1], \tag{2}$$

where that branch of the square root is chosen which is positive for $x > 1$, respectively,

$$x = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad |z| < 1. \tag{3}$$

Obviously the unit circle is mapped to the twice covered interval $[-1, 1]$.

Next let us introduce Bernstein's rational functions in a simpler way than in [**2**, Chapter 1, Section 5]. The following approach was certainly known to members of Bernstein's circle such as Akhiezer, but we did not find it carried out in the literature. Bernstein introduced a more general class of rational functions but we restrict ourselves to rational functions with denominator degree one which is all that will be needed in the following. The extension to the case of higher denominator degree is obvious. Let $c \in (1, \infty)$. Then, by (2) and (3), the polynomial $c - x$ can be represented in the form

$$c - x = \frac{(1 - \tau z)(z - \tau)}{2\tau z}, \tag{4}$$

where we have put

$$\tau = c - \sqrt{c^2 - 1}. \tag{5}$$

Note that for $x \in [-1, 1]$, that is, $z = e^{i\varphi}$, $\varphi \in [0, \pi]$,

$$c - x = (2\tau)^{-1}|z - \tau|^2 = (2\tau)^{-1}|1 - z\tau|^2. \tag{6}$$

Now let us consider

$$2R_{n+1}(x) := z^n \frac{1 - \tau z}{z - \tau} + z^{-n}\frac{z - \tau}{1 - \tau z}. \tag{7}$$

Using the well-known representation of the Chebyshev polynomial

$$T_n(x) = (z^n + z^{-n})/2, \quad x \in \mathbb{C}, \tag{8}$$

it follows that

$$R_{n+1}(x) = \frac{\tau^2 T_{n+1}(x) - 2\tau T_n(x) + T_{n-1}(x)}{(2\tau)(c - x)}. \tag{9}$$

Note that by (8) the numerator of R_{n+1} has a double zero at $z = \tau$, that is, at $x = c$ asymptotically with respect to n. For $z = e^{i\varphi}$, $\varphi \in [0, \pi]$,

$$R_{n+1}(\cos\varphi) = \mathrm{Re}\left\{ z^n \frac{1 - \tau z}{z - \tau} \right\}, \tag{10}$$

which implies immediately that

$$|R_{n+1}(x)| \leqslant 1 \quad \text{for } x \in [-1, 1]. \tag{11}$$

For $n \geqslant n_0$ the argument of $z^n(1 - \tau z)/(z - \tau)$, $z = e^{i\varphi}$, is monotone increasing with respect to φ, where the change of the argument is $2(n - 1)\pi$, when φ varies from 0 to 2π (since $\tau \in (-1, 1)$). Hence $\mathrm{Im}\{z^n(1 - \tau z)/(z - \tau)\}$ has $n - 2$ simple zeros in $(0, \pi)$, taking into consideration the symmetry with respect to π. Thus, by the relation

$$\left(\mathrm{Re}\left\{ z^n \frac{1 - \tau z}{z - \tau} \right\}\right)^2 + \sin^2\varphi \left(\frac{\mathrm{Im}\{z^n(1 - \tau z)/(z - \tau)\}}{\sin\varphi} \right)^2 = 1, \tag{12}$$

$R_{n+1}(x)$ has n a-points on $[-1, 1]$, including the boundary points ± 1.

Next, using (5) and

$$\frac{1}{z} = x + \sqrt{x^2-1} \quad \text{and} \quad \frac{1}{\tau} = c + \sqrt{c^2-1}, \tag{13}$$

we obtain that

$$\frac{(1-\tau z)^2}{2\tau z} = \frac{1}{2}\left(\tau z + \frac{1}{\tau z}\right) - 1 = cx - 1 + \sqrt{(c^2-1)(x^2-1)}, \tag{14}$$

and thus by (4)

$$\frac{1-\tau z}{z-\tau} = \frac{cx - 1 + \sqrt{(c^2-1)(x^2-1)}}{c-x} \tag{15}$$

and

$$\frac{z-\tau}{1-\tau z} = \frac{c-x}{cx - 1 + \sqrt{(c^2-1)(x^2-1)}} = \frac{cx - 1 - \sqrt{(c^2-1)(x^2-1)}}{c-x}. \tag{16}$$

Moreover inspection of the left-hand side of (15) shows that the right-hand expression in (15) is precisely the so-called complex Green's function of $\bar{\mathbb{C}} \setminus [-1,1]$ with pole at c. This fact is not needed explicitly in the following.

Finally, inserting the derived expressions in (7) we obtain Bernstein's formula for $R_{n+1}(x)$:

$$\begin{aligned} R_{n+1}(x) = {} & (x - \sqrt{x^2-1})^n \frac{cx - 1 + \sqrt{(c^2-1)(x^2-1)}}{2(c-x)} \\ & + (x + \sqrt{x^2-1})^n \frac{cx - 1 - \sqrt{(c^2-1)(x^2-1)}}{2(c-x)}. \end{aligned} \tag{17}$$

In the following we mainly use the representations (7), (9) and (10). Bernstein used the above rational function in a suitable way to give the asymptotic minimum deviation of the Zolotarev polynomial [**2**, p. 23]. We shall prove that it in fact represents the Zolotarev polynomial asymptotically.

3. *Asymptotic representation of Zolotarev polynomials*

The following lemma can be found in [**5**, Section 2] (see also [**15**]).

LEMMA 3.1. *Let p_n be a polynomial of degree n with $\|p_n\|_{[-1,1]} = 1$ which has exactly n a-points on $[-1,1]$ including the points ± 1, which are supposed to satisfy additionally the conditions $p_n'(\pm 1) \neq 0$. Then*

$$p_n^{-1}([-1,1]) = [-1,1] \cup [B_n, C_n] \quad \text{with } [-1,1] \cap [B_n, C_n] = \emptyset \tag{18}$$

and p_n satisfies a polynomial equation of the form

$$p_n^2 - (x^2-1)(x-C_n)(x-B_n)q_{n-2}^2 = 1, \tag{19}$$

where

$$p_n' = n(x-A_n)q_{n-2}, \tag{20}$$

assuming that the leading coefficients of p_n and of q_{n-2} have the same sign. Furthermore,

$$\int_{B_n}^{C_n} \frac{|x-A_n|}{\sqrt{|(x-B_n)(x-C_n)(x^2-1)|}}\, dx = \frac{\pi}{n} \tag{21}$$

and

$$\int_{-1}^{1} \frac{|x-A_n|}{\sqrt{|(x-B_n)(x-C_n)(x^2-1)|}}\,dx = \frac{(n-1)}{n}\pi. \tag{22}$$

The polynomial p_n has a representation of the form

$$p_n(x) = \pm\cosh\left(n\int_1^x \frac{t-A_n}{\sqrt{(t-B_n)(t-C_n)(t^2-1)}}\,dt\right). \tag{23}$$

In the following we choose that branch of the square root in the cut plane $\bar{\mathbb{C}} \setminus ([-1,1] \cup [B_n, C_n])$ which is positive for large positive values. We note that the choice of the branch of the square root in (23) is irrelevant.

LEMMA 3.2. *Let (n_ν), $\nu \in \mathbb{N}$, be a subsequence in $\mathbb{N}$ and suppose that the sequence of polynomials (p_{n_ν}) satisfies the assumptions of Lemma* 3.1 *and is such that (C_{n_ν}), respectively (B_{n_ν}), from Lemma* 3.1 *is bounded. Then*

$$|C_{n_\nu} - B_{n_\nu}| = O(1/n_\nu^2). \tag{24}$$

Proof. For simplicity assume that $1 < B_{n_\nu} < C_{n_\nu}$, $\nu \in \mathbb{N}$, and let $T_{n_\nu}^{\mathrm{tr}}(x) = T_{n_\nu}((2x-(C_{n_\nu}-1))/(C_{n_\nu}+1))$ be the transformed Chebyshev polynomial on $[-1, C_{n_\nu}]$. Further let

$$y_{n_\nu-1} = \frac{C_{n_\nu}-1}{2} + \frac{C_{n_\nu}+1}{2}\cos\left(\frac{2\pi}{n_\nu}\right)$$

be the third largest a-point of $T_{n_\nu}^{\mathrm{tr}}$. We claim that

$$[B_{n_\nu}, C_{n_\nu}] \subseteq [y_{n_\nu-1}, C_{n_\nu}]. \tag{25}$$

Suppose indeed that $B_{n_\nu} < y_{n_\nu-1}$. Considering $T_{n_\nu}^{\mathrm{tr}} - p_{n_\nu}$ on $[-1,1]$ at the n_ν a-points of p_{n_ν} and on $[B_{n_\nu}, C_{n_\nu}]$ at the a-points of $T_{n_\nu}^{\mathrm{tr}}$ it follows that $T_{n_\nu}^{\mathrm{tr}} - p_{n_\nu}$ has at least $n_\nu + 1$ zeros on $[-1, C_{n_\nu}]$ and this is a contradiction. □

NOTATION 3.3. Let $n \in \mathbb{N}, \sigma \in \mathbb{R}$. By $\hat{Z}_n(x,\sigma) = -x^n + \sigma x^{n-1} + \dots$ we denote the unique monic Zolotarev polynomial. Thus $\hat{Z}_n(.,\sigma)$ is such that

$$\|\hat{Z}_n(x,\sigma)\| = \inf_{q\in\mathbb{P}_{n-2}} \|-x^n + \sigma x^{n-1} + q\|. \tag{26}$$

Then $Z_n(x,\sigma) = \hat{Z}_n(x,\sigma)/\|\hat{Z}_n(x,\sigma)\|$ denotes the normalized Zolotarev polynomial.

Note that $(-1)^n Z_n(x;-\sigma) = Z_n(-x,\sigma)$ and thus it suffices to consider the case $\sigma \in [0,\infty)$; more precisely $\sigma > n\tan^2(\pi/2n)$. For $\sigma = 0$ it is, as is well known [**1**, p. 281], the Chebyshev polynomial and for $\sigma \in (0, n\tan^2(\pi/2n)]$ it is a linearly transformed Chebyshev polynomial.

THEOREM 3.4. *Let $\sigma \in (0,\infty)$ and put $\sigma = \sqrt{c^2-1}$, $\tau = c - \sqrt{c^2-1}$. Then on any compact subset of $\mathbb{C} \setminus \{c\}$*

$$\begin{aligned} Z_n(x,\sigma) &= \frac{1}{2}\left\{z^n\frac{1-\tau z}{z-\tau} + z^{-n}\frac{z-\tau}{1-\tau z}\right\}\left(1+O\left(\frac{1}{n}\right)\right) \\ &= \frac{\tau^2 T_{n+1}(x) - 2\tau T_n(x) + T_{n-1}(x)}{(2\tau)(c-x)}\left(1+O\left(\frac{1}{n}\right)\right). \end{aligned} \tag{27}$$

Proof. It is well known (see, for example, [**1**]) that for $\sigma > n\tan^2(\pi/2n)$ the polynomial Z_n has exactly n a-points on $[-1,1]$. The points ± 1 belong to the a-points and are not critical points. Thus Z_n satisfies the assumptions of Lemma 3.1, and so, in particular, there exist A_n, B_n and C_n with

$$1 < A_n < B_n < C_n \tag{28}$$

such that

$$Z_n^{-1}([-1,1]) = [-1,1] \cup [B_n, C_n] \tag{29}$$

and

$$Z_n'(x) = n(x - A_n)q_{n-2}. \tag{30}$$

Comparing the second leading coefficient in (19) we obtain that

$$A_n = \left(\frac{C_n + B_n}{2} - \frac{\sigma}{n}\right). \tag{31}$$

By relation (21) and by estimating the integrand we obtain

$$\begin{aligned} &\frac{1}{\sqrt{C_n^2 - 1}} \int_{B_n}^{C_n} \frac{x - A_n}{\sqrt{(x - B_n)(C_n - x)}}\, dx \\ &\quad \leqslant \frac{\pi}{n} = \int_{B_n}^{C_n} \frac{x - A_n}{\sqrt{(x - B_n)(C_n - x)}} \frac{dx}{\sqrt{x^2 - 1}} \\ &\quad \leqslant \frac{1}{\sqrt{B_n^2 - 1}} \int_{B_n}^{C_n} \frac{x - A_n}{\sqrt{(x - B_n)(C_n - x)}}\, dx. \end{aligned} \tag{32}$$

Using (31) and

$$\int_{B_n}^{C_n} \frac{x - (C_n + B_n)/2}{\sqrt{(x - B_n)(C_n - x)}}\, dx = 0 \quad \text{and} \quad \int_{B_n}^{C_n} \frac{dx}{\sqrt{(x - B_n)(C_n - x)}} = \pi, \tag{33}$$

it follows by (32) that

$$\sqrt{B_n^2 - 1} \leqslant \sigma \leqslant \sqrt{C_n^2 - 1}. \tag{34}$$

Since $\sigma > 0$, the point 1 is not an accumulation point of (C_n). The same holds for ∞, since otherwise by (31) $A_{n_\nu} \to \infty$ as $\nu \to \infty$. Hence by (28) $B_{n_\nu} \to \infty$ which is not possible by (34).

Thus (C_n) is bounded and therefore, by Lemma 3.2,

$$B_n = C_n + O(1/n^2), \tag{35}$$

which implies by (34) that each convergent subsequence (C_{n_ν}) of (C_n) and (B_{n_ν}) of (B_n) has the limit $\sqrt{\sigma^2 + 1} = c$. Hence (C_n) and (B_n) are convergent with

$$C_n = c + O(1/n^2), \tag{36}$$

and thus by (31)

$$nA_n = nc - \sqrt{c^2 - 1} + O(1/n). \tag{37}$$

Next we use the fact that by (35), on compact subsets of $\mathbb{C} \setminus \{c\}$,

$$\frac{1}{\sqrt{(x - B_n)(x - C_n)}} = \frac{1}{x - c} + O(1/n^2), \tag{38}$$

where in accordance with the remark after Lemma 3.1 we have chosen that branch of the square root which is positive for large positive values. By (37) we obtain

$$\int_1^x \frac{n(t-A_n)}{\sqrt{(t-C_n)(t-B_n)(t^2-1)}}\,dt$$
$$= n\int_1^x \frac{dt}{\sqrt{t^2-1}} - \sqrt{c^2-1}\int_1^x \frac{1}{(c-t)}\frac{dt}{\sqrt{t^2-1}} + O(1/n). \tag{39}$$

Since

$$-\sqrt{c^2-1}\int_1^x \frac{1}{c-t}\frac{dt}{\sqrt{t^2-1}} = \ln\left(\frac{cx-1-\sqrt{(c^2-1)(x^2-1)}}{c-x}\right) \tag{40}$$

and

$$\int_1^x \frac{dt}{\sqrt{t^2-1}} = \ln(x+\sqrt{x^2-1}), \tag{41}$$

it follows by (23) that $Z_n(x,\sigma)$ is asymptotically equal to Bernstein's rational function from (17) and by (7) the theorem is proved. □

COROLLARY 3.5. (a) *For* $x=\cos\varphi$, $\varphi\in[0,\pi]$, $z=e^{i\varphi}$,

$$Z_n(x,\sigma) = \mathrm{Re}\left\{z^n\frac{1-\tau z}{z-\tau}\right\} + O\left(\frac{1}{n}\right)$$
$$= \frac{\tau^2 T_{n+1}(x) - 2\tau T_n(x) + T_{n-1}(x)}{(2\tau)(c-x)} + O\left(\frac{1}{n}\right)$$
$$= -\tau T_n(x) + (1-\tau^2)\sum_{j=1}^{n}{}' \tau^{j-1}T_{n-j}(x) + O\left(\frac{1}{n}\right), \tag{42}$$

where the prime on $\sum$ *signifies that the coefficient of* T_0 *is to be halved.*

(b) *On any compact subset of* $\mathbb{C}\setminus([-1,1]\cup\{c\})$,

$$2z^n Z_n(x,\sigma) = \frac{z-\tau}{1-\tau z} + O(1/n). \tag{43}$$

Proof. (a) The first and second representations follow immediately; see (10). To prove the third representation, let $\sum_{j=0}^{n} a_{n-j}T_{n-j}$ be such that

$$\tau^2 T_{n+1} - 2\tau T_n + T_{n-1} = (1+\tau^2-2\tau T_1)\sum_{j=0}^{n} a_{n-j}T_{n-j}, \tag{44}$$

which implies that $a_n=-\tau$, $a_{n-j}=\tau^{j-1}(1-\tau^2)$ for $j=1,\dots,n-1$, and $a_0=(1-\tau^2)\tau^{n-1}/2$. Since by the second representation in (42)

$$Z_n(x,\sigma) - \sum_{j=0}^{n} a_{n-j}T_{n-j}(x) = O(1/n), \tag{45}$$

the assertion is proved.

(b) For compact subsets of $\mathbb{C}\setminus([-1,1]\cup\{c\})$ in (27) the first expression on the right-hand side in the first line is of order $O(|z^n|)$, where $|z|<1$. Hence part (b) follows. □

REMARK 3.6. (a) The third asymptotic representation in (42) can be written in the form

$$\mathrm{Re}\left\{-\tau z^n + (1-\tau^2) z^{-(n-1)} \left(\frac{1-(\tau z)^n}{1-\tau z}\right) - \frac{\tau^{n-1}}{2}\right\}, \tag{46}$$

where $z = e^{i\varphi}$.

(b) To improve Bernstein's bounds on the minimum deviation Haussmann and Zeller [**8**] derived by the Carathéodory–Fejér method the following approximation of the normalized Zolotarev polynomial, written in a slightly modified way,

$$\mathrm{Re}\left\{-\tau z^n + (1-\tau^2) z^{-(n-1)} \frac{1-(\tau z)^{2n-2}}{1-\tau z}\right\}, \tag{47}$$

which is a polynomial of degree n of the form

$$-\tau\left(T_n + \left(\tau - \frac{1}{\tau}\right) T_{n-1} + \dots\right).$$

By Corollary 3.5, estimating the difference of (46) and (47), it follows that (47) also represents the normalized Zolotarev polynomial on $[-1,1]$ asymptotically. Since by (47) (see [**8**, Theorem 5]) the maximum norm on $[-1,1]$ is less than or equal to $1+\tau^{n+1}(1+\tau)$; this is an even better approximation than (46).

(c) By the second representation in (42) and the well-known fact that $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ the Zolotarev polynomials satisfy asymptotically the three term recurrence relation

$$Z_{n+1}(x;\sigma) \simeq 2xZ_n(x;\sigma) - Z_{n-1}(x;\sigma) \ \text{ on } [-1,1]. \tag{48}$$

Next let us express the Zolotarev polynomials in terms of orthonormal polynomials.

COROLLARY 3.7. *Let $\sigma > 0$, put $\sigma = \sqrt{c^2-1}$ and let $p_{n-1}(x;(c-x)^2/\sqrt{1-x^2})$ be the polynomial of degree $n-1$ orthonormal on $[-1,1]$ to $\mathbb{P}_{n-2}$ with respect to the weight function $(c-x)^2/\sqrt{1-x^2}$. Then on $[-1,1]$*

$$Z_n(x;\sigma) = (c-x)\sqrt{2\pi}\, p_{n-1}(x;(c-x)^2/\sqrt{1-x^2}) + O(1/n). \tag{49}$$

Proof. Since

$$D(z) = (1-\tau z)^2/2\tau \tag{50}$$

satisfies

$$|D(e^{i\varphi})|^2 = (c-\cos\varphi)^2, \tag{51}$$

D is the so-called Szegő function, which implies by the asymptotic representation of orthonormal polynomials (taking into consideration the fact that D is analytic on $|z| \leqslant 1$, see, for example, [**23**]) that

$$\sqrt{2\pi}\, p_n(\cos\varphi) = \mathrm{Re}\{e^{-in\varphi}/D(e^{i\varphi})\} + O(q^n), \tag{52}$$

where $|q| < 1$. We mention that relation (52) could be derived in an elementary way with the help of Christoffel's formula [**23**, Section 2.5] and the asymptotics for the Chebyshev polynomials. Now (49) follows by (6) and Corollary 3.5(a). □

4. *Minimal polynomials with a given local extremal point outside* $[-1,1]$

As mentioned above the following minimal polynomials are of importance in the iterative solution of large sparse inconsistent linear systems of equations [**4**, **7**].

THEOREM 4.1. *Let $A \in (1,\infty)$ and let $m_n(x)$, $n \in \mathbb{N}$, be the polynomial of degree at most n which is minimal on $[-1,1]$ with respect to the maximum norm among all polynomials $q \in \mathbb{P}_n$ satisfying the constraints $q(A) = 1$ and $q'(A) = 0$. For $n \in \mathbb{N}$ put*

$$c_n = A + \frac{\sqrt{A^2-1}}{n} \quad \text{and} \quad \tau_n = c_n - \sqrt{c_n^2 - 1}.$$

Then the minimal polynomial $M_n(x)$, normalized by $\|M_n\|_{[-1,1]} = 1$, on any compact subset of $\mathbb{C} \setminus \{A\}$ has a representation of the form

$$\begin{aligned} M_n(x) &= \frac{1}{2}\left\{ z^n \frac{1-\tau_n z}{z-\tau_n} + z^{-n}\frac{z-\tau_n}{1-\tau_n z}\right\}(1+o(1)) \\ &= \frac{\tau_n^2 T_{n+1}(x) - 2\tau_n T_n(x) + T_{n-1}(x)}{(2\tau_n)(c_n - x)}(1+o(1)). \end{aligned} \tag{53}$$

Furthermore for $x = \cos\varphi, \varphi \in [0,\pi], z = e^{i\varphi}$,

$$M_n(x) = \operatorname{Re}\left\{ e^{in\varphi}\frac{1-\tau_n e^{i\varphi}}{e^{i\varphi}-\tau_n}\right\} + o(1). \tag{54}$$

Proof. By the Alternation Theorem it follows that $M_n(x)$ is a minimal polynomial on

$$M_n^{-1}([-1,1]) = [-1,1] \cup [B_n, C_n] \tag{55}$$

where

$$1 < A < B_n < C_n \tag{56}$$

and, by Lemma 3.1,

$$\int_{-1}^{1} \frac{A-x}{\sqrt{|(x-B_n)(x-C_n)|}} \frac{dx}{\sqrt{1-x^2}} = \left(\frac{n-1}{n}\right)\pi. \tag{57}$$

Hence (C_n) is bounded. By Lemma 3.2

$$C_n = B_n + O(1/n^2) \tag{58}$$

and thus on $[-1,1]$

$$\frac{1}{\sqrt{(x-B_n)(x-C_n)}} = \frac{1}{x-C_n} + O(1/n^2). \tag{59}$$

Now it follows by (57) and $\int_{-1}^{1}(1/\sqrt{1-x^2})\,dx = \pi$ that for any convergent subsequence (C_{n_ν})

$$\lim_\nu C_{n_\nu} = \lim_\nu B_{n_\nu} = A. \tag{60}$$

Furthermore we know by Lemma 3.1 that

$$\int_{B_n}^{C_n} \frac{x-A}{\sqrt{(B_n-x)(C_n-x)}} \frac{dx}{\sqrt{|x^2-1|}} = \frac{\pi}{n}. \tag{61}$$

Proceeding as in the proof of Theorem 3.4 we obtain from (61) that

$$\frac{((C_n+B_n)/2-A)\pi}{\sqrt{C_n^2-1}} \leqslant \frac{\pi}{n} \leqslant \frac{((C_n+B_n)/2-A)\pi}{\sqrt{B_n^2-1}} \tag{62}$$

and thus by (58) and (60)

$$C_n - A = \frac{\sqrt{A^2-1}}{n} + O\Big(\frac{1}{n^\gamma}\Big), \tag{63}$$

where $\gamma > 1$. Using the fact that relation (59) holds on compact subsets of $\mathbb{C}\setminus\{A\}$ we obtain

$$\begin{aligned}\int_1^x \frac{t-A_n}{\sqrt{(t-B_n)(t-C_n)(t^2-1)}}\,dt \\ = n\int_1^x \frac{dt}{\sqrt{t^2-1}} - \sqrt{A^2-1}\int_1^x \frac{1}{(C_n-t)}\frac{dt}{\sqrt{t^2-1}} + o(1).\end{aligned} \tag{64}$$

Replacing C_n by $c_n = A + (\sqrt{A^2-1})/n$ and $\sqrt{A^2-1}$ by $\sqrt{c_n^2-1}$, the assertion follows by (63), (40) and (41). □

Remark 4.2. Bernstein [**2**, p. 22] has shown that

$$\|m_n(x)\|_{[-1,1]} \simeq \frac{4n\sqrt{A^2-1}}{(A+\sqrt{A^2-1})^n}. \tag{65}$$

We mention that this fact can be proved more transparently by using the representation in the first line of (53).

Corollary 4.3. *Let $p_{n-1}(x;(A-x)^2/\sqrt{1-x^2})$ denote the polynomial of degree $n-1$ orthonormal on $[-1,1]$ with respect to the weight function $(A-x)^2/\sqrt{1-x^2}$. Then on $[-1,1]$*

$$M_n(x) = (A-x)\sqrt{2\pi}\,p_{n-1}(x;(A-x)^2/\sqrt{1-x^2}) + o(1). \tag{66}$$

Proof. Put $\tau = A - \sqrt{A^2-1}$. Since by (63)

$$\tau_n = \tau + o(1), \tag{67}$$

it follows by (53) that

$$M_n(x) = \mathrm{Re}\Big\{e^{-in\varphi}\frac{1-\tau e^{i\varphi}}{e^{i\varphi}-\tau}\Big\} + o(1). \tag{68}$$

The assertion follows now as in Corollary 3.7. □

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