Intervalized Iterative Learning Control for Monotone Convergence in the Sense of Sup-norm

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Abstract

In this paper, the monotone convergence in the sense of sup-norm is studied. First, it is pointed out that, when a typical D-type iterative learning control (ILC) algorithm is applied to LTI systems, some huge overshoot in the sense of sup-norm may be observed even though the exponential convergence is guaranteed in the sense of λ-norm. Then, an ILC law using an intervalized learning scheme is proposed to resolve such an undesirable phenomenon, and it is shown that the learning gain affects the exponential convergence in the sense of sup-norm.

1. Introduction

Since the iterative learning control (ILC) method has been developed by Arimoto et al. [1] to overcome the limitations of conventional controllers against uncertainty due to inaccurate modeling and/or parameter variations, it has been further studied by many researchers [2-6] and the λ-norm has been adopted as a measure in the proof of convergence of the proposed ILC algorithms. For a vector function $h:[0,T]\to\mathbf{R}^n, h(t)=\left(h^1(t),\cdots,h^n(t)\right)^T$ and a real number $\lambda>0$, the λ-norm is defined as

$$\|h(\cdot)\|_\lambda = \sup_{0\le t\le T} e^{-\lambda t}\|h(t)\|_\infty .$$

Here, $\|h(t)\|_\infty$ denotes the ∞-norm defined by

$$\|h(t)\|_\infty = \sup_{1\le i\le n}\left|h^i(t)\right| .$$

From the definition of λ-norm, it is easily shown that

$$\|h(\cdot)\|_\lambda \le \sup_{0\le t\le T}\|h(t)\|_\infty \le e^{\lambda T}\|h(\cdot)\|_\lambda$$

and the λ-norm is equivalent to the sup-norm defined as

$$\|h[0,T]\|_{\sup} = \sup_{0\le t\le T}\|h(t)\|_\infty .$$

Then, the convergence property in the sense of sup-norm seems to be equivalently obtained in the sense of λ-norm. We can, however, observe some huge overshoot in the sense of sup-norm even though the exponential convergence is guaranteed in the sense of λ-norm. This undesirable phenomenon is first observed by Lee and Bien [5], and it is reported that the pure error term of PD-type ILC law plays an important role in a bound of the interval where the exponential convergence is guaranteed in the sense of sup-norm. To be more specific, consider the linear system described by (1) and the PD-type ILC law described by (2).

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)\\ y(t) &= Cx(t)\end{aligned} \tag{1}$$

$$u_{k+1}(t) = u_k(t)+\Gamma\left(\dot{e}_k(t)-Re_k(t)\right) \tag{2}$$

Here,

$$e_k(t) = y_d(t)-y_k(t),$$

and $x\in\mathbf{R}^n, u\in\mathbf{R}^r$ and $y\in\mathbf{R}^m$ denote the state, the input and the output, respectively. A, B and C are matrices with appropriate dimensions and it is assumed that CB is a full rank matrix. Let $y_d(\cdot)$ be the desired output trajectory, and $u_d(\cdot)$ and $x_d(\cdot)$ be the corresponding input trajectory and state trajectory, respectively. It is shown by Lee and Bien that if the desired output trajectory is given on the interval $t\in[0,T_{\sup}]$ where $T_{\sup}$ is bounded by

$$T_{\sup} < \frac{1}{a}\ln\left(1+\frac{a(1-\rho)}{\gamma b}\right)$$

where

$$a = \|A\|_\infty, b = \|B\|_\infty, \gamma = \|\Gamma(CA-RC)\|_\infty$$

then, there exists $\rho_0 < 1$ such that

$$\|\Delta u_{k+1}[0,T_{\sup}]\|_{\sup} \le \rho_0 \|\Delta u_k[0,T_{\sup}]\|_{\sup}$$

where

$$\Delta u_k(\cdot) = u_d(\cdot) - u_k(\cdot) .$$

Therefore, one can easily find that if R is chosen such that $\|CA-RC\|_\infty \to 0$, then $T_{\sup}$ becomes very large implying that the interval for exponential convergence in the sense of sup-norm becomes wider. For this end, however, we have to achieve accurate model of the plant.

In this paper, an ILC law using an intervalized learning scheme is proposed, and it is shown that exponential convergence in the sense of sup-norm is affected by the learning gain. It is shown that we can obtain the exponential convergence of output error in the sense of sup-norm by appropriately choosing the learning gain.

In the sequel, for n-dimensional Euclidean space $\mathbf{R}^n$, $\|x\|$ denotes the Euclidean norm of a vector $x = (x^1,\cdots,x^n)^T$. For a matrix A, $\|A\|$ denotes its induced matrix norm. $\|x\|_\infty$ denotes the ∞-norm defined by

$$\|x\|_\infty = \sup_{1\le i\le n}|x^i| .$$

For an $n\times r$ matrix A with components a^{ij}, $\|A\|_\infty$ is defined by

$$\|A\|_\infty = \sup_{1\le i\le n}\sum_{j=1}^{r}|a^{ij}| .$$

As in the notation $u_k(t)$, the subscript k denotes the iteration number.

2. Main Result

In this section, the monotone convergence of output error in the sense of sup-norm is shown for LTI systems. Consider LTI system described by (1).

First, we show that the convergence can be proved not using λ-norm. For this end, consider the following D-type ILC law.

$$u_{k+1}(t) = u_k(t) + \Gamma\dot{e}_k(t) \tag{3}$$

Before showing the convergence of the ILC law (3), we need the following Lemma 1, whose result is utilized in the proof of the convergence in the sense of sup-norm.

Lemma 1: *Let a_k and b_k be nonnegative real values for every integer $k \ge 0$. Suppose $\lim_{k\to\infty} b_k = 0$ and $0 \le \rho < 1$. Then the inequality*

$$a_{k+1} \le \rho a_k + b_k$$

implies

$$\lim_{k\to\infty} a_k = 0 .$$

Now, the convergence of the ILC law (3) will be shown.

Theorem 1: *Suppose that the update law (3) is applied to the system (1) and that the initial state at each iteration is same as the desired initial state, i.e., $x_k(0) = x_d(0)$ for $k = 0,1,2,\cdots$. If*

$$\|I - CB\Gamma\|_\infty \le \rho < 1$$

then,

$$\lim_{k\to\infty}\|e_k[0,T]\|_{\sup} = 0 . \tag{4}$$

Proof:

It follows from (3) that

$$\begin{aligned}
e_{k+1}(t) &= y_d(t) - y_k(t)\\
&= y_d(t) - \left[Ce^{At}x_d(0) + C\int_0^t e^{A(t-\tau)}Bu_{k+1}(\tau)d\tau\right]\\
&= y_d(t) - \left[Ce^{At}x_d(0) + C\int_0^t e^{A(t-\tau)}Bu_k(\tau)d\tau\right]\\
&\qquad - C\int_0^t e^{A(t-\tau)}B[u_{k+1}(\tau) - u_k(\tau)]d\tau\\
&= e_k(t) - C\int_0^t e^{A(t-\tau)}B\Gamma\dot{e}_k(\tau)d\tau\\
&= (I - CB\Gamma)e_k(t) - CA\int_0^t e^{A(t-\tau)}B\Gamma e_k(\tau)d\tau .
\end{aligned} \tag{5}$$

Let h be a real number such that

$$0 < h < \frac{1}{a}\ln\left[1 + \frac{1-\rho}{2\|C\|_\infty \|B\|_\infty \|\Gamma\|_\infty}\right]. \quad (6)$$

From (5), we find that, for $t \in [ih,(i+1)h]$,

$$\begin{aligned} e_{k+1}(t) &= (I - CB\Gamma)e_k(t) \\ &- CA\int_0^{t_1} e^{A(t-\tau)} B\Gamma e_k(\tau)d\tau \\ &- CA\int_{t_1}^{t_2} e^{A(t-\tau)} B\Gamma e_k(\tau)d\tau \\ &- \cdots \\ &- CA\int_{t_i}^{t} e^{A(t-\tau)} B\Gamma e_k(\tau)d\tau \end{aligned} \quad (7)$$

Here,

$$\begin{aligned} t_{i+1} &= t_i + h,\ i = 0,1,\cdots,N \\ t_0 &= 0, \quad t_{N+1} = T. \end{aligned}$$

Taking the sup-norm on both sides of (7), we find that

$$\begin{aligned} &\|e_{k+1}[t_i,t_{i+1}]\|_{\sup} \le \rho\|e_k[t_i,t_{i+1}]\|_{\sup} \\ &+ \|CA\|_\infty \|B\Gamma\|_\infty \sup_{t_i \le t \le t_{i+1}} \int_0^{t_1} e^{a(t-\tau)}d\tau \|e_k[0,t_1]\|_{\sup} \\ &+ \|CA\|_\infty \|B\Gamma\|_\infty \sup_{t_i \le t \le t_{i+1}} \int_{t_1}^{t_2} e^{a(t-\tau)}d\tau \|e_k[t_1,t_2]\|_{\sup} \\ &+ \cdots \\ &+ \|CA\|_\infty \|B\Gamma\|_\infty \sup_{t_i \le t \le t_{i+1}} \int_{t_i}^{t} e^{a(t-\tau)}d\tau \|e_k[t_i,t_{i+1}]\|_{\sup} \end{aligned} \quad (8)$$

where

$$a = \|A\|_\infty .$$

From (8), we further find that

$$\begin{aligned} \|e_{k+1}[t_i,t_{i+1}]\|_{\sup} &\le \rho\|e_k[t_i,t_{i+1}]\|_{\sup} \\ &+ \|C\|_\infty \|B\Gamma\|_\infty \Big[e^{a(ih)}\left(e^{ah}-1\right)\|e_k[0,t_1]\|_{\sup} \\ &+ e^{a(i-1)h}\left(e^{ah}-1\right)\|e_k[t_1,t_2]\|_{\sup} \\ &+ \cdots \\ &+ \left(e^{ah}-1\right)\|e_k[t_i,t_{i+1}]\|_{\sup}\Big] \\ = \rho\|e_k[t_i,t_{i+1}]\|_{\sup} &+ \|C\|_\infty \|B\Gamma\|_\infty \\ &\times \sum_{j=0}^{i} e^{a(i-j)h}\left(e^{ah}-1\right)\|e_k[t_j,t_{j+1}]\|_{\sup} \end{aligned} \quad (9)$$

For each $i \ge 0$, let P_i be the statement that

$$\lim_{k\to\infty}\|e_k[t_i,t_{i+1}]\|_{\sup} = 0 .$$

From (6) and (9), we can obtain

$$\begin{aligned} \|e_{k+1}[0,t_1]\|_{\sup} &\le \rho\|e_k[0,t_1]\|_{\sup} \\ &+ \|C\|_\infty \|B\Gamma\|_\infty \left(e^{ah}-1\right)\|e_k[0,t_1]\|_{\sup} \\ &< \frac{1}{2}(1+\rho)\|e_k[0,t_1]\|_{\sup} \end{aligned}$$

Since $0 \le \rho < 1$ from assumption, the statement P_0 is true. That is

$$\lim_{k\to\infty}\|e_k[0,t_1]\|_{\sup} = 0 .$$

Now, suppose the statement P_j is true for every integer j with $0 \le j < i$. Then, it is easily seen that

$$\begin{aligned} &\|e_{k+1}[t_i,t_{i+1}]\|_{\sup} \le \rho\|e_k[t_i,t_{i+1}]\|_{\sup} \\ &\quad + \|C\|_\infty \|B\Gamma\|_\infty \left(e^{ah}-1\right)\|e_k[t_i,t_{i+1}]\|_{\sup} \\ &\quad + \|C\|_\infty \|B\Gamma\|_\infty \sum_{j=0}^{i-1} e^{a(i-j)h}\left(e^{ah}-1\right)\|e_k[t_j,t_{j+1}]\|_{\sup} \\ &< \frac{1}{2}(1+\rho)\|e_k[t_i,t_{i+1}]\|_{\sup} \\ &\quad + \|C\|_\infty \|B\Gamma\|_\infty \sum_{j=0}^{i-1} e^{a(i-j)h}\left(e^{ah}-1\right)\|e_k[t_j,t_{j+1}]\|_{\sup} \end{aligned}$$

Since $0 \le \rho < 1$, from Lemma 1, we can conclude that

$$\lim_{k\to\infty}\|e_k[t_i,t_{i+1}]\|_{\sup} = 0$$

which establishes the truth of the statement P_i.

By mathematical induction, (4) is true. This completes the proof. ■

Now, we propose a new type of ILC law, which guarantees the monotone convergence of the output error. The problem is formulated as follows. Suppose a desired output trajectory $y_d(t), t \in [0,T]$, is given and the initial state at each iteration is the same as the desired initial state, i.e., $x_k(0) = x_d(0)$ for $k = 0,1,2,\cdots$. The problem is to find a control input $u(t), 0 \le t \le T$, such that the output of the given linear dynamic system (1) exponentially converges to the desired output trajectory $y_d(\cdot)$ in the sense of sup-norm.

To solve this problem, the following D-type ILC law using an intervalized learning scheme is proposed.

$$\begin{aligned} &u_{k+1}[0,t_1] = u_k[0,t_1] + \Gamma\dot{e}_k[0,t_1] \\ &u_{k+1}[t_i,t_{i+1}] = u_k[t_i,t_{i+1}] + \Gamma\dot{e}_k[t_i,t_{i+1}] \\ &\qquad \text{if } \|e_k[t_{j-1},t_j]\|_{\sup} \le \frac{1}{2}e^{-ah}\|e_k[t_j,t_{j+1}]\|_{\sup}, \forall j \le i \\ &u_{k+1}[t_i,t_{i+1}] = u_k[t_i,t_{i+1}] \quad \text{otherwise} \\ &\quad , i = 1,2,\cdots,N \end{aligned} \quad (10)$$

Monotone convergence of the proposed ILC law is presented in the following theorem.

Theorem 2: *Suppose that the update law (10) is applied to the system (1) and that the initial state at each iteration is same as the desired initial state, i.e.,* $x_k(0) = x_d(0)$ *for* $k = 0,1,2,\cdots$. *If*

$$\|I - CB\Gamma\|_\infty \le \rho < 1$$

and

$$h < \frac{1}{a}\ln\left[1 + \frac{1-\rho}{2\|C\|_\infty \|B\|_\infty \|\Gamma\|_\infty}\right]$$

then, there exists a constant $\rho_0, 0 \le \rho_0 < 1$, *such that*

$$\|e_{k+1}[t_i, t_{i+1}]\|_{\sup} \le \rho_0 \|e_k[t_i, t_{i+1}]\|_{\sup}, \forall i \in \{0,1,\cdots,N\}$$

and therefore

$$\lim_{k\to\infty} y_k(t) = y_d(t), \forall t \in [0,T].$$

Proof:

From (9), we obtain that

$$\begin{aligned}\|e_{k+1}[t_i, t_{i+1}]\|_{\sup} &\le \rho \|e_k[t_i, t_{i+1}]\|_{\sup} \\ &+ \|C\|_\infty \|B\Gamma\|_\infty \Big[e^{a(ih)}\left(e^{ah} - 1\right)\|e_k[0, t_1]\|_{\sup} \\ &+ e^{a(i-1)h}\left(e^{ah} - 1\right)\|e_k[t_1, t_2]\|_{\sup} \\ &+ \cdots \\ &+ \left(e^{ah} - 1\right)\|e_k[t_i, t_{i+1}]\|_{\sup}\Big]\end{aligned}$$

Since

$$h < \frac{1}{a}\ln\left[1 + \frac{1-\rho}{2\|C\|_\infty \|B\|_\infty \|\Gamma\|_\infty}\right]$$

and

$$\|e_k[t_{i-1}, t_i]\|_{\sup} \le \frac{1}{2}e^{-ah}\|e_k[t_i, t_{i+1}]\|_{\sup},$$

we can obtain

$$\begin{aligned}\|e_{k+1}[t_i, t_{i+1}]\|_{\sup} &\le \rho \|e_k[t_i, t_{i+1}]\|_{\sup} \\ &+ 2\|C\|_\infty \|B\Gamma\|_\infty \left(e^{ah} - 1\right)\|e_k[t_i, t_{i+1}]\|_{\sup} \qquad (11)\end{aligned}$$

and

$$\begin{aligned}\rho_0 &= \rho + 2\|C\|_\infty \|B\Gamma\|_\infty \left(e^{ah} - 1\right) \\ &< \rho + 2\|C\|_\infty \|B\Gamma\|_\infty \left(\left[1 + \frac{1-\rho}{2\|C\|_\infty \|B\Gamma\|_\infty}\right] - 1\right) \\ &= 1. \qquad (12)\end{aligned}$$

From (11) and (12), we can find

$$\|e_{k+1}[t_i, t_{i+1}]\|_{\sup} \le \rho_0 \|e_k[t_i, t_{i+1}]\|_{\sup}. \qquad (13)$$

From (13), we can finally conclude the following:

$$\lim_{k\to\infty} y_k(t) = y_d(t).$$

This completes the proof. ■

Theorem 2 implies that, if the time interval h can be estimated and the sufficient condition of Theorem 2 is satisfied, then the output trajectory of the system can be exponentially converged to the desired output trajectory by adopting the proposed intervalized learning scheme.

Note that the time interval h is bounded by ρ and learning gain Γ. Thus, if we more accurately estimate the system parameters, the wider time interval can be obtained. This means that the faster convergence can be achieved.

3. Concluding Remarks

In this paper, we first showed that the convergence of the D-type ILC law could be proved not by means of λ-norm but by means of sup-norm. Then, we proposed a new type of ILC law adopting intervalized learning scheme, by which the monotone convergence of the output trajectory can be guaranteed in the sense of sup-norm.

However, since the performance of the ILC laws also depends on the convergence speed, a rigorous analysis on the convergence rate should be a challenging problem.

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