POINT-FREE CARRIER SPACE TOPOLOGY FOR COMMUTATIVE BANACH ALGEBRAS

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DIT PROEFSCHRIFT IS GOEDGEKEURD DOOR DE PROMOTOR
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Introduction; preliminaries

0.1. It seems that the axiom of choice has become a standard tool in certain branches of mathematics, including functional analysis. Nevertheless, in many cases proofs can also be given without the axiom, although it must be admitted that these proofs are usually less elegant and more cumbersome.

Since the time Cohen obtained his results (cf. e.g. [7]) on the axiom of choice, it is no longer a matter of taste to use it or not: results obtained by the axiom are definitely weaker than the corresponding ones obtained without it.

We remind the reader of Wiener's theorem on absolutely convergent Fourier series: if $f(t) := \Sigma_{\nu=-\infty}^{\infty} \gamma_\nu e^{i\nu t}$, $\Sigma_{\nu=-\infty}^{\infty} |\gamma_\nu| < \infty$ and $\forall_t\ f(t) \neq 0$, then $f(t)^{-1}$ has an absolutely convergent Fourier series $\Sigma_{\nu=-\infty}^{\infty} \gamma'_\nu\, e^{i\nu t}$.

Originally published in 1932 (cf. [26], p. 14 or [27], p. 91) and provided with another proof in 1938 by Beurling ([3]; see also [22], pp. 422 - 426), in both cases without the aid of the axiom, it became a standard example of Banach algebra methods since 1941, when Gelfand published his famous papers; ([10], [11]; cf. also [12], [13]).

Wiener's theorem gained a short proof (in [11]), in contrast with the lengthy ones of Wiener and Beurling; this shortness, however, was effectuated by the uncontrollable machinary of the axiom of choice.

De Bruijn, during a seminar on functional analysis held at Eindhoven Technological University over the years 1961-1964, suggested an approach to the theory of commutative Banach algebras avoiding the axiom, leading to a theory entirely parallel to Gelfand's, such that it is possible at every stage to reach the corresponding stage in Gelfand's theory by a simple application of the axiom of choice. Needless to say, he was not the only one to be unsatisfied with the state of affairs; Cohen published as early as 1961 constructive proofs of theorems related to Wiener's (cf. [6]); Bishop reported on constructive analysis in La Jolla and Moscow ([1], [2]); Garnir, de Wilde and Schmets proved for separable algebras the existence of maximal ideals, containing some given proper ideal ([9]).

De Bruijn's suggestion was inspired by an old idea outlined already in Menger's "Dimensionstheorie" in 1928, and more extensively published in 1940 (cf. [20] and the literature cited there); this promising "topology without points", which did not find applications until now, appears to provide an appropriate setting for discussion of Gelfand's theory.

The first part of the problem is to construct a lattice of sets of ideals of the Banach algebra, featuring the main properties of the lattice of all closed sets in a topological space. Since maximal ideals correspond to the points of a topological space, if one wishes to avoid discussing maximal ideals (whose existence depends on the axiom of choice), one has to avoid discussing the topology as based upon the underlying point set.

Next, de Bruijn intended to describe elements of the algebra as mappings from the point-free topology into (the lattice of closed sets of) the complex number field, in such a way that they should degenerate into continuous functions when restricted (in one way or another) to the minimal elements of the point - free topology.

Since 1963 we have co-operated on the subject; this thesis is mainly an account of results concerning the first part of the problem.

Not only the principal idea, which can be found in sections 3 and 4, but also several details are due to de Bruijn; this particularly applies to the complicated proofs in section 5.

The axiom of choice is not employed in sections 1, 2, 3 and 4; in section 5, we use the restricted (i.e. the countable) version of the axiom; section 6 gives connections with the space of maximal ideals, and there of course, several conclusions depend on the unrestricted axiom.

A very important theorem in the sequel is 2.8; it provides the Banach algebra with ideals acting, in a way, as maximal ideals; this theorem is also the key to a new proof of Wiener's theorem to be published elsewhere ([5]).

0.2. Although the reader is supposed to be more or less familiar with the concepts of "Banach algebra" and "lattice" we list here the axioms for these structures to facilitate references; further information on Banach algebras can be found in [12] and [21], on lattices in [4] and [23].

A non-empty set $\mathcal{H}$ is called a Banach algebra over the complex number field Cm if mappings

$+ \quad : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, called addition

$* \quad : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$, called multiplication

$\cdot \quad : \mathrm{Cm} \times \mathcal{H} \to \mathcal{H}$, called multiplication by scalars

and

$\| \ \| : \mathcal{H} \to \mathrm{R}\ell$ ($\mathrm{R}\ell$ denoting the field of real numbers),

called norm

are defined in such a way that

i: $(\mathcal{H}, +, *)$ is a ring (its zero-element denoted by σ);

ii: $(\mathcal{H}, \mathrm{Cm}, +, \cdot)$ is a linear space over Cm;

iii: $\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} \forall_{\lambda \in \mathrm{Cm}}\ \lambda \cdot (x * y) = (\lambda \cdot x) * y = x * (\lambda \cdot y)$;

iv: $\forall_{x \in \mathcal{H}}\ \|x\| \geq 0$; $\forall_{x \in \mathcal{H}} [\|x\| = 0 \Rightarrow x = \sigma]$;

$\forall_{\lambda \in \mathrm{Cm}} \forall_{x \in \mathcal{H}}\ \|\lambda \cdot x\| = |\lambda| \cdot \|x\|$;

$\forall_{x \in \mathcal{H}} \forall_{y \in \mathcal{H}} [\|x + y\| \leq \|x\| + \|y\| \ \& \ \|x * y\| \leq \|x\| \cdot \|y\|]$;

v: The metric topology induced by the norm $\| \ \|$ is complete.

A non-empty set L is called a lattice if mappings

$\wedge \quad : L \times L \to L$, called meet (or simply cap)

 and

$\vee \quad : L \times L \to L$, called join (or cup)

are defined in such a way that

$L_1 \quad : \forall_{a \in L} \forall_{b \in L} \forall_{c \in L} \; (a \wedge b) \wedge c = a \wedge (b \wedge c)$;

$L_2 \quad : \forall_{a \in L} \forall_{b \in L} \forall_{c \in L} \; (a \vee b) \vee c = a \vee (b \vee c)$;

$L_3 \quad : \forall_{a \in L} \forall_{b \in L} \; a \wedge b = b \wedge a$;

$L_4 \quad : \forall_{a \in L} \forall_{b \in L} \; a \vee b = b \vee a$;

$L_5 \quad : \forall_{a \in L} \forall_{b \in L} \; a \wedge (a \vee b) = a$;

$L_6 \quad : \forall_{a \in L} \forall_{b \in L} \; a \vee (a \wedge b) = a$.

In a lattice L a partial order is defined by

$a \overset{L}{\leqslant} b : \Longleftrightarrow a \wedge b = a$

or equivalently $a \overset{L}{\leqslant} b : \Longleftrightarrow a \vee b = b$.

In view of this order we may define:

If $D \subset L$ and if the greatest lower bound $\inf D$ exists, then

$\bigwedge_{a \in D} a := \inf D$;

analogously $\bigvee_{a \in D} a := \sup D$ if $\sup D$ exists;

we occasionally write $\wedge D$ and $\vee D$ respectively.

A lattice L is called $\wedge$-complete if $\wedge D$ exists for every subset D of L; a $\wedge$-complete lattice is called $\vee$-distributive if

$\forall_{a \in L} \forall_{D \in P(L)} \; a \vee (\wedge D) = \bigwedge_{d \in D} (a \vee d)$, $P(L)$ denoting the set of subsets of L.

$\vee$ - completeness and $\wedge$ - distributiveness are defined analogously.

A lattice L is called distributive if

$$\forall_{a \in L} \forall_{b \in L} \forall_{c \in L}\ a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

or equivalently, if

$$\forall_{a \in L} \forall_{b \in L} \forall_{c \in L}\ a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$$

A lattice is called complete if it is $\wedge$ - complete as well as $\vee$ - complete.

0.3. A lattice L is called a point-free topology if it has the following properties:

PT_1 : L contains elements v and w, $v < w$ and $\forall_{a \in L}\ v \leq a \leq w$;

PT_2 : L is $\wedge$ - complete ;

PT_3 : L is $\vee$ - distributive ;

PT_4 : $\forall_{a \in L} \forall_{b \in L} [b \not\leq a \Rightarrow \exists_{c \in L} \{v \neq c \leq b \;\&\; a \wedge c = v\}]$.

By virtue of well-known theorems of lattice theory, a point-free topology is distributive and $\vee$ - complete as well.

An element m of L is called minimal if $m \neq v$ and if $\forall_{a \in L} [v \neq a \leq m \Rightarrow a = m]$.

Denote the set of minimal elements of L by M;
define $F_a := \{m \in M \mid m \leq a\}$ for every $a \in L$.

The point-free topology L is called atomic if $\forall_{a \in L \setminus \{v\}}\ F_a \neq \emptyset$.

 If the point-free topology L is atomic, then obviously

$F_v = \emptyset$, $F_w = M$, $a \leqslant b \Rightarrow F_a \subset F_b$, $F_{a \wedge b} = F_a \cap F_b$,

$F_{a \vee b} = F_a \cup F_b$, $m \in M \Rightarrow F_m = \{m\}$;

moreover, if $D \in P(L) \setminus \{\emptyset\}$, then $F_{\wedge D} = \bigcap_{d \in D} F_d$.

If $a \neq b$ then, without loss of generality assuming $a \not\geqslant b$,

$\exists_{c \in L}\ v \neq c \leqslant b \ \&\ a \wedge c = v$, by PT_4 ; hence $F_c \neq \emptyset$,

implying $\exists_{m \in M} [m \leqslant c \leqslant b \ \&\ a \wedge m = v]$, whence $m \in F_b \setminus F_a$ and $F_a \neq F_b$.

All these facts prove that $\{F_a\}_{a \in L}$ is a topology (in terms of closed sets) for M. It can be considered as a lattice with set-intersection and set-union as operations. As a lattice it is lattice-isomorphic with L. It satisfies, moreover, the Fréchet separation condition (i.e. sets consisting of a single point are closed).

We call the topology $\{F_a\}_{a \in L}$ the companion topology of the point-free topology L.

The point-free topology L is called regular if it satisfies

PT_5 : If $a \in L$, $b \in L$, and $b \not\leqslant a$, there exist elements $c, d, e \in L$ with the following properties:

i: $v \neq c \leqslant b$;

ii: $c \wedge d = v$;

iii: $a \wedge e = v$;

iv: $a \leqslant d$;

v: $c \leqslant e$;

vi: $d \vee e = w$.

PT_4 is clearly implied by PT_5; if L is atomic and satisfies PT_5, then M is a regular topological space in the usual sense.

The point-free topology L is called compact if every subset D with "finite L-intersection property" has $\wedge D \neq v$.

Compactness of L implies compactness of the companion topology.

It is also possible to define the Hausdorff property in terms of the point-free topology (cf. [5]). We do not discuss this here, since the point-free topology to be constructed in this dissertation possesses the stronger property of regularity.

Unfortunately, we did not succeed in proving the normality of our point-free topology. The usual argument leading to normality involves compactness, but the kind of compactness we could obtain (section 5) seems too weak for that purpose. On the other hand the description of normality for a point-free topology is very close to, and even easier than, that of regularity. So at first sight there seems to be no reason to expect the same kind of difficulties one meets when trying to prove compactness, which is logically of a much more intricate nature.

0.4. The text, sections 1 - 7, consists of theorems, definitions (preceded by the symbol :) and further statements (corollaries, remarks, etc.; preceded by the symbol ,). Every statement is preceded in the usual way by a pair of positive integers to facilitate references.

The * for multiplication we drop almost immediately hereafter; both kinds of multiplication in the Banach algebra are only quite incidentally denoted by $\cdot$, usually by nothing. Indeed, we use standard set-theoretical, algebraic and topological notations without further comment; for a list of symbols we refer to the end of this section.

In order to reduce the frequency of words like "hence", "consequently", we occasionally replace them by $\therefore$.

The symbols $\forall$ and $\exists$ are used frequently, but somewhat unsystematically; logicians will find several pages where they may insert or omit a few of them.

From ALGOL we borrowed the symbol := for definitions; from Gillman and Jerison ([14]. p. 1) we adopt $\overleftarrow{\varphi}$ to describe the inverse image produced by a function φ :

If $\varphi : A \to B$ is a function, then for every $y \in B$

$$\overleftarrow{\varphi}(y) := \{x \in A \mid \varphi(x) = y\} \ ;$$

denoting by $P(B)$ the set of all subsets of B then for every $S \in P(B)$

$$\overleftarrow{\varphi}(S) := \{x \in A \mid \varphi(x) \in S\} \ .$$

If proofs of standard results are omitted (particularly in sections 1 and 2) the reader may find them in [12], [21], [24] or [25] or in modern textbooks on functional analysis; in most cases we do not give explicit references.

0.5. In this dissertation, the fields of complex numbers and real numbers are denoted by Cm and $R\ell$ respectively; complex numbers are denoted by α, β, $\gamma, \ldots$; subsets of $\mathbb{C}m$ are denoted by Ψ, Φ, $\ldots$.

$\mathbb{N}t$ denotes the set of positive integers; integers are denoted by i, j, k, ℓ, m, n .

I, K, Λ, M, N denote index sets, their elements are denoted by ι, κ, λ, μ, ν .

$\mathcal{H}$ denotes a Banach algebra, its elements are denoted by x, y, z, $\ldots$.

$\mathfrak{N}$ denotes the set of proper closed ideals in $\mathcal{H}$; ideals are denoted by $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{r}$, $\ldots$; subsets of $\mathfrak{N}$ are denoted by $\mathfrak{A}$, $\mathcal{B}$, $\mathcal{L}$, $\ldots$.

$P(\ \ )$ is the class of subsets of a set.

$F(\ \ )$ is the class of finite subsets of a set.

The following symbols have a specified meaning; cf. the cited article.

A, B, C	elements of Q, subsets of F
a, b, c	elements of F
$D := \mathbb{C}m^{\mathbb{N}t}$	;5.10
e	identity of $\mathcal{H}$
$F := F(\mathcal{H})$	
θ	zero-element of $\mathcal{H}$
$Q := P(F(\mathcal{H}))$	
R, $R_{\mathfrak{n}}$	sets of regular elements in $\mathcal{H}$ or $\varphi_{\mathfrak{n}}(\mathcal{H})$ respectively; 1.3, 1.19

S, $S_{\mathscr{m}}$	sets of singular elements in $\mathscr{X}$ or $\varphi_{\mathscr{m}}(\mathscr{X})$ respectively; 1.3, 1.19
Z	set of topological divisors of zero in $\mathscr{X}$; 1.17
Γ, Δ, E	subsets of $P(\mathfrak{M})$; 3.2, 3.3, 4.10
$v(\ ,\ )$, $v(\ )$	spectral radius; 1.21
$\sigma(\ ,\ )$, $\sigma(\ )$	spectrum; 1.23
Φ, $\Phi(\ ,\ )$	special subsets of Cm; 4.1
$\varphi_{\mathscr{m}}$	homomorphism corresponding with an ideal $\mathscr{m}$; 1.11
$\mathscr{L}(\ )$, $\mathscr{L}_x$, $\mathscr{L}_a$, $\mathscr{L}_A$	elements of Γ, clusters; 4.5, 4.7, 4.10, 4.14
$\mathfrak{M}$	set of maximal ideals in $\mathscr{X}$; 6.1
$\mathfrak{M}(\ )$	bonnet; 2.1
$\mathscr{W}$, $\mathscr{W}_{\mathscr{m}}$	radical of the algebra $\mathscr{X}$ or $\varphi_{\mathscr{m}}(\mathscr{X})$ respectively; 1.25
$\mathscr{S}$	set of strong ideals in $\mathscr{X}$; 1.30
$\mathscr{S}_a$	subset of $\mathfrak{M}$ with the property $\mathscr{S}_a^* = \mathscr{L}_a$; 4.10
$\mathscr{Y} := \mathscr{X}^{Nt}$	; 5.10
$\mathscr{Z}$	the ideal $\{\sigma\}$ in $\mathscr{X}$

Intersections and unions of sets of a class $\mathscr{A}$ will be denoted sometimes as $\cap\mathscr{A}$ and $\cup\mathscr{A}$.

General properties of commutative Banach algebras with identity element

1.1. A Banach algebra $\mathscr{X}$ is said to be a B_1 - algebra if the ring $(\mathscr{X}, +, \times)$ is commutative and contains an identity element e with the property $\|e\| = 1$.

1.2. We start with a B_1 - algebra $\mathscr{X}$; its identy element e is unique.

1.3.
$$R := \{x \in \mathscr{X} \mid \exists_{y \in \mathscr{X}}\ xy = e\}.$$
$$S := \mathscr{X} \setminus R.$$

Elements of R are called regular, those of S are called singular.

1.4. If $x \in R$ then the element y for which $xy = e$ is unique; it is called the inverse of x, belongs to R and is denoted by x^{-1}.

1.5. $\|e - x\| < 1 \Rightarrow [x \in R \ \&\ x^{-1} = e + \sum_{n=1}^{\infty} (e - x)^n]$.

1.6. R is open and S is closed.

Proof: $\forall_{x \in R}\ \forall_{y \in \mathscr{X}}\ \|x^{-1}y - e\| \leq \|x^{-1}\|\ \|y - x\|$.

Hence, if $\|y - x\| < \|x^{-1}\|^{-1}$, then $x^{-1}y \in R$ by 1.5, implying $y \in R$.

1.7. A non-empty subset $\mathfrak{a}$ of $\mathscr{X}$ is called an ideal if $\mathfrak{a} + \mathfrak{a} \subset \mathfrak{a}$ and $\mathscr{X}.\mathfrak{a} \subset \mathfrak{a}$; the ideal $\mathfrak{a}$ is called a proper ideal if $\mathfrak{a} \neq \mathscr{X}$.

1.8. If $\mathfrak{a}$ is an ideal then $\mathcal{C}m \cdot \mathfrak{a} \subset \mathfrak{a}$.

1.9. If $\mathfrak{a}$ is a proper ideal then the closure $\overline{\mathfrak{a}}$ of $\mathfrak{a}$ is a proper ideal.

1.10. If $\mathfrak{a}$ is an ideal and $\mathfrak{a} = \overline{\mathfrak{a}}$ then $\mathfrak{a}$ is called a closed ideal; moreover, if $\mathfrak{a}$ is a proper ideal, then $\mathfrak{a}$ is called a proper closed ideal. The set of proper closed ideals of $\mathcal{X}$ is denoted by $\mathfrak{N}$; the null ideal $\{o\}$ belongs to $\mathfrak{N}$ and is denoted by $\mathfrak{z}$.

1.11. For every $\mathfrak{n} \in \mathfrak{N}$ we denote by $\varphi_{\mathfrak{n}}$ the canonical homomorphism $\mathcal{X} \to \mathcal{X}/\mathfrak{n}$; if we take

$\|\varphi_{\mathfrak{n}}(x)\| := \inf\{\|y\| \mid y \in \overleftarrow{\varphi_{\mathfrak{n}}}\varphi_{\mathfrak{n}}(x)\}$ then it turns out that $\mathcal{X}/\mathfrak{n}$ together with this norm $\| \ \|$ is a B_1-algebra with identity $\varphi_{\mathfrak{n}}(e)$; in particular $\|\varphi_{\mathfrak{n}}(e)\| = \inf\{\|e + y\| \mid y \in \mathfrak{n}\} = 1$.

For a proof see [21], p. 44.

We prefer to write $\varphi_{\mathfrak{n}}(\mathcal{X})$ for $\mathcal{X}/\mathfrak{n}$.

1.12. If $\mathfrak{m} \in \mathfrak{N}$ and $\mathfrak{n} \in \mathfrak{N}$ then

i: $\varphi_{\mathfrak{n}}(\mathfrak{m})$ is an ideal in $\varphi_{\mathfrak{n}}(\mathcal{X})$,

ii: $\overleftarrow{\varphi_{\mathfrak{n}}}\varphi_{\mathfrak{n}}(\mathfrak{m}) = \mathfrak{m} + \mathfrak{n}$.

Proof:

i: Trivially $\varphi_{\mathfrak{n}}(\mathfrak{m}) + \varphi_{\mathfrak{n}}(\mathfrak{m}) \subset \varphi_{\mathfrak{n}}(\mathfrak{m})$ and $\varphi_{\mathfrak{n}}(\mathcal{X}) \cdot \varphi_{\mathfrak{n}}(\mathfrak{m}) \subset$ $\subset \varphi_{\mathfrak{n}}(\mathfrak{m})$ hence $\varphi_{\mathfrak{n}}(\mathfrak{m})$ is an ideal in $\varphi_{\mathfrak{n}}(\mathcal{X})$.

ii: For every $x \in \mathcal{X}$ the following statements are easily seen to be equivalent:

$x \in \overleftarrow{\varphi_{\mathfrak{n}}}\varphi_{\mathfrak{n}}(\mathfrak{m})$

$\varphi_{\mathfrak{n}}(x) \in \varphi_{\mathfrak{n}}(\mathfrak{m})$

$\exists_{y \in \mathfrak{m}} \ \varphi_{\mathfrak{n}}(x) = \varphi_{\mathfrak{n}}(y)$

$\exists_{y \in \mathcal{m}} \; x - y \in \mathcal{n}$

$x \in \mathcal{m} + \mathcal{n}$.

1.13. If $\mathcal{m} \in \mathcal{N}$, $\mathcal{n} \subset \mathcal{N}$ and $\mathcal{n} \subset \mathcal{m}$ then

i: $\mathcal{m} = \overleftarrow{\varphi}_{\mathcal{n}} \varphi_{\mathcal{n}}(\mathcal{m})$,

ii: $\varphi_{\mathcal{n}}(\mathcal{m})$ is a proper closed ideal in $\varphi_{\mathcal{n}}(\mathcal{H})$.

Proof:

i: $\mathcal{n} \subset \mathcal{m} \Rightarrow \mathcal{m} = \mathcal{m} + \mathcal{n}$; the result follows from 1.12.ii.

ii: $\varphi_{\mathcal{n}}(\mathcal{m})$ is an ideal in $\varphi_{\mathcal{n}}(\mathcal{H})$ by 1.12.i.

If $\varphi_{\mathcal{n}}(e) \in \varphi_{\mathcal{n}}(\mathcal{m})$ then $e \in \overleftarrow{\varphi}_{\mathcal{n}} \varphi_{\mathcal{n}}(\mathcal{m})$; hence by 1.13.i $e \in \mathcal{m}$ contrary to the assumption $\mathcal{m} \in \mathcal{N}$; thus $\varphi_{\mathcal{n}}(\mathcal{m})$ is a proper ideal.

If $\varphi_{\mathcal{n}}(x) \subset \overline{\varphi_{\mathcal{n}}(\mathcal{m})}$ and $\varepsilon > 0$ then $\exists_{y \in \mathcal{m}} \|\varphi_{\mathcal{n}}(x) - \varphi_{\mathcal{n}}(y)\| < \varepsilon$;

$\therefore \|\varphi_{\mathcal{n}}(x - y)\| < \varepsilon$; by the definition of $\| \;\|$ in $\varphi_{\mathcal{n}}(\mathcal{H})$ it follows that $\exists_{z \in \overleftarrow{\varphi}_{\mathcal{n}} \varphi_{\mathcal{n}}(x-y)} \|z\| < \varepsilon$.

Now $x - y - z \in \mathcal{n}$ and $y \in \mathcal{m}$ imply $x - z \in \mathcal{m} + \mathcal{n} = \mathcal{m}$; writing $x - z = w$, we obtain

$\forall_{x \in \overleftarrow{\varphi}_{\mathcal{n}} \overline{\varphi_{\mathcal{n}}(\mathcal{m})}} \; \forall_{\varepsilon > 0} \; \exists_{w \in \mathcal{m}} \|x - w\| < \varepsilon$ or

$\overleftarrow{\varphi}_{\mathcal{n}} \overline{\varphi_{\mathcal{n}}(\mathcal{m})} \subset \overline{\mathcal{m}} = \mathcal{m}$ whence $\overline{\varphi_{\mathcal{n}}(\mathcal{m})} \subset \varphi_{\mathcal{n}}(\mathcal{m})$

which proves that $\varphi_{\mathcal{n}}(\mathcal{m})$ is closed.

1.14. If $\mathcal{m} \in \mathcal{N}$, $\mathcal{n} \in \mathcal{N}$ and $\mathcal{n} \subset \mathcal{m}$ then $\varphi_{\mathcal{m}}(\mathcal{H})$ and $\varphi_{\mathcal{n}}(\mathcal{H}) \,/\, \varphi_{\mathcal{n}}(\mathcal{m})$ are isomorphic and isometric.

Proof: 1.11 and 1.13 imply that $\varphi_{\mathcal{m}}(\mathcal{H})$, $\varphi_{\mathcal{n}}(\mathcal{H})$ and $\varphi_{\mathcal{n}}(\mathcal{H}) \,/\, \varphi_{\mathcal{n}}(\mathcal{m})$ are B_1 - algebras.

The algebraic part of the statement is a well - known algebraic property.

Let ψ be the homomorphism $\varphi_{\mathcal{n}}(\mathcal{X}) \to \varphi_{\mathcal{n}}(\mathcal{X}) \,/\, \varphi_{\mathcal{n}}(\mathcal{m})$, then the isomorphism between $\varphi_{\mathcal{m}}(\mathcal{X})$ and $\varphi_{\mathcal{n}}(\mathcal{X}) \,/\, \varphi_{\mathcal{n}}(\mathcal{m})$ implies that $\forall_{x \in \mathcal{X}}\ \overset{\leftarrow}{\varphi}_{\mathcal{m}}\varphi_{\mathcal{m}}(x) = (\psi \circ \varphi_{\mathcal{n}})^{\leftarrow}(\psi \circ \varphi_{\mathcal{n}})(x)$;

$\therefore\ \|(\psi \circ \varphi_{\mathcal{n}})(x)\| = \|\psi(\varphi_{\mathcal{n}}(x))\| = \inf\{\|\varphi_{\mathcal{n}}(y)\| \mid \varphi_{\mathcal{n}}(y) \in \overset{\leftarrow}{\psi}\psi\varphi_{\mathcal{n}}(x)\}$

with $\|\varphi_{\mathcal{n}}(y)\| = \inf\{\|z\| \mid z \in \overset{\leftarrow}{\varphi}_{\mathcal{n}}\varphi_{\mathcal{n}}(y)\} = \inf\{\|z\| \mid \varphi_{\mathcal{n}}(z) = \varphi_{\mathcal{n}}(y)\}$

hence
$$\begin{aligned}\|(\psi \circ \varphi_{\mathcal{n}})(x)\| &= \inf\{\|z\| \mid \varphi_{\mathcal{n}}(z) \in \overset{\leftarrow}{\psi}\psi\varphi_{\mathcal{n}}(x)\} = \\ &= \inf\{\|z\| \mid \psi\varphi_{\mathcal{n}}(z) = \psi\varphi_{\mathcal{n}}(x)\} = \\ &= \inf\{\|z\| \mid z \in (\psi \circ \varphi_{\mathcal{n}})^{\leftarrow}(\psi \circ \varphi_{\mathcal{n}})(x)\} = \\ &= \inf\{\|z\| \mid z \in \overset{\leftarrow}{\varphi}_{\mathcal{m}}\varphi_{\mathcal{m}}(x)\} = \\ &= \|\varphi_{\mathcal{m}}(x)\| .\end{aligned}$$

1.15. $\varphi_{\mathcal{m}}$, as a linear operator $\mathcal{X} \to \varphi_{\mathcal{m}}(\mathcal{X})$ has a norm $\|\varphi_{\mathcal{m}}\|$ defined by $\|\varphi_{\mathcal{m}}\| := \sup\{\|\varphi_{\mathcal{m}}(x)\| \mid \|x\| = 1\}$.
We have $\|\varphi_{\mathcal{m}}(x)\| \leq \|x\|$ for all $x \in \mathcal{X}$, and $\|\varphi_{\mathcal{m}}(e)\| = 1 = \|e\|$.
Hence $\|\varphi_{\mathcal{m}}\| = 1$.

1.16. An element $z \in \mathcal{X}$ is called a topological divisor of zero if $\mathcal{X}$ contains a sequence $\{z_n\}_{n \in \mathbb{N}t}$ with the properties

i: $\forall_{n \in \mathbb{N}t}\ \|z_n\| = 1$, and

ii: $\lim\limits_{n \to \infty} z z_n = \sigma$.

The set of topological divisors of zero is denoted by Z.

1.17. $Z \subset S$.

1.18. i: If $x \in S$ then $\overline{x\mathcal{X}} \in \mathcal{M}$.

ii: If $x \in S \setminus Z$ then $x\mathcal{X} \in \mathcal{M}$.

Proof:

i: If $x \in S$ then $x\mathcal{H}$ is a proper ideal; hence by 1.9 $\overline{x\mathcal{H}} \in \mathcal{M}$.

ii: If $x \in S \setminus Z$ and $y \in \overline{x\mathcal{H}}$ then there exists a sequence $\{y_n\}_{n \in \mathbb{N}t}$ in $\mathcal{H}$ with $\lim_{n \to \infty} xy_n = y$.

Suppose that $\{y_n\}_n$ is not bounded; then it contains a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}t}$ with $\|y_{n_k}\| \uparrow \infty$;

since $\| \, \|y_{n_k}\|^{-1} \cdot xy_{n_k} \| \leq \| \, \|y_{n_k}\|^{-1} (xy_{n_k} - y) \| + \| \, \|y_{n_k}\|^{-1} \cdot y \|$

this implies $\lim_{k \to \infty} \|y_{n_k}\|^{-1} \cdot y_{n_k} x = \sigma$;

by $\| \, \|y_{n_k}\|^{-1} \cdot y_{n_k} \| = 1$ we infer that $x \in Z$, contrary to the assumptions.

Consequently $\{y_n\}_n$ is bounded, and so it contains a convergent subsequence $\{y_{n_k}\}_k$.

Denote the limit of $\{y_{n_k}\}_k$ by y', then

$\|x(y_{n_k} - y')\| \leq \|x\| \cdot \|y_{n_k} - y'\|$ implies that

$y = \lim_{k \to \infty} xy_{n_k} = xy'$, hence $y \in x\mathcal{H}$;

this proves that $x\mathcal{H} = \overline{x\mathcal{H}}$ and $x\mathcal{H} \in \mathcal{M}$.

Remark. The employment of the Bolzano-Weierstrasz property in the foregoing argument does not involve the axiom of choice.

1.19. If $m \in \mathcal{M}$ then

$$R_m := \{\varphi_m(x) \in \varphi_m(\mathcal{H}) \mid \exists_{y \in \mathcal{H}} \; \varphi_m(xy) = \varphi_m(e)\}$$

 and $$S_m := \varphi_m(\mathcal{H}) \setminus R_m .$$

In other words, $R_{\mathfrak{n}}$ is the set of regular elements of $\varphi_{\mathfrak{n}}(\mathcal{X})$ and $S_{\mathfrak{n}}$ the set of singular elements of $\varphi_{\mathfrak{n}}(\mathcal{X})$.

The set of proper closed ideals in $\varphi_{\mathfrak{n}}(\mathcal{X})$ will be denoted by $\mathcal{N}_{\mathfrak{n}}$.

1.20. If $\mathfrak{m} \in \mathcal{N}$, $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \supset \mathfrak{n}$ then

$$\overset{\leftarrow}{\varphi}_{\mathfrak{m}}(S_{\mathfrak{m}}) \subset \overset{\leftarrow}{\varphi}_{\mathfrak{n}}(S_{\mathfrak{n}}) \quad \text{and} \quad \overset{\leftarrow}{\varphi}_{\mathfrak{m}}(R_{\mathfrak{m}}) \supset \overset{\leftarrow}{\varphi}_{\mathfrak{n}}(R_{\mathfrak{n}}) .$$

Proof: The statements

$$x \in \overset{\leftarrow}{\varphi}_{\mathfrak{n}}(R_{\mathfrak{n}})$$

$$\exists_{y \in \mathcal{X}}\ \varphi_{\mathfrak{n}}(xy - e) = \varphi_{\mathfrak{n}}(\theta)$$

$$\exists_{y \in \mathcal{X}}\ xy - e \in \mathfrak{n}$$

are equivalent; since $\mathfrak{n} \subset \mathfrak{m}$, the latter of them implies

$$\exists_{y \in \mathcal{X}}\ xy - e \in \mathfrak{m}$$

and this, in turn, is equivalent to

$$x \in \overset{\leftarrow}{\varphi}_{\mathfrak{m}}(R_{\mathfrak{m}}) ;$$

this completes the proof.

1.21. The limit $\nu(\mathfrak{n},x) := \lim\limits_{k \to \infty} \|\varphi_{\mathfrak{n}}(x^k)\|^{\frac{1}{k}}$ exists for every $x \in \mathcal{X}$ and every $\mathfrak{n} \in \mathcal{N}$, and has the following properties:

i: $\nu(\mathfrak{n},x) = \inf\{\|\varphi_{\mathfrak{n}}(x^k)\|^{\frac{1}{k}} \mid k \in \mathbb{N}^{+}\}$,

ii: $0 \leqslant \nu(\mathfrak{n},x) \leqslant \|\varphi_{\mathfrak{n}}(x)\|$,

iii: $\nu(\mathfrak{n},\alpha x) = |\alpha| \cdot \nu(\mathfrak{n},x)$,

iv: $\nu(\mathfrak{n},x^k) = \nu(\mathfrak{n},x)^k \qquad (k \in \mathbb{N}^{+})$,

v: $\nu(\mathfrak{n},x+y) \leqslant \nu(\mathfrak{n},x) + \nu(\mathfrak{n},y)$,

vi: $\nu(\mathfrak{n},xy) \leqslant \nu(\mathfrak{n},x) \cdot \nu(\mathfrak{n},y)$.

Proof: [21], pp. 10, 11 is applicable to $\varphi_{\mathfrak{m}}(\mathcal{X})$ without any difficulty.

1.22. If $\nu(\mathfrak{m}, e-x) < 1$ then $\varphi_{\mathfrak{m}}(x) \in R_{\mathfrak{m}}$ and

$$\varphi_{\mathfrak{m}}(x)^{-1} = \varphi_{\mathfrak{m}}[e + \sum_{k=1}^{\infty} (e-x)^k] .$$

Proof: Apply the proof in [21], p. 12 to $\varphi_{\mathfrak{m}}(\mathcal{X})$.

:1.23. $\sigma(\mathfrak{m}, x) := \{\gamma \in Cm \mid \varphi_{\mathfrak{m}}(x - \gamma e) \in S_{\mathfrak{m}}\}$.

$\sigma(\mathfrak{m}, x)$ is called the spectrum of x modulo $\mathfrak{m}$.

1.24. $\sigma(\mathfrak{m}, x)$ is a non-vacuous closed set in Cm for every $\mathfrak{m} \in \mathfrak{N}$ and every $x \in \mathcal{X}$; moreover, $\max\{|\gamma| \mid \gamma \in \sigma(\mathfrak{m}, x)\} = \nu(\mathfrak{m}, x)$.

Proof: Apply [21], pp. 28–30 to $\varphi_{\mathfrak{m}}(\mathcal{X})$; this proof is independent of the axiom of choice.

:1.25. $\mathcal{W}_{\mathfrak{m}} := \{\varphi_{\mathfrak{m}}(x) \mid \nu(\mathfrak{m}, x) = 0\}$.

$\mathcal{W}_{\mathfrak{m}}$ is called the radical of $\varphi_{\mathfrak{m}}(\mathcal{X})$.

The elements of $\mathcal{W}_{\mathfrak{m}}$ are called topologically nilpotent in $\varphi_{\mathfrak{m}}(\mathcal{X})$.

1.26. $\mathcal{W}_{\mathfrak{m}} \in \mathfrak{N}_{\mathfrak{m}}$.

Proof: $\mathcal{W}_{\mathfrak{m}}$ is an ideal in $\varphi_{\mathfrak{m}}(\mathcal{X})$ by 1.21.v and 1.21.vi; since $\nu(\mathfrak{m}, e) = 1$ is $\varphi_{\mathfrak{m}}(e) \notin \mathcal{W}_{\mathfrak{m}}$; hence $\mathcal{W}_{\mathfrak{m}}$ is a proper ideal. 1.21.v and 1.21.ii imply $|\nu(\mathfrak{m}, x) - \nu(\mathfrak{m}, y)| \leq \nu(\mathfrak{m}, x - y) \leq$ $\leq \|\varphi_{\mathfrak{m}}(x - y)\| = \|\varphi_{\mathfrak{m}}(x) - \varphi_{\mathfrak{m}}(y)\|$;

$\therefore$ $\nu(\mathfrak{m}, x)$ is a continuous function; $\mathcal{W}_{\mathfrak{m}}$, as the zero-set of a continuous function, is a closed set in $\varphi_{\mathfrak{m}}(\mathcal{X})$.

An analogous result can be found in [19], p. 52.

1.27. $\mathcal{W}_{\mathscr{m}} = \{\varphi_{\mathscr{m}}(x) \mid \forall_{y\in\mathcal{X}}\ \varphi_{\mathscr{m}}(xy-e) \in R_{\mathscr{m}}\}$.

Proof: $\varphi_{\mathscr{m}}(x) \in \mathcal{W}_{\mathscr{m}}$ is by definition equivalent to $\nu(\mathscr{m},x) = 0$, implying $\forall_{y\in\mathcal{X}}\ \nu(\mathscr{m},xy) = 0$ by 1.21.vi; hence $\forall_{y\in\mathcal{X}}\ \varphi_{\mathscr{m}}(xy-e) \in R_{\mathscr{m}}$ by 1.22; and conversely the latter formula implies $\forall_{\lambda\in \mathrm{Cm}}\ \varphi_{\mathscr{m}}(\lambda x-e) \in R_{\mathscr{m}}$ or, equivalently, $\forall_{\lambda\neq 0}\ \varphi_{\mathscr{m}}(x-\lambda e) \in R_{\mathscr{m}}$, whence, by 1.24, $\sigma(\mathscr{m},x) = \{0\}$ and therefore $\nu(\mathscr{m},x) = 0$, $\varphi_{\mathscr{m}}(x) \in \mathcal{W}_{\mathscr{m}}$.

1.28. Since $\mathcal{X}$ can be identified with $\varphi_{\mathfrak{z}}(\mathcal{X})$, the expressions $\nu(\mathfrak{z},x)$, $\sigma(\mathfrak{z},x)$ and $\mathcal{W}_{\mathfrak{z}}$ will be denoted by $\nu(x)$, $\sigma(x)$ and $\mathcal{W}$ respectively.

1.29. $\varphi_{\mathscr{m}}(\mathcal{W}) \subset \mathcal{W}_{\mathscr{m}}$ for every $\mathscr{m} \in \mathcal{N}$.

Proof: If $x \in \mathcal{W}$ then $\forall_{y\in\mathcal{X}}\ xy - e \in R$; by 1.20 this implies $\forall_{\mathscr{m}\in\mathcal{N}}\ \forall_{y\in\mathcal{X}}\ \varphi_{\mathscr{m}}(xy-e) \in R_{\mathscr{m}}$ and by 1.27 $\forall_{\mathscr{m}\in\mathcal{N}}\ \varphi_{\mathscr{m}}(x) \in \mathcal{W}_{\mathscr{m}}$.

1.30. If $\mathscr{m} \in \mathcal{N}$ and $\mathcal{W}_{\mathscr{m}} = \{\varphi_{\mathscr{m}}(e)\}$ then $\mathscr{m}$ is called a strong ideal in $\mathcal{X}$; the set of strong ideals in $\mathcal{X}$ is denoted by $\mathcal{Y}$; analogously $\mathcal{Y}_{\mathscr{m}}$ is defined as the set of strong ideals in $\varphi_{\mathscr{m}}(\mathcal{X})$ for every $\mathscr{m} \in \mathcal{N} \setminus \{\mathfrak{z}\}$. An algebra with $\mathcal{W} = \mathfrak{z}$ is usually called semi-simple.

1.31. If $\mathscr{m} \in \mathcal{Y}$ then $\mathscr{m} \supset \mathcal{W}$.

Proof: $\mathscr{m} = \overleftarrow{\varphi}_{\mathscr{m}}\varphi_{\mathscr{m}}(e) = \overleftarrow{\varphi}_{\mathscr{m}}(\mathcal{W}_{\mathscr{m}}) \supset \overleftarrow{\varphi}_{\mathscr{m}}\varphi_{\mathscr{m}}(\mathcal{W})$ by 1.30 and 1.29, and $\overleftarrow{\varphi}_{\mathscr{m}}\varphi_{\mathscr{m}}(\mathcal{W}) = \mathscr{m} + \mathcal{W}$ by 1.12; hence $\mathscr{m} \supset \mathcal{W}$.

1.32. $w \in \mathcal{T}$, and analogously $\mathcal{W}_m \in \mathcal{T}_m$.

Proof: $w \in \mathcal{T}$ if and only if $\mathcal{W}_w = \{\varphi_w(o)\}$ or $\mathcal{W}_w = \varphi_w(w)$.

Now, $\varphi_w(w) \subset \mathcal{W}_w$ by 1.26 and 1.29.

Conversely, if $\varphi_w(x) \in \mathcal{W}_w$ then by 1.27

$\forall_{y \in \mathscr{X}}\ \varphi_w(xy - e) \in R_w$ or

$\forall_{y \in \mathscr{X}}\ \exists_{z \in \mathscr{X}}\ \varphi_w[(xy - e)z - e] = \varphi_w(o)$.

If we take $u := (xy - e)z - e$ then $u \in w$; hence by 1.27

$u + e \in R$

from which we conclude

$\forall_{y \in \mathscr{X}}\ \exists_{z \in \mathscr{X}}\ (xy - e)z \in R$

implying $\forall_{y \in \mathscr{X}}\ xy - e \in R$

whence $x \in w$ by 1.27 and

$\varphi_w(x) \in \varphi_w(w)$.

1.33. $m \in \mathcal{T} \iff \overset{\leftarrow}{\varphi}_m(\mathcal{W}_m) = m$.

Proof: $m \in \mathcal{T}$ means by definition $\mathcal{W}_m = \{\varphi_m(o)\}$ and $\overset{\leftarrow}{\varphi}_m(\mathcal{W}_w) = m$ is equivalent to $\mathcal{W}_w = \varphi_m(m) = \{\varphi_m(o)\}$; hence the result.

Properties of spectra

2.1. If $\mathfrak{n} \in \mathcal{N}$ then $\mathcal{M}(\mathfrak{n}) := \{\mathfrak{m} \in \mathcal{N} \mid \mathfrak{m} \supset \mathfrak{n}\}$.
$\mathcal{M}(\mathfrak{n})$ is called the bonnet of $\mathfrak{n}$.

2.2. If $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \in \mathcal{M}(\mathfrak{n})$ then $\forall_{x \in \mathcal{X}}\ \sigma(\mathfrak{m},x) \subset \sigma(\mathfrak{n},x)$.

Proof: If $\lambda \in \sigma(\mathfrak{m},x)$ then $\varphi_{\mathfrak{m}}(x-\lambda e) \in S_{\mathfrak{m}}$ and by 1.20
$x - \lambda e \in \overleftarrow{\varphi}_{\mathfrak{m}}(S_{\mathfrak{m}}) \subset \overleftarrow{\varphi}_{\mathfrak{n}}(S_{\mathfrak{n}})$ hence $\varphi_{\mathfrak{n}}(x-\lambda e) \in S_{\mathfrak{n}}$ and $\lambda \in \sigma(\mathfrak{n},x)$.
An analogous result can be found in [16], p. 698.

2.3. If $\mathfrak{m} \in \mathcal{N}$ and $\mathfrak{n} \in \mathcal{N}$ then $\forall_{x \in \mathcal{X}}\ \sigma(\mathfrak{m} \cap \mathfrak{n},x) =$
$= \sigma(\mathfrak{m},x) \cup \sigma(\mathfrak{n},x)$.

Proof: $\mathfrak{m} \cap \mathfrak{n} \in \mathcal{N}$; hence
$\forall_{x \in \mathcal{X}}\ \sigma(\mathfrak{m} \cap \mathfrak{n},x) \supset \sigma(\mathfrak{m},x) \cup \sigma(\mathfrak{n},x)$ by 2.2.
Suppose $\lambda \notin \sigma(\mathfrak{m},x) \cup \sigma(\mathfrak{n},x)$, then
$\varphi_{\mathfrak{m}}(x-\lambda e) \in R_{\mathfrak{m}}$ and $\varphi_{\mathfrak{n}}(x-\lambda e) \in R_{\mathfrak{n}}$, or
$\exists_{y \in \mathcal{X}}\ \varphi_{\mathfrak{m}}[(x-\lambda e)y - e] = \varphi_{\mathfrak{m}}(\theta)$ and $\exists_{z \in \mathcal{X}}\ \varphi_{\mathfrak{n}}[(x-\lambda e)z - e] =$
$= \varphi_{\mathfrak{n}}(\theta)$, whence
$(x-\lambda e)y - e \in \mathfrak{m}$ and $(x-\lambda e)z - e \in \mathfrak{n}$;
from $(x-\lambda e)[(y+z) - (x-\lambda e)yz] - e =$
$= -[(x-\lambda e)y - e] \cdot [(x-\lambda e)z - e] \in \mathfrak{m} \cap \mathfrak{n}$
we infer that

$\varphi_{\mathcal{m}\cap\mathcal{n}}[(x-\lambda e)((y+z)-(x-\lambda e)yz)] = \varphi_{\mathcal{m}\cap\mathcal{n}}(e)$

which by 1.19 means $\varphi_{\mathcal{m}\cap\mathcal{n}}(x-\lambda e) \in R_{\mathcal{m}\cap\mathcal{n}}$

and consequently $\lambda \notin \sigma(\mathcal{m}\cap\mathcal{n},x)$.

This proves the desired result.

2.4. If $\mathcal{n} \in \mathcal{N}$ then $\mathcal{N} \in \mathcal{M}(\mathcal{n})$ if and only if $\forall_{x\in\mathcal{X}}\ \sigma(x) = \sigma(\mathcal{n},x)$.

Proof:

i: Sufficiency. $\forall_{x\in\mathcal{X}}\ \sigma(x) = \sigma(\mathcal{n},x)$ implies

$\forall_x \forall_\lambda\ [x-\lambda e \in S \Rightarrow \varphi_{\mathcal{n}}(x-\lambda e) \in S_{\mathcal{n}}]$,

$\forall_{x\in\mathcal{n}} \forall_\lambda [x-\lambda e \in S \Rightarrow \varphi_{\mathcal{n}}(x-\lambda e) \in S_{\mathcal{n}}]$,

$\forall_{x\in\mathcal{n}} \forall_\lambda [x-\lambda e \in S \Rightarrow \lambda \cdot \varphi_{\mathcal{n}}(e) \in S_{\mathcal{n}}]$,

$\forall_{x\in\mathcal{n}} \forall_\lambda [x-\lambda e \in S \Rightarrow \lambda = 0]$,

$\forall_{x\in\mathcal{n}}\ \sigma(x) = \{0\}$ and

$\forall_{x\in\mathcal{n}}\ \nu(x) = 0$;

hence $\mathcal{n} \subset \mathcal{N}$.

ii: Necessity. Assume $\mathcal{n} \subset \mathcal{N}$. Since $\varphi_{\mathcal{n}}(x) \in R_{\mathcal{n}}$ implies

$\exists_{y\in\mathcal{X}}\ \varphi_{\mathcal{n}}(xy-e) = \varphi_{\mathcal{n}}(e)$ and hence $xy-e \in \mathcal{n} \subset \mathcal{N}$, we have $\sigma(xy-e) = \{0\}$, $xy \in R$ and consequently $x \in R$.

Therefore $\forall_x[\varphi_{\mathcal{n}}(x) \in R_{\mathcal{n}} \Rightarrow x \in R]$,

$\forall_x[x \in S \Rightarrow \varphi_{\mathcal{n}}(x) \in S_{\mathcal{n}}]$,

$\forall_x \forall_\lambda\ [x-\lambda e \in S \Rightarrow \varphi_{\mathcal{n}}(x-\lambda e) \in S_{\mathcal{n}}]$ and

$\forall_x\ \sigma(x) \subset \sigma(\mathcal{n},x)$;

 hence by 2.2 $\forall_x\ \sigma(x) = \sigma(\mathcal{n},x)$.

2.5. As a corollary of 2.4 we infer that, if $\mathfrak{n} \in \mathcal{N}$ and $\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})$ then $\mathfrak{m} \subset \overset{\leftarrow}{\varphi}_{\mathfrak{n}}(\mathcal{W}_{\mathfrak{n}})$ if and only if $\forall_x\, \sigma(\mathfrak{m},x) = \sigma(\mathfrak{n},x)$.

Proof: If χ is the isomorphism $\varphi_{\mathfrak{m}}(\mathcal{H}) \to \varphi_{\mathfrak{n}}(\mathcal{H}) \,/\, \varphi_{\mathfrak{n}}(\mathfrak{m})$ described in 1.14 and if ψ, as in 1.14, is the homomorphism $\varphi_{\mathfrak{n}}(\mathcal{H}) \to \varphi_{\mathfrak{n}}(\mathcal{H}) \,/\, \varphi_{\mathfrak{n}}(\mathfrak{m})$, then obviously $\chi \circ \varphi_{\mathfrak{m}} = \psi \circ \varphi_{\mathfrak{n}}$.

$\sigma(\mathfrak{m},x) = \sigma(\varphi_{\mathfrak{m}}(\mathfrak{m}), \varphi_{\mathfrak{m}}(x))$ and $\sigma(\mathfrak{n},x) = \sigma(\varphi_{\mathfrak{n}}(\mathfrak{n}), \varphi_{\mathfrak{n}}(x))$ by the definition of the spectrum modulo an element of $\mathcal{N}$;

moreover

$$\sigma(\varphi_{\mathfrak{m}}(\mathfrak{m}), \varphi_{\mathfrak{m}}(x)) = \sigma((\chi \circ \varphi_{\mathfrak{m}})(\mathfrak{m}),\ (\chi \circ \varphi_{\mathfrak{m}})(x))$$

by the definition of χ.

It is now obvious that the conditions

$$\forall_x\, \sigma(\mathfrak{m},x) = \sigma(\mathfrak{n},x)$$

and $\forall_x\, \sigma((\psi \circ \varphi_{\mathfrak{n}})(\mathfrak{m}),\ (\psi \circ \varphi_{\mathfrak{n}})(x)) = \sigma(\varphi_{\mathfrak{n}}(\mathfrak{n}), \varphi_{\mathfrak{n}}(x))$

are equivalent, the latter can also be read

$\forall_x\, \sigma(\psi(\varphi_{\mathfrak{n}}(\mathfrak{m})),\ \psi(\varphi_{\mathfrak{n}}(x))) = \sigma(\varphi_{\mathfrak{n}}(\mathfrak{n}), \varphi_{\mathfrak{n}}(x))$

which happens to be the condition of 2.4 applied to the algebra $\varphi_{\mathfrak{n}}(\mathcal{H})$ instead of $\mathcal{H}$ and the ideal $\varphi_{\mathfrak{n}}(\mathfrak{m})$ instead of $\mathfrak{n}$; hence the condition is by 2.4 equivalent to

$$\mathcal{W}_{\mathfrak{n}} \supset \varphi_{\mathfrak{n}}(\mathfrak{m})$$

or $\overset{\leftarrow}{\varphi}_{\mathfrak{n}}(\mathcal{W}_{\mathfrak{n}}) \supset \mathfrak{m}$.

2.6. If $\mathfrak{m} \in \mathcal{N}$ and $\mathfrak{n} \in \mathcal{N}$ then $\forall_x\, \sigma(\mathfrak{m},x) = \sigma(\mathfrak{n},x)$ is equivalent to $\overline{\mathfrak{m} + \mathfrak{n}} \subset \overset{\leftarrow}{\varphi}_{\mathfrak{m} \cap \mathfrak{n}}(\mathcal{W}_{\mathfrak{m} \cap \mathfrak{n}})$.

Proof: If $\sigma(\mathfrak{m},x) = \sigma(\mathfrak{n},x)$, then they are both equal to $\sigma(\mathfrak{m} \cap \mathfrak{n}, x)$ by 2.3; hence

$\forall_x\ \sigma(\mathfrak{m},x) = \sigma(\mathfrak{n},x)$

implies

$\mathfrak{m} \subset \overleftarrow{\varphi}_{\mathfrak{m}\cap\mathfrak{n}}(\mathfrak{W}_{\mathfrak{m}\cap\mathfrak{n}})$ and $\mathfrak{n} \subset \overleftarrow{\varphi}_{\mathfrak{m}\cap\mathfrak{n}}(\mathfrak{W}_{\mathfrak{m}\cap\mathfrak{n}})$ by 2.5.

Since $\mathfrak{W}_{\mathfrak{m}\cap\mathfrak{n}} \in \mathfrak{M}_{\mathfrak{m}\cap\mathfrak{n}}$ and $\varphi_{\mathfrak{m}\cap\mathfrak{n}}$ is continuous,

$\overline{\mathfrak{m}+\mathfrak{n}} \subset \overleftarrow{\varphi}_{\mathfrak{m}\cap\mathfrak{n}}(\mathfrak{W}_{\mathfrak{m}\cap\mathfrak{n}})$.

Conversely $\mathfrak{m} \subset \overline{\mathfrak{m}+\mathfrak{n}} \subset \overleftarrow{\varphi}_{\mathfrak{m}\cap\mathfrak{n}}(\mathfrak{W}_{\mathfrak{m}\cap\mathfrak{n}})$ implies by 2.5

$\forall_x\ \sigma(\mathfrak{m},x) = \sigma(\mathfrak{m}\cap\mathfrak{n},x)$

and analogously $\forall_x\ \sigma(\mathfrak{n},x) = \sigma(\mathfrak{m}\cap\mathfrak{n},x)$.

2.7. If $\mathfrak{n} \in \mathfrak{N}$ then $\mathfrak{W} \in \mathfrak{M}(\mathfrak{n})$ implies $\mathfrak{W} = \overleftarrow{\varphi}_{\mathfrak{n}}(\mathfrak{W}_{\mathfrak{n}})$.

Proof: If $\mathfrak{W} \in \mathfrak{M}(\mathfrak{n})$ then by 2.4 $\forall_x\ \sigma(x) = \sigma(\mathfrak{n},x)$. This implies $\forall_x\ \nu(x) = \nu(\mathfrak{n},x)$ and $\mathfrak{W} = \overleftarrow{\varphi}_{\mathfrak{n}}(\mathfrak{W}_{\mathfrak{n}})$.

2.8. If $x \in \mathfrak{X}$, $\mathfrak{n} \in \mathfrak{N}$ and $\lambda \in \sigma(\mathfrak{n},x)$, and if we define

$\mathfrak{m} := \overline{\overleftarrow{\varphi}_{\mathfrak{n}}\varphi_{\mathfrak{n}}[(x-\lambda e)\mathfrak{X}]}$

then $\mathfrak{m}$ has the following properties:

i: $\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})$,

ii: $x-\lambda e \in \mathfrak{m}$,

iii: $\sigma(\mathfrak{m},x) = \{\lambda\}$.

Proof:

i: By the definition of $\sigma(\mathfrak{n},x)$ we have $\varphi_{\mathfrak{n}}(x-\lambda e) \in S_{\mathfrak{n}}$; therefore, $\varphi_{\mathfrak{n}}[(x-\lambda e)\mathfrak{X}]$ is a proper ideal in $\varphi_{\mathfrak{n}}(\mathfrak{X})$ and consequently

$\overleftarrow{\varphi}_{\mathfrak{m}}\varphi_{\mathfrak{n}}[(x-\lambda e)\mathcal{X}]$ is a proper ideal in $\mathcal{X}$, whence $\mathfrak{m}\in\mathcal{M}$ by 1.9; obviously $\mathfrak{m}\supset\mathfrak{n}$; hence $\mathfrak{m}\in\mathcal{M}(\mathfrak{n})$.

ii: $x-\lambda e = (x-\lambda e)e \in \overleftarrow{\varphi}_{\mathfrak{m}}\varphi_{\mathfrak{m}}[(x-\lambda e)e] \subset \mathfrak{m}$.

iii: $x-\lambda e\in\mathfrak{m}$ implies $\varphi_{\mathfrak{m}}(x-\lambda e) = \varphi_{\mathfrak{m}}(o)\in S_{\mathfrak{m}}$; hence $\lambda\in\sigma(\mathfrak{m},x)$.

If $\mu\in\sigma(\mathfrak{m},x)$, then $\varphi_{\mathfrak{m}}(x-\mu e)\in S_{\mathfrak{m}}$; since

$$(\lambda-\mu)\varphi_{\mathfrak{m}}(e) = \varphi_{\mathfrak{m}}(x-\mu e) - \varphi_{\mathfrak{m}}(x-\lambda e) = \varphi_{\mathfrak{m}}(x-\mu e) - \varphi_{\mathfrak{m}}(o) =$$

$$= \varphi_{\mathfrak{m}}(x-\mu e)$$

we have $(\lambda-\mu)\varphi_{\mathfrak{m}}(e)\in S_{\mathfrak{m}}$, whence $\lambda=\mu$.

2.9. Obvious consequences of 2.8. are

i: if $x\in\mathcal{X}$, $\{\lambda_i\}_{i=1,\dots,n}\subset\sigma(x)$ and $\mathfrak{m} := \bigcap_{i=1}^{n}\overline{(x-\lambda_i e)\mathcal{X}}$ then $\sigma(\mathfrak{m},x) = \{\lambda_i\}_{i=1,\dots,n}$.

ii: if $x\in\mathcal{X}$ and $\mathfrak{q}(x) := \bigcap_{\lambda\in Cm}\overline{(x-\lambda e)\mathcal{X}}$ then $\sigma(x) = \sigma(\mathfrak{q}(x),x)$.

iii: $\bigcap_{x\in\mathcal{X}}\mathfrak{q}(x) = \mathfrak{Z}$.

iv: if $y\in\mathcal{W}$ then $\mathfrak{q}(y) = \overline{y\mathcal{X}}$.

Proof:

i: If $\mathfrak{m}_i := \overline{(x-\lambda_i e)\mathcal{X}}$ $(i=1,\dots,n)$ then by 2.8, taking $\mathfrak{n}=\mathfrak{Z}$, $\sigma(\mathfrak{m}_i,x) = \{\lambda_i\}$ $(i=1,\dots,n)$ and by a corollary of 2.3

$$\sigma(\mathfrak{m},x) = \sigma(\bigcap_{i=1}^{n}\mathfrak{m}_i,x) = \bigcup_{i=1}^{n}\sigma(\mathfrak{m}_i,x) = \{\lambda_i\}_{i=1,\dots,n}.$$

ii: $\mathfrak{q}(x)$ is clearly a proper closed ideal in $\mathcal{X}$, hence by 2.2 $\sigma(\mathfrak{q}(x),x)\subset\sigma(x)$.

If $\lambda \in \sigma(x)$ then $\mathfrak{m} := \overline{(x-\lambda e)\mathcal{X}}$ has the properties of 2.8 (with $\mathfrak{n} = \mathfrak{z}$), moreover $\mathfrak{m} \supset \mathfrak{q}(x)$; hence, again by 2.2 $\lambda \in \sigma(\mathfrak{q}(x), x)$; this entails that $\sigma(x) \subset \sigma(\mathfrak{q}(x), x)$.

iii: If $y \in \bigcap_{x \in \mathcal{X}} \mathfrak{q}(x)$, then

$\forall_{x \in \mathcal{X}} \; \forall_{\lambda \in \mathbb{C}\mathrm{m}} \; y \in \overline{(x-\lambda e)\mathcal{X}}$,

whence $y \in \overline{\sigma\mathcal{X}} = \mathfrak{z}$.

iv: If $y \in \mathcal{W}$ then $\sigma(y) = \{0\}$; hence, if $\lambda \neq 0$, then $y - \lambda e \in R$ and $(y-\lambda e)\mathcal{X} = \mathcal{X}$.
Now $\mathfrak{q}(y) = \overline{(y-0e)\mathcal{X}} = \overline{y\mathcal{X}}$.

2.10. If $x \in \mathcal{X}$, $y \in \mathcal{X}$ and $\mathfrak{n} \in \mathcal{N}$ then

i: $\sigma(\mathfrak{n}, x+y) \subset \sigma(\mathfrak{n}, x) + \sigma(\mathfrak{n}, y)$;

ii: $\sigma(\mathfrak{n}, xy) \subset \sigma(\mathfrak{n}, x) \cdot \sigma(\mathfrak{n}, y)$.

Proof:

i: If $\nu \in \sigma(\mathfrak{n}, x+y)$ then $\mathfrak{m} := \overline{\overleftarrow{\varphi}_{\mathfrak{n}} \varphi_{\mathfrak{n}}[(x+y-\nu e)\mathcal{X}]}$ has properties which, according to 2.8 and 2.2, guarantee

$\varphi_{\mathfrak{m}}(x+y-\nu e) = \varphi_{\mathfrak{m}}(\sigma)$, $\sigma(\mathfrak{m}, x+y) = \{\nu\}$ and $\sigma(\mathfrak{m}, x) \subset \sigma(\mathfrak{n}, x)$;

since $\sigma(\mathfrak{m}, x)$ is non-vacuous by 1.24, we can take $\lambda \in \sigma(\mathfrak{m}, x)$.

$\varphi_{\mathfrak{m}}[y - (\nu-\lambda)e] = \varphi_{\mathfrak{m}}(x+y-\nu e) - \varphi_{\mathfrak{m}}(x-\lambda e) = -\varphi_{\mathfrak{m}}(x-\lambda e) \in S_{\mathfrak{m}}$,

hence $\nu - \lambda \in \sigma(\mathfrak{m}, y)$ and

$\nu = \lambda + (\nu - \lambda) \in \sigma(\mathfrak{m}, x) + \sigma(\mathfrak{m}, y) \subset \sigma(\mathfrak{n}, x) + \sigma(\mathfrak{n}, y)$

which shows that $\sigma(\mathfrak{n}, x+y) \subset \sigma(\mathfrak{n}, x) + \sigma(\mathfrak{n}, y)$.

ii: Analogously, assume $\nu \in \sigma(\mathfrak{n}, xy)$; then take

$\mathfrak{m} := \overline{\overset{\leftarrow}{\varphi}_{\mathfrak{n}} \varphi_{\mathfrak{n}}[(xy-\nu e)\mathcal{X}]}$ and $\lambda \in \sigma(\mathfrak{m}, x)$.

Since $xy-\nu e = (x-\lambda e)y + (\lambda y - \nu e)$ and $\varphi_{\mathfrak{m}}[(x-\lambda e)y] \in S_{\mathfrak{m}}$ we obtain $\varphi_{\mathfrak{m}}(\lambda y - \nu e) \in S_{\mathfrak{m}}$.

If $\lambda = 0$ this implies $\nu = 0$ and $\nu = \lambda\mu$ for every $\mu \in \sigma$; if $\lambda \neq 0$ then $\nu\lambda^{-1} \in \sigma(\mathfrak{m}, y)$. In both cases

$\nu \in \sigma(\mathfrak{m}, x) \cdot \sigma(\mathfrak{m}, y) \subset \sigma(\mathfrak{n}, x) \cdot \sigma(\mathfrak{n}, y)$ for all $\nu \in \sigma(\mathfrak{n}, xy)$.

2.11. If $\alpha \in \mathrm{Cm}$, $x \in \mathcal{X}$ and $\mathfrak{n} \in \mathcal{N}$ then $\sigma(\mathfrak{n}, \alpha x) = \alpha \cdot \sigma(\mathfrak{n}, x)$.

Proof: Trivial from 1.23.

2.12. Theorems 2.8 and 2.10 can be used for proving Wiener's theorem on absolutely convergent Fourier series by Banach algebra methods, without using the axiom of choice; see [5].

2.13. Another consequence of 2.8 is the following theorem:

If $x \in \mathcal{X}$, $\mathfrak{n} \in \mathcal{N}$ and $\forall_{\mathfrak{m} \in \mathcal{M}(\mathfrak{n})} \exists_{k \in \mathcal{M}(\mathfrak{m})}\ \varphi_k(x) \in R_k$ then $\varphi_{\mathfrak{n}}(x) \in R_{\mathfrak{n}}$.

Proof: If $\varphi_{\mathfrak{n}}(x) \notin R_{\mathfrak{n}}$ or, equivalently, $\varphi_{\mathfrak{n}}(x) \in S_{\mathfrak{n}}$ then $\mathfrak{m} := \overline{\overset{\leftarrow}{\varphi}_{\mathfrak{n}} \varphi_{\mathfrak{n}}(x\mathcal{X})}$ has the properties of 2.8, particularly $x \in \mathfrak{m}$ and $\mathfrak{m} \in \mathcal{M}(\mathfrak{n})$; hence by assumption $\exists_{k \in \mathcal{M}(\mathfrak{m})}\ \varphi_k(x) \in R_k$, contradicting the fact that, since $x \in \mathfrak{m} \subset k$, $\varphi_k(x) = \varphi_k(o)$.

This proves that $\varphi_{\mathfrak{n}}(x) \in R_{\mathfrak{n}}$.

2.14. If $\mathfrak{n} \in \mathcal{N}$ then $\overset{\leftarrow}{\varphi}_{\mathfrak{n}}(S_{\mathfrak{n}}) = \bigcup_{\mathfrak{m} \in \mathcal{M}(\mathfrak{n})} \overset{\leftarrow}{\varphi}_{\mathfrak{m}}(\mathcal{W}_{\mathfrak{m}})$.

Proof: If $x \in \overset{\leftarrow}{\varphi}_{\mathfrak{n}}(S_{\mathfrak{n}})$ then $\varphi_{\mathfrak{n}}(x) \in S_{\mathfrak{n}}$, implying $0 \in \sigma(\mathfrak{n}, x)$;

according to 2.8 there exists an $\mathfrak{m} \in \mathcal{M}(\mathfrak{n})$ with $\sigma(\mathfrak{m}, x) = \{0\}$; hence $\nu(\mathfrak{m}, x) = 0$, $\varphi_{\mathfrak{m}}(x) \in \mathcal{W}_{\mathfrak{m}}$ and $x \in \overleftarrow{\varphi}_{\mathfrak{m}}(\mathcal{W}_{\mathfrak{m}})$.

Conversely, if $\varphi_{\mathfrak{m}}(x) \in \mathcal{W}_{\mathfrak{m}}$ for any $\mathfrak{m} \in \mathcal{M}(\mathfrak{n})$ then $\nu(\mathfrak{m}, x) = 0$; this implies $\sigma(\mathfrak{m}, x) = \{0\}$, $0 \in \sigma(\mathfrak{m}, x)$ by 2.2 and $\varphi_{\mathfrak{m}}(x) \in S_{\mathfrak{m}}$.

The *-operation and the lattice of *-invariant sets of closed ideals

,3.1. Recall that $\mathfrak{M}(\mathfrak{n}) := \{\mathfrak{m} \in \mathfrak{N} \mid \mathfrak{m} \supset \mathfrak{n}\}$.

Recall that if S is a set, P(S) denotes the class of subsets of S.

:3.2. $E := \{\mathfrak{A} \in P(\mathfrak{N}) \mid \forall_{\mathfrak{n} \in \mathfrak{A}}\ \mathfrak{M}(\mathfrak{n}) \subset \mathfrak{A}\}$.

E is clearly a subset of $P(\mathfrak{N})$.

:3.3. We define the mapping $^*: P(\mathfrak{N}) \to P(\mathfrak{N})$ by:

if $\mathfrak{A} \in P(\mathfrak{N})$ then $\mathfrak{A}^* := \{\mathfrak{n} \in \mathfrak{N} \mid \forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \exists_{\mathfrak{k} \in \mathfrak{M}(\mathfrak{m})}\ \mathfrak{k} \in \mathfrak{A}\}$.

By Δ we denote the subset of $P(\mathfrak{N})$ which contains the *-invariant subsets of $\mathfrak{N}$:

$\Delta := \{\mathfrak{A} \in P(\mathfrak{N}) \mid \mathfrak{A} = \mathfrak{A}^*\}$.

3.4. i: $\emptyset^* = \emptyset$, hence $\emptyset \in \Delta$;

ii: $\mathfrak{N}^* = \mathfrak{N}$, hence $\mathfrak{N} \in \Delta$;

iii: $\mathfrak{A} \subset \mathfrak{B} \Rightarrow \mathfrak{A}^* \subset \mathfrak{B}^*$.

Proof:

i: Since $\forall_{\mathfrak{k} \in \mathfrak{N}}\ \mathfrak{k} \notin \emptyset$, we have $\forall_{\mathfrak{n}}\ \forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \forall_{\mathfrak{k} \in \mathfrak{M}(\mathfrak{m})}\ \mathfrak{k} \notin \emptyset$ and consequently $\forall_{\mathfrak{n}}\ \mathfrak{n} \notin \emptyset^*$.

ii: Since $\forall_{\mathfrak{n}}\ \forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \forall_{k \in \mathfrak{M}(\mathfrak{m})}\ k \in \mathfrak{N}$, we have $\forall_{\mathfrak{n}}\ \mathfrak{n} \in \mathfrak{N}^*$, whence $\mathfrak{N}^* = \mathfrak{N}$.

iii: If $\mathfrak{A} \subset \mathfrak{B}$ and $\mathfrak{n} \in \mathfrak{A}^*$ then

$\forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \exists_{k \in \mathfrak{M}(\mathfrak{m})}\ k \in \mathfrak{A} \subset \mathfrak{B}$, hence $\mathfrak{A}^* \subset \mathfrak{B}^*$.

3.5. $\forall_{\mathfrak{A} \in P(\mathfrak{N})}\ \mathfrak{A}^* \in E$.

Proof: If $\mathfrak{n} \in \mathfrak{A}^*$ and $\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})$ we have to prove $\mathfrak{m} \in \mathfrak{A}^*$ or equivalently $\forall_{k \in \mathfrak{M}(\mathfrak{m})}\ \exists_{\mathfrak{q} \in \mathfrak{M}(k)}\ \mathfrak{q} \in \mathfrak{A}$.

But this is trivial since $\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})$ and $k \in \mathfrak{M}(\mathfrak{m})$ imply $k \in \mathfrak{M}(\mathfrak{n})$, and $\mathfrak{n} \in \mathfrak{A}^*$.

3.6. $\Delta \subset E$.

Proof: Consequence of 3.3 and 3.5.

3.7. $\mathfrak{A} \in E \Rightarrow \mathfrak{A} \subset \mathfrak{A}^*$.

Proof: If $\mathfrak{A} \in E$ and $\mathfrak{n} \in \mathfrak{A}$ then $\mathfrak{M}(\mathfrak{n}) \subset \mathfrak{A}$ and $\forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \mathfrak{M}(\mathfrak{m}) \subset \mathfrak{A}$; hence $\forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \exists_{k \in \mathfrak{M}(\mathfrak{m})}\ k \in \mathfrak{A}$ viz. $k := \mathfrak{m}$; hence $\mathfrak{n} \in \mathfrak{A}^*$.

3.8. $\forall_{\mathfrak{A} \in P(\mathfrak{N})}\ \mathfrak{A}^{**} = \mathfrak{A}^*$.

Proof: $\mathfrak{A}^* \in E$ by 3.5, whence $\mathfrak{A}^* \subset \mathfrak{A}^{**}$ by 3.7.
Conversely, if $\mathfrak{n} \in \mathfrak{A}^{**}$ then $\forall_{\mathfrak{m} \in \mathfrak{M}(\mathfrak{n})}\ \exists_{k \in \mathfrak{M}(\mathfrak{m})}\ k \in \mathfrak{A}^*$; if $k \in \mathfrak{A}^*$ then trivially $k \in \mathfrak{M}(k)$ and moreover $\exists_{\mathfrak{q} \in \mathfrak{M}(k)}\ \mathfrak{q} \in \mathfrak{A}$. Now $\mathfrak{q} \in \mathfrak{M}(\mathfrak{m})$ and consequently $\mathfrak{n} \in \mathfrak{A}^*$.

3.9. If $\mathfrak{A} \in P(\mathfrak{N})$ then $\mathfrak{A}^* \in \Delta$.

 Proof: Consequence of 3.8 and 3.3.

3.10. If $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda} \in P(E)$ then

i: $\bigcup_\lambda \mathfrak{A}_\lambda \in E$ and $(\bigcup_\lambda \mathfrak{A}_\lambda)^* \supset \bigcup_\lambda \mathfrak{A}_\lambda^*$

ii: $\bigcap_\lambda \mathfrak{A}_\lambda \in E$ and $(\bigcap_\lambda \mathfrak{A}_\lambda)^* \subset \bigcap_\lambda \mathfrak{A}_\lambda^*$

iii: If, moreover, Λ is finite, then $(\bigcap_\lambda \mathfrak{A}_\lambda)^* = \bigcap_\lambda \mathfrak{A}_\lambda^*$.

Proof: i and ii are trivial consequences of 3.2 and 3.4.iii.

iii: Since Λ is finite we have that $\Lambda = \{\lambda_j\}_{j=1,\dots,\ell}$.
If $\mathfrak{n} \in \bigcap_{j=1}^{\ell} \mathfrak{A}_{\lambda_j}^*$ then $\forall_{j=1,\dots,\ell}\ \forall_{\mathfrak{m}\in\mathfrak{M}(\mathfrak{n})}\ \exists_{\mathfrak{k}_j\in\mathfrak{M}(\mathfrak{m})}\ \mathfrak{k}_j \in \mathfrak{A}_{\lambda_j}$.

Take $\mathfrak{m}\in\mathfrak{M}$ and $\mathfrak{m}_1 \in \mathfrak{M}(\mathfrak{m}) \cap \mathfrak{A}_{\lambda_1}$, then $\mathfrak{m}_1 \in \mathfrak{M}(\mathfrak{n})$; hence there exists an $\mathfrak{m}_2 \in \mathfrak{M}(\mathfrak{m}_1) \cap \mathfrak{A}_{\lambda_2}$ and since $\mathfrak{A}_{\lambda_1} \in E$ we have also $\mathfrak{m}_2 \in \mathfrak{A}_{\lambda_1}$

Proceeding in this way we construct a sequence

$\mathfrak{m}_1 \subset \mathfrak{m}_2 \subset \mathfrak{m}_3 \dots \subset \mathfrak{m}_\ell$
and we see at once that $\mathfrak{m}_\ell \in \mathfrak{M}(\mathfrak{m})$ and $\mathfrak{m}_\ell \in \bigcap_{j=1}^{\ell} \mathfrak{A}_{\lambda_j}$; hence

$\mathfrak{n} \in \left(\bigcap_{j=1}^{\ell} \mathfrak{A}_{\lambda_j}\right)^*$.

This proves the theorem.

3.11. If $\mathfrak{A} \in P(\mathfrak{N})$ and $\mathfrak{B} \in \Sigma$ then $\mathfrak{A}^* \cap \mathfrak{B} \subset (\mathfrak{A} \cap \mathfrak{B})^*$.

Proof: The assumptions together with $\mathfrak{n} \in \mathfrak{A}^* \cap \mathfrak{B}$ and $\mathfrak{m}\in\mathfrak{M}(\mathfrak{n})$ imply $\mathfrak{n} \in \mathfrak{A}^*$, $\exists_{\mathfrak{k}\in\mathfrak{M}(\mathfrak{m})}\ \mathfrak{k} \in \mathfrak{A}$, $\mathfrak{n} \in \mathfrak{B}$ and $\forall_{\mathfrak{k}\in\mathfrak{M}(\mathfrak{m})}\ \mathfrak{k} \in \mathfrak{B}$; hence $\exists_{\mathfrak{k}\in\mathfrak{M}(\mathfrak{m})}\ \mathfrak{k} \in \mathfrak{A} \cap \mathfrak{B}$; thus by definition $\mathfrak{n} \in (\mathfrak{A} \cap \mathfrak{B})^*$.

3.12. If $\{\mathfrak{A}_\lambda\}_{\lambda\in\Lambda} \in P(\Delta)$ then $(\bigcap_\lambda \mathfrak{A}_\lambda)^* = \bigcap_\lambda \mathfrak{A}_\lambda^*$, whence $\bigcap_\lambda \mathfrak{A}_\lambda \in \Delta$.

Proof: By 3.10.ii $(\bigcap_\lambda \mathfrak{A}_\lambda)^* \subset \bigcap_\lambda \mathfrak{A}_\lambda^*$ whence by assumption

$(\bigcap_\lambda \mathfrak{A}_\lambda)^* \subset \bigcap_\lambda \mathfrak{A}_\lambda$; the converse inclusion is true by 3.10.ii and 3.7.

3.13. If $\mathfrak{A} \in P(\mathfrak{N})$, $\mathfrak{B} \in E$ and $(\mathfrak{A} \cup \mathfrak{B})^* \subset \mathfrak{A}$ then $\mathfrak{B} \subset \mathfrak{A}$.

Proof: $n \notin \mathfrak{A}$ implies by assumption $n \notin (\mathfrak{A} \cup \mathfrak{B})^*$ and by definition this entails $\exists_{m \in \mathfrak{N}(n)} \mathfrak{N}(m) \cap (\mathfrak{A} \cup \mathfrak{B}) = \emptyset$; particularly $m \notin \mathfrak{A} \cup \mathfrak{B}$ and $m \notin \mathfrak{B}$; since $\mathfrak{B} \in E$ we conclude $n \notin \mathfrak{B}$.

3.14. The *-operation is not a closure operator in $\mathfrak{N}$ in the usual topological sense, since both $\mathfrak{A} \subset \mathfrak{A}^*$ and $(\mathfrak{A} \cup \mathfrak{B})^* = \mathfrak{A}^* \cup \mathfrak{B}^*$ do not always hold in $P(\mathfrak{N})$; the first of these conditions by 3.7 holds in E, the latter does not hold always even in Δ; for proofs see section 7.

3.15. If $\{\mathfrak{A}_\mu\}_{\mu \in M} \in P(\Delta)$ and $\mathfrak{B} \in \Delta$ then

i: $\forall_{\lambda \in M}\, \mathfrak{A}_\lambda \supset \bigcap_{\mu \in M} \mathfrak{A}_\mu$,

ii: $[\forall_{\lambda \in M}\, \mathfrak{A}_\lambda \supset \mathfrak{B}] \Rightarrow \bigcap_{\mu \in M} \mathfrak{A}_\mu \supset \mathfrak{B}$,

iii: $\forall_{\lambda \in M}\, \mathfrak{A}_\lambda \subset \left(\bigcup_{\mu \in M} \mathfrak{A}_\mu\right)^*$,

iv: $[\forall_{\lambda \in M}\, \mathfrak{A}_\mu \subset \mathfrak{B}] \Rightarrow \left(\bigcup_{\mu \in M} \mathfrak{A}_\mu\right)^* \subset \mathfrak{B}$.

Proof: i and ii are trivial, iii is a consequence of 3.4.iii and the assumption $\mathfrak{B} \in \Delta$; iv follows from the fact that

$[\forall_{\lambda \in M}\, \mathfrak{A}_\lambda \subset \mathfrak{B}] \Rightarrow \bigcup_{\mu \in M} \mathfrak{A}_\mu \subset \mathfrak{B}$, hence, by 3.4.iii and the assumption $\mathfrak{B} \in \Delta$, $\left(\bigcup_{\mu \in M} \mathfrak{A}_\mu\right)^* \subset \mathfrak{B}^* = \mathfrak{B}$.

3.16. If $\{\mathfrak{A}_\mu\}_{\mu \in M} \in P(\Delta)$ then

i: $\bigwedge_{\mu \in M} \mathfrak{A}_\mu := \bigcap_{\mu \in M} \mathfrak{A}_\mu$

ii: $\bigvee_{\mu \in M} \mathfrak{A}_\mu := \left(\bigcup_{\mu \in M} \mathfrak{A}_\mu \right)^*$.

If $M = \emptyset$ then $\bigcap_{\mu \in M} \mathfrak{A}_\mu = \mathfrak{N}$ and $\bigcup_{\mu \in M} \mathfrak{A}_\mu = \emptyset$; hence $\bigwedge_{\mu \in \emptyset} \mathfrak{A}_\mu = \mathfrak{N}$ and $\bigvee_{\mu \in \emptyset} \mathfrak{A}_\mu = \emptyset$; if M consists of only one index then we have $\wedge \mathfrak{A} = \mathfrak{A} = \vee \mathfrak{A}$; if M consists of a finite number ℓ of indices then we occasionally write

$\mathfrak{A}_1 \wedge \mathfrak{A}_2 \ldots \wedge \mathfrak{A}_\ell$ for $\bigwedge_{j=1}^{\ell} \mathfrak{A}_j$ and $\mathfrak{A}_1 \vee \mathfrak{A}_2 \ldots \vee \mathfrak{A}_\ell$ for $\bigvee_{j=1}^{\ell} \mathfrak{A}_j$.

3.17 $(\Delta, \wedge, \vee)$ is a complete, $\wedge$-distributive, $\vee$-distributive lattice.

Proof:

i: $\wedge$ and $\vee$ have the required property of $\mathfrak{A} \in \Delta$ and $\mathfrak{B} \in \Delta$ implying $\mathfrak{A} \wedge \mathfrak{B} \in \Delta$ and $\mathfrak{A} \vee \mathfrak{B} \in \Delta$; this is shown for $\wedge$ in 3.12, for $\vee$ it follows from the definition 3.16.ii and 3.8.

ii: Properties L_3 and L_4 of section 0.2 on commutativity and L_1 on associativity of $\wedge$ follow trivially from the definitions; if $\forall_{\mu \in M} \forall_{\nu \in N} \mathfrak{A}_{\mu\nu} \in \Delta$ then

$(\bigcup_\nu \mathfrak{A}_{\mu\nu})^* \supset \bigcup_\nu \mathfrak{A}_{\mu\nu}$ for every $\mu \in M$, by 3.10.i,

hence $\bigcup_\mu (\bigcup_\nu \mathfrak{A}_{\mu\nu})^* \supset \bigcup_\mu \bigcup_\nu \mathfrak{A}_{\mu\nu} = \bigcup_{\mu,\nu} \mathfrak{A}_{\mu\nu}$;

since $[\bigcup_\mu (\bigcup_\nu \mathfrak{A}_{\mu\nu})^*]^* \supset (\bigcup_{\mu,\nu} \mathfrak{A}_{\mu\nu})^*$ by 3.4.iii we have $\bigvee_{\mu}(\bigvee_{\nu} \mathfrak{A}_{\mu\nu}) \supset \bigvee_{\mu,\nu} \mathfrak{A}_{\mu\nu}$ by 3.16.ii ;

we now prove the converse inclusion:

Putting $\mathfrak{L} := \bigvee_{\mu,\nu} \mathfrak{A}_{\mu\nu}$ we have

$\mathfrak{A}_{\mu\nu} \subset \mathfrak{L}$ for all μ,ν by 3.4.iii,

hence $(\bigcup_{\nu} \mathfrak{A}_{\mu\nu})^{*} \subset \mathfrak{L}$ for all μ

and $[\bigcup_{\mu}(\bigcup_{\nu} \mathfrak{A}_{\mu\nu})^{*}]^{*} \subset \mathfrak{L}$ by 3.15.iv.

This means $\bigvee_{\mu}(\bigvee_{\nu} \mathfrak{A}_{\mu\nu}) \subset \bigvee_{\mu,\nu} \mathfrak{A}_{\mu\nu}$.

The results combine to the extended law of associativity for $\vee$:

$\bigvee_{\mu} (\bigvee_{\nu} \mathfrak{A}_{\mu\nu}) = \bigvee_{\mu,\nu} \mathfrak{A}_{\mu\nu} = \bigvee_{\nu}(\bigvee_{\mu} \mathfrak{A}_{\mu\nu})$,

implying that $\mathfrak{A} \vee (\mathfrak{L} \vee \mathfrak{L}) = (\mathfrak{A} \vee \mathfrak{L}) \vee \mathfrak{L}$; this establishes property L_4 of section 0.2 .

Before we are allowed to conclude that $(\Delta, \wedge, \vee)$ is a lattice we should have verified L_5 and L_6 that for every $\mathfrak{A} \in \Delta$ and for every $\mathfrak{L} \in \Delta$ $\mathfrak{A} \wedge (\mathfrak{A} \vee \mathfrak{L}) = \mathfrak{A} = \mathfrak{A} \vee (\mathfrak{A} \wedge \mathfrak{L})$; this, however, is a consequence of the laws of distributivity, see vi below.

iii: Since $\mathfrak{A} \wedge \mathfrak{L} = \mathfrak{A} \cap \mathfrak{L}$ we have $\mathfrak{A} \subset \mathfrak{L} \Longleftrightarrow \mathfrak{A} \wedge \mathfrak{L} = \mathfrak{A}$, implying that the lattice-order of $(\Delta, \wedge, \vee)$ coincides with set-inclusion in $P(\mathcal{H})$, which together with 3.12 and 3.15.i and ii implies that Δ is $\wedge$-complete.

The $\vee$-completeness of Δ follows analogously from 3.8 and 3.15.iii and iv.

 iv: Let $\{\mathfrak{A}_{\mu}\}_{\mu \in M} \in P(\Delta)$ and $\mathfrak{L} \in \Delta$; then by 3.11

$\mathcal{L} \wedge (\bigvee_\mu \mathfrak{A}_\mu) = \mathcal{L} \cap (\bigcup_\mu \mathfrak{A}_\mu)^* \subset [\mathcal{L} \cap (\bigcup_\mu \mathfrak{A}_\mu)]^* = [\bigcup_\mu(\mathcal{L} \cap \mathfrak{A}_\mu)]^*$

hence $\mathcal{L} \wedge (\bigvee_\mu \mathfrak{A}_\mu) \subset \bigvee_\mu(\mathcal{L} \wedge \mathfrak{A}_\mu)$.

Conversely, if $m \in \bigvee_\mu(\mathcal{L} \wedge \mathfrak{A}_\mu)$ then

$\forall_{m \in \mathfrak{M}(m)} \exists_{k \in \mathfrak{M}(m)}\ k \in \bigcup_\mu(\mathcal{L} \wedge \mathfrak{A}_\mu) = \mathcal{L} \cap (\bigcup_\mu \mathfrak{A}_\mu)$

implying $m \in \mathcal{L}^* = \mathcal{L}$ and $m \in (\bigcup_\mu \mathfrak{A}_\mu)^* = \bigvee_\mu \mathfrak{A}_\mu$

hence $m \in \mathcal{L} \wedge (\bigvee_\mu \mathfrak{A}_\mu)$.

This proves that $(\Delta, \wedge, \vee)$ is $\wedge$-distributive.

v: Again we take $\{\mathfrak{A}_\mu\}_{\mu \in M} \in P(\Delta)$ and $\mathcal{L} \in \Delta$; now

$\mathcal{L} \vee (\bigwedge_\mu \mathfrak{A}_\mu) = [\mathcal{L} \cup (\bigcap_\mu \mathfrak{A}_\mu)]^* = [\bigcap_\mu(\mathcal{L} \cup \mathfrak{A}_\mu)]^* \subset$

$\bigcap_\mu(\mathcal{L} \cup \mathfrak{A}_\mu)^*$ by 3.10.ii since $\mathcal{L} \cup \mathfrak{A}_\mu \in E$;

hence $\mathcal{L} \vee (\bigwedge_\mu \mathfrak{A}_\mu) \subset \bigwedge_\mu(\mathcal{L} \vee \mathfrak{A}_\mu)$.

If, conversely, $m \notin \mathcal{L} \vee (\bigwedge_\mu \mathfrak{A}_\mu) = [\mathcal{L} \cup (\bigcap_\mu \mathfrak{A}_\mu)]^*$ then

$\exists_{m_1 \in \mathfrak{M}(m)} \forall_{k \in \mathfrak{M}(m_1)}\ k \notin \mathcal{L} \cup (\bigcap_\mu \mathfrak{A}_\mu)$ or equivalently

$\exists_{m_1 \in \mathfrak{M}(m)}\ \mathfrak{M}(m_1) \cap [\mathcal{L} \cup (\bigcap_\mu \mathfrak{A}_\mu)] = \emptyset$

implying $\mathfrak{M}(m_1) \cap \mathcal{L} = \emptyset$ and $\mathfrak{M}(m_1) \cap (\bigcap_\mu \mathfrak{A}_\mu) = \emptyset$.

In particular $\exists_{\nu \in M}\ m_1 \notin \mathfrak{A}_\nu = \mathfrak{A}_\nu^*$ and consequently

$\exists_{m_2 \in \mathfrak{M}(m_1)}\ \mathfrak{M}(m_2) \cap \mathfrak{A}_\nu = \emptyset$

implying, since $\mathfrak{M}(m_2) \subset \mathfrak{M}(m_1)$

$\mathfrak{M}(m_2) \cap (\mathfrak{A}_\nu \cup \mathfrak{B}) = \emptyset$;

since $m_2 \in \mathfrak{M}(n)$ we can conclude

$n \notin (\mathfrak{A}_\nu \cup \mathfrak{B})^* = \mathfrak{A}_\nu \vee \mathfrak{B}$

whence $n \notin \bigcap_{\mu \in M} (\mathfrak{A}_\mu \vee \mathfrak{B}) = \bigwedge_\mu (\mathfrak{A}_\mu \vee \mathfrak{B})$.

This shows that $\mathfrak{B} \vee (\bigwedge_\mu \mathfrak{A}_\mu) = \bigwedge_\mu (\mathfrak{B} \vee \mathfrak{A}_\mu)$

which connotes that $(\Delta, \wedge, \vee)$ is $\vee$-distributive.

vi: If $\mathfrak{A} \in \Delta$, $\mathfrak{B} \in \Delta$ then

$$\mathfrak{A} \wedge (\mathfrak{A} \vee \mathfrak{B}) = (\mathfrak{A} \wedge \mathfrak{A}) \vee (\mathfrak{A} \wedge \mathfrak{B}) = [\mathfrak{A} \cup (\mathfrak{A} \cap \mathfrak{B})]^* = \mathfrak{A}^* = \mathfrak{A}$$

and

$$\mathfrak{A} \vee (\mathfrak{A} \wedge \mathfrak{B}) = [\mathfrak{A} \cup (\mathfrak{A} \cap \mathfrak{B})]^* = \mathfrak{A}^* = \mathfrak{A}$$

(cf. final part of ii in this proof) are evident.

This concludes the proof of the theorem.

3.18. i: $(\bigwedge_\mu \mathfrak{A}_\mu) \vee (\bigwedge_\nu \mathfrak{B}_\nu) = \bigwedge_{\mu,\nu} (\mathfrak{A}_\mu \vee \mathfrak{B}_\nu)$,

ii: $(\bigvee_\mu \mathfrak{A}_\mu) \wedge (\bigvee_\nu \mathfrak{B}_\nu) = \bigvee_{\mu,\nu} (\mathfrak{A}_\mu \vee \mathfrak{B}_\nu)$.

Proof:

i: $(\bigwedge_\mu \mathfrak{A}_\mu) \vee (\bigwedge_\nu \mathfrak{B}_\nu) = \bigwedge_\mu [\mathfrak{A}_\mu \vee (\bigwedge_\nu \mathfrak{B}_\nu)] =$

$= \bigwedge_\mu [\bigwedge_\nu (\mathfrak{A}_\mu \vee \mathfrak{B}_\nu)] = \bigwedge_{\mu,\nu} (\mathfrak{A}_\mu \vee \mathfrak{B}_\nu)$.

ii: Analogously.

A point-free topology : the lattice of clusters

:4.1. If $\alpha \in \mathrm{Cm}$ and $\varepsilon > 0$ then $\Phi(\alpha,\varepsilon) := \{\gamma \in \mathrm{Cm} \mid |\gamma - \alpha| \geq \varepsilon\}$; $\Phi := \Phi(0,1)$.

4.2. $\{\Phi(\alpha,\varepsilon) \mid \alpha \in \mathrm{Cm},\ \varepsilon > 0\}$ is a base for the closed sets in Cm.

:4.3. By $F(\mathcal{H})$ or F we denote the class of finite subsets of (the B_1 - algebra) $\mathcal{H}$; by Q we denote the class $P(F(\mathcal{H}))$ of subsets of F.

,4.4. Recall that if $x \in \mathcal{H}$ then $\{x\} \in F$; and if $a \in F$ then $\{a\} \in Q$; particularly $\{\{x\}\} \in Q$ for every $x \in \mathcal{H}$. The mapping $\omega : \mathcal{H} \to Q$, defined by $\omega(x) := \{\{x\}\}$, is evidently an injection.

:4.5. If $A \in \omega(\mathcal{H})$, $\alpha \in \mathrm{Cm}$ and $\varepsilon > 0$ then

$$\mathcal{L}(A,\alpha,\varepsilon) := \{m \in \mathcal{M} \mid \sigma(m,\omega^{\leftarrow}(A)) \subset \Phi(\alpha,\varepsilon)\} .$$

If we substitute x for $\omega^{\leftarrow}(A)$ in this expression we get

$$\mathcal{L}(\{\{x\}\},\alpha,\varepsilon) = \{m \in \mathcal{M} \mid \sigma(m,x) \subset \Phi(\alpha,\varepsilon)\} \quad \text{for every } x \in \mathcal{H}.$$

4.6. $\mathcal{L}(\{\{x\}\},\alpha,\varepsilon) = \mathcal{L}(\{\{\frac{x-\alpha e}{\varepsilon}\}\},0,1)$.

Proof: The following statements are equivalent:

$\lambda \in \sigma(m,x)$; $\varphi_m(x-\lambda e) \in S_m$; $\varphi_m(\frac{x-\alpha e}{\varepsilon} - \frac{\lambda-\alpha}{\varepsilon}e) \in S_m$;

$\frac{\lambda-\alpha}{\varepsilon} \in \sigma(m, \frac{x-\alpha e}{\varepsilon})$.

Hence the result.

4.7. For every $x \in \mathscr{X}$ we denote $\mathcal{L}(\{\{x\}\},0,1)$ by $\mathcal{L}_x$.

4.8. If $\mathscr{m} \in \mathcal{L}_x$ and $\mathscr{n} \in \mathcal{L}_x$ then $\mathscr{m} \cap \mathscr{n} \in \mathcal{L}_x$.

Proof: Corollary of 2.3.

4.9. If $x \in \mathscr{X}$ then $\mathcal{L}_x \in \Delta$.

Proof: We observe that $\mathcal{L}_x \in E$ since $\mathscr{n} \in \mathcal{L}_x$ as a corollary of 2.2 implies $\mathcal{M}(\mathscr{n}) \subset \mathcal{L}_x$; consequently $\mathcal{L}_x \subset \mathcal{L}_x^*$ by 3.7.
Now suppose $\mathscr{n} \in \mathcal{L}_x^*$, then $\forall_{\mathscr{m} \in \mathcal{M}(\mathscr{n})} \exists_{\mathfrak{k} \in \mathcal{M}(\mathscr{m})} \mathfrak{k} \in \mathcal{L}_x$ or equivalently, $\forall_{\mathscr{m} \in \mathcal{M}(\mathscr{n})} \exists_{\mathfrak{k} \in \mathcal{M}(\mathscr{m})} \sigma(\mathfrak{k},x) \subset \Phi$, or

$$\forall_{\mathscr{m} \in \mathcal{M}(\mathscr{n})} \exists_{\mathfrak{k} \in \mathcal{M}(\mathscr{m})} \forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \varphi_{\mathfrak{k}}(x - \lambda e) \in R_{\mathfrak{k}}] .$$

Now by 2.13 $\forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \varphi_{\mathscr{n}}(x - \lambda e) \in R_{\mathscr{n}}]$

or equivalently $\sigma(\mathscr{n},x) \subset \Phi$,

whence $\mathscr{n} \in \mathcal{L}_x$.

Consequently $\mathcal{L}_x \in \Delta$.

4.10. If $a \in F$ then $\gamma_a := \bigcup_{x \in a} \mathcal{L}_x$,

$$\mathcal{L}_a := \gamma_a^* = \bigvee_{x \in a} \mathcal{L}_x$$

and if $A \in Q$ then $\mathcal{L}_A := \bigcap_{a \in A} \mathcal{L}_a = \bigwedge_{a \in A} \mathcal{L}_a$;

$\Gamma := \{\mathcal{L}_A \mid A \in Q\}$.

The sets $\mathcal{L}_A$ are called clusters.

4.11. i: $\emptyset \in \Gamma$, $\mathcal{N} \in \Gamma$, $\forall_{x \in \mathscr{X}} \mathcal{L}_x \in \Gamma$, $\forall_{a \in F} \mathcal{L}_a \in \Gamma$;

 ii: $\Gamma \subset \Delta$;

iii: Γ is closed in Δ with respect to $\wedge$: if $\{A_\mu\}_{\mu\in M} \in P(Q)$ then $\bigwedge_{\mu\in M} \mathcal{L}_A \in \Gamma$;

iv: Γ is finitely closed in Δ with respect to $\vee$:
if $\{A_j\}_{j=1,\dots,\ell} \in P(Q)$ then $\bigvee_{j=1}^{\ell} \mathcal{L}_{A_j} \in \Gamma$.

Proof:

i: If $x \in \mathcal{X}$ and $a \in F$ then $\{\{x\}\} \in Q$ and $\{a\} \in Q$;
since $\mathcal{L}_x = \mathcal{L}_{\{x\}} = \mathcal{L}_{\{\{x\}\}}$ and $\mathcal{L}_a = \mathcal{L}_{\{a\}}$ we have $\mathcal{L}_a \in \Gamma$ and $\mathcal{L}_a \in \Gamma$.

Now, denoting as before the zero and identity of $\mathcal{X}$ by σ and e, we have

$$\mathcal{L}_\sigma = \{m \in \mathcal{M} \mid \sigma(m,\sigma) \subset \Phi\} = \{m \in \mathcal{M} \mid 0 \geq 1\} = \emptyset \text{ and}$$

$$\mathcal{L}_e = \{m \in \mathcal{M} \mid \sigma(m,e) \subset \Phi\} = \{m \in \mathcal{M} \mid 1 \geq 1\} = \mathcal{M} .$$

ii: Since $\mathcal{L}_x \in \Delta$ for every $x \in \mathcal{X}$, also $\mathcal{L}_a \in \Delta$ for every $a \in F$ as a consequence of the fact that Δ is a lattice; and from the completeness of Δ it follows that $\bigwedge_{a\in A} \mathcal{L}_a \in \Delta$ for every $A \in Q$.

iii: If $\{A_\mu\}_{\mu\in M} \in P(Q)$ then

$$\bigwedge_{\mu\in M} \mathcal{L}_A = \bigcap_{\mu\in M}\Big(\bigcap_{a\in A_\mu} \mathcal{L}_a\Big) = \bigcap_{a\in \bigcup_{\mu\in M} A_\mu} \mathcal{L}_a = \mathcal{L}_{\bigcup_{\mu\in M} A_\mu}, \quad \text{since}$$

$\bigcup_{\mu\in M} A_\mu \in Q$.

iv: Let $A \in Q$ and $B \in Q$; suppose first that $A := \{a\}$, $B := \{b\}$ with $a \in F$ and $b \in F$.

Then $\mathcal{L}_A \vee \mathcal{L}_B = \mathcal{L}_a \vee \mathcal{L}_b = \Big(\bigvee_{x\in a} \mathcal{L}_x\Big) \vee \Big(\bigvee_{y\in b} \mathcal{L}_y\Big) =$

$$= \bigvee_{x\in a\cup b} \mathcal{L}_x \quad \text{by the associativity of } \vee \text{;}$$

hence $\mathcal{L}_a \vee \mathcal{L}_b = \mathcal{L}_{a \cup b}$.

In the general case we have

$\mathcal{L}_A \vee \mathcal{L}_B = \left(\bigwedge_{a \in A} \mathcal{L}_a \right) \vee \left(\bigwedge_{b \in B} \mathcal{L}_b \right)$, and by 3.18.i we obtain

$$\mathcal{L}_A \vee \mathcal{L}_B = \bigwedge_{\substack{a \in A \\ b \in B}} \left(\mathcal{L}_a \vee \mathcal{L}_b \right) = \bigwedge_{(a,b) \in A \times B} \mathcal{L}_{a \cup b} \quad .$$

Now, if $A \in Q$ and $B \in Q$, then $\{a \cup b\}_{(a,b) \in A \times B} \in Q$;
hence $\mathcal{L}_A \vee \mathcal{L}_B \in \Gamma$.

From this result it follows by induction that, if
$\{A_j\}_{j=1,\ldots,\ell} \in P(Q)$, then

$$\bigvee_{j=1}^{\ell} \mathcal{L}_{A_j} = \bigwedge_{(a_1,\ldots,a_\ell) \in A_1 \times \ldots \times A_\ell} \mathcal{L}_{a_1 \cup \ldots \cup a_\ell} \quad ,$$

and this again is an element of Γ.

4.12. From 4.11 it will be clear that Γ is a sublattice of Δ ; since Γ is $\wedge$-complete and bounded, it is complete by a well-known theorem of lattice theory ([23], p. 68). Since Δ is $\vee$-distributive and Γ is $\wedge$-closed in Δ we see that Γ is $\vee$-distributive. Thus Γ, as a lattice, has the properties PT_1, PT_2 and PT_3 of point-free topologies as stated in section 0.3 . We now proceed to show that Γ fulfils the conditions PT_4 and PT_5 as well. Though PT_5 implies PT_4, the case of PT_4 will be treated separately, mainly because part of its proof serves as a lemma for the proof of PT_5 (4.18 and 4.19); 4.17 stays a little apart as a minor result.

46 **4.13.** If $x \in \mathfrak{X}$ and Ψ is a closed set in Cm, then

$\{\mathcal{n} \in \mathcal{N} \mid \sigma(\mathcal{n},x) \subset \Psi\} \in \Gamma$.

Proof: By 4.2 Ψ is the intersection of a certain family $\{\Phi(\alpha,\varepsilon)\}_{(\alpha,\varepsilon)}$; now

$$\{\mathcal{n} \in \mathcal{N} \mid \sigma(\mathcal{n},x) \subset \Psi\} = \{\mathcal{n} \in \mathcal{N} \mid \sigma(\mathcal{n},x) \subset \bigcap_{(\alpha,\varepsilon)} \Phi(\alpha,\varepsilon)\} =$$

$$= \bigcap_{(\alpha,\varepsilon)} \{\mathcal{n} \in \mathcal{N} \mid \sigma(\mathcal{n},x) \subset \Phi(\alpha,\varepsilon)\} = \bigwedge_{(\alpha,\varepsilon)} \mathcal{L}_{(x-\alpha e)\varepsilon^{-1}} \quad \text{by 4.6.}$$

This proves the theorem.

4.14. If $\mathcal{n} \in \mathcal{N}$, then $\mathcal{L}(\mathcal{n}) := \{\mathcal{q} \in \mathcal{N} \mid \forall_{x \in \mathcal{X}}\, \sigma(\mathcal{q},x) \subset \sigma(\mathcal{n},x)\}$.

4.15. i: $\mathcal{L}(\mathcal{n}) \in \Gamma$ for every $\mathcal{n} \in \mathcal{N}$.

ii: $\mathcal{M}(\mathcal{n}) \subset \mathcal{L}(\mathcal{n})$ for every $\mathcal{n} \in \mathcal{N}$.

iii: If $x \in \mathcal{X}$ and $\mathcal{n} \in \mathcal{L}_x$, then $\mathcal{L}(\mathcal{n}) \subset \mathcal{L}_x$.

Proof:

i: $\mathcal{L}(\mathcal{n}) = \bigcap_{x \in \mathcal{X}} \{\mathcal{q} \in \mathcal{N} \mid \sigma(\mathcal{q},x) \subset \sigma(\mathcal{n},x)\}$; since $\sigma(\mathcal{n},x)$ is closed in Cm for every $\mathcal{n}$ and x, $\{\mathcal{q} \in \mathcal{N} \mid \sigma(\mathcal{q},x) \subset \sigma(\mathcal{n},x)\} \in \Gamma$ by 4.13; hence the result.

ii: Trivial consequence of 2.2 and 4.14.

iii: Trivial consequence of 4.6 (and 4.7) and 4.14.

4.16. If $y \in S$ then $\overline{y\mathcal{X}} \in \mathcal{N}$ by 1.18; we call $\mathcal{L}(\overline{y\mathcal{X}})$ the zero-cluster of y.

4.17. If $\{y_j\}_{j=1,\ldots,\ell} \subset S$ and $\bigwedge_{j=1}^{\ell} \mathcal{L}(\overline{y_j\mathcal{X}}) = \emptyset$ then $e \in \sum_{j=1}^{\ell} y_j\mathcal{X}$.

Proof: $\sum_{j=1}^{\ell} y_j\mathcal{H}$ is an ideal in $\mathcal{H}$; if we assume

$e \notin \sum_{j=1}^{\ell} y_j\mathcal{H}$,

then $e \notin \overline{\Sigma_j\ y_j\mathcal{H}}$ by 1.9;

hence $\overline{\Sigma_j\ y_j\mathcal{H}} \in \mathcal{M}(\overline{y_i\mathcal{H}})$ for every $i = 1,\dots,\ell$;

we infer that $\overline{\Sigma_j\ y_j\mathcal{H}} \in \mathcal{L}(\overline{y_i\mathcal{H}})$ $(i = 1,\dots,\ell)$ by 4.14.ii, whence

$\bigwedge_{i=1}^{\ell} \mathcal{L}(\overline{y_i\mathcal{H}}) \neq \emptyset$.

This proves the theorem.

4.18 If $\mathcal{L}_A \in \Gamma$, $\mathcal{L}_B \in \Gamma$ and $\mathcal{L}_B \not\subset \mathcal{L}_A$ then

$\exists_{\mathcal{L}_C \in \Gamma} [\emptyset \neq \mathcal{L}_C \subset \mathcal{L}_B \ \&\ \mathcal{L}_C \wedge \mathcal{L}_A = \emptyset]$.

Proof: The assumption $\mathcal{L}_B \not\subset \mathcal{L}_A$ implies that $\mathcal{L}_B \neq \emptyset$; since

$\mathcal{L}_A = \bigwedge_{a \in A} \mathcal{L}_a$, $\exists_{a \in A}\ \mathcal{L}_B \not\subset \mathcal{L}_a$.

Thus $\exists_{\mathfrak{n} \in \mathfrak{N}}\ \mathfrak{n} \in \mathcal{L}_B \setminus \mathcal{L}_a$.

Writing $\mathcal{L}_a = \bigvee_{j=1}^{\ell} \mathcal{L}_{y_j}$ we have $\mathfrak{n} \notin \left(\bigcup_{j=1}^{\ell} \mathcal{L}_{y_j}\right)^*$; hence

$\exists_{\mathfrak{n}_1 \in \mathcal{M}(\mathfrak{n})}\ \forall_{\mathfrak{k} \in \mathcal{M}(\mathfrak{n}_1)}\ \mathfrak{k} \notin \bigcup_{j=1}^{\ell} \mathcal{L}_{y_j}$ or

$\exists_{\mathfrak{n}_1 \in \mathcal{M}(\mathfrak{n})}\ \forall_{\mathfrak{k} \in \mathcal{M}(\mathfrak{n}_1)}\ \forall_{j=1,\dots,\ell}\ \exists_{\xi \in \sigma(\mathfrak{k},y_j)}\ |\xi| < 1$.

According to this statement we take $\mathfrak{n}_1 \in \mathcal{M}(\mathfrak{n})$ and $\xi_1 \in \sigma(\mathfrak{n}_1,y_1)$ with $|\xi_1| < 1$; next, according to 2.8, we take $\mathfrak{n}_2 \in \mathcal{M}(\mathfrak{n}_1)$ to the effect that $\sigma(\mathfrak{n}_2,y_1) = \{\xi_1\}$.

Now we can select $\xi_2 \in \sigma(\mathfrak{n}_2,y_2)$ with $|\xi_2| < 1$ and construct

$\mathcal{n}_3 \in \mathcal{M}(\mathcal{n}_2)$ such that $\sigma(\mathcal{n}_3, y_2) = \{\xi_2\}$; proceeding in this way we get the sequences

$\mathcal{n}_1 \subset \mathcal{n}_2 \dots \subset \mathcal{n}_{\ell+1} := \mathcal{m}$

and $\xi_1, \xi_2, \dots, \xi_\ell$

with the properties $\forall_{j=1,\dots,\ell} |\xi_j| < 1$

and $\forall_{j=1,\dots,\ell} \forall_{i=1,\dots,j} \sigma(\mathcal{n}_{j+1}, y_i) = \{\xi_i\}$,

whence $\mathcal{m} \in \mathcal{M}(\mathcal{n})$ and $\forall_{j=1,\dots,\ell} [\sigma(\mathcal{m}, y_i) = \{\xi_j\}$ & $|\xi_j| < 1]$.

We now define $\mathcal{L}_C := \mathcal{L}(\mathcal{m}) \cap \mathcal{L}_B$ and will prove that $\mathcal{L}_C$ has the desired properties:

Since $\mathcal{n} \in \mathcal{L}_B$ and $\mathcal{m} \in \mathcal{M}(\mathcal{n})$ also $\mathcal{m} \in \mathcal{L}_B$; $\mathcal{m} \in \mathcal{L}(\mathcal{m})$ by 4.15.ii;

hence $\mathcal{m} \in \mathcal{L}_C$ and $\mathcal{L}_C \neq \emptyset$; that $\mathcal{L}_C \subset \mathcal{L}_B$ is obvious.

Finally, $\mathcal{L}_C \wedge \mathcal{L}_A \subset \mathcal{L}_C \wedge \mathcal{L}_a = \mathcal{L}_C \wedge \left(\bigvee_{j=1}^{\ell} \mathcal{L}_{y_j} \right) = \bigvee_{j=1}^{\ell} (\mathcal{L}_C \wedge \mathcal{L}_{y_j})$;

since $\alpha \in \mathcal{L}_C$ implies $\alpha \in \mathcal{L}(\mathcal{m})$ and consequently

$\forall_{j=1,\dots,\ell} \sigma(\alpha, y_j) = \{\xi_j\}$ and $\forall_j \alpha \notin \mathcal{L}_{y_j}$ we have $\forall_j \mathcal{L}_C \wedge \mathcal{L}_{y_j} = \emptyset$;

this proves that $\mathcal{L}_C \wedge \mathcal{L}_A = \emptyset$.

4.19. If $\mathcal{L}_A \in \Gamma$, $\mathcal{L}_B \in \Gamma$ and $\mathcal{L}_B \not\subset \mathcal{L}_A$, then there exist $\mathcal{L}_C$, $\mathcal{L}_D$ and $\mathcal{L}_E$ in Γ with the following properties:

i: $\emptyset \neq \mathcal{L}_C \subset \mathcal{L}_B$

ii: $\mathcal{L}_C \wedge \mathcal{L}_D = \emptyset$

iii: $\mathcal{L}_A \wedge \mathcal{L}_E = \emptyset$

iv: $\mathcal{L}_A \subset \mathcal{L}_D$

v: $\mathcal{L}_C \subset \mathcal{L}_E$

vi: $\mathcal{L}_D \vee \mathcal{L}_E = \mathcal{N}$.

Proof: Taking $a = \{y_j\}_{j=1,\dots,\ell}, \mathcal{M}, m$ and $\{\xi_j\}_{j=1,\dots,\ell}$ as in the proof of 4.18 we now proceed as follows:

Let δ be a positive number with $\delta < 1 - \max\{|\xi_j| \; | j = 1,\dots,\ell\}$;

then $\Phi_1 := \{\lambda \in Cm| \; |\lambda| \leq 1 - \frac{\delta}{2}\}$

$\Phi_2 := \{\lambda \in Cm| \; |\lambda| \geq 1 - \delta\}$

and for every $x \in \mathcal{X}$ and $i = 1,2$

$$\mathcal{L}_{i,x} := \{\mathcal{M} \in \mathcal{N} \mid \sigma(\mathcal{M},x) \subset \Phi_i\}.$$

It will be clear that $\mathcal{L}_{2,x} \supset \mathcal{L}_x$ for every $x \in \mathcal{X}$, since $\Phi_2 \supset \Phi$; and also that $\mathcal{L}_{1,x} \in \Gamma$ and $\mathcal{L}_{2,x} \in \Gamma$ since Φ_1 and Φ_2 are closed in Cm.

Thus $\mathcal{L}_A \subset \mathcal{L}_a = \bigvee_{j=1}^{\ell} \mathcal{L}_{y_j} \subset \bigvee_{j=1}^{\ell} \mathcal{L}_{2,y_j}$.

We now define

$\mathcal{L}_C := \mathcal{L}(m) \cap \mathcal{L}_B$ as in 4.18; hence $\mathcal{L}_C \subset \mathcal{L}_B$ and $\mathcal{L}_C \neq \emptyset$.

$\mathcal{L}_D := \bigvee_{j=1}^{\ell} \mathcal{L}_{2,y_j}$; hence $\mathcal{L}_A \subset \mathcal{L}_D$.

$\mathcal{L}_E := \bigwedge_{j=1}^{\ell} \mathcal{L}_{1,y_j}$.

Properties i and iv are established by now; we have to prove the remaining ones: ii, iii, v and vi.

 ii: $\mathcal{M} \in \mathcal{L}_D$ implies

$$\forall_{\mathfrak{q}\in\mathcal{M}(\mathfrak{n})}\ \exists_{\mathfrak{k}\in\mathcal{M}(\mathfrak{q})}\ \mathfrak{k}\in\bigcup_{j=1}^{\ell}\mathcal{L}_{2,y_j}\quad\text{or}$$

$$\forall_{\mathfrak{q}\in\mathcal{M}(\mathfrak{n})}\ \exists_{\mathfrak{k}\in\mathcal{M}(\mathfrak{n})}\ \exists_{j=1,\dots,\ell}\ \sigma(\mathfrak{k},y_j)\subset\Phi_2$$

and $\mathfrak{n}\in\mathcal{L}(\mathfrak{m})$ implies

$$\forall_{j=1,\dots,\ell}\ \sigma(\mathfrak{n},y_j)=\{\xi_j\}\ \text{and consequently}$$

$$\forall_{\mathfrak{q}\in\mathcal{M}(\mathfrak{n})}\ \forall_{j=1,\dots,\ell}\ \sigma(\mathfrak{q},y_j)=\{\xi_j\}\ .$$

Since $\forall_j\ \xi_j\not\subset\Phi_2$ we can conclude that $\mathcal{L}(\mathfrak{m})\cap\mathcal{L}_D=\emptyset$;
hence $\mathcal{L}_C\wedge\mathcal{L}_D=\emptyset$.

iii: $$\mathcal{L}_a\wedge\mathcal{L}_E=\left(\bigvee_{j=1}^{\ell}\mathcal{L}_{y_j}\right)\wedge\left(\bigwedge_{i=1}^{\ell}\mathcal{L}_{1,y_i}\right)=$$

$$=\bigvee_{j=1}^{\ell}\left[\mathcal{L}_{y_j}\wedge\left(\bigwedge_{i=1}^{\ell}\mathcal{L}_{1,y_i}\right)\right]$$

by the $\vee$-distributivity of Γ, and

$$\mathcal{L}_a\wedge\mathcal{L}_E\subset\bigvee_{j=1}^{\ell}\left[\mathcal{L}_{y_j}\wedge\mathcal{L}_{1,y_j}\right]\ .$$

Since $\mathcal{L}_{y_j}\wedge\mathcal{L}_{1,y_j}=\{\mathfrak{n}\in\mathcal{N}\mid\sigma(\mathfrak{n},y_j)\subset\Phi\ \&\ \sigma(\mathfrak{n},y_j)\subset\Phi_1\}$ and $\Phi\cap\Phi_1=\emptyset$ we conclude that $\mathcal{L}_a\wedge\mathcal{L}_E=\emptyset$; but then also $\mathcal{L}_A\wedge\mathcal{L}_E=\emptyset$, since $\mathcal{L}_A\subset\mathcal{L}_a$.

v: If $\mathfrak{n}\in\mathcal{L}(\mathfrak{m})$ then $\forall_j\ \sigma(\mathfrak{n},y_j)\subset\sigma(\mathfrak{m},y_j)$, whence $\forall_j\ \sigma(\mathfrak{n},y_j)=\{\xi_j\}$; since $\forall_j\ \xi_j\in\Phi_1$ we have

$\forall_j\ \mathfrak{n}\in\mathcal{L}_{1,y_j}$ and consequently $\mathfrak{n}\in\bigwedge_{j=1}^{\ell}\mathcal{L}_{1,y_j}=\mathcal{L}_E$.

This proves $\mathcal{L}(\mathfrak{m})\subset\mathcal{L}_E$ and, by the definition of $\mathcal{L}_C$, $\mathcal{L}_C\subset\mathcal{L}_E$.

vi: $\mathcal{L}_D \vee \mathcal{L}_E = \left(\bigvee_{i=1}^{\ell} \mathcal{L}_{2,y_i} \right) \vee \left(\bigwedge_{j=1}^{\ell} \mathcal{L}_{1,y_j} \right) =$

$$= \bigwedge_{j=1}^{\ell} \left[\mathcal{L}_{1,y_j} \vee \left(\bigvee_{i=1}^{\ell} \mathcal{L}_{2,y_i} \right) \right] \supset$$

$$\supset \bigwedge_{j=1}^{\ell} \left[\mathcal{L}_{1,y_j} \vee \mathcal{L}_{2,y_j} \right] .$$

Putting for simplicity's sake $\mathfrak{A} := \mathcal{L}_{1,y_j} \cup \mathcal{L}_{2,y_j}$ then

$\mathfrak{A} = \{ \mathfrak{m} \in \mathfrak{N} \mid \sigma(\mathfrak{m}, y_j) \subset \Phi_1$ or $\sigma(\mathfrak{m}, y_j) \subset \Phi_2 \}$ and

$\mathfrak{A}^* = \{ q \subset \mathfrak{N} \mid \forall_{k \in \mathfrak{M}(q)} \exists_{\mathfrak{m} \in \mathfrak{M}(k)}\ \sigma(\mathfrak{m}, y_j) \subset \Phi_1$

or $\sigma(\mathfrak{m}, y_j) \subset \Phi_2 \}$;

since $\Phi_1 \cup \Phi_2 = \mathbb{C}m$ and for every $k \notin \mathfrak{N}$ there exists an $\mathfrak{m} \in \mathfrak{M}(k)$ with the spectrum $\sigma(\mathfrak{m}, y_j)$ consisting of one single point, we get $\mathfrak{A}^* = \mathfrak{N}$; consequently $\mathcal{L}_D \vee \mathcal{L}_E = \mathfrak{N}$.

This concludes the proof of the theorem.

On compactness

,5.1. A topological space X is said to be compact if every collection $\{F_\alpha\}_\alpha$ of closed sets in X with the finite intersection property has a non-void intersection; accordingly we define: a point-free topology L is said to be compact if every subset D of L has the property

$$[\ \forall_{A \in F(D)} \ \bigwedge_{a \in A} a \neq v] \Rightarrow [\ \bigwedge_{a \in D} a \neq v] \ .$$

It is easily seen that compactness of L implies compactness of its companion topology (compare section 0.3) in M.

The companion topology of the lattice Γ of clusters is, as will be shown in section 6 below, the Gelfand topology in the space $\mathfrak{M}$ of maximal ideals in the B_1 -algebra $\mathcal{X}$. Since in this case compactness is a consequence of Tychonoff's theorem, and hence of the axiom of choice, it will be clear that compactness properties for Γ will be hard to derive.

This section features results yielding certain compactness properties, but they depend on the restricted axiom of choice. They are preceded by a few introductory remarks on topological spaces.

,5.2. A topological space X is called first countable if the neighbourhood system of every point of X has a countable base. In a first countable topological space X a point x is an accumulation point of the subset Y if and only if there is a sequence in $Y\setminus\{x\}$ which converges to x .

5.3. If $\{X_\nu\}_{\nu\in N}$ is a family of spaces, then we denote by $\times_{\nu\in N} X_\nu$ the cartesian product of these spaces; if N is not finite then $\times_{\nu\in N} X_\nu \neq \emptyset$ is equivalent to the restricted or unrestricted axiom of choice.

By p_μ $(\mu \in N)$ we denote the projection $\times_{\nu\in N} X_\nu \to X_\mu$, defined by $p_\mu(\{x_\nu\}_{\nu\in N}) := x_\mu$.

If every X_ν is a topological space, the "natural" topology for $\times_\nu X_\nu$ is the weak topology for the family of functions $\{p_\nu\}_{\nu\in N}$ which we shall refer to as the "product topology"; it has as a subbase for the open sets $\{p_\mu^{-}(U_\nu) \mid U_\nu$ open in $X_\nu;\ \nu \in N\}$; the mappings p_ν are continuous and open.

If N is countable then $\times_{\nu\in N} X_\nu$ is first countable if and only if X_ν is first countable for every $\nu \in N$.

5.4. If the topological spaces X_j $(j \in Nt)$ are first countable and compact, then $\times_{j\in Nt} X_j$ is compact.

Proof: If $\times_{j\in Nt} X_j = \emptyset$ then its compactness is trivial.

If $\{\sigma_\beta\}_{\beta\in B}$ is an open covering of $\times_{j\in Nt} X_j$, it might happen that $\exists_{\beta_o\in B}\ \sigma_{\beta_o} = \times_{j\in Nt} X_j$; in this case $\{\sigma_{\beta_o}\}$ is a finite subcovering of $\{\sigma_\beta\}_\beta$.

Suppose $\times_j X_j \neq \emptyset$ and $\forall_{\beta\in B}\ \sigma_\beta \neq \times_j X_j$; consider the refinement

 $\{W_\alpha\}_{\alpha\in A}$ of $\{\sigma_\beta\}_{\beta\in B}$, consisting of the basic sets W_α which constitute the $\mathcal{O}_\beta$'s; hence $\{W_\alpha\}_{\alpha\in A}$ is an open covering of $\times_j X_j$ and $\forall_{\alpha\in A}\ W_\alpha \neq \times_j X_j$.

We denote the family of open sets in X_j by $\mathcal{T}_j$, $(j \in Nt)$ and define $\gamma(\alpha) := \max\{j \in Nt \mid \exists_{U\in \mathcal{T}_j\setminus\{X_j\}} : W_\alpha \subset p_j^{\leftarrow}(U)\}$. W_α, as a basic set, is a finite intersection of sets $p_j^{\leftarrow}(U)$ with $U \in \mathcal{T}_j\setminus\{X_j\}$; and since $W_\alpha \neq \times_j X_j$, this finite number is not zero; hence $\gamma(\alpha) \in Nt$.

We will now prove $\exists_{n_0\in Nt} \bigcup_{\gamma(\alpha)\leq n_0} W_\alpha = \times_j X_j$.

To this end we assume the opposite

$$\forall_{n\in Nt}\ \exists_{y_n\in \times_j X_j}\ y_n \notin \bigcup_{\gamma(\alpha)\leq n} W_\alpha .$$

As X_1 is first-countable and compact, $\{y_n\}_{n\in Nt}$ contains a monotonic subsequence $\{y_n'\}_n$ such that $\{p_1(y_n')\}_n$ converges to a point η_1 in X_1; as X_2 is first-countable and compact, $\{y_n'\}_n$ contains a monotonic subsequence $\{y_n''\}_n$ such that $\{p_2(y_n'')\}_n$ converges to a point $\eta_2 \in X_2$; of course, $\{p_1(y_n'')\}_n$ converges to η_1.

Proceeding in this way we construct by the principle of induction sequences $\{y_n^{(j)}\}_n$, $j \in Nt$

with the properties

$\{y_n^{(j+1)}\}_n$ is a monotonic subsequence of $\{y_n^{(j)}\}_n$, $j \in Nt$

and

$\{p_i(y_n^{(j)})\}_n$ converges to a point $\eta_i \in X_i$; $j \in Nt$, $i \leq j$.

Now consider the sequence $\{y_n^{(n)}\}_{n\in Nt}$.

For every $j \in Nt$ $\{p_j(y_n^{(n)})\}_{n\geq j}$ is a subsequence of $\{p_j(y_n^{(j)})\}$, from which we infer that for every $j \in Nt$ $\{p_j(y_n^{(n)})\}_n$ converges to η_j.

Take $y := \{\eta_j\}_{j\in Nt}$, then $y \in \times_j X_j$; consequently there exists an $\alpha_0 \in A$ such that $y \in W_{\alpha_0}$; hence $\forall_j\ \eta_j \in p_j(W_{\alpha_0})$ and, since p_j is open, every $p_j(W_{\alpha_0})$ is a neighbourhood of η_j.

But $n \geqslant \gamma(\alpha_0)$ implies by the definition of y_n

$y_n \notin W_{\alpha_0}$

and consequently $y_n^{(n)} \notin W_{\alpha_0}$.

Since $W_{\alpha_0} \neq \times_j X_j$ there are finitely many indices, say $j_1,\ldots,j_\ell$, such that $W_{\alpha_0} = \bigcap_{i=1}^{\ell} \overleftarrow{p}_{j_i}(U_i)$ with U_i open in X_{j_i} and $U_i \neq X_{j_i}$; for these $j_1,\ldots,j_\ell$ $U_i = p_{j_i}(W_{\alpha_0})$, $(i=1,\ldots,\ell)$, and for at least one of these, say j_0, there are countably many elements of $\{y_n^{(n)}\}_n$ such that $p_{j_0}(y_n^{(n)}) \notin U_{j_0} = p_{j_0}(W_{\alpha_0})$. This contradicts the earlier observation that $p_{j_0}(W_{\alpha_0})$ is a neighbourhood of the limit η_{j0} of the convergent sequence $\{p_{j_0}(y_n^{(n)})\}_n$.

We conclude that

$$\exists_{n_0 \in Nt} \bigcup_{\gamma(\alpha) \leqslant n_0} W_\alpha = \times_j X_j$$

as stated before.

Hence $\{W_\alpha\}_{\gamma(\alpha)\leqslant n_0}$ is a subcovering of $\{W_\alpha\}_{\alpha\in A}$.

If p is the projection of $\times_j X_j$ onto $Y := \times_{j=1}^{n_0+1} X_j$ then

 $\{p(W_\alpha)\}_{\gamma(\alpha)\leqslant n_0}$

is an open covering of Y and since Y as a finite product of compact spaces is compact, there exist $\alpha_1,\dots,\alpha_m$

with $\forall_{i=1,\dots,m}\ \gamma(\alpha_i) \leq n_0$ and

$$\bigcup_{i=1}^{m} p(W_{\alpha_i}) = Y \ ;$$

hence $\bigcup_{i=1}^{m} p^{\leftarrow} p(W_{\alpha_i}) = \bigtimes_j X_j$

and since $\gamma(\alpha_i) \leq n_0$ by definition implies $p^{\leftarrow} p(W_{\alpha_i}) = W_{\alpha_i}$,

$$\bigcup_{i=1}^{m} W_{\alpha_i} = \bigtimes_j X_j \ .$$

If we now replace the W_{α_i}'s by the corresponding σ_{β_i}'s from the original covering, we get

$$\bigcup_{i=1}^{m} \sigma_{\beta_i} = \bigtimes_j X_j \ .$$

:5.5. If $\{\mathfrak{a}_\iota\}_{\iota\in I}$ is a collection of (not necessarily closed) ideals in the B_1-algebra $\mathcal{X}$, then

$$\Sigma_{\iota\in I}\, \mathfrak{a}_\iota := \{\Sigma_{\varkappa\in K}\, x_\varkappa \mid K \in F(I)\ \&\ \forall_{\varkappa\in K}\ x_\varkappa \in \mathfrak{a}_\varkappa\} \ ,$$

$F(I)$ denoting the class of finite subsets of I.

5.6. $\Sigma_{\iota\in I}\, \mathfrak{a}_\iota$ is an ideal in $\mathcal{X}$, and consequently

$$\overline{\Sigma_{\iota\in I}\, \mathfrak{a}_\iota} \in \mathcal{N} \cup \{\mathcal{X}\} \ .$$

5.7. If $\{\mathfrak{m}_i\}_{i\in \mathbb{N}t} \in P(\mathcal{N})$ and $\forall_{i\in \mathbb{N}t}\ \mathfrak{m}_{i+1} \in \mathcal{M}(\mathfrak{m}_i)$, then

$$\Sigma_{i=1}^{\infty}\, \mathfrak{m}_i = \bigcup_{i=1}^{\infty} \mathfrak{m}_i \quad \text{and} \quad \overline{\Sigma_{i=1}^{\infty}\, \mathfrak{m}_i} \in \mathcal{N} \ .$$

Proof: The first part of the statement is trivial; since $\forall_{i \in Nt}\ e \notin \mathfrak{m}_i$, also $e \notin \bigcup_{i=1}^{\infty} \mathfrak{m}_i$; hence, in connection with the first part of the statement, 5.6 and 1.9

$$\overline{\Sigma_{i=1}^{\infty} \mathfrak{m}_i} \in \mathcal{N} .$$

5.8. If $\forall_{i \in Nt}\ x_i \in \mathcal{H}$ and $\Sigma_{i \in Nt}\ x_i \mathcal{H} = \mathcal{H}$, then there is a finite subset $\{i_1,\dots,i_\ell\}$ in Nt with the property $\Sigma_{j=1}^{\ell}\ x_{i_j} \mathcal{H} = \mathcal{H}$.

Proof: $\Sigma_{i \in Nt}\ x_i \mathcal{H} = \mathcal{H}$ implies $e \in \Sigma_{i \in Nt}\ x_i \mathcal{H}$; hence, by definition, $\exists_{K \in F(Nt)}\ e \in \Sigma_{i \in K}\ x_i \mathcal{H}$, and the theorem follows.

5.9. If $\forall_{i \in Nt}\ x_i \in \mathcal{H}$ and $\Sigma_{i \in Nt}\ x_i \mathcal{H} = \mathcal{H}$, then there exist

i: a number $m \in Nt$,

ii: a finite subset $\{i_1,\dots,i_m\}$ in Nt ,

iii: positive numbers $\delta_1,\dots,\delta_m$

with the property:

If $\{\lambda_1,\dots,\lambda_m\} \subset Cm$ and $\forall_{j=1,\dots,m}\ |\lambda_j| < \delta_j$, then

$$\Sigma_{j=1}^{m}\ (x_{i_j} - \lambda_j\, e)\mathcal{H} = \mathcal{H} .$$

Proof: The assumption yields by 5.8 the existence of a finite set $\{i_1,\dots,i_m\}$ with $\Sigma_{j=1}^{m}\ x_{i_j} \mathcal{H} = \mathcal{H}$; hence $e = \Sigma_{j=1}^{m}\ x_{i_j}\ y_j$ with $\{y_1,\dots,y_m\} \subset \mathcal{H}$. Let $\delta_1,\dots,\delta_m$ be chosen in such a way that

 $\Sigma_{j=1}^{m}\ \delta_j\ \|y_j\| < 1$.

Then $\forall_{j=1,\dots,m}\ |\lambda_j| < \delta_j$ implies $\|\Sigma_{j=1}^{m}\ \lambda_j\ y_j\| < 1$, whence by 1.5

$e - \Sigma_{j=1}^{m}\ \lambda_j\ y_j \in R$;

since $e - \Sigma_j\ \lambda_j\ y_j = \Sigma_j(x_{i_j} - \lambda_j\, e)y_j \in \Sigma_j(x_{i_j} - \lambda_j\, e)\mathcal{H}$, the ideal $\Sigma_j(x_{i_j} - \lambda_j\, e)\mathcal{H}$ contains a regular element; therefore it equals $\mathcal{H}$.

:5.10. By $\mathcal{Y}$ we denote the set of sequences $\{x_n\}_{n\in Nt}$ with $\forall_n\ x_n \in \mathcal{H}$; elements of $\mathcal{Y}$ will be denoted by symbols like $\underline{x}$.

By D we denote the set of sequences $\{\xi_n\}_{n\in Nt}$ with $\forall_n\ \xi_n \in Cm$; elements of D will be denoted by symbols like $\underline{\xi}$.

We now define for every $\underline{x}$

$T(\underline{x}) := \{\underline{\xi} \in D \mid \Sigma_{i=1}^{\infty}(x_i - \xi_i\, e)\mathcal{H} \neq \mathcal{H}\}$.

5.11. If $\underline{\xi} \in T(\underline{x})$ then $\forall_{n\in Nt}\ [\xi_n \in \sigma(x_n)\ \&\ |\xi_n| \leq \nu(x_n)]$.

Proof: $\xi_n \notin \sigma(x_n)$ would imply that $x_n - \xi_n\, e \in R$ and $(x_n - \xi_n\, e)\mathcal{H} = \mathcal{H}$; hence the result.

5.12. If $\underline{\xi} \in T(\underline{x})$ then $\forall_{\underline{\eta}\in D}\ \underline{\xi} + \underline{\eta} \in T(\underline{x} + \underline{\eta}e)$

($\underline{\xi} + \underline{\eta}$ and $\underline{\eta}e$ being defined in the obvious "pointwise" way).

Proof: $x_n - \xi_n\, e = (x_n + \eta_n\, e) - (\xi_n + \eta_n)e$; hence the result.

5.13. If $\underline{x} \in \mathcal{Y}$ then $T(\underline{x})$, as a subset of D, is closed in the product topology of D.

Proof:

i: Suppose that the zero sequence $\underline{0}$ satisfies $\underline{0} \notin T(\underline{x})$, then by definition $\Sigma_{i=1}^{\infty} x_i \mathcal{H} = \mathcal{H}$; we apply 5.9 and get a subset $\{x_{i_1}, \ldots, x_{i_\ell}\}$ and positive numbers $\delta_1, \ldots, \delta_\ell$.

Now, if $j \in \{i_1, \ldots, i_\ell\}$, take $\Omega_j := \{\gamma \in Cm \mid |\gamma| < \delta_j\}$

and otherwise $\Omega_j := Cm$;

then, with $\Omega := \bigtimes_{j=1}^{\infty} \Omega_j$,

Ω is a neighbourhood of $\underline{0}$ in D and

$$\forall_{\underline{\xi} \in \Omega} \; \Sigma_{i=1}^{\infty} (x_i - \xi_i e)\mathcal{H} \supset \sum_{j=1}^{\ell} (x_{i_j} - \xi_{i_j} e)\mathcal{H} = \mathcal{H};$$

hence, by definition, $\Omega \cap T(\underline{x}) = \emptyset$.

ii: If $\underline{\eta} \notin T(\underline{x})$ then $\underline{0} \notin T(\underline{x} - \underline{\eta} e)$ by virtue of 5.12; hence $\Omega \cap T(\underline{x} - \underline{\eta} e) = \emptyset$ and $(\Omega + \underline{\eta}) \cap T(\underline{x}) = \emptyset$.

Since $(\Omega + \underline{\eta})$ is a neighbourhood of $\underline{\eta}$, we see that $D \setminus T(\underline{x})$ is open and consequently $T(\underline{x})$ is closed in D.

:5.14. If $x \in \mathcal{H}$ then $G(x) := \{\gamma \in Cm \mid |\gamma| \leqslant \nu(x)\}$;

if $\underline{x} \in \mathcal{H}$ then $G(\underline{x}) := \bigtimes_{j=1}^{\infty} G(x_j)$.

5.15. If $\underline{x} \in \mathcal{H}$ then $G(\underline{x})$, as a subset of D, is compact in the product topology of D.

Proof: $G(x)$, as a bounded and closed subset of Cm, is compact for every $x \in \mathcal{H}$; Cm, and therefore also $G(x)$, are first countable; hence $G(\underline{x})$ is compact according to 5.4.

5.16. If $\underline{x} \in \mathcal{H}$ then $T(\underline{x})$ is compact in D.

Proof: If $\underline{x} \in \mathcal{H}$ and $\xi \in T(\underline{x})$ then $\xi \in G(\underline{x})$ by 5.11; hence $T(\underline{x})$ as a closed subset of the compact set $G(\underline{x})$ is compact.

5.17. If $x \in \mathcal{X}$ then

$$\mathcal{L}_x = \{\mathfrak{m} \in \mathcal{N} \mid \forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \overleftarrow{\varphi}_{\mathfrak{m}} \varphi_{\mathfrak{m}}((x - \lambda e)\mathcal{X}) = \mathcal{X}]\} .$$

Proof: The following statements are successively equivalent:

$\mathfrak{m} \in \mathcal{L}_x$

$\sigma(\mathfrak{m}, x) \subset \Phi$

$\forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \lambda \notin \sigma(\mathfrak{m}, x)]$

$\forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \varphi_{\mathfrak{m}}(x - \lambda e) \in R_{\mathfrak{m}}]$

$\forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \varphi_{\mathfrak{m}}((x - \lambda e)\mathcal{X}) = \varphi_{\mathfrak{m}}(\mathcal{X})]$

$\forall_{\lambda \in Cm} [|\lambda| < 1 \Rightarrow \overleftarrow{\varphi}_{\mathfrak{m}} \varphi_{\mathfrak{m}}((x - \lambda e)\mathcal{X}) = \mathcal{X}]$;

hence the result.

5.18. If $\underline{x} \in \mathcal{H}$ and $\sum_{i=1}^{\infty} x_i \mathcal{X} \neq \mathcal{X}$, then $\mathfrak{m} := \overline{\sum_{i=1}^{\infty} x_i \mathcal{X}}$ has the property $\forall_{i \in Nt}\ \sigma(\mathfrak{m}, x_i) = \{0\}$.

Proof: $\forall_{i \in Nt}\ \mathfrak{m} \supset x_i \mathcal{X}$ and consequently $\mathfrak{m} \supset \overline{x_i \mathcal{X}}$;
since $\mathfrak{m} \in \mathcal{N}$ by 1.9 we can apply 2.2 with $\mathfrak{n} = \mathfrak{n}_i := \overline{x_i \mathcal{X}}$ to show $\forall_i\ \sigma(\mathfrak{m}, x_i) \subset \sigma(\mathfrak{n}_i, x_i)$.

Since $x_i \in S$, $0 \in \sigma(x_i)$; and now $\sigma(\mathfrak{n}_i, x_i) = \{0\}$ by 2.8;
hence the result.

5.19. If $\underline{x} \in \mathcal{Y}$ and $\mathfrak{n} \in \mathcal{N}$ then there exist

i: $\mathfrak{m} \in \mathcal{N}$, and

ii: for every $i \in Nt$ a $\xi_i \in \sigma(\mathfrak{n}, x_i)$

with the property

$$\forall_{i \in Nt} [x_i - \xi_i e \in \mathfrak{m} \,\&\, \sigma(\mathfrak{m}, x_i) = \{\xi_i\}] .$$

Proof: We take $\mathfrak{n}_1 := \mathfrak{n}$; let $\xi_1 \in \sigma(\mathfrak{n}_1, x_1)$, then

$\mathfrak{n}_2 := \overline{\overleftarrow{\varphi}_{\mathfrak{n}_1} \varphi_{\mathfrak{n}_1} ((x_1 - \xi_1 e)\mathcal{H})}$ has the properties of 2.8:

$\mathfrak{n}_2 \in \mathcal{M}(\mathfrak{n}_1)$, $x_1 - \xi_1 e \in \mathfrak{n}_2$ and $\sigma(\mathfrak{n}_2, x_1) = \{\xi_1\}$.

Let $\xi_2 \in \sigma(\mathfrak{n}_2, x_2)$, then $\mathfrak{n}_3 := \overline{\overleftarrow{\varphi}_{\mathfrak{n}_2} \varphi_{\mathfrak{n}_2} ((x_2 - \xi_2 e)\mathcal{H})}$ and again

$\mathfrak{n}_3 \in \mathcal{M}(\mathfrak{n}_2)$, $x_2 - \xi_2 e \in \mathfrak{n}_3$, $\sigma(\mathfrak{n}_3, x_2) = \{\xi_2\}$; $\sigma(\mathfrak{n}_3, x_1) = \{\xi_1\}$.

Proceeding in this way we construct sequences $\{\xi_i\}_{i \in Nt}$ and $\{\mathfrak{n}_i\}_{i \in Nt}$ with properties

$$\forall_{i \in Nt} \; x_i - \xi_i e \in \mathfrak{n}_i$$

$$\forall_{i \in Nt} \; \mathfrak{n}_{i+1} \in \mathcal{M}(\mathfrak{n}_i) \text{ and}$$

$$\forall_{i \geq 2} \forall_{j < i} \; \sigma(\mathfrak{n}_i, x_j) = \{\xi_j\} .$$

Take $\mathfrak{m} := \overline{\bigcup_{i \in Nt} \mathfrak{n}_i}$, then $\mathfrak{m} \in \mathcal{N}$ by 5.7 and

$$\forall_{i \in Nt} \; x_i - \xi_i e \in \mathfrak{m} .$$

Since $\forall_i \, \mathfrak{m} \in \mathcal{M}(\mathfrak{n}_i)$, also $\forall_i \, \sigma(\mathfrak{m}, x_i) = \{\xi_i\}$.

5.20. We now choose an $\underline{x} \in \mathcal{Y}$ and define

 $F_j := \{\gamma \in Cm \mid 1 \leq |\gamma| \leq \nu(x_j)\}$,

$$H_j := (\times_{i=1}^{j} F_i) \times (\times_{i=j+1}^{\infty} G(x_i)) \ ,$$

$$I_j := T(\underline{x}) \cap H_j \ .$$

5.21. If $\underline{x} \in \mathcal{Y}$ and $\{\mathcal{L}_{x_j}\}_{j \in Nt}$ has the property

$\forall_{k \in Nt} \bigcap_{i=1}^{k} \mathcal{L}_{x_i} \neq \emptyset$, then $\bigcap_{j=1}^{\infty} \mathcal{L}_{x_j} \neq \emptyset$.

Proof: If $\bigcap_{i=1}^{k} \mathcal{L}_{x_i} \neq \emptyset$, then there exists a proper closed ideal $\mathcal{n}_k$ with $\forall_{i=1,\dots,k} \mathcal{n}_k \in \mathcal{L}_{x_i}$; this implies
$\forall_{i=1,\dots,k} \sigma(\mathcal{n}_k, x_i) \subset F_i$.

Applying 5.19 to $\mathcal{n}_k$ we find a proper closed ideal $\mathcal{m}_k$ and an element $\underline{\xi}_k = \{\xi_{k_j}\}_{j \in Nt}$ of D with the property
$\forall_{j \in Nt} x_j - \xi_{k_j} e \in \mathcal{m}_k$ implying $\sum_{j=1}^{\infty} (x_j - \xi_{k_j})\mathcal{H} \subset \mathcal{m}_k$, from which we infer $\underline{\xi}_k \in T(\underline{x})$ by definition 5.10.

Another consequence of 5.19 is that $\underline{\xi}_k$ can be chosen such that $\underline{\xi}_k \in \times_{j=1}^{\infty} \sigma(\mathcal{n}_k, x_j)$, but this set is a subset of the set H_k defined in 5.20; we conclude that $\underline{\xi}_k \in T(\underline{x}) \cap H_k = I_k$.

Since H_k and $T(\underline{x})$ are compact by 5.4 and 5.16 respectively, I_k is compact.

The sequence $\{I_k\}_{k \in Nt}$ is a descending sequence of non-vacuous compact sets in the Hausdorff-space D; hence

$\bigcap_{k \in Nt} I_k \neq \emptyset$; let $\underline{\xi} \in \bigcap_{k \in Nt} I_k$, $\underline{\xi} = \{\xi_j\}_{j \in Nt}$,

then $\underline{\xi} \in T(\underline{x})$ and $\forall_{k \in Nt} \underline{\xi} \in H_k$ being equivalent to

$\sum_{j \in Nt} (x_j - \xi_j e)\mathcal{H} \neq \mathcal{H}$ and $\forall_{j \in Nt} \ 1 \leq |\xi_j| \leq \nu(x_j)$.

If $\mathcal{m} := \overline{\sum_{j \in Nt} (x_j - \xi_j e)\mathcal{H}}$, then by 5.18

$\forall_{j \in Nt} \ \sigma(\mathcal{m}, x_j) = \{\xi_j\} \subset \Phi$ and consequently

$\forall_{j \in Nt} \ \mathcal{m} \in \mathcal{L}_{x_j}$, or $\mathcal{m} \in \bigcap_{j \in Nt} \mathcal{L}_{x_j}$.

This completes the proof.

5.22. If $\{a_j\}_{j \in Nt}$ is a sequence in F with the property $\forall_{j \in Nt} \ \bigwedge_{i=1}^{j} \mathcal{L}_{a_i} \neq \emptyset$, then $\bigwedge_{j \in Nt} \mathcal{L}_{a_j} \neq \emptyset$.

Proof: Recall that $\gamma_a := \bigcup_{x \in a} \mathcal{L}_x$ and $\mathcal{L}_a = \gamma_a^*$.

We first prove $\forall_{j \in Nt} \ \bigcap_{i=1}^{j} \gamma_{a_i} \neq \emptyset$.

Take $j \in Nt$ and $\mathcal{n} \in \bigcap_{i=1}^{j} \mathcal{L}_{a_i}$.

Since $\mathcal{n} \in \mathcal{L}_{a_1} = \gamma_{a_1}^*$ we can choose an $\mathcal{m}_1 \in \mathcal{M}(\mathcal{n}) \cap \gamma_{a_1}$ by 3.3; notice that $\mathcal{M}(\mathcal{m}_1) \subset \gamma_{a_1}$ by 3.10.i and 3.2.

Since $\mathcal{n} \in \mathcal{L}_{a_2}$ and $\mathcal{L}_{a_2} \in \Delta \subset E$, $\mathcal{m}_1 \in \mathcal{L}_{a_2}$ and we can choose an $\mathcal{m}_2 \in \mathcal{M}(\mathcal{m}_1) \cap \gamma_{a_2}$; now, $\mathcal{m}_2 \in \mathcal{M}(\mathcal{m}_1) \cap \gamma_{a_1} \cap \gamma_{a_2}$ by the preceding remark. Continuation of this process shows that $\mathcal{M}(\mathcal{n}) \cap \left(\bigcap_{i=1}^{j} \gamma_{a_i} \right) \neq \emptyset$ and in particular that $\bigcap_{i=1}^{j} \gamma_{a_i} \neq \emptyset$; as a consequence also $\forall_{j \in Nt} \ a_j \neq \emptyset$.

 Next, we consider the cartesian product $\times_{j=1}^{\infty} a_j$; providing every

a_j with the discrete topology, we infer that $\times_{j=1}^{\infty} a_j$ is compact in its product topology by 5.4.

An element α of $\times_{j=1}^{\infty} a_j$ can be considered a mapping on Nt, assigning an element $\alpha(j) \in a_j \subset \mathcal{X}$ to every $j \in$ Nt.

Accordingly,

$$V := \{\alpha \in \times_{j=1}^{\infty} a_j \mid \bigcap_{j=1}^{\infty} \mathcal{L}_{\alpha(j)} \neq \emptyset\}$$

and $V_j := \{\alpha \in \times_{i=1}^{\infty} a_i \mid \bigcap_{i=1}^{j} \mathcal{L}_{\alpha(i)} \neq \emptyset\}$, $j \in$ Nt.

$\forall_{j \in Nt}\ V_j \neq \emptyset$ as a consequence of the fact that $\bigcap_{i=1}^{j} \mathcal{Y}_{a_i} \neq \emptyset$, the latter set being a union of sets like $\bigcap_{i=1}^{j} \mathcal{L}_{\alpha(i)}$.

Since in the definition of V_j no conditions are imposed on $\alpha(i)$ for $i > j$, V_j is a closed set;

$V_1 \supset V_2 \supset V_3 \ldots$ is obvious; consequently, $\{V_j\}_{j \in Nt}$ having the finite intersection property and $\times_j a_j$ being compact, $\bigcap_{j=1}^{\infty} V_j \neq \emptyset$.

But $V = \bigcap_{j=1}^{\infty} V_j$ by 5.21; hence $V \neq \emptyset$.

If $\beta \in V$ then $\forall_{j \in Nt}\ \mathcal{L}_{\beta(j)} \subset \mathcal{Y}_{a_j} \subset \mathcal{L}_{a_j}$ by 5.7; hence $\bigcap_{j=1}^{\infty} \mathcal{L}_{\beta(j)} \neq \emptyset$ implies $\bigcap_{j=1}^{\infty} \mathcal{L}_{a_j} \neq \emptyset$. This completes the proof.

,5.23. The selection of a sequence $\{\beta(j)\}_{j \in Nt}$ from the sets a_j, $j \in$ Nt, as in 5.22, is related to a theorem of König, known as the "Unendlichkeitslemma" (cf. [18], pp. 81 - 82); and, in fact, it is possible to give a proof of 5.22 based on König's lemma; but this theorem still depends on the restricted axiom of choice as well as 5.3, 5.21 and 5.22 do.

5.24. $\{\mathcal{L}_a\}_{a \in F}$ can be considered as a base for Γ in the following sense: Every element $\mathcal{L}_A \in \Gamma$ can be written as $\bigwedge_{a \in A} \mathcal{L}_a$; this is simply the definition of $\mathcal{L}_A$.

5.22 can be reformulated in terms of this base: Γ, as a point-free topology, has a countably compact base.

To inform the reader about the position we hold at this moment, we mention two well-known theorems of general topology (cf. [17], p. 139):

If a topological space has a compact base, it is compact.

If a topological space has a compact subbase, it is compact. This result depends again on the axiom of choice.

Maximal ideals

,6.1. If $\mathfrak{m}$ is a maximal ideal in the B_1 - algebra $\mathcal{H}$, then $\mathfrak{m}$ is closed as a consequence of 1.9; we denote the set of maximal ideals in $\mathcal{H}$ by $\mathfrak{M}$ and have $\mathfrak{M} \subset \mathfrak{N}$. In this chapter we suppose $\mathfrak{M} \neq \emptyset$.

If $x \in \mathcal{H}$, $\mathfrak{m} \in \mathfrak{M}$ and $\lambda \in \sigma(\mathfrak{m}, x)$, then as a consequence of 2.8 we have

$$\mathfrak{m} = \overset{\leftarrow}{\varphi}_{\mathfrak{m}} \varphi_{\mathfrak{m}} [(x - \lambda e)\mathcal{H}]$$

$$x - \lambda e \in \mathfrak{m}$$

and $\sigma(\mathfrak{m}, x) = \{\lambda\}$.

$\sigma(\mathfrak{m}, x)$ consists of one complex number for every $x \in \mathcal{H}$, and this number is, as usual, denoted by $\hat{x}(\mathfrak{m})$: $\sigma(\mathfrak{m}, x) = \{\hat{x}(\mathfrak{m})\}$.

By 2.10 and 2.11 we have

$$\widehat{x+y}(\mathfrak{m}) = \hat{x}(\mathfrak{m}) + \hat{y}(\mathfrak{m})$$

$$\widehat{xy}(\mathfrak{m}) = \hat{x}(\mathfrak{m}) \cdot \hat{y}(\mathfrak{m})$$

and $\widehat{\alpha x}(\mathfrak{m}) = \alpha \cdot \hat{x}(\mathfrak{m})$,

whence $x \to \hat{x}(\mathfrak{m})$ is a multiplicative linear functional of $\mathcal{H}$ onto Cm for every $\mathfrak{m} \in \mathfrak{M}$.

Since $\|\varphi_{\mathfrak{m}}(x - \hat{x}(\mathfrak{m})e)\| = \|\varphi_{\mathfrak{m}}(o)\| = 0$ we have that

$$\|\varphi_{\mathfrak{m}}(x) - \hat{x}(\mathfrak{m})\varphi_{\mathfrak{m}}(e)\| = 0$$

or $\|\varphi_{\mathfrak{m}}(x)\| = |\hat{x}(\mathfrak{m})|$. $\|\varphi_{\mathfrak{m}}(e)\| = |\hat{x}(\mathfrak{m})|$

and moreover: the kernel of the homomorphism $x \to \hat{x}(\mathfrak{m})$ is $\mathfrak{m}$.
This means that $\varphi_{\mathfrak{m}}(x) \to \hat{x}(\mathfrak{m})$ is an isometric isomorphism from $\varphi_{\mathfrak{m}}(\mathcal{X})$ onto Cm for every $\mathfrak{m} \in \mathfrak{M}$.

6.2. $\mathfrak{M} \subset \mathcal{Y}$.

(Recall that $\mathcal{Y}$ is the set of strong ideals in $\mathcal{X}$, compare 1.30).

Proof: Since for every $\mathfrak{m} \in \mathfrak{M}$ $\varphi_{\mathfrak{m}}(\mathcal{X})$ can be identified with Cm, we have for every $x \in \mathcal{X}$.

$\|\varphi_{\mathfrak{m}}(x)^n\|^{\frac{1}{n}} = |\hat{x}(\mathfrak{m})^n|^{\frac{1}{n}} = |\hat{x}(\mathfrak{m})|$; .'. $\mathcal{N}_{\mathfrak{m}} = \{\varphi_{\mathfrak{m}}(x) | \ \hat{x}(\mathfrak{m}) = 0\}$

$= \{\varphi_{\mathfrak{m}}(o)\}$; this proves the theorem.

6.3. i: $\mathcal{N} \subset \cap \mathfrak{M}$.

ii: If $S = \cup \mathfrak{M}$ then $\mathcal{N} = \cap \mathfrak{M}$.

Proof:

i: Trivial consequence of 1.31 and 6.2.

ii: If $x \notin \mathcal{N}$ then by 1.27 (with $\mathfrak{n} = \mathfrak{z}$)

$\exists_{y \in \mathcal{X}}$ $xy = e \in S$;

hence by assumption

$\exists_{\mathfrak{m} \in \mathfrak{M}}$ $xy - e \in \mathfrak{m}$

implying $xy \notin \mathfrak{m}$ and $x \notin \mathfrak{m}$.'. $x \notin \cap \mathfrak{M}$.

,6.4. If $\mathfrak{n} \in \mathcal{N}$ and if we denote the set of maximal ideals in

 $\varphi_{\mathfrak{n}}(\mathcal{X})$ by $\mathfrak{M}_{\mathfrak{n}}$ then (again assuming $S_{\mathfrak{n}} = \cup \mathfrak{M}_{\mathfrak{n}}$) we have

$\mathfrak{v}_{\mathfrak{n}} = \cap\, \mathfrak{M}_{\mathfrak{n}}$; observe that $S_{\mathfrak{n}} = \cup\, \mathfrak{M}_{\mathfrak{n}}$ implies that $\mathfrak{M}_{\mathfrak{n}} \neq \emptyset$.

It is easy to check that $\varphi_{\mathfrak{n}}^{\leftarrow}(\mathfrak{M}_{\mathfrak{n}}) = \mathfrak{M}(\mathfrak{n}) \cap \mathfrak{M}$, hence

$\varphi_{\mathfrak{n}}^{\leftarrow}(\mathfrak{M}_{\mathfrak{n}}) = \cap\ (\mathfrak{M}(\mathfrak{n}) \cap \mathfrak{M})$.

We now have by 1.33 that for every $\mathfrak{n} \in \mathcal{V}$

$\mathfrak{n} = \cap\ (\mathfrak{M}(\mathfrak{n}) \cap \mathfrak{M})$.

This reveals that, with the definition of kernel as in [21] p.115, $\mathcal{V}$ is the set of kernels of $\mathcal{X}$.

6.5. If $\{A_\lambda\}_{\lambda\in\Lambda} \in P(Q)$ then

i: $\bigcap_{\lambda\in\Lambda} (\mathcal{L}_{A_\lambda} \cap \mathfrak{M}) = \left(\bigwedge_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right) \cap \mathfrak{M}$

ii: $\bigcup_{\lambda\in\Lambda} (\mathcal{L}_{A_\lambda} \cap \mathfrak{M}) = \left(\bigvee_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right) \cap \mathfrak{M}$.

Proof: The first part of the proof is trivial; in the second we have

$$\left(\bigvee_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right) \cap \mathfrak{M} = \left(\bigcup_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right)^{*} \cap \mathfrak{M} \supset \left(\bigcup_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right) \cap \mathfrak{M}$$

by 3.7; and if, conversely, $\mathfrak{m} \in \left(\bigvee_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right) \cap \mathfrak{M}$ then

$\mathfrak{m} \in \left(\bigcup_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}\right)^{*}$ which by definition 3.3, observing that $\mathfrak{m} \in \mathfrak{M}$ implies $\mathfrak{M}(\mathfrak{m}) = \{\mathfrak{m}\}$, entails $\mathfrak{m} \in \bigcup_{\lambda\in\Lambda} \mathcal{L}_{A_\lambda}$; hence the result.

6.6. For every $x \in \mathcal{X}$ we consider $\hat{x}$ as a function from $\mathfrak{M}$ into Cm. The weakest topology in $\mathfrak{M}$ with the property that $\hat{x}$ is continuous for every $x \in \mathcal{X}$ is usually called the Gelfand topology. We define $\Gamma \,|\, \mathfrak{M} := \{\mathcal{L}_A \cap \mathfrak{M}\}_{A\in Q}$.

6.7. $\Gamma|\mathfrak{M}$ is the class of Gelfand closed sets in $\mathfrak{M}$.

Proof: The class of closed sets has a subbase

$\{\hat{x}^{\leftarrow}(\phi) \mid x \in \mathcal{H}, \ \phi \text{ closed in } Cm\}$.

Now

$$\hat{x}^{\leftarrow}(\phi) = \{m \in \mathfrak{M} \mid \hat{x}(m) \subset \phi\} = \mathfrak{M} \cap \{n \in \mathfrak{N} \mid \sigma(n,x) \subset \phi\}$$

is an element of $\Gamma|\mathfrak{M}$ by 4.13.

As a consequence of this and 6.5 the Gelfand closed sets are contained in $\Gamma|\mathfrak{M}$.

The converse is also true: if $A \in Q$ then

$\mathcal{L}_A \cap \mathfrak{M} = \left\{ \bigwedge_{a \in A} \left(\bigvee_{x \in a} \mathcal{L}_x \right)\right\} \cap \mathfrak{M}$ and again by 5.5

$$\begin{aligned}\mathcal{L}_A \cap \mathfrak{M} &= \bigcap_{a \in A} \left(\bigcup_{x \in a} \mathcal{L}_x \cap \mathfrak{M}\right) = \\ &= \bigcap_{a \in A} \left(\bigcup_{x \in a} \{m \in \mathfrak{M} \mid \hat{x}(m) \in \Phi\}\right) = \\ &= \bigcap_{a \in A} \left(\bigcup_{x \in a} \hat{x}^{\leftarrow}(\Phi) \right)\end{aligned}$$

which, since $a \in F(\mathcal{H})$, is a Gelfand closed set.

6.8. If $\forall_{n \in \mathfrak{N}} \ \mathfrak{M}(n) \cap \mathfrak{M} \neq \emptyset$ then

$$\forall_{A \in Q} \forall_{B \in Q} \ [[\mathcal{L}_A \cap \mathfrak{M} = \mathcal{L}_B \cap \mathfrak{M}] \Rightarrow [\mathcal{L}_A = \mathcal{L}_B]] \quad .$$

Proof: Suppose $n \notin \mathcal{L}_A$; since $\mathcal{L}_A = \mathcal{L}_A^*$

$$\exists_{q \in \mathfrak{M}(n)} \ \mathfrak{M}(q) \cap \mathcal{L}_A = \emptyset$$

 implying by virtue of Zorn's lemma

$\exists_{m \in \mathfrak{M}(n) \cap \mathfrak{M}}\ m \notin \mathcal{L}_A$ and $m \notin \mathcal{L}_A \cap \mathfrak{M}$

whence $m \notin \mathcal{L}_B \cap \mathfrak{M}$ and $m \notin \mathcal{L}_B$.

Since $\mathcal{L}_B \in \Gamma \subset E$, we see that

$n \notin \mathcal{L}_B$;

hence $\mathcal{L}_A \supset \mathcal{L}_B$ and analogously $\mathcal{L}_B \supset \mathcal{L}_A$,

which proves the theorem.

6.9. If $m \in \mathfrak{M}$, $\mathcal{L} \in \Gamma$ and $m \in \mathcal{L}$, then $\mathcal{L}(m) \subset \mathcal{L}$.

Proof: There is no loss of generality if we suppose $\exists_{a \in F}\ \mathcal{L} = \mathcal{L}_a$;
$m \in \mathcal{L}_a = \gamma^*_a$ is equivalent to $\forall_{q \in \mathfrak{M}(m)}\ \exists_{k \in \mathfrak{M}(q)}\ k \in \gamma_a$;
since $\mathfrak{M}(m) = \{m\}$, $m \in \gamma_a$, implying $\exists_{x_0 \in a}\ m \in \mathcal{L}_{x_0}$; hence
$\mathcal{L}(m) \subset \mathcal{L}_{x_0}$ by 4.15.iii and
$\mathcal{L}(m) \subset \gamma_a \subset \mathcal{L}_a$.

6.10. If $m \in \mathfrak{M}$ then $\mathcal{L}(m) \cap \mathfrak{M} = \{m\}$.

Proof: $m \in \mathcal{L}(m) \cap \mathfrak{M}$ is obvious.

If $q \in \mathfrak{M}$ then either $q = m$ or $\exists_{x \in \mathcal{X}}\ x \in q \setminus m$; the latter case implies $\hat{x}(q) = 0$, $\hat{x}(m) \neq 0$; hence $\sigma(q, x) \neq \sigma(m, x)$ and $q \notin \mathcal{L}(m)$. This completes the proof.

6.11. If $m \in \mathfrak{M}$, then by Zorn's lemma $\mathcal{L}(m)$ is a minimal element of Γ.

Proof: If $\emptyset \neq \mathcal{L} \subset \mathcal{L}(m)$ then $\exists_{n \in \mathfrak{N}}\ n \in \mathcal{L}$; by Zorn's lemma $\exists_{q \in \mathfrak{N}}\ q \in \mathfrak{M}(n) \cap \mathfrak{M}$, implying $q \in \mathcal{L}$ since $\mathcal{L} \in \Gamma \subset E$; hence

$\mathcal{L}(\mathfrak{q}) \subset \mathcal{L}$ by 6.9 and $\mathcal{L}(\mathfrak{q}) \subset \mathcal{L}(m)$ by assumption; by virtue of 6.10 $\mathcal{L}(\mathfrak{q}) = \mathcal{L}(m)$, and consequently $\mathcal{L} = \mathcal{L}(m)$; this proves that $\mathcal{L}(m)$ is minimal.

6.12. If $\mathcal{L}$ is a minimal element of Γ then by Zorn's lemma

$\exists_{m \in \mathfrak{M}}\ \mathcal{L} = \mathcal{L}(m)$.

Proof: $\mathcal{L} \neq \emptyset$ by assumption, whence $\exists_{n \in \mathfrak{N}}\ n \in \mathcal{L}$.

By Zorn's lemma $\exists_{m \in \mathfrak{N}}\ m \in \mathfrak{M} \cap \mathfrak{M}(n)$, implying $m \in \mathcal{L}$ and, by 6.9, $\mathcal{L}(m) \subset \mathcal{L}$; consequently $\mathcal{L}(m) = \mathcal{L}$ by assumption.

6.13. The Gelfand topology $\Gamma \,|\, \mathfrak{M}$ and the companion topology of Γ are homeomorphic by virtue of the axiom of choice.

Proof: The mapping $\theta : \mathfrak{M} \to \Gamma$ defined by

$\theta(m) := \mathcal{L}(m)$ is injective by 6.10; $\theta(\mathfrak{M})$ is the set of minimal elements of Γ by 6.11 and 6.12.

Since $m \in \mathcal{L} \cap \mathfrak{M}$ if and only if $m \in \mathfrak{M}$ and $\mathcal{L}(m) \subset \mathcal{L}$ we have

$\theta(\mathcal{L} \cap \mathfrak{M}) = \{\mathcal{L}(m) \mid \mathcal{L}(m) \subset \mathcal{L}\}$.

The left-hand side of this equality is the θ-image of a Gelfand closed set in $\mathfrak{M}$ by virtue of 6.7, the right-hand side is a closed set in the companion topology; this explains that θ is a homeomorphism.

6.14. Assuming Zorn's lemma we have: If $\Sigma_{j=1}^{\ell}\ y_j \mathcal{H} \neq \mathcal{H}$ for a set of elements $y_1, \ldots, y_\ell$ of $\mathcal{H}$, then $\exists_{m \in \mathfrak{M}}\ \forall_{j=1,\ldots,}\ \hat{y}_j(m) = 0$.

Proof: If $\Sigma_{j=1}^{\ell}\ y_j \mathcal{H} \neq \mathcal{H}$ then $e \notin \Sigma_j\ y_j \mathcal{H}$, $\forall_{j=1,\ldots,\ell}\ y_j \in S$ and, by 4.17, $\bigwedge_{j=1}^{\ell} \mathcal{L}(\overline{y_j \mathcal{X}}) \neq \emptyset$;

hence $\exists_{n \in \mathfrak{N}} \forall_{j=1,\dots,\ell}\ n \in \mathcal{L}(\overline{y_j \mathcal{X}})$.

Applying Zorn's lemma we get

$$\exists_{m \in \mathfrak{M} \cap \mathfrak{M}(n)} \forall_{j=1,\dots,\ell}\ m \in \mathcal{L}(\overline{y_j \mathcal{X}})$$

equivalent to

$$\exists_{m \in \mathfrak{M} \cap \mathfrak{M}(n)} \forall_{j=1,\dots,\ell} \forall_{x \in \mathcal{X}}\ \hat{x}(m) \in \sigma(\overline{y_j \mathcal{X}}, x) .$$

Particularly

$$\forall_{j=1,\dots,\ell}\ \hat{y}_j(m) \in \sigma(\overline{y_j \mathcal{X}}, y_j) = \{0\} .$$

Examples

Let T be a set consisting of three elements a, b, c; T is topologised with the discrete topology, and C(T) denotes the ring of continuous functions on T, which turns into a B_1 - algebra if $\|f\| := \max\{|f(a)|, |f(b)|, |f(c)|\}$ for every $f \in C(T)$.

The set $\mathfrak{M}$ of proper closed ideals consists of

$\mathfrak{a} := \{f \in C(T) \mid f(a) = 0\}$,

$\mathfrak{b} := \{f \in C(T) \mid f(b) = 0\}\}$,

$\mathfrak{c} := \{f \in C(T) \mid f(c) = 0\}$,

$\mathfrak{a}' := \mathfrak{b} \cap \mathfrak{c}$, $\mathfrak{b}' := \mathfrak{a} \cap \mathfrak{c}$, $\mathfrak{c}' := \mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{z} = \mathfrak{a} \cap \mathfrak{b} \cap \mathfrak{c}$.

If $\mathfrak{A} := \{\mathfrak{z}, \mathfrak{a}\}$ then $\mathfrak{A}^* = \{\mathfrak{a}\}$; this shows that 3.7 does not always hold in $P(\mathfrak{M})$.

If $\mathfrak{A} := \{\mathfrak{z}, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\}$ then $\mathfrak{A}^* = \{\mathfrak{z}, \mathfrak{a}', \mathfrak{b}', \mathfrak{c}', \mathfrak{a}, b, \mathfrak{c}\}$, showing that the converse of 3.7 is untrue.

If $\mathfrak{A} := \{\mathfrak{a}\}$ and $\mathfrak{B} := \{\mathfrak{b}\}$ then $(\mathfrak{A} \cup \mathfrak{B})^* = \{\mathfrak{c}', \mathfrak{b}, \mathfrak{a}\}$ and $\mathfrak{A}^* \cup \mathfrak{B}^* = \{\mathfrak{a}, \mathfrak{b}\}$; this explains why equality in 3.10.i does not always hold, not even if $\{\mathfrak{A}_\lambda\}_\lambda \in P(\Delta)$.

$x := (0,2,2)$ and $y := (2,0,0)$ are elements of C(T);

 $\mathfrak{L}_x = \{\mathfrak{a}', \mathfrak{b}, \mathfrak{c}\}$, $\mathfrak{L}_y = \{\mathfrak{a}\}$ and $(\mathfrak{L}_x \cup \mathfrak{L}_y)^* = \mathfrak{M}$;

since $\mathcal{L}_x = \mathcal{L}_x^*$ and $\mathcal{L}_y = \mathcal{L}_y^*$, $\mathcal{L}_x^* \cup \mathcal{L}_y^* \neq (\mathcal{L}_x \cup \mathcal{L}_y)^*$;

hence, in 3.10.i equality even does not always hold if $\{\mathcal{O}\!\mathcal{l}_\lambda\}_\lambda \in P(\Gamma)$.

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Samenvatting

In dit proefschrift wordt beschreven hoe men voor een kommutatieve Banach algebra met eenheidselement in de verzameling der echte gesloten idealen een klasse van deelverzamelingen kan aangeven die de eigenschappen heeft van een topologie; bij de beschrijving van deze klasse spelen de maximale idealen (waarvan de existentie op het gebruik van het keuze axioma berust) geen rol; dit wil zeggen dat een topologie is beschreven zonder dat van een puntenverzameling wordt gebruik gemaakt en die daarom een "puntvrije topologie" kan worden genoemd.

Deze puntvrije topologie impliceert bij toepassing van het keuze-axioma achteraf een topologie voor de ruimte der maximale idealen zoals die door Gelfand in 1941 is beschreven.

De in dit proefschrift gebruikte methode is te danken aan de Bruijn; het beginsel van een puntvrije topologie is reeds in 1928 door Menger beschreven, maar vond tot nog toe weinig weerklank.

Curriculum vitae

De schrijver van dit proefschrift, geboren op 23 november 1928 te Schiedam, bezocht aldaar de lagere school en behaalde aan de Christelijke hogere burgerschool te Vlaardingen in 1946 het B-diploma.

Hij studeerde wis- en natuurkunde aan de Rijksuniversiteit te Leiden en volgde colleges van van der Woude, Droste, Kloosterman, Haantjes, Gorter, van den Handel en Korringa; in 1954 legde hij het doctoraal examen af.

Van 1954 tot 1956 werd hij opgeleid tot en te werk gesteld als luchtvaart meteoroloog bij de Marine Luchtvaart Dienst.

In september 1956 werd hij tot wiskunde leraar aan de Christelijke h.b.s. "Charlois" te Rotterdam benoemd; vanaf augustus 1959 is hij als wetenschappelijk medewerker verbonden aan de onderafdeling wiskunde van de Technische Hogeschool te Eindhoven.

Stellingen

behorende bij het proefschrift van W. van der Meiden

1. Het vaak gemaakte onderscheid tussen "zuivere wiskunde" en "toegepaste wiskunde" is verouderd en als middel om onderdelen van de wiskunde te karakteriseren onbruikbaar.

2. In de ontwikkeling van sommige delen van de wiskunde zijn drie fasen te onderscheiden.
Ten eerste worden problemen opgelost die hetzij elementair wiskundig formuleerbaar of uit een andere discipline afkomstig zijn; dit is de heuristische fase.
Ten tweede worden bereikte resultaten geordend en in hun onderlinge samenhang (vaak vanuit een axiomatisch gezichtspunt) beschreven; dit is de beschrijvende fase.
Ten derde worden problemen geformuleerd die rijzen naar aanleiding van of in verband met de verkregen axiomatisering; dit is de abstracte fase.

3. Bij het onderwijs aan hen die wiskunde als hulpwetenschap zullen gaan hanteren dient op de heuristische aspecten der wiskunde de nadruk te worden gelegd; het onderwijs aan aanstaande wiskundige ingenieurs behoort zo weinig mogelijk abstract te zijn.

4. Het onderwijs in beschrijvende meetkunde wordt door de belanghebbenden vaak, en niet ten onrechte, verdedigd met een beroep op de bevordering van het ruimtelijk inzicht; bij de algebraïsering van het meetkundeonderwijs wordt deze functie op onvoldoende wijze overgenomen.

5. Het invoeren van zogenaamde "objectieve studietoetsen" (in plaats van proefwerken en tentamens van klassieke makelij) wordt vaak verdedigd met een beroep op de grotere rechtvaardigheid van de op grond daarvan te nemen beslissingen. De woorden objectief en rechtvaardig bezitten dan evenwel slechts propagandistische betekenis.

6. De cirkels door de voetpunten van de loodlijnen, die uit een vast punt van een kegelsnede worden neergelaten op de zijden van driehoeken waarvan de hoekpunten worden bepaald door een derdegraads involutie op de kegelsnede, vormen een bundel.

7. De begrensde lineaire operatoren B_n die op $C([0,1])$ volgens een van Bernstein afkomstig procédé worden gedefinieerd door

$$\forall_{f \in C} \forall_{x \in [0,1]} (B_n f)(x) := \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) \cdot \binom{n}{\nu} x^\nu (1-x)^{n-\nu}$$

hebben als spectrum $\left\{1, \frac{n-1}{n}, \frac{(n-1)(n-2)}{n^2}, \ldots, \frac{(n-1)!}{n^{n-1}}\right\}$; $(n \in Nt)$.

8. De door Lundgren en Chiang (Quart. Appl. Math. 24 (1966), 303 - 313) gegeven oplossingen van de integraalvergelijkingen

$$\int_0^1 F(y)|x-y|^{-\alpha}\,dy = f(x), \qquad 0 < x < 1$$

en

$$\int_0^1 F(y)|x-y|^{-\alpha}\,\mathrm{sgn}(x-y)\,dy = f(x), \qquad 0 < x < 1,$$

waarin $0 < \alpha < 1$ is, kunnen op meer elementaire wijze worden afgeleid met behulp van een door Peters (Comm. Pure Appl. Math. 16 (1963), 57 - 61) aangegeven methode; de aldus verkregen oplossingen van de eerste en de tweede integraalvergelijking blijken bovendien geldig te zijn voor $-1 < \alpha < 1$, $\alpha \neq 0$, respectievelijk $0 < \alpha < 2$.

9. Men kan op het in het proefschrift in hoofdstuk 4 beschreven lattice Γ van clusters afbeeldingen in de klasse P(Om) definiëren door

$$\tilde{x}(\mathcal{L}) := \{\lambda \in \mathrm{Om} \mid \exists_{\mathfrak{m} \in \mathcal{L}}\ \lambda \in \sigma(\mathfrak{m}, x)\}$$

voor alle $x \in \mathfrak{X}$ en voor alle $\mathcal{L} \in \Gamma$.
Deze afbeeldingen hebben de volgende eigenschappen:

a: $\tilde{x}(\mathcal{L}) = \bigcup_{\mathfrak{m} \in \mathcal{L}} \sigma(\mathfrak{m}, x)$

b: $\tilde{x}(\bigvee_{\alpha} \mathcal{L}_{\alpha}) = \bigcup_{\alpha} \tilde{x}(\mathcal{L}_{\alpha})$

c: $\tilde{x}(\mathfrak{A} \wedge \mathfrak{B}) \subset \tilde{x}(\mathfrak{A}) \cap \tilde{x}(\mathfrak{B})$

d: $\widetilde{x+y}(\mathcal{L}) \subset \tilde{x}(\mathcal{L}) + \tilde{y}(\mathcal{L})$

e: $\widetilde{xy}(\mathcal{L}) \subset \tilde{x}(\mathcal{L}) \cdot \tilde{y}(\mathcal{L})$.

Als $\hat{x}$ op de gebruikelijke wijze wordt gedefinieerd (vergelijk 6.1 in het proefschrift) kan met behulp van het keuze - axioma worden bewezen:

f: $\tilde{x}(\mathcal{L}) = \hat{x}(\mathcal{L} \cap \mathfrak{M})$ en

g: $\forall_{\mathfrak{m} \in \mathfrak{M}}\ \tilde{x}(\mathcal{L}(\mathfrak{m})) = \{\hat{x}(\mathfrak{m})\}$.

De afbeeldingen $\tilde{x}$ gaan dus bij restrictie tot $\mathfrak{M}$ over in de continue functies $\hat{x}$ uit de theorie van Gelfand.

10. Uitdrukkingen zoals "this result is best - possible" en "this condition cannot be weakened" geven in wiskundige teksten vaak aanleiding tot verwarring. Veelal zijn dergelijke mededelingen niet voldoende gespecificeerd om betekenis te hebben.

11. Zolang de equivalentie van het (tot een aftelbare klasse) beperkte keuze-axioma en het onbeperkte keuze - axioma niet bewezen is dienen bij de toepassing van het keuze-axioma in bewijzen deze gevallen duidelijk onderscheiden te worden.

12. Het kerkelijk ambt van predikant heeft de neiging te ontaarden tot het beroep van (geestelijk) verzorgingstechnicus; het ware aan te bevelen dominees te ontlasten van hun administratieve en organisatorische taken en deze over te dragen aan daartoe opgeleide, bezoldigde ouderlingen.

Eindhoven, 7 november 1967