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isomorphism) as the least fixed point `Rec (Square (MyRec f))` of the covariant functor `Square (MyRec f)`.

Another way of showing that covariant functors suffice is to first eliminate the use of mutual recursion in the definitions of `cata` and `ana` (using standard techniques), and then construct datatypes on which the resulting functions are catamorphisms and anamorphisms.

5 Discussion

In this paper we have explained how the recursion functionals `cata` and `ana` can be generalised from polynomial datatypes to those involving exponentials. An important area for future research is in experimenting with the use of the generalised operators and laws in writing and transforming programs. Another interesting topic for study is non-regular datatypes [28], i.e. datatypes in which the recursive calls in the body are not all of the form of the head of a definition. Examples are the datatype `Twist a b` of lists of alternating elements of type `a` and type `b`, and the datatype `Nest a` of lists of nested lists:

```
data Twist a b = Nil | Cons a (Twist b a)

data Nest a = Block a (Nest [a])
```

To our knowledge, it is not in general known how to express non-regular datatypes as fixed points of functors (or difunctors). Note however that it is possible, with some effort, to express non-regular datatypes as fixed points of type constructors, by using a `Rec` of kind `((* -> *) -> (* -> *)) -> (* -> *)` instead of kind `(* -> *) -> *`.

Our final remarks concern the Gofer type system. In a number of places we had to hack around the limitations of type synonyms. First of all, since standard type synonyms cannot be partially applied we were forced in some cases to make use of restricted type synonyms, which can be partially applied. Secondly, since type synonyms cannot be recursive, we were forced to use a data declaration in defining `Rec`, leading to the introduction of the fictitious constructor `In`. Both these problems don't seem to be inherent to type synonyms, but are rather artifacts of the treatment of type synonyms as macros in earlier functional languages; see [19] for further discussion on this point.

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```
type Cont = (Closure -> Closure) -> Closure

instance Retract Cont (Cont -> Cont) where
   up f =
      \a -> \cont ->
         f (\(In (Clos f)) -> f a cont)
   down f =
      \cont -> cont (In (Clos f))

instance Reflexive Cont

instance LambdaModel Cont
```

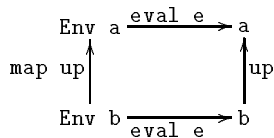where `Closure` is the fixed point of difunctor `C`:

```
data C c c' =
   Clos (((c -> c') -> c)
      -> ((c' -> c) -> c'))

instance Difunctor (->) => Difunctor C where
   (f 'dimap' g) (Clos h) =
      Clos (((((f 'dimap' g) 'dimap' f)
         'dimap' ((g 'dimap' f) 'dimap' g)) h)

type Closure = Rec C
```

Suppose now that we want to show that the continuation-based interpreter is correct with respect to the standard interpreter [30]. An interpreter based upon a reflexive type b is correct with respect to an interpreter based upon a reflexive type a if there exists a retract from b to a such that the following diagram commutes for all expressions e:

$$
\begin{array}{ccc}
\text{Env a} & \xrightarrow{\text{eval e}} & \text{a} \\
\text{map up} \uparrow & & \uparrow \text{up} \\
\text{Env b} & \xrightarrow[\text{eval e}]{} & \text{b}
\end{array}
$$

That is, evaluating an expression using an environment binding variables to b-values and then converting the result to an a-value is equivalent to first converting the bindings to a-values and then interpreting the expression.

It turns out now that the free theorem for the polymorphic λ-interpreter `eval :: Reflexive a => Expr -> Env a -> a` gives tractable conditions that establish the correctness of one interpreter with respect to another:

$$
\begin{array}{l}
\quad \text{h (f 'apply' a) = (h f) 'apply' (h a)} \\
\wedge \\
\quad \text{h.g = f.h} \Rightarrow \text{h (abstr g) = abstr f} \\
\Rightarrow \\
\quad \text{h.(eval e) = (eval e).map h}
\end{array}
$$

(See [26] for a general discussion of free theorems in the context of class restrictions.) This result is a generalisation to the untyped λ-calculus of Reynolds' (functional) *abstraction theorem* for the typed λ-calculus [31].

Appealing to the abstraction theorem with a = Scott, b = Cont and h = up, we are required to find a retract from Cont to Scott satisfying the preconditions of the theorem. Using the fact that Scott is of the form Rec S', together with the fact that cata and ana preserve retracts (previous section), we are motivated to look for a retract from Cont to S' Cont Cont. It is easy to prove that the following definitions for up and down give such a retract, thus establishing a retract from Cont to Scott:

```
instance Retract Cont (S' Cont Cont) where
   up f             = Func (up f)
   down (Func f) = down f

instance Retract Cont Scott
```

It remains to show that up :: Cont -> Scott satisfies the preconditions of the abstraction theorem. Verifying the second condition is straightforward. However, we have not yet been successful in establishing the first condition, namely that up (f 'apply' a) = (up f) 'apply' (up a).

4.5 Covariant functors suffice

Freyd [12] shows that, somewhat surprisingly, the generalisation from functors to difunctors is not technically necessary to handle exponentials: fixed points of difunctors can be expressed in terms of fixed points of covariant functors. The result is mainly of theoretical interest, but it is instructive to see how the translation from difunctors to functors works. The present class system of Gofer isn't quite powerful enough to let us implement all aspects of the translation directly, so here we just give an outline.

As we have seen previously for the case of (->), a difunctor f can be made into a covariant functor (f a) by fixing its contravariant argument to a specific type a:

```
instance Difunctor f => Functor (f a) where
   map g = id 'dimap' g
```

Consider now the mapping on types MyRec f which sends a type a to the fixed point of the covariant functor (f a):

```
type MyRec f a = Rec (f a)
```

By using the cata operator for functors of section 2.2, the mapping MyRec f can be extended to a mapping on functions, such that MyRec f is then a contravariant functor:

```
mycomap :: (Difunctor f, Functor (f b)) =>
   (a -> b) -> (MyRec f b -> MyRec f a)
mycomap g = cata (In . (g 'dimap' id))
```

For technical reasons concerning type classes in Gofer, MyRec f cannot directly be made into an instance of Gofer class Cofunctor of contravariant functors.

A contravariant functor f can be made into a covariant functor Square f by composing f with itself:

```
type Square f a = f (f a) in sqrmap

sqrmap :: Cofunctor f =>
   (a -> b) -> (Square f a -> Square f b)
sqrmap g = comap (comap g)

instance Cofunctor f => Functor (Square f)
   where map = sqrmap
```
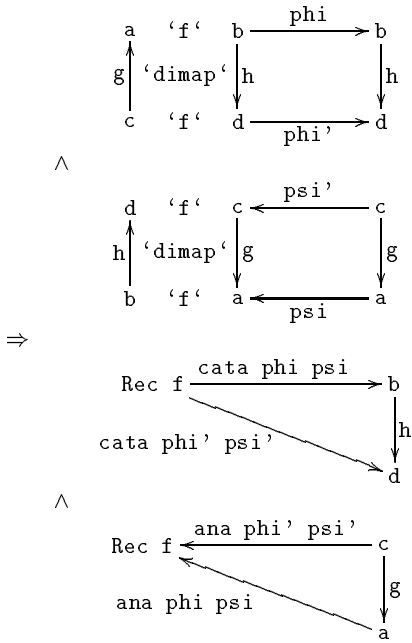
Again for technical reasons, sqrmap cannot be defined directly within the instance declaration above.

Freyd shows now that fixed points of difunctors can be reduced to fixed points of covariant functors in two steps. First of all, the least fixed point Rec f of a difunctor f is isomorphic to the least fixed point Rec (MyRec f) of the contravariant functor MyRec f (viewed as a difunctor independent of its second argument). And secondly, the least fixed point Rec f of a contravariant functor is isomorphic to the least fixed point Rec (Square f) of the covariant functor Square f. Combining the two steps, we see that the least fixed point Rec f of a difunctor f can be obtained (up to

or in diagrammatic form,

```
        'f'         phi
    a ------- b --------> b
    |         |           |
  g | 'dimap' | h         | h
    |         |           |
    v   'f'   v   phi'    v
    c ------- d --------> d

    /\

        'f'         psi'
    d ------- c <-------- c
    |         |           |
  h | 'dimap' | g         | g
    |         |           |
    v   'f'   v   psi     v
    b ------- a <-------- a

    =>

            cata phi psi
    Rec f -----------------> b
        \                    |
         \  cata phi' psi'   | h
          \                  v
           ----------------> d

    /\

            ana phi' psi'
    Rec f <----------------- c
        ^                    |
         \   ana phi psi     | g
          \                  v
           <---------------- a
```

It is interesting to note that the above fusion law turns out to be the specialisation to functions of Pitt's relational induction principle for recursive datatypes [29, Prop 2.10].

Let us consider an example of the use of fusion. A *retract* from a type b to a type a is a pair of functions up :: b -> a and down :: a -> b such that up.down = id :: a -> a. In other words, down is an injective function with up as a left-inverse. In Gofer, the notion of a retract can be encapsulated as a type class, as follows:

```
class Retract b a where
    up   :: b -> a
    down :: a -> b
```

Using fusion it can be shown that given a difunctor f, if up and down form a retract from (f a a) to a, then (ana up down) and (cata up down) form a retract from a to (Rec f). In Gofer, this can be implemented as follows:

```
instance (Difunctor f, Retract (f a a) a) =>
    Retract a (Rec f) where
        up = ana (up :: Retract (f a a) a =>
                 f a a -> a) (down :: Retract
                 (f a a) a => a -> f a a)
        down = cata (up :: Retract (f a a) a =>
                 f a a -> a) (down :: Retract
                 (f a a) a => a -> f a a)
```

This result will be used in the next section.

4.4 Interpreters for the λ-calculus

We illustrate the generalised theory by defining a class of interpreters for the untyped λ-calculus, and taking some steps towards formally relating such interpreters. We begin by defining a datatype Expr of λ-expressions:

```
data Expr = Var String
          | Lambda String Expr
          | Apply Expr Expr
```

The datatype Expr could of course be expressed as the fixed point of a functor, but we don't do this here, preferring instead to concentrate on the use of fixed points in defining the semantic domains for λ-interpreters.

A datatype can serve as such a semantic domain if it is reflexive [4]. Formally, a type a is *reflexive* if there is a retract from a to (a -> a). (The notion of a retract was defined in the previous section.) Intuitively then, a type is reflexive if it is large enough to faithfully represent its own function-space. In Gofer, the notion of a reflexive type can be encapsulated as a type class, as follows:

```
class Retract a (a -> a) => Reflexive a where
    apply :: a -> (a -> a)
    abstr :: (a -> a) -> a

    f 'apply' a = up f a
    abstr f     = down f
```

We can now define a class of interpreters (one for each reflexive type a) that map a λ-expression to its value in the semantic domain a; as usual, an environment carries the values of the free variables in the expression:

```
class Reflexive a => LambdaModel a where
    eval :: Expr -> Env a -> a

    eval (Var x) env =
        env 'lookup' x
    eval (Lambda x b) env =
        abstr (\a -> eval b (env 'update' (x,a)))
    eval (Apply f a) env =
        (eval f env) 'apply' (eval a env)
```

Environments are represented as functions from identifiers to values, and are equipped with two operations:

```
type Env a = String -> a in
                    lookup, update, mapEnv

lookup :: Env a -> String -> a
update :: Env a -> (String,a) -> Env a

env 'lookup' x = env x

env 'update' (x,a) =
    \y -> if y==x then a else env 'lookup' y
```

Later on in this section, we will use the fact that Env can be extended to a functor, as follows:

```
mapEnv :: (a -> b) -> (Env a -> Env b)
mapEnv = (.)

instance Functor Env where
    map = mapEnv
```

The standard (call by name) interpreter for the untyped λ-calculus is obtained by taking the reflexive datatype Scott of section 3.3 as the semantic domain:

```
instance Retract Scott (Scott -> Scott) where
    up   (In (Func f)) = f
    down f             = In (Func f)

instance Reflexive Scott

instance LambdaModel Scott
```

An appropriate reflexive type Cont for a (call by name) continuation-based semantics for λ-expressions is defined by:

4.1 Difunctors and recursive datatypes

Given a difunctor f, its induced recursive datatype Rec f is defined as the simultaneous fixed point of f in both arguments. In Gofer this definition for Rec f can be implemented as follows (as previously, strictness of the constructor In is necessary to obtain an isomorphism):
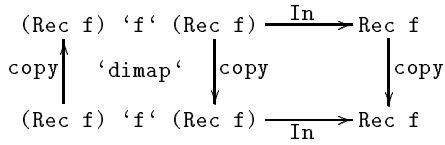
```
data Rec f =
    In (f (Rec f) (Rec f)) {- #STRICT# -}
```

4.2 Catamorphisms and anamorphisms

An isomorphism between types f a a and a is called an f-invariant. An example of an f-invariant is In :: f (Rec f) (Rec f) -> Rec f. It is also the minimal f-invariant, in the sense that copy = id, where

```
copy :: Difunctor f => (Rec f -> Rec f)
copy (In x) = In ((copy 'dimap' copy) x)
```

This definition can be expressed in diagrams by

Note that by drawing the arrows g :: a -> b and h :: c -> d of a difunctor g 'dimap' h :: (b 'f' c) -> (a 'f' d) separately, both the contravariance and typing assumptions of dimap are made explicit.

For datatypes expressed as fixed points of functors, the notion of a catamorphism arose by abstracting on In in the body of the definition of copy. Let us now try to play the same game for the difunctors version of copy. As a first attempt, abstracting (naively) on In in the body of the difunctors version of copy gives the definition

```
cata phi (In x) =
    phi (((cata phi) 'dimap' (cata phi)) x)
```

However this definition is too restrictive, since it forces the argument function phi to have type f (Rec f) (Rec f) -> Rec f, and cata phi itself to have type Rec f -> Rec f.

The problem is the use of cata phi as *both* the covariant and contravariant argument of dimap in the definition. The covariant use of cata phi requires that the argument function phi have type f b a -> a; a function of this type is called an f-*dialgebra* [12]. The additional contravariant use of cata phi then requires that a = b = Rec f, i.e. that phi have type f (Rec f) (Rec f) -> Rec f.

As a first step to solving this problem, let us assume the existence of a function g :: b -> Rec f to use as the contravariant argument of dimap in the body of cata phi, rather than cata phi itself. This assumption leads to a definition for cata phi with sufficiently general typing requirements, as illustrated by the following diagram:
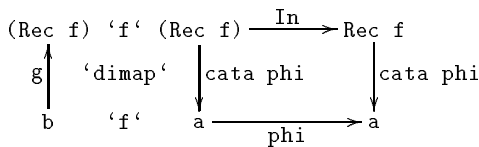
A similar problem occurs with the naive generalisation of copy to obtain an anamorphism functional:

```
ana psi x =
    In (((ana psi) 'dimap' (ana psi)) (psi x))
```

The covariant use of ana psi here requires that psi have type b -> f a b; a function of this type is called an f-*codialgebra*. The additional contravariant use of ana psi then requires that a = b = Rec f, i.e. that psi have type Rec f -> f (Rec f) (Rec f), and hence that ana psi have type Rec f -> Rec f. However, a definition for ana psi with sufficiently general typing requirements can be obtained by assuming the existence of a function h :: Rec f -> a to use as the contravariant argument of dimap:
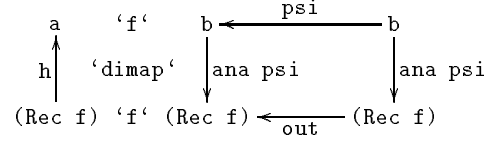
Let us now consider the above diagrams for cata phi and ana psi simultaneously. We observe that a function g :: b -> Rec f required to define cata phi can be obtained simply as g = ana psi, and similarly, a function h :: Rec f -> a required to define ana psi can be obtained as h = cata phi. Thus we are naturally led to the following mutually recursive definitions for cata and ana on datatypes expressed as fixed points of difunctors:

```
cata :: Difunctor f =>
    (f b a -> a) -> (b -> f a b) -> (Rec f -> a)
ana :: Difunctor f =>
    (f b a -> a) -> (b -> f a b) -> (b -> Rec f)

cata phi psi (In x) =
    phi (((ana phi psi)
        'dimap' (cata phi psi)) x)
ana phi psi x =
    In (((cata phi psi)
        'dimap' (ana phi psi)) (psi x))
```

Note that the difunctor versions of cata and ana above are proper generalisations of the functor versions from Section 2, in the sense that if the difunctor f is independent of its contravariant argument, the definitions reduce to the standard definitions for functors.

4.3 Free theorems and fusion

Just as was the case for functors, the cata and ana functionals for difunctors satisfy a fusion law, which arises as a free theorem. Because the difunctors versions of cata and ana are defined mutually recursively, we get a simultaneous fusion law for the two functionals, rather than two separate laws as was the case previously: for strict functions h,

$$
\begin{array}{l}
\text{h.phi = phi'.(g 'dimap' h)}\\
\wedge\\
\text{psi.g = (h 'dimap' g).psi'}\\
\Rightarrow\\
\text{h.(cata phi psi) = cata phi' psi'}\\
\wedge\\
\text{(ana phi psi).g = ana phi' psi'}
\end{array}
$$

```
exec :: Functor (S a b) => State a b -> [b]
exec = cata (\x -> case x of
                    Done n    -> [n]
                    Pause n h -> concat (h n))
```

As a simple application, exec and ana can be used to define a function n2d that extracts the list of digits from a number:

```
n2d :: Int -> [Int]
n2d = exec . ana (\x ->
        if x <= 9 then Done x
        else Pause x (\n -> [n `div` 10,
                            n `mod` 10]))
```

For example, n2d $1234 = [1, 2, 3, 4]$.

A more practical use of the type State is the extension of a library of parsing combinators [15] with a combinator for parallel composition of parsers [9].

3.2 Contravariant uses of (->)

An example where (->) is used contravariantly is in the definition of a fixed point combinator by using recursion on types rather than recursion on functions. We first define a type Inf a of functions that yield a result of type a from an infinitely nested argument of such functions:

```
type I a i = (i -> a) in inI, outI

inI  :: (i -> a) -> I a i
outI :: I a i -> (i -> a)

inI  = id
outI = id

type Inf a = Rec (I a)
```

The functions inI and outI above play the rôle of constructor and destructor functions for the type I a i.

Using Inf a we can define Curry's fixed point combinator $Yf = gg$ where $g = \lambda h.f(hh)$ from the untyped λ-calculus in a typed functional language:

```
fix :: (a -> a) -> a
fix f = g (In (inI g))
        where g (In h) = f (outI h (In h))
```

We would like to be able to express recursive functions on Inf a using recursion functionals such as cata, but it is not possible to define a map that makes I a into a covariant functor. However, the following definition of a functional comap makes I a into a contravariant functor:

```
comap :: (b -> c) -> (I a c -> I a b)
comap g h = inI.(h.g).outI
```

A *contravariant* functor is like a covariant functor, except that the functional comap lifts a function g ::   a -> b to a function comap g ::   f b -> f a where the argument and result types have been interchanged. As a consequence, such functors must distribute contravariantly over function composition: comap (g.h) = (comap h).(comap g).

In Gofer, the concept of a contravariant functor can be encapsulated as a constructor class, as follows:

```
class Cofunctor f where
    comap :: (a -> b) -> (f b -> f a)
```

It is possible, with some effort, to define versions of cata, ana, para, and hylo on datatypes expressed as fixed points of contravariant functors, but this would only be a partial solution to the problem. In general, a type constructor involving function-spaces can be of mixed variance.

3.3 Mixed variant uses of (->)

An example where (->) is used both covariantly and contravariantly is in the definition of a type Scott for modelling the untyped (lazy) λ-calculus [1]:

```
data S s = Func (s -> s)

type Scott = Rec S
```

An occurrence of a type variable in a type expression is said to be contravariant if it occurs to the left of an odd number of nested arrows (->), and covariant otherwise. The argument s to S above occurs both covariantly (s -> s̲) and contravariantly (s̲ -> s). The effect is that S cannot be made into a functor, either covariant or contravariant.

We can however make the distinction between the two kinds of occurrences of the argument s in the definition of S explicit by defining a binary type constructor S':

```
data S' s s' = Func (s -> s')
```

By fixing its first argument, S' can be made into a covariant functor; by fixing its second argument, S' can be made into a contravariant functor. In general, a binary type constructor with this property is called a difunctor.

Formally, a *difunctor* [12] is a binary type constructor f that assigns to each pair of types a and b a type f a b, together with a polymorphic functional dimap that lifts a pair of functions g ::   a -> b and h ::   c -> d to a function g `dimap` h ::   f b c -> f a d. A difunctor must also preserve the identity function and distribute over function composition in the following way:

$$
\begin{aligned}
\text{id} \ \text{`dimap`} \ \text{id} \ &= \ \text{id} \\
(g.h) \ \text{`dimap`} \ (i.j) \ &= \ (h \ \text{`dimap`} \ i).(g \ \text{`dimap`} \ j)
\end{aligned}
$$

In Gofer the concept of a difunctor can be encapsulated as a constructor class, as follows:

```
class Difunctor f where
    dimap :: (a -> b) -> (c -> d) ->
                (f b c -> f a d)
```

One can verify now that the following definition for dimap makes the type constructor S' into a difunctor:

```
instance Difunctor S' where
    (f `dimap` g) (Func h) = Func (g.h.f)
```

In the above, the Func constructor only plays an auxiliary rôle. In fact, S' is a difunctor because the function-space constructor (->) is itself a difunctor:

```
instance Difunctor (->) where
    (f `dimap` g) h = g.h.f
```

In general, by separating the covariant and contravariant occurrences of the argument a in the body of a non-recursive datatype declaration data F a = ..., every such type constructor F induces a difunctor F', such that F can be recovered from F' by diagonalising, i.e. F a = F' a a.

4   General datatypes

We have seen in the previous section that (non-recursive) type constructors involving exponentials do not in general induce functors, but do induce difunctors. Freyd [12] presents a categorical theory of recursive datatypes modelled as fixed points of difunctors. In this section we explain how Freyd's work shows how to generalise the recursion functionals cata and ana, together with their associated fusion rules. As was the case previously, cata and ana are obtained by suitably generalising a simple copy function.

In this case there is no strictness requirement on h, since dualising h.bot = bot gives bot.h = bot, which is true for all functions h. Using fusion for anamorphisms, together with the fact that ana out is the identity function on Rec f, we can show that ana psi is in fact the unique function satisfying its defining equation.

2.6 Primitive and general recursion

Meertens has shown that every primitive recursive function, i.e. *paramorphism* [23], can be expressed as an ana followed by a cata. Let us briefly show how paramorphisms can be implemented in Gofer. The first step is to define a family of functors P f, one for each functor f:

```
type P f a = f (Rec f,a) in mapP, para, pp

mapP :: Functor f =>
    (a -> b) -> (P f a -> P f b)
mapP g = map (\(x,a) -> (x, g a))

instance Functor f => Functor (P f) where
    map = mapP
```

Note that P above is defined as a restricted type synonym [16] so that it can be partially applied. As a consequence, the functional mapP cannot be defined directly within the instance declaration for P f. A functional para that builds paramorphisms is defined now by:

```
para :: Functor (P f) =>
    (f (Rec f, a) -> a) -> (Rec f -> a)
para phi = cata phi . preds

preds :: Functor (P f) => Rec f -> Rec (P f)
preds = ana pp

pp :: Functor f => Rec f -> P f (Rec f)
pp (In x) = map (\a -> (a,a)) x
```

Again for technical reasons concerning types, the definition for para above has to be split up into parts.

It came as somewhat of a surprise to the authors to discover that a general fixed point operator can also be defined as the composition of an ana followed by a cata, thus providing the full power of recursion. (We have since discovered that this observation has already been made by Freyd [11].) The effect is that algebraic languages that provide cata and ana as the *only* means to define recursive functions are not limited in expressive power.

Using cata and ana, the least fixed point fix f of a function f can be computed as the infinite application f (f (f ...)) in the following way: first use an anamorphism to build an infinite list In (Cons f (In (Cons f (In (Cons f ...)))))), and then use a catamorphism to replace each constructor Cons by function application.

```
fix :: Functor (L (a -> a)) => (a -> a) -> a
fix = cata (\(Cons f x) -> f x)
        . ana (\f -> Cons f f)
```

In general, many functions can be naturally expressed as the composition of an ana and a cata, so it seems useful to name this idiom. Functions expressed in this way are known as *hylomorphisms* [27]:

```
hylo :: Functor f =>
    (f a -> a) -> (b -> f b) -> (b -> a)
hylo phi psi = cata phi . ana psi
```

A straightforward fixed point induction shows that the two constituents of a hylomorphism can be fused together to give a direct recursive definition that avoids building an intermediate [37] (or virtual [35]) value:

```
hylo phi psi = phi . map (hylo phi psi) . psi
```

For example, if we express fix as a hylomorphism rather than the composition of a cata and an ana,

```
fix f =
    hylo (\(Cons f x) -> f x) (\f -> Cons f f)
```

then by unfolding using the more efficient definition of hylo we find that fix f = f (fix f), as expected.

3 Problems with exponentials

In the previous section we reviewed how the functionals foldr and unfold are generalised from lists to polynomial datatypes. While such datatypes are sufficient for many programming tasks, a central aspect of functional programming is that functions are first-class values.

However, exponentials (function-spaces) are problematic because the type constructor (->) is contravariant in its first argument. The effect is that certain type constructors defined using (->) cannot be made into functors, and as a result, functionals such as cata and ana cannot always be used to define functions on recursive datatypes involving exponentials. This section gives a number of examples of recursive datatypes involving exponentials, and elaborates on the problems with such datatypes.

3.1 Covariant uses of (->)

An example in which function-spaces are used covariantly is that of non-deterministic computations [32]. An element of datatype State a b is either a final value of type b, or an intermediate state of type a together with a non-deterministic continuation of type a -> [State a b]:

```
data S a b s = Done b | Pause a (a -> [s])

type State a b = Rec (S a b)
```

To make the type constructor S a b into a functor, we first observe that the sub-component (a ->) can itself be made into a functor. That is, fixing the first argument of (->) to a specific type a yields a functor ((->) a). The required type of the map functional for ((->) a) is (b -> c) -> ((->) a b) -> ((->) a c). Using familiar infix notation we recognise (b -> c) -> (a -> b) -> (a -> c) as the type of function composition (.). One can easily verify that (.) indeed makes ((->) a) into a functor.

```
instance Functor ((->) a) where
    map = (.)
```

Now S a b can be made into a functor, as follows:

```
instance (Functor ((->) a), Functor []) =>
    Functor (S a b) where
      map g =
        \x -> case x of
                Done n    -> Done n
                Pause n h ->
                    Pause n (map (map g) h)
```

Note that map for S a b is not recursive; the uses of map in its definition are those for ((->) a) and lists [].

A function exec that forces evaluation of a state to its set of final values can now be defined as a catamorphism:

functor E for arithmetic expressions. The following function (which replaces the constructors `Num` and `Add` by the functions `id` and `(+)`) is an E-algebra of type `E Int -> Int`:

```
\x -> case x of
          Num n    -> id n
          Add e e' -> e + e'
```

Applying `cata` to the above algebra gives the standard evaluator for arithmetic expressions:

```
eval :: Expr -> Int
eval = cata (\x -> case x of
                       Num n    -> id n
                       Add e e' -> e + e')
```

This definition says that expressions can be evaluated by simultaneously replacing all `Num` constructors by the `id` function on integers, and all `Add` constructors by `(+)` on integers. Unfolding the definition to eliminate the use of `cata` and `map` makes clear that it has the expected behaviour:

```
eval (In x) =
    case x of
        Num n    -> n
        Add e e' -> (eval e) + (eval e')
```

2.3 Free theorems and fusion

A useful heuristic in functional programming is to inspect the "free theorem" [36] that comes from the type of a polymorphic function. The free theorem for `cata :: Functor f => (f a -> a) -> (Rec f -> a)` is the well-known *fusion* law [27]: for strict functions h,

$$h.phi = phi'.(map\ h) \implies h.(cata\ phi) = cata\ phi'$$

If we only consider finite elements of `Rec f` the strictness condition on h can be removed. Fusion can also be proved directly using a simple fixed point induction [27], for which it is also necessary that h be strict.

The hidden type information in the fusion law is exposed when using commuting diagrams instead of equations:
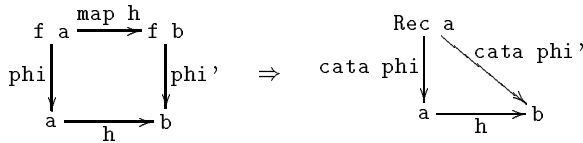
Fusion captures a common pattern of inductive proof on programs expressed as catamorphisms, in a similar way to that in which `cata` itself captures a common pattern of recursion over polynomial datatypes. Minimality and fusion can together be used to show that `cata phi` satisfies a universal property, namely that `cata phi` is the *unique* function satisfying its defining equation.

Returning to our running example, an alternative way to evaluate arithmetic expressions is to use a stack of type `[Int]` to store intermediate values. Such a stack-based evaluator can be defined as follows:

```
eval' :: Expr -> ([Int] -> [Int])
eval' = cata (\x -> case x of
                        Num n    -> push n
                        Add e e' -> add.e'.e)
```

where `push a as = a:as` pushes a number onto the stack and `add (a:b:cs) = (b+a):cs` adds the top two values.

The fact that the stack-based evaluator leaves the expected value on top of the stack, i.e. for all finite expressions `e :: Expr` we have `push (eval e) = eval' e`, can easily be proved using fusion and the distribution of `push` over addition: `push (a+b) = add.(push a).(push b)` [25].

2.4 Coalgebras and anamorphisms

Using `cata` we can define functions with recursive datatypes as their source. Dually, it is also useful to have a functional for defining functions with recursive datatypes as their target. Let us begin by re-writing the function `copy` from which catamorphisms arose in the equivalent form

```
copy :: Functor f => (Rec f -> Rec f)
copy x = In (map copy (out x))
```

where `out (In x) = x` is the inverse of the isomorphism `In`. If we now generalise this version of `copy` by replacing the occurrence of `out :: Rec f -> f (Rec f)` in its definition by an arbitrary function `psi : a -> f a` (an f-*coalgebra*), we obtain the notion of an *anamorphism* [27]:

```
ana :: Functor f => (a -> f a) -> (a -> Rec f)
ana psi x = In (map (ana psi) (psi x))
```

The functional `ana`—written as "lens" brackets ⟦ ⟧ in the Squiggol literature—is the generic version of the recursion functional `unfold` [8, p173] on lists. The Greek preposition $\alpha\nu\alpha$ means upwards, and its use here reflects the fact that `ana psi` recursively builds up its result by decomposing its argument using the function `psi`.

We illustrate the notion of an anamorphism by defining a function `n2b` that converts natural numbers to their binary representation. The first step is to define a type `Bin` of binary numbers as the fixed point of a functor B:

```
data B b = Empty | Zero b | One b

instance Functor B where
    map g = \x -> case x of
                      Empty  -> Empty
                      Zero b -> Zero (g b)
                      One b  -> One (g b)

type Bin = Rec B
```

The binary representation of a natural number is built by recursively splitting off its least significant bit:

```
n2b :: Int -> Bin
n2b = ana (\x -> case x of
                     0     -> Empty
                     2*n   -> Zero n
                     2*n+1 -> One n)
```

For example, `n2b 2 = In (Zero (In (One (In Empty))))`. The dual function `b2n` that converts a binary number back to a natural number can be defined as a catamorphism:
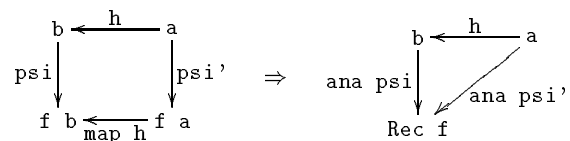
```
b2n :: Bin -> Int
b2n = cata (\x -> case x of
                      Empty  -> 0
                      Zero b -> 2*b
                      One b  -> 1+2*b)
```

2.5 Free theorems and fusion

The free theorem for the functional `ana :: Functor f => (a -> f a) -> (a -> Rec f)` is also a fusion theorem:

$$psi.h = (map\ h).psi' \implies (ana\ psi).h = ana\ psi'$$

or in diagrammatic form,

map g :: f a -> f b. In Gofer, the concept of a functor can be encapsulated as a constructor class, as follows:

```
class Functor f where
    map :: (a -> b) -> (f a -> f b)
```

Such a declaration is not possible using the standard class system, because the parameter f of the class Functor is a type constructor rather than a type.

A familiar example of a functor is the type constructor [] (not to be confused with the empty list []) for lists:

```
instance Functor [] where
    map f xs = [f x | x <- xs]
```

Technically, a functor must also preserve the identity function id and distribute over function composition (.), i.e. the following two equations must hold:

$$\begin{aligned} \text{map} \quad \text{id} &= \text{id} \\ \text{map} \quad (g.h) &= (\text{map } g).(\text{map } h) \end{aligned}$$

However it is not possible to express these extra requirements directly in the Gofer class definition of a functor. It is the responsibility of the programmer to check that they indeed hold for each instance of the class.

Given a functor f, its induced recursive datatype Rec f is defined as the fixed point of f. In Gofer this can be implemented as follows:

```
data Rec f = In (f (Rec f)) {- #STRICT# -}
```

Since Rec f is recursive, we have been forced to define it using data rather than type, and as a consequence have been required to introduce the fictitious strict constructor In. Strictness of In is necessary to obtain an isomorphism between Rec f and f (Rec f). If In was not strict, there would be no value in f (Rec f) that corresponds to the "undefined" value bot in Rec f, defined by bot = bot.

The strictness pragma in the definition of Rec f is not currently permitted in Gofer. However, a number of Haskell implementations permit such constraints in datatype definitions (e.g. [2]), as will future releases of Gofer [18].

Consider a simple datatype of arithmetic expressions, built out of numbers and binary addition:

```
data Expr = Num Int | Add Expr Expr
```

To express this datatype as the fixed point of a functor, we first define a functor E which captures the recursive structure of arithmetic expressions:

```
data E e = Num Int | Add e e

instance Functor E where
    map g = \x -> case x of
                      Num n    -> Num n
                      Add e e' -> Add (g e) (g e')
```

It is a simple exercise to verify that map satisfies the two equations required of a functor. The type Expr of expressions can now be defined as the fixed point of functor E:

```
type Expr = Rec E
```

Some illustrative values of type Expr are

```
In bot
In (Num 3)
In (Add bot bot)
In (Add (In (Num 1)) bot)
In (Add bot (In (Num 5)))
In (Add (In (Num 7)) (In (Num 2)))
...
let e = In (Add e e) in e
```

It is clear from these examples that In plays no essential rôle, except as an explicit type coercion between E Expr and Expr, and in general, between f (Rec f) and Rec f. It is also clear that the type Expr defined using Rec is isomorphic to the original Gofer definition using recursion. If Rec f could be defined as a recursive type synonym, the two types would in fact be identical.

Parameterised datatypes can also be defined as fixed points of functors. The general method is to partially parameterise a binary type constructor with a type variable to give a functor. For example, the datatype

```
data List a = Nil | Cons a (List a)
```

of lists with elements of type a can be defined as follows:

```
data L a l = Nil | Cons a l

instance Functor (L a) where
    map g = \x -> case x of
                      Nil      -> Nil
                      Cons a l -> Cons a (g l)

type List a = Rec (L a)
```

(In the remainder of this paper, only the Rec definition of most recursive datatypes used will be given. Such definitions can be translated to normal Gofer recursive definitions simply by unfolding the definition of Rec.)

Mutually recursive datatypes can be reduced to direct recursive datatypes in a similar way to that in which mutually recursive functions can be reduced to direct recursive functions [5, 10]. So no generality is lost by restricting our attention to direct recursive datatypes.

Note that only the type constructor part of a functor is necessary to express datatypes as fixed points of functors. As we shall see in the next section, the map part comes into play when recursion functionals on datatypes are defined.

2.2 Invariants, algebras and catamorphisms

In Freyd's terminology [12], an isomorphism between types f a and a is an f-*invariant*. An example of an f-invariant is In :: f (Rec f) -> Rec f. Among all possible f-invariants, In is special in the sense that it is the *minimal* f-invariant. Minimality expresses that the function

```
copy :: Functor f => (Rec f -> Rec f)
copy (In x) = In (map copy x)
```

which recursively replaces the constructor In by itself is the identity function on the datatype Rec f. That copy = id holds is easily proved by structural induction.

Suppose now that we generalise copy to replace In not by itself but by an arbitrary function phi :: f a -> a. In this way we obtain the notion of a *catamorphism* [27]:

```
cata :: Functor f => (f a -> a) -> (Rec f -> a)
cata phi (In x) = phi (map (cata phi) x)
```

The functional cata—written as "banana" brackets $(\!|\ \ |\!)$ in the Squiggol literature—is the generic version of the familiar recursion functional foldr on lists, generic in the sense that it can be used with any polynomial datatype. The term catamorphism comes from the Greek preposition $\kappa\alpha\tau\alpha$, meaning downwards, and reflects the fact that cata phi recursively walks down its argument replacing each occurrence of In by a function phi along the way.

Given a functor f and a specific type a, a function phi :: f a -> a is known as an f-*algebra*. Consider again the

Bananas in Space:
Extending Fold and Unfold to Exponential Types

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Abstract

Fold and unfold are general purpose functionals for process-
ing and constructing lists. By using the categorical approach
of modelling recursive datatypes as fixed points of functors,
these functionals and their algebraic properties were gener-
alised from lists to polynomial (sum-of-product) datatypes.
However, the restriction to polynomial datatypes is a serious
limitation: it precludes the use of exponentials (function-
spaces), whereas it is central to functional programming that
functions are first-class values, and so exponentials should
be able to be used freely in datatype definitions. In this
paper we explain how Freyd's work on modelling recursive
datatypes as fixed points of difunctors shows how to gen-
eralise fold and unfold from polynomial datatypes to those
involving exponentials. Knowledge of category theory is not
required; we use Gofer throughout as our meta-language,
making extensive use of constructor classes.

1 Introduction

During the 1980s, Bird and Meertens [6, 22] developed a cal-
culus (nicknamed *Squiggol*) of recursion functionals on lists,
using which efficient functional programs can be derived
from specifications by using equational reasoning. Squiggol
was subsequently generalised from lists to polynomial (sum-
of-product) datatypes [20] by using the categorical approach
of modelling recursive datatypes as fixed points of functors
[21, 14]. This approach allows `foldr`, `unfold` and other re-
cursion functionals to be uniformly generalised from lists to
polynomial datatypes. The generalised functionals are given
special names (such as catamorphism and anamorphism),
and are written symbolically using special brackets (such as
"banana" brackets ⦇ ⦈ and "lens" brackets ⟦ ⟧.) The cat-
egorical approach also provides a number of algebraic laws
that can be used to derive, transform and reason about pro-
grams expressed using these functionals. The theory and
practice of such generic functionals has been explored by
many authors, e.g. [3, 7, 10, 13, 14, 24, 33].

The aim of the *bananas* paper of Meijer, Fokkinga and
Paterson [27] was to bring the ideas of Squiggol closer to lazy
functional languages. This was achieved by moving from the
category *set* of sets and total functions (the world of stan-
dard category theory and Squiggol) to the category *cpo* of
cpos and continuous functions (the world of *cpo*-categories
[12] and lazy functional programming). However, a serious
deficiency of the bananas paper — and more generally, the
work of the Squiggol community — is its limitation to poly-
nomial datatypes [20]. This precludes the use of exponen-
tials (function-spaces), whereas it is central to functional
programming that functions are first-class values, and so
exponentials should be able to be used freely in datatype
definitions. So to truly bring Squiggol closer to functional
programming, the theory must be extended to deal with
datatypes that involve exponentials.

Technically, exponentials are problematic because the
exponential functor is contravariant in its first argument.
A standard solution to the problem is to move from the
category *cpo* to the category cpo^{ep} of cpos and embedding-
projection pairs, on which category the exponential functor
can be made covariant [34]. But while the setting of cpo^{ep}
is technically sufficient, from a practical point of view it is
not a convenient category upon which to base a program-
ming calculus for reasoning about datatypes and recursion
functionals, because the arrows in cpo^{ep} do not naturally
correspond to programs.

An alternative solution that allows us to stay within *cpo*
has been proposed by Freyd [12]. His key idea is to model
recursive datatypes as fixed points of *difunctors*, functors
on two variables, contravariant on the first, covariant on
the second. In the present article (but see also [29, 28]) we
explain to functional programmers how Freyd's work shows
how to generalise fold and unfold from polynomial datatypes
to those involving exponentials.

We use Gofer throughout as our meta-language, making
extensive use of the constructor classes extension to the stan-
dard Gofer (or Haskell) class system [17, 19]. Using Gofer
rather than category theory as our meta-language makes the
concepts more accessible as well as executable, and elimi-
nates the gap between theory and practice.

2 Polynomial datatypes

We begin in this section by reviewing the theory introduced
in the bananas paper, by implementing it in Gofer. In par-
ticular, we implement the generic versions `cata` and `ana` of
the recursion functionals `foldr` and `unfold`.

2.1 Functors and recursive datatypes

A (*covariant*) *functor* is a type constructor `f` that assigns a
type `f a` to each type `a`, together with a polymorphic func-
tional `map` that lifts a function `g :: a -> b` to a function