Hunting Forbidden Subgraphs of Regular-Stable Graphs

Philippe Kriz
3300 N.E. 192st apt. 1001
Aventura, FL. 33180
philippekriz@hotmail.com

Mentor: Felix Goldberg, Department of Mathematics

Abstract

We have studied the properties of λ-regular-stability and well-covered λ-regular-stability, introduced by Barbosa and Cardoso. We have found new families of forbidden induced subgraphs for the property of well-covered λ-regular-stability, for every λ>2. We have also shown that for a generalized line graph the two properties are equivalent.

1. Introduction

A graph is a finite nonempty set of *points* (also called *vertices*) linked by *lines* (also called *edges*). Graphs appear in other fields of study such as chemistry, computer science, and mathematics as well.

As we started studying Graph Theory, we took up the subject of λ-regular-stable graphs. First, let us define an *independent set*: it is a set of vertices such that no two of them are adjacent in a graph G. The *independence number* $\alpha(G)$ is the number of vertices in a maximum independent set.

Barbosa and Cardoso [BC] introduced the concept of λ-regular-stable graphs: a graph G is *λ-regular-stable* if there is at least one maximum independent set of vertices, *I*, such that all vertices outside of *I* have the same number, λ, of neighbors inside *I*. We say that *I* has the λ property and that G is λ-regular-stable.

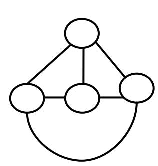


Figure 1: a 1-regular-stable graph.

Another class of graphs that we studied is the class of *well-covered λ-regular-stable graphs*. A graph G is *well covered* if every independent sets of vertices in G can be extended to a maximum independent set. Now, a *well-covered λ-regular-stable graph* is a well covered graph in which every maximum independent set has the λ property. (This definition is also due to [BC]).

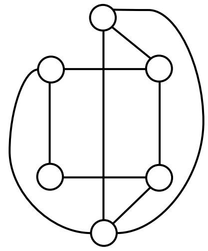


Figure 2: G is well-covered 3-regular-stable.

However note that if a graph is well-covered and λ-regular-stable it does not have to be well-covered λ-regular-stable graph (see figure 3).

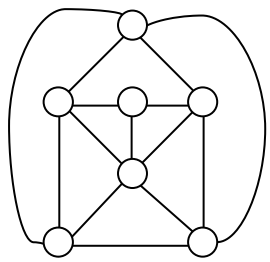


Figure 3: G is 2-regular-stable, but it has a maximum independent set without the λ=2 property.

2. Forbidden Subgraphs

An *induced subgraph* of a graph is formed by choosing some vertices and then taking all edges between the chosen vertices that are present in the original graph. For example graph 2 in figure 4 is an induced subgraph of graph 1 , while graph 3 is not. We denote by $N_G(v)$ the subgraph induced by all the neighbors of the vertex *v*.

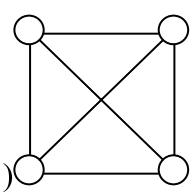
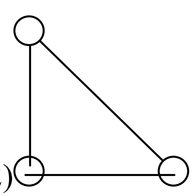
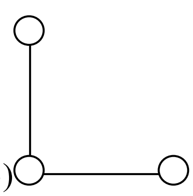
1)  2)  3) 

Figure 4

Graph 2 in figure 4 is induced, while graph 3 is not.

When we have a property P (such being a well-covered λ-regular-stable graph), a graph H is *forbidden for property P* if every graph G that has H as an induced subgraph cannot have property P.

While studying forbidden subgraphs, Barbosa and Cardoso proved the following theorem:

Theorem 1 *Let G be a well-covered λ-regular-stable graph. Then for every vertex v in G:* $\alpha(N_G(v)) = \lambda$

As an immediate consequence of the above theorem Barbosa and Cardoso [BC] gave the following corollary:

Corollary 2 *Let G be a well-covered 2-regular-stable graph. Then G does not have* $K_{1,3}$ *as an induced subgraph.*

They then proceeded to prove the:

Theorem 3 *The graphs given in figure 5 are forbidden induced subgraphs for well-covered 2-regular-stable graphs.*

Let us reproduce the argument from [BC] that shows that the graph (2) in fig 5 is forbidden: In this case, see figure 6, let I be a maximum independent set that contains $\{a, b\}$, then c must have 2 neighbors in I and thus must have a point $w \in I$ as a neighbor. However, since x is not connected neither to z nor to y (as λ=2), we would find an induced $K_{1,3}$ formed by the points *xayz*, a contradiction. ■

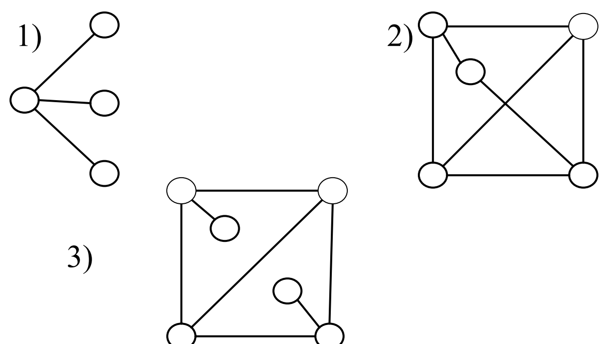


Figure 5: The forbidden induced subgraphs of Theorem 3.

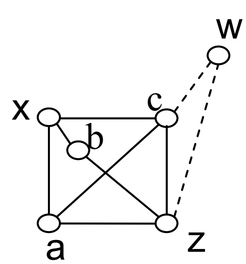


Figure 6: an illustration of the proof of theorem 3

Barbosa and Cardoso posed 2 questions:

The first question asked about whether the graphs in figure 5 are the only forbidden graphs for the family of well-covered 2-regular-stable graphs.

The second question asked to determine the properties of well-covered λ-regular-stable graphs.

We tackled these two questions.

We tried to find more forbidden subgraphs for λ=2 using a construction method that generalizes the argument we reproduced above. The idea was to start with an independent set *J* and a vertex *v* that is connected to less than 2 vertices inside *J*, so that if our subgraph is a subgraph of a well-covered 2-regular-stable graph G, there has to be some a neighbor *w* of *v* that is part of a maximum independent set *I* that contains *J*. We were then looking for an induced $K_{1,3}$ formed by *v, w* and other vertices.

What we found was that each time we sought for a forbidden subgraph in this way we got one of the 3 subgraphs shown in figure 2 or a graph containing one of them as an induced subgraph. We conclude that with this method there is no way to find another minimal forbidden subgraph for λ=2. We think that there are indeed no other minimal forbidden subgraphs for λ=2.

Then we applied our method to look for forbidden subgraphs in the case λ>2. We started out with the following straightforward generalization of Corollary 2:

Proposition 4 *Let G be a well-covered λ-regular-stable graph. Then G does not have* $K_{1,\lambda+1}$ *as an induced subgraph.*

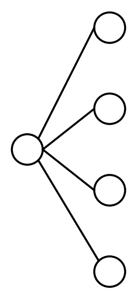


Figure 7: $K_{1,4}$ is a forbidden subgraph for any well-covered 3-regular-stable graph.

Using our method we were able to find new minimum forbidden subgraphs for 3-regular-stable graphs.

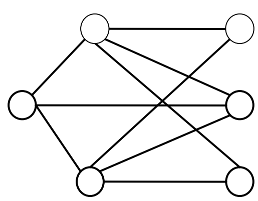


Figure 8: a forbidden subgraph for any well-covered 3-regular-stable graph

Theorem 5 *The graphs given in figures 7 and 8 are forbidden induced subgraphs for well-covered 3-regular-stable graphs.*

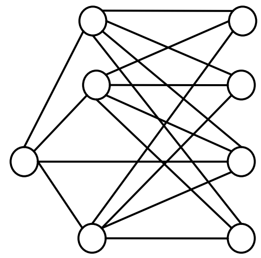


Figure 9: forbidden graph for any 4-regular-stable graph

Proof:
For $K_{1,\,4}$ see Proposition 4. Now we consider the graph in figure 8
In this case, see figure 10, when we choose a maximal independent set I that contains $\{a,b,c\}$, then f must have two vertices $z,w \in I$ as neighbors since I has the $\lambda=3$ property. However, since neither z nor w may be adjacent to e (as $\lambda=3$), we would then have the induced $K_{1,\,4}$ formed by the points $fxed$, a contradiction. ■

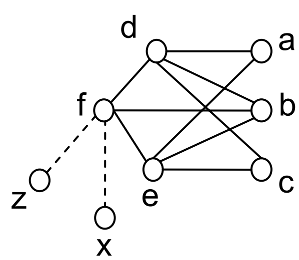


Figure 10: A forbidden induced subgraph

This is just a special case of a more general construction that works for any $\lambda>1$, with essentially the same proof:

Theorem 6 *The graphs given in figures 11, is a forbidden induced subgraphs for well-covered λ-regular-stable graphs.*

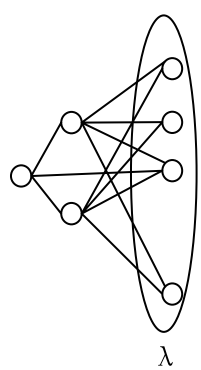


Figure 11: the generalized forbidden subgraph for any $\lambda>1$

The construction of the graph in figure 11 proceeds as follows: first take λ independent vertices, then take two new vertices and connect each to all λ vertices and finally take another new vertex and connect it to the two vertices and one of the λ vertices.

In fact a family of λ-1 different constructions is possible for each $\lambda>1$. Suppose $0<m<\lambda$. First take λ independent vertices, then add m+1 vertices each of which is connected to all λ original vertices, and finally you take a new vertex and connect it to all m+1 vertices chosen in the previous stage, and to m vertices in the independent set. This graph will be called $G_{\lambda, m}$.

Theorem 7 *For any $\lambda>1$ and $0<m<\lambda$, $G_{\lambda, m}$ is a forbidden induced subgraphs for well-covered λ-regular-stable graphs.*

Note that for $\lambda=2$ we get from this construction the graph (2) of figure 5.

3. How many sets with the λ property force well-covered λ-regular-stability?

The property of λ-regular-stability is a local one, whereas well-covered λ- regular-stability is a global one. We naturally wondered under what conditions the local property implied the global one.

First, we found out that no simple numerical condition on the number of maximum independent sets with the λ property is possible. Figure 12 show that all maximum independent sets but one may have the λ property.

The graph in Figure 12 has a feature which we suspect might be responsible for the failure of the global property, in spite of the local property being abundantly present. We formulate this as a conjecture:

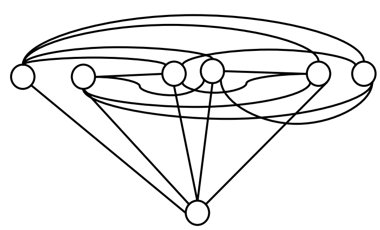


Figure 12: a counter example

Conjecture 8 *If at least two maximum independent sets have property λ, and no vertex belongs to two different maximum independent set, then the graph is well-covered λ-regular-stable.*

For a special class of graphs, the generalized line graphs (see chapter 3 in [CvL]) we have been able to show that the local property implies the global one.

Theorem 9 *If G is a generalized line graph and G is λ-regular-stable, then G is well-covered λ-regular-stable (for $\lambda \geq 2$).*

Proof: By an argument in [BC, p.152], if $\lambda \geq -\lambda_{\min}(A_G)$ (where $\lambda_{\min}(A_G)$ is the smallest eigenvalue of the adjacency matrix A_G of the graph G), then λ-regular-stability of G implies well-covered λ-regular-stable of G. Now, by the well-known result of Cameron, Goethals, Siedel and Shult [CGSS], the generalized line graphs are precisely those graphs with $\lambda_{\min}(A_G) \geq -2$. This completes the proof. ■

4. Acknowledgments

I would like to thank the following people for their support and contribution in this investigation:
Felix Goldberg, Gregory Shapiro, Orr Shalit, Professor Abraham Berman, Stern Family, Kriz Family.

5. References

Barbosa, R. & Cardoso, D. M., On regular-stable graphs, *Ars Combinatoria* 70(2004), pp. 149-159.

Chartrand, G. & Lesniak, L., *Graphs and Digraphs*, 3rd ed., Chapman and Hall, (1996).

Read, R. C. & Wilson, R. J., *An Atlas of Graphs*, Clarendon Press (1998).

P. J. Cameron, J.-M. Goethals, J.J Seidel & E. Shult (1976), Line graphs, root systems, and elliptic geometry, *J. Algebra*, 43, 305-327.

P.J. Cameron & J. H. Van Lint, *Designs, Graphs, Codes and their links*, Cambridge University Press (1991).