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Mixture Representations of Reliability in Coherent Systems and Preservation Results under Double Monitoring

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The mixture representations of the reliability functions of the residual life and inactivity time of a coherent system with n independent and identically distributed components are obtained, given that before time t_1 $(t_1 \geq 0)$, exactly r $(r < n)$ components have failed and at time t_2 $(t_2 \geq t_1)$, the system is either still working or has failed. Based on the stochastically ordered coefficient vectors *between systems, some preservation results of the* residual life *and the* inactivity time *of the system are obtained. The results in this paper extend previous results in the literature and are useful for comparing similar systems that have different structure functions.*

Keywords Mixture representation; Double monitoring; Stochastic order; Signature; Inactivity time; Residual life

Mathematics Subject Classification: 60K10 60E15

1. Introduction and Preliminaries

Let T be the lifetime of an unit with distribution function F, density function f and reliability function $\bar{F}$. Then, for any $t \geq 0$, the reliability function and density function of the residual life $(T - t \mid T > t)$ can be expressed as $\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t), h_t(x) = f(t+x)/\bar{F}(t)$, given $\bar{F}(t) > 0$. The reliability function and density function of the inactivity time $(t - T \mid T \leq t)$ can be expressed as $\bar{F}_{(t)}(x) = F(t-x)/F(t), f_{(t)}(x) = f(t-x)/F(t)$, given $F(t) > 0$, for $x > 0$.

Residual life and inactivity time are important concepts in reliability theory, survival analysis, and auctions. Over the last ten years, much research has been done on the distribution of residual lives and inactivity times for certain kinds of coherent systems and especially k-out-of-n systems.

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Some examples include Bairamov, Ahsanullah, and Akhundov (2002), Asadi and Bairamov (2005), Asadi and Bairamov (2006), Navarro and Eryilmaz (2007), Khaledi and Shaked (2007), Gurler and Bairamov (2009), Hu, Jin, and Khaledi (2007), Navarro and Hernandez (2008), Li and Zhao (2008), Wang, Zhuang, and Hu (2010), Tavangar and Asadi (2010), and Hashemi, Tavangar, and Asadi (2010).

The signature of a coherent system is an important tool for investigating the performance of a system structure and comparing different structures. It has been proven to be a useful metric for a system design, as it is a distribution-free measure that efficiently captures important features of a system performance. For example often a poor system design with good components will outperform a good system design with poor components. If, however, two different systems have common components, then any difference between them must be attributable to the system design.

Let $X_1, ..., X_n$ denote absolutely continuous lifetimes of n independent and identically distributed (i.i.d.) components of a coherent system, and let $X_{1:n}, ..., X_{n:n}$ be the corresponding order statistics. The lifetime of a coherent system can be expressed as $T = \tau(X_1, ..., X_n)$, where τ is the coherent system life function. Samaniego (1985) first defined the signature of a coherent system as a probability vector $\mathbf{p} = (p_1, ..., p_n)$ whose element j is the probability that the system fails when component j. That is, $p_j = \Pr(T = X_{j:n})$ for $j = 1, 2, ..., n$, such that $\sum_{j=1}^{n} p_j = 1$. Samaniego (1985) and subsequently Kochar, Mukerjee, and Samaniego (1999) showed that the reliability of a coherent system having n i.i.d. components can be expressed as a discrete a mixture of the reliability functions of k-out-of-n systems with weights p_k for $k = 1, 2, ..., n$. Navarro, Ruiz, and Sandoval (2005) found that a similar result holds when the components of the system are exchangeable (i.e. the joint survival function $R(x_1, ..., x_n)$ is symmetric in $x_1, ..., x_n$). Khaledi and Shaked (2007) further showed that the reliability functions of the residual life (and inactivity time) of a coherent system with n i.i.d. components can be expressed as similar mixtures of the reliability functions (and inactivity time) of k-out-of-n systems. These mixture representations have proven to be useful in the comparison of the performance of competing systems. For some examples, see Kochar, Mukerjee, and Samaniego (1999), Li and Zhang (2008), Zhang (2010), and Navarro, Samaniego, Balakrishnan, and Bhattacharya (2008).

Navarro, Balakrishnan, and Samaniego (2008) showed that if $T = \tau(X_1, ..., X_n)$ is the lifetime of a coherent system with n i.i.d. component lifetimes $X_1, ..., X_n$, each distributed according to a continuous distribution F, then the distribution of the system lifetime T given $T > t$ is a mixture of the residual lifetimes of k-out-of-n systems. That is, for all $t \geq 0$ and $x \geq 0$,

$$\Pr(T - t > x \mid T > t) = \sum_{i=1}^{n} p_i(t) \Pr(X_{i:n} - t > x \mid X_{i:n} > t), \tag{1}$$

where coefficients $p_i(t) = p_i \Pr(X_{i:n} > t)/\bar{F}(t)$ for $i = 1, ..., n$, and $\bar{F}(t) > 0$.

In order to properly state our results, we will use the following stochastic orderings concepts. Let X and Y be the lifetimes of two components, with respective distribution functions F and G, and survival functions $\bar{F}$ and $\bar{G}$. Denote their probability density functions by f and g, respectively. Then X is said to be less than Y in the: (a) usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all x; (b) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{F}(x)/\bar{G}(x)$ is *decreasing* in x; (c) reversed hazard rate order (denoted by $X \leq_{rh} Y$) if $F(x)/G(x)$ is *decreasing* in x; (d) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $f(x)/g(x)$ is *decreasing* in the union of their *supports*. The details of these stochastic orders can be found in Shaked and Shanthikumar (2007). Throughout the paper, the notions of *increasing* and *decreasing* are used in the weaker sense of *non-decreasing* and *non-increasing*, respectively.

By using the mixture representation (1), Navarro, Balakrishnan, and Samaniego (2008) proved the following preservation theorems for system signatures.

Theorem 1 (Navarro, Balakrishnan, and Samaniego 2008) Let $T_1 = \tau_1(X_1, ..., X_n)$ and $T_2 = \tau_2(X_1, ..., X_n)$ be the lifetimes of two coherent systems, both based on n components with i.i.d. lifetimes distributed according to the continuous distribution F. And, for all $t \geq 0$, let $\mathbf{p}(t) = (p_1(t), ..., p_i(t), ..., p_n(t))$ and $\mathbf{q}(t) = (q_1(t), ..., q_i(t), ..., q_n(t))$ be their corresponding coefficient vectors.

(a) If $\mathbf{p}(t) \leq_{st} \mathbf{q}(t)$, then, $(T_1 - t \mid T_1 > t) \leq_{st} (T_2 - t \mid T_2 > t)$;

(b) If $\mathbf{p}(t) \leq_{hr} \mathbf{q}(t)$, then, $(T_1 - t \mid T_1 > t) \leq_{hr} (T_2 - t \mid T_2 > t)$;

(c) If $\mathbf{p}(t) \leq_{lr} \mathbf{q}(t)$, then, $(T_1 - t \mid T_1 > t) \leq_{lr} (T_2 - t \mid T_2 > t)$.

Samaniego, Balakrishnan, and Navarro (2009) define the dynamic signature of a used system with lifetime T. Specifically, let $\mathbf{p}$ be the signature of a coherent system based on n i.i.d. components. Suppose that the system is operating at time t, and the event $\{T > t\} \bigcap \{X_{i:n} < t < X_{i+1:n}\}$ is noted. Then at time t, there are $n - i$ working components. The dynamic signature of the system at time t is the $n - i$ dimensional vector $\mathbf{p}$ whose k-th element is $p_k = \Pr(T = \{X_{k:n} | T > t\} \bigcap \{X_{i:n} < t < X_{i+1:n}\})$ for $k = i + 1, ..., n$. By means of this definition, some existing representations and preservation theorems for system signatures are generalized to dynamic versions. Mahmoudi and Asadi (2011) provide more information about the dynamic signature of coherent systems and some corresponding results.

To obtain more information about system state or to assure safe operation of a system, systems can be inspected. Because continuous inspection may be expensive or impossible (e.g., inspection for crack initiation of n fan blades in an aircraft engine), inspections may be scheduled at times. In a simple situation, two inspections might be scheduled during the system's lifetime. This is called *double monitoring*. For this situation and a $(n-k+1)$-out-of-n system, conditional on the event that exactly r $(r < k)$ components have failed before time t_1 $(t_1 \geq 0)$, Poursaeed (2010) provided some ordering properties for the expected value of the residual lifetime

$$(X_{k:n} - t_2 | X_{r:n} < t_1 < X_{r+1:n}, X_{k:n} > t_2) \tag{2}$$

if the $(n-k+1)$-out-of-n system is still working at time t_2 $(t_2 \geq t_1)$ and for the expected value of the inactivity time

$$(t_2 - X_{k:n} | X_{r:n} < t_1 < X_{r+1:n}, X_{k:n} < t_2), \tag{3}$$

if the $(n-k+1)$-out-of-n system has failed. These results are extensions of those given in Poursaeed and Nematollahi (2008) where the special case of parallel systems $(k = n)$ was studied. Recently, Zhang and Yang (2010) obtained some more general order properties and some stochastic comparisons of residual life (2) and inactivity time (3) for $(n-k+1)$-out-of-n systems having i.i.d. components.

The purpose of this paper is to extend the results described in the previous paragraph to more general coherent systems. Specifically, in Section 2 we obtain a mixture representation of the reliability function of the residual life of a system under double monitoring, some stochastic properties, and preservation theorems for coefficient vectors. In Section 3 we provide similar results for inactivity times.

2. Results for Residual Lifetime

In this section, we investigate the residual lifetime of a coherent system, under the condition that before time t_1 $(t_1 \geq 0)$, exactly r $(r < n)$ components have failed and at time t_2 $(t_2 \geq t_1)$, the system is still working.

Let $T_1 = \tau_1(X_1, X_2, ..., X_n)$, $T_2 = \tau_2(X_1, X_2, ..., X_n)$ be the lifetimes of two coherent systems of size n, both based on components with i.i.d. lifetimes $X_1, X_2, ..., X_n$ having a distribution function F, where τ_1 and τ_2 are coherent life functions. Assume that $\mathbf{p}$ and $\mathbf{q}$ are the signatures of τ_1 and τ_2, respectively. Under double monitoring, if both systems are still working at time t_2 with probability one, then the corresponding signatures must have the forms $\mathbf{p} = (0, ..., 0, p_s, ..., p_n)$, and $\mathbf{q} = (0, ..., 0, q_s, ..., q_n)$, $r < s \leq n$.

In the following theorem we give the mixture representation of the reliability function of the residual life of a coherent system under double monitoring.

Theorem 2 For $t_2 \geq t_1 \geq 0$, $r < s \leq n$ and any $x > 0$,

$$\Pr(T_1 - t_2 > x \mid A_r(t_1), T_1 > t_2) = \sum_{i=s}^{n} p_i(r, t_1, t_2) \Pr(X_{i:n} - t_2 > x, A_r(t_1) | X_{i:n} > t_2, A_r(t_1)),$$

where $A_r(t_1)$ indicates the event $(X_{r:n} < t_1 < X_{r+1:n})$, and

$$p_i(r, t_1, t_2) = \frac{p_i \Pr(X_{i:n} > t_2, A_r(t_1))}{\sum_{i=s}^{n} p_i \Pr(X_{i:n} > t_2, A_r(t_1))} \tag{4}$$

such that $\sum_{i=s}^{n} p_i(r, t_1, t_2) = 1$.

Proof. Note that, for $t_2 \geq t_1 \geq 0$, and $r < i \leq n$,

$$\Pr(X_{i:n} - t_2 > x, A_r(t_1)) = \Pr(X_{i:n} - t_2 > x, A_r(t_1) | X_{i:n} > t_2, A_r(t_1)) \Pr(X_{i:n} > t_2, A_r(t_1)),$$

hence, for $r < s \leq n$ and any $x > 0$,

$$\begin{aligned}
\Pr(T_1 - t_2 > x \mid A_r(t_1), T_1 > t_2) &= \frac{\Pr(T_1 - t_2 > x, A_r(t_1))}{\Pr(A_r(t_1), T_1 > t_2)} \\
&= \frac{\sum_{i=s}^{n} \Pr(T_1 = X_{i:n}, T_1 - t_2 > x, A_r(t_1))}{\sum_{i=s}^{n} \Pr(A_r(t_1), T_1 > t_2, T_1 = X_{i:n})} \\
&= \frac{\sum_{i=s}^{n} \Pr(T_1 = X_{i:n}) \Pr(T_1 - t_2 > x, A_r(t_1) \mid T_1 = X_{i:n})}{\sum_{i=s}^{n} \Pr(T_1 = X_{i:n}) \Pr(A_r(t_1), T_1 > t_2 \mid T_1 = X_{i:n})} \\
&= \frac{\sum_{i=s}^{n} p_i \Pr(T_1 - t_2 > x, A_r(t_1) \mid T_1 = X_{i:n})}{\sum_{i=s}^{n} p_i \Pr(A_r(t_1), T_1 > t_2 \mid T_1 = X_{i:n})} \\
&= \frac{\sum_{i=s}^{n} p_i \Pr(X_{i:n} - t_2 > x, A_r(t_1))}{\sum_{i=s}^{n} p_i \Pr(X_{i:n} > t_2, A_r(t_1))} \\
&= \sum_{i=s}^{n} p_i(r, t_1, t_2) \Pr(X_{i:n} - t_2 > x, A_r(t_1) | X_{i:n} > t_2, A_r(t_1)).
\end{aligned}$$

This completes the proof.

Remark 1. The function $p_i(r, t_1, t_2)$ is $\Pr(T_1 = X_{i:n} | T_1 > t_2, A_r(t_1))$ as follows:

$$\begin{aligned}
\Pr(T_1 = X_{i:n} | T_1 > t_2, A_r(t_1)) &= \frac{\Pr(T_1 = X_{i:n}, T_1 > t_2, A_r(t_1))}{\Pr(T_1 > t_2, A_r(t_1))} \\
&= \frac{\Pr(T_1 = X_{i:n}) \Pr(T_1 > t_2, A_r(t_1) | T_1 = X_{i:n})}{\Pr(T_1 > t_2, A_r(t_1))} \\
&= \frac{\Pr(T_1 = X_{i:n}) \Pr(X_{i:n} > t_2, A_r(t_1) | T_1 = X_{i:n})}{\Pr(T_1 > t_2, A_r(t_1))} \\
&= \frac{p_i \Pr(X_{i:n} > t_2, A_r(t_1))}{\Pr(T_1 > t_2, A_r(t_1))} \\
&= p_i(r, t_1, t_2),
\end{aligned}$$

where the fourth equality follows from the fact that the events $\{T_1 = X_{i:n}\}$ and $\{X_{i:n} > t_2, A_r(t_1)\}$ are independent when the component lifetimes are i.i.d.

Remark 2. It follows from the result of Theorem 2, that the residual lifetime $(T_1 - t_2 \mid A_r(t_1), T_1 > t_2)$ of the system under double monitoring is a mixture of the residual lifetimes $(X_{i:n} - t_2, A_r(t_1)|X_{i:n} > t_2, A_r(t_1))$ of an $(n-i+1)$-out-of-n system under double monitoring with coefficients $p_i(r, t_1, t_2)$ for $i = s, s+1, ..., n$. The coefficients vector

$$\mathbf{p}(r, t_1, t_2) = (0, ..., 0, p_s(r, t_1, t_2), ..., p_n(r, t_1, t_2)), \quad r < s \le n$$

is the conditional distribution of the ordered component lifetimes that would cause the system to fail given $(A_r(t_1), T_1 > t_2)$. The coefficients depend both on the system structure function τ_1 and F. It should be noted that $\mathbf{p}(r, t, t) = (0, ..., 0, p_s, ..., p_n) \equiv \mathbf{p}, \quad r < s \le n$.

The following result shows that any coherent system with signature $\mathbf{p} = (0, ..., 0, p_s, ..., p_n)$ has a tail stochastic behavior similar to that of parallel system.

Theorem 3 Assume that if a coherent system has lifetime T_1 and signature $\mathbf{p} = (0, ..., 0, p_s, ..., p_n)$, then $\lim_{t_2 \to \infty} \mathbf{p}(r, t_1, t_2) = (\underbrace{0, 0, ..., 0}_{n-1 \text{ times}}, 1)$ for fixed $t_1 > 0$.

Proof. Note that, for $t_2 \ge t_1 \ge 0$, and $r < i \le n$,

$$\begin{aligned}
\Pr(X_{i:n} > t_2, A_r(t_1)) &= \binom{n}{r} F^r(t_1) \sum_{j=n-i+1}^{n-r} \binom{n-r}{j} \bar{F}^j(t_2)(\bar{F}(t_1) - \bar{F}(t_2))^{n-r-j} \\
&= \binom{n}{r} F^r(t_1)\bar{F}^{n-r}(t_1) \sum_{j=n-i+1}^{n-r} \binom{n-r}{j} \frac{\bar{F}^j(t_2)}{\bar{F}^j(t_1)} \left(1 - \frac{\bar{F}(t_2)}{\bar{F}(t_1)}\right)^{n-r-j} \\
&= \binom{n}{r} F^r(t_1)\bar{F}^{n-r}(t_1) \sum_{j=n-i+1}^{n-r} (-1)^{j-n+i-1} \binom{j-1}{n-i} \binom{n-r}{j} \bar{F}_{t_1}^j(\Delta t),
\end{aligned}$$

where the last equality follows from David and Nagaraja (2003), and $\Delta t = t_2 - t_1$. Thus it can be easily obtained that

$$\lim_{t_2 \to \infty} \frac{\Pr(X_{i:n} > t_2, A_r(t_1))}{\Pr(X_{k:n} > t_2, A_r(t_1))} = \begin{cases} +\infty, & \text{if } i > k, \\ 1, & \text{if } i = k, \\ 0, & \text{if } i < k. \end{cases}$$

It follows that

$$\lim_{t_2 \to \infty} p_k(r, t_1, t_2) = \lim_{t_2 \to \infty} \frac{p_k \Pr(X_{k:n} > t_2, A_r(t_1))}{\sum_{i=s}^{n} p_i \Pr(X_{i:n} > t_2, A_r(t_1))} = \begin{cases} 1, & \text{if } k = n, \\ 0, & \text{if } k < n, \end{cases}$$

Hence the result holds.

Theorem 4 Let T_1, T_2 be the lifetimes of two coherent systems of size n, both based on components with i.i.d. lifetimes $X_1, X_2, ..., X_n$ distributed according to a distribution function F. Let $\mathbf{p} = (0, ..., 0, p_s, ..., p_n)$ and $\mathbf{q} = (0, ..., 0, q_s, ..., q_n)$ be the corresponding system signatures. For any $t_2 \geq t_1 \geq 0$, suppose $\mathbf{p}(r, t_1, t_2) = (0, ..., 0, p_s(r, t_1, t_2), ..., p_n(r, t_1, t_2))$, $\mathbf{q}(r, t_1, t_2) = (0, ..., 0, q_s(r, t_1, t_2), ..., q_n(r, t_1, t_2))$ are the corresponding coefficient vectors, for $r < s \leq n$. Then the following results hold.

(a) If $\mathbf{p}(r, t_1, t_2) \leq_{st} \mathbf{q}(r, t_1, t_2)$, then, for any $t_2 \geq t_1 \geq 0$,

$$(T_1 - t_2 \mid A_r(t_1), T_1 > t_2) \leq_{st} (T_2 - t_2 \mid A_r(t_1), T_2 > t_2);$$

(b) If $\mathbf{p}(r, t_1, t_2) \leq_{hr} \mathbf{q}(r, t_1, t_2)$, then, for any $t_2 \geq t_1 \geq 0$,

$$(T_1 - t_2 \mid A_r(t_1), T_1 > t_2) \leq_{hr} (T_2 - t_2 \mid A_r(t_1), T_2 > t_2);$$

(c) If $\mathbf{p}(r, t_1, t_2) \leq_{lr} \mathbf{q}(r, t_1, t_2)$, then, for any $t_2 \geq t_1 \geq 0$,

$$(T_1 - t_2 \mid A_r(t_1), T_1 > t_2) \leq_{lr} (T_2 - t_2 \mid A_r(t_1), T_2 > t_2).$$

Proof. From Shaked and Shanthikumar (2007) it is well-known that $X_{i:n} \leq_{lr} X_{i+1:n}$ for $i = 1, ..., n-1$, which implies that

$$(X_{i:n} - t_2, A_r(t_1) | X_{i:n} > t_2, A_r(t_1)) \leq_{lr} (X_{i+1:n} - t_2, A_r(t_1) | X_{i+1:n} > t_2, A_r(t_1))$$

(see Theorem 1.C.6 of Shaked and Shanthikumar 2007), and hence

$$(X_{i:n} - t_2, A_r(t_1) | X_{i:n} > t_2, A_r(t_1)) \leq_{st} (\leq_{hr})(X_{i+1:n} - t_2, A_r(t_1) | X_{i+1:n} > t_2, A_r(t_1)).$$

Using the conditions and Theorems 1.A.6, 1.B.14, and 1.C.17 of Shaked and Shanthikumar (2007), respectively, it is easy to show (a), (b), and (c). This completes the proof.

Example 5 Consider the two coherent systems of order 5 depicted in Figure 1. The signature of the system on the left with lifetime $\max\{X_1, X_2, \min\{X_3, X_4, X_5\}\}$ is $\mathbf{q} = \left(0, 0, \frac{3}{10}, \frac{3}{10}, \frac{2}{5}\right)$. With some computations, one can show that, for any $t_2 \geq t_1 \geq 0$ and $r = 1$ (or 2), the corresponding coefficient vector is

$$\mathbf{q}(r, t_1, t_2) = \left(0, 0, \frac{3\bar{F}^2(t_2)}{A(t_1, t_2)}, \frac{9\bar{F}(t_1)\bar{F}(t_2) - 6\bar{F}^2(t_2)}{A(t_1, t_2)}, \frac{4\bar{F}^2(t_2) - 12\bar{F}(t_1)\bar{F}(t_2) + 12\bar{F}^2(t_1)}{A(t_1, t_2)}\right),$$

where $A(t_1, t_2) = \bar{F}^2(t_2) + 12\bar{F}^2(t_1) - 3\bar{F}(t_1)\bar{F}(t_2)$. The signature of the system on the right with lifetime $\max\{\min\{X_1, X_2\}, \min\{X_3, X_4\}, X_5\}$ is $\mathbf{p} = \left(0, 0, \frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$, and for any $t_2 \geq t_1 \geq 0$ and $r = 1$ (or 2), the corresponding coefficient vector is

$$\mathbf{p}(r, t_1, t_2) = \left(0, 0, \frac{2\bar{F}^2(t_2)}{B(t_1, t_2)}, \frac{6\bar{F}(t_1)\bar{F}(t_2) - 4\bar{F}^2(t_2)}{B(t_1, t_2)}, \frac{\bar{F}^2(t_2) - 3\bar{F}(t_1)\bar{F}(t_2) + 3\bar{F}^2(t_1)}{B(t_1, t_2)}\right),$$

where $B(t_1, t_2) = -\bar{F}^2(t_2) + 3\bar{F}^2(t_1) + 3\bar{F}(t_1)\bar{F}(t_2)$. It can be verified that $\mathbf{p}(r, t_1, t_2) \leq_{lr} \mathbf{q}(r, t_1, t_2)$ and hence $\mathbf{p}(r, t_1, t_2) \leq_{hr} (\leq_{st})\mathbf{q}(r, t_1, t_2)$. By Theorem 4 the system on the left is better in the sense that it has a stochastically longer general residual life under the condition that at time t_1 ($t_1 \geq 0$), given that exactly 1 (or 2) component(s) has (have) failed and at time t_2 ($t_2 \geq t_1$), the systems are still working.

Figure 1. Two coherent systems with likelihood ratio ordered coefficient vectors

Remark 3. Samaniego, Balakrishnan, and Navarro (2009) have shown that if a system of order n is operating and is inspected at time t ($t \geq 0$) and it is noted that precisely r failures have occurred, then the vector $\mathbf{p} \in [0,1]^{n-r}$ whose jth element is the probability that the $(r+j)$th component failure is fatal to the system for $j = 1, 2, ..., n-r$, is a distribution-free measure of the design of the residual life of the system. Based on this fact they defined the dynamic signature of a working but used system having age t ($t \geq 0$). If $t_2 = t_1 = t$ in Theorem 2, then, by Balakrishnan, and Navarro (2009), the residual reliability function of working but used system can be represented as a mixture of the distribution of order statistic from a random sample of size $n - r$ from the same distribution as $(X_1 - t | X_1 > t)$ with the dynamic signatures being weights.

3. Results for Inactivity Times

In this section, we investigate the inactivity times of coherent systems, under the condition that before time t_1 ($t_1 \geq 0$), exactly r ($r < n$) components have failed and at time t_2 ($t_2 \geq t_1$), the systems have failed.

Let $T_1 = \tau_1(X_1, X_2, ..., X_n)$ and $T_2 = \tau_2(X_1, X_2, ..., X_n)$ be the lifetimes of two coherent systems of size n, both based on components with i.i.d. lifetimes $X_1, X_2, ..., X_n$ having distribution function F. Assume that $\mathbf{p}$ and $\mathbf{q}$ are the signatures corresponding to τ_1 and τ_2, respectively. Given that at time t_1 ($t_1 \geq 0$), exactly r ($r < n$) components have failed and at time t_2 ($t_2 \geq t_1$), both systems have failed, the corresponding signatures must have the forms $\mathbf{p} = (0, ..., 0, p_l, ..., p_m, 0, ..., 0)$, $\mathbf{q} = (0, ..., 0, q_l, ..., q_m, 0, ..., 0)$, for $r < l \leq m \leq n$.

Similar to Theorem 2, it can be shown that if T_1 is the lifetime of a coherent system with signature $\mathbf{p} = (0, ..., 0, p_l, ..., p_m, 0, ..., 0)$ for $r < l \le m \le n$, then the reliability function of the inactivity time of the coherent system is a mixture of the reliability functions of the inactivity time of i-out-of-n systems under double monitoring, as described in the following theorem.

Theorem 6 For $r < l \le m \le n$, $0 \le t_1 \le t_2$ and any $x > 0$,

$$\Pr(t_2 - T_1 > x \mid A_r(t_1), T_1 < t_2) = \sum_{i=l}^{m} p_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1)),$$

where

$$p_i(r, t_1, t_2) = \frac{p_i \Pr(X_{i:n} < t_2, A_r(t_1))}{\sum_{i=l}^{m} p_i \Pr(X_{i:n} < t_2, A_r(t_1))} \tag{5}$$

for $i = l, ..., m$, such that $\sum_{i=l}^{m} p_i(r, t_1, t_2) = 1$.

Remark 4. The function $p_i(r, t_1, t_2)$ can be shown to be the probability of $\Pr(T_1 = X_{i:n}|T_1 < t_2, A_r(t_1))$ as follows:

$$\begin{aligned}
\Pr(T_1 = X_{i:n}|T_1 < t_2, A_r(t_1)) &= \frac{\Pr(T_1 = X_{i:n}, T_1 < t_2, A_r(t_1))}{\Pr(T_1 < t_2, A_r(t_1))} \\
&= \frac{\Pr(T_1 = X_{i:n}) \Pr(T_1 < t_2, A_r(t_1)|T_1 = X_{i:n})}{\Pr(T_1 < t_2, A_r(t_1))} \\
&= \frac{\Pr(T_1 = X_{i:n}) \Pr(X_{i:n} < t_2, A_r(t_1)|T_1 = X_{i:n})}{\Pr(T_1 < t_2, A_r(t_1))} \\
&= \frac{p_i \Pr(X_{i:n} < t_2, A_r(t_1))}{\Pr(T_1 < t_2, A_r(t_1))} \\
&= p_i(r, t_1, t_2),
\end{aligned}$$

where the forth equality follows from the fact that the events $\{T_1 = X_{i:n}\}$ and $\{X_{i:n} < t_2, A_r(t_1)\}$ are independent when the components lifetimes are i.i.d.

Remark 5. It follows from the result of Theorem 6, that the inactivity time $(t_2 - T_1 \mid A_r(t_1), T_1 < t_2)$ of the system under double monitoring is a mixture of the inactivity times $(t_2 - X_{i:n}, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$ of i-out-of-n systems under double monitoring with coefficients $p_i(r, t_1, t_2)$ for $i = l, l + 1, ..., m$. Here the vector of coefficients

$$\mathbf{p}(r, t_1, t_2) = (0, ..., 0, p_l(r, t_1, t_2), ..., p_m(r, t_1, t_2), 0, ..., 0), \quad n \ge r > s$$

is the conditional distribution of the ordered component lifetimes that are fatal to the system given $(A_r(t_1), T_1 < t_2)$. These coefficients depend on both the coherent system life function τ_1 and on F.

The result below shows that any coherent system with signature $\mathbf{p} = (0, ..., 0, p_l, ..., p_m, 0, ..., 0)$ has stochastic behavior that is similar to a $(n - l + 1)$-out-of-n system.

Theorem 7 Assume that a coherent system has signature $\mathbf{p} = (0, ..., 0, p_l, ..., p_m, 0, ..., 0)$ and lifetime T_1, then $\lim_{t_2 \to \infty} \mathbf{p}(r, t_1, t_2) = (\underbrace{0, ..., 0,}_{l-1 \text{ times}}\ 1,\ \underbrace{0, ..., 0}_{n-l \text{ times}})$ for fixed $t_1 > 0$.

The proof of Theorem 7 is the same as that of Theorem 3, and hence is omitted.

Recall that a bivariate function $f(x, y) \geq 0$ is said to be *totally positive of order two*, abbreviated as TP2, if $f(x, y)f(x', y') \geq f(x', y)f(x, y')$ for all $x \leq x', y \leq y'$. When the inequality above is reversed, $f(x, y)$ is said to be *reverse regular of order two*, abbreviated as RR2. For more details about TP2 and RR2, see Karlin (1968).

The following lemma is useful in proving Theorem 9.

Lemma 8 (Shaked and Shanthikumar (2007)) Let α and β be real valued functions such that α is nonnegative and β/α and α are non-increasing. If X_i has distribution F_i, $i = 1, 2$ and $X_1 \leq_{rh} X_2$, then

$$\frac{\int_{-\infty}^{\infty} \beta(x)\, \mathrm{d}F_1(x)}{\int_{-\infty}^{\infty} \alpha(x)\, \mathrm{d}F_1(x)} \geq \frac{\int_{-\infty}^{\infty} \beta(x)\, \mathrm{d}F_2(x)}{\int_{-\infty}^{\infty} \alpha(x)\, \mathrm{d}F_2(x)}.$$

Theorem 9 Suppose $\mathbf{p} = (0, ..., 0, p_l, ..., p_m, 0, ..., 0)$, $\mathbf{q} = (0, ..., 0, q_l, ..., q_m, 0, ..., 0)$ are the respective signatures of two coherent systems with lifetimes T_1, T_2 having common i.i.d. component lifetimes $X_1, X_2, ..., X_n$, for $r < l \leq m \leq n$. For any $0 \leq t_1 \leq t_2$,

$$\mathbf{p}(r, t_1, t_2) = (0, ..., 0, p_l(r, t_1, t_2), ..., p_m(r, t_1, t_2), 0, ..., 0),$$

$$\mathbf{q}(r, t_1, t_2) = (0, ..., 0, q_l(r, t_1, t_2), ..., q_m(r, t_1, t_2), 0, ..., 0)$$

are the corresponding vectors of coefficients, for $r < s \leq n$. The following are true:

(a) If $\mathbf{p}(r, t_1, t_2) \leq_{st} \mathbf{q}(r, t_1, t_2)$, then, for $t_2 \geq t_1 \geq 0$,

$$(t_2 - T_1 \mid A_r(t_1), T_1 < t_2) \geq_{st} (t_2 - T_2 \mid A_r(t_1), T_2 < t_2);$$

(b) If $\mathbf{p}(r, t_1, t_2) \leq_{rh} \mathbf{q}(r, t_1, t_2)$, then, for $t_2 \geq t_1 \geq 0$,

$$(t_2 - T_1 \mid A_r(t_1), T_1 < t_2) \geq_{hr} (t_2 - T_2 \mid A_r(t_1), T_2 < t_2);$$

(c) If $\mathbf{p}(r, t_1, t_2) \leq_{lr} \mathbf{q}(r, t_1, t_2)$, then, for $t_2 \geq t_1 \geq 0$,

$$(t_2 - T_1 \mid A_r(t_1), T_1 < t_2) \geq_{lr} (t_2 - T_2 \mid A_r(t_1), T_2 < t_2).$$

Proof. (a) Note that $(t_2 - X_{i:n}, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$ is decreasing in i in the likelihood ratio order, which implies that $\Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$ is decreasing in $i = l, ..., m$. Hence

$$\begin{aligned}\Pr(t_2 - T_1 > x \mid A_r(t_1), T_1 < t_2) &= \sum_{i=l}^{m} p_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1)) \\ &\geq \sum_{i=l}^{m} q_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1)) \\ &= \Pr(t_2 - T_2 > x \mid A_r(t_1), T_2 < t_2).\end{aligned}$$

The inequality follows from Shaked and Shanthikumar (2007). This proves the result of (a).

(b) To prove this part, it is enough to show that

$$\frac{\Pr(t_2 - T_1 > x \mid A_r(t_1), T_1 < t_2)}{\Pr(t_2 - T_2 > x \mid A_r(t_1), T_2 < t_2)} = \frac{\sum_{i=l}^{m} p_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))}{\sum_{i=l}^{m} q_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))}$$

is increasing in $x \geq 0$, that is, for all $x_2 \geq x_1 \geq 0$,

$$\begin{aligned}&\frac{\sum_{i=l}^{m} p_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x_2, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))}{\sum_{i=l}^{m} p_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x_1, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))} \\ &\geq \frac{\sum_{i=l}^{m} q_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x_2, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))}{\sum_{i=l}^{m} q_i(r, t_1, t_2) \Pr(t_2 - X_{i:n} > x_1, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))}.\end{aligned}$$

Let $\alpha(i) = \Pr(t_2 - X_{i:n} > x_1, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$ and $\beta(i) = \Pr(t_2 - X_{i:n} > x_2, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$. Note that $\alpha(i)$ is decreasing in $i = l, ..., m$, and it is easy to show that $\Pr(t_2 - X_{i:n} > x, A_r(t_1)|X_{i:n} < t_2, A_r(t_1))$ is RR2 in $i = l, ..., m$ and $x \geq 0$, which implies that $\beta(i)/\alpha(i)$ is decreasing in i. Hence from Lemma 8, the desired result of (b) follows.

(c) Let $f_{r:n}^{t_1,(t_2)}(x)$, $g_{r:n}^{t_1,(t_2)}(x)$ denote the density functions of $(t_2 - T_1 \mid A_r(t_1), T_1 < t_2)$ and $(t_2 - T_2 \mid A_r(t_1), T_2 < t_2)$, respectively. By Theorem 6, we have

$$f_{r:n}^{t_1,(t_2)}(x) = \sum_{i=l}^{m} p_i(r, t_1, t_2) f_{X_{i:r:n}^{t_1,(t_2)}}(x),$$

$$g_{r:n}^{t_1,(t_2)}(x) = \sum_{i=l}^{m} q_i(r, t_1, t_2) f_{X_{i:r:n}^{t_1,(t_2)}}(x),$$

where $f_{X_{i:r:n}^{t_1,(t_2)}}(x)$ is the density function of the random variable $(t_2 - X_{i:n}, A_r(t_1) | X_{i:n} < t_2, A_r(t_1))$.

For $c > 0$, let us consider the function

$$h(x) = f_{r:n}^{t_1,(t_2)}(x) - c g_{r:n}^{t_1,(t_2)}(x) = \sum_{i=l}^{m} [p_i(r, t_1, t_2) - c q_i(r, t_1, t_2)] f_{X_{i:r:n}^{t_1,(t_2)}}(x).$$

From Theorem 1.C.37 of Shaked and Shanthikumar (2007), again, for fixed r and n, $f_{X_{i:r:n}^{t_1,(t_2)}}(x)$ is RR2 in $x \in \Re^+$ and $i = l, ..., m$. Because $\mathbf{p}(r, t_1, t_2) \le_{lr} \mathbf{q}(r, t_1, t_2)$, $p_i(r, t_1, t_2)/q_i(r, t_1, t_2)$ is decreasing in $i = l, ..., m$, the sequence $\{p_i(r, t_1, t_2) - c q_i(r, t_1, t_2)\}$ has at most one change of sign from positive to negative as i ranges from l to m. By Karlin (1968), we have $h(x)$ has at most one change of sign from positive to negative as x increases. This proves the result of (c).

Example 10 Consider the two systems of order 5 depicted in Figure 2. The signature of the system on the left with lifetime $\max\{\min\{X_1, X_2\}, \min\{X_3, X_4, X_5\}\}$ is $\mathbf{p} = \left(0, \frac{3}{5}, \frac{3}{10}, \frac{1}{10}, 0\right)$, and for any $t_2 \ge t_1 \ge 0$ and $r = 1$, the corresponding coefficient vector is

$$\mathbf{p}(1, t_1, t_2) = \left(0, \frac{3\Pr(X_{2:5} < t_2, E_1(t_1))}{5B(t_1, t_2)}, \frac{3\Pr(X_{3:5} < t_2, E_1(t_1))}{10B(t_1, t_2)}, \frac{\Pr(X_{4:5} < t_2, E_1(t_1))}{10B(t_1, t_2)}, 0\right),$$

where $B(t_1, t_2) = \sum_{i=2}^{4} p_i \Pr(X_{i:5} < t_2, E_1(t_1))$. The signature of the system on the right with lifetime $\max\{\min\{X_1, X_2\}, \min\{X_3, \max\{X_4, X_5\}\}\}$ is $\mathbf{q} = \left(0, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, 0\right)$, and for any $t_2 \ge t_1 \ge 0$, the corresponding coefficient vector is

$$\mathbf{q}(1, t_1, t_2) = \left(0, \frac{\Pr(X_{2:5} < t_2, A_1(t_1))}{5C(t_1, t_2)}, \frac{3\Pr(X_{3:5} < t_2, A_1(t_1))}{5C(t_1, t_2)}, \frac{\Pr(X_{4:5} < t_2, A_1(t_1))}{5C(t_1, t_2)}, 0\right),$$

where $C(t_1, t_2) = \sum_{i=2}^{4} q_i \Pr(X_{i:5} < t_2, E_1(t_1))$. It can be verified that $\mathbf{p}(1, t_1, t_2) \le_{lr} \mathbf{q}(1, t_1, t_2)$, and hence $\mathbf{p}(1, t_1, t_2) \le_{rh} (\le_{st}) \mathbf{q}(1, t_1, t_2)$. By the Theorem 9, the system in the left side is better than the system on the right in the sense of inactivity time given that at time t_1 ($t_1 \ge 0$), exactly 1 component has failed and at time t_2 ($t_2 \ge t_1$), both systems have failed.

Figure 2. Two coherent systems with likelihood ratio ordered coefficient vectors

Conclusions

In this paper, the mixture representations of the reliability functions of the residual life and inactivity time of a coherent system with n i.i.d. components are obtained, given some particular information on the state of the coherent system at inspection at times t_1 and t_2 ($t_2 \geq t_1$). Some preservation results of the *residual live* and the *inactivity time* of the system are obtained. The application of these results is illustrated using examples in which the system's reliabilities are computed and compared. The results extend previous ones in the literature and are useful for comparing similar systems that have different structure functions.

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References

Asadi, M., and Bairamov, I. (2005). A note on the mean residual life function of a parallel system. *Communications in Statistics-Theory and Methods* 34: 475-484.

Asadi, M., and Bairamov, I. (2006). On the mean residual life function of a k-out-of-n structure at the system level. *IEEE Transactions on Reliability* 55: 314-318.

Bairamov, I., Ahsanullah, M., and Akhundov, I. (2002). A residual life function of a system having parallel or series structures. *Journal of Statistical Theory and Applications* 1: 119-131.

David, H. A., and Nagaraja, H. N. (2003). *Order Statistics*, 3rd edition, Wiley, Hoboken.

Gurler, S., and Bairamov, I. (2009). Parallel and k-out-of-n: G system with nonidentical components and their mean residual life functions. *Applied Mathematical Modeling* 33: 1116-1125.

Hashemi, M., Tavangar, M., and Asadi, M. (2010). Some properties of the residual lifetime of progressively Type-II right censored order statistics. *Statistics and Probability Letters* 80, 845-859.

Hu, T., Jin, W., and Khaledi, B. E. (2007). Ordering conditional distributions of generalized order statistics. *Probability in the Engineering and Informational Sciences* 21: 401-417.

Karlin, S. (1968). *Total Positivity.* California: Stanford University Press.

Khaledi, B., and Shaked, M. (2007). Ordering conditional lifetimes of coherent systems. *Journal of Statistical Planning and Inference* 137: 1173-1184.

Kochar, S. C., Mukerjee, H., and Samaniego, F. J. (1999). The signature of a coherent system and its application to comparisons among systems. *Naval Research Logistics* 46: 507-523.

Li, X., and Zhang, Z. (2008). Some stochastic comparisons of conditional coherent systems. *Applied Stochastic Models in Business and Industry* 24: 541-549.

Li, X., and Zhao, P. (2008). Stochastic comparison on general inactivity time and general residual life of k-out-of-n systems. *Communications in Statistics-Simulation and Computation* 37: 1005-1019.

Mahmoudi, M., and Asadi, M. (2011). The dynamic signature of coherent systems. *IEEE Transactions on Reliability* 60: 817-822.

Navarro, J., Balakrishnan, N., and Samaniego, F. J. (2008). Mixture representations of residual lifetimes of used systems. *Journal of Applied Probability* 45: 1097-1112.

Navarro, J., and Eryilmaz, S. (2007). Mean residual lifetimes of consecutive-k-out-of-n systems. *Journal of Applied Probability* 44: 82-98.

Navarro, J., and Hernandez, P. J. (2008). Mean residual life functions of finite mixtures, order statistics and coherent systems. *Metrika* 67: 277-298.

Navarro, J., Ruiz, J. M., and Sandoval, C. J. (2005). A note on comparisons among coherent systems with dependent components using signatures. *Statistics and Probability Letters* 72: 179-185.

Navarro, J., Samaniego, F. J., Balakrishnan, N., and Bhattacharya, D. (2008). On the application and extension of system signatures in engineering reliability. *Naval Research Logistics* 4: 313-327.

Poursaeed, M. H. (2010). A note on the mean past and the mean residual life of a $(n-k+1)$-out-of-n system under multi monitoring. *Statistical Papers* 51, 409-419.

Poursaeed, M. H., and Nematollahi, A. R. (2008). On the mean past and the mean residual life under double monitoring. *Communications in Statistics-Theory and Methods* 37: 1119-1133.

Samaniego, F. J. (1985). On closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability* 34: 69-72.

Samaniego, F. J., Balakrishnan, N., and Navarro, J. (2009). Dynamic signatures and their use in comparing the reliability of new and used systems. *Naval Research Logistics* 56: 577-591.

Shaked, M., and Shanthikumar, J. G. (2007). *Stochastic Orders and Their Applications.* San Diego, Academic Press.

Tavangar, M., and Asadi, M. (2010). A study on the mean past lifetime of the components of $(n-k+1)$-out-of-n system at the system level. *Metrika* 72: 59-73.

Wang, Y., Zhuang, W., and Hu, T. (2010). Conditionally stochastic domination of generalized order statistics from two samples. *Statistical Papers* 51: 369-373.

Zhang, Z. (2010). Ordering conditional general coherent systems with exchangeable components. *Journal of Statistical Planning and Inference* 140: 454-460.

Zhang, Z., and Yang, Y. (2010). Ordered properties on the residual life and inactivity time of $(n-k+1)$-out-of-n systems under double monitoring. *Statistics and Probability Letters* 80: 711-717.