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BLOW-UP AND STABILITY OF SEMILINEAR PDE'S WITH GAMMA GENERATORS

José Alfredo López-Mimbela   Nicolas Privault

Abstract

We investigate finite-time blow-up and stability of semilinear partial differential equations of the form $\partial w_t/\partial t = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}$, $w_0(x) = \varphi(x) \geq 0$, $x \in \mathbb{R}_+$, where Γ is the generator of the standard gamma process and $\nu > 0$, $\sigma \in \mathbb{R}$, $\beta > 0$ are constants. We show that any initial value satisfying $c_1 x^{-a_1} \leq \varphi(x)$, $x > x_0$ for some positive constants x_0, c_1, a_1, yields a non-global solution if $a_1\beta < 1 + \sigma$, or if $a_1\beta = 1 + \sigma$ and $\beta > 1$. If $\varphi(x) \leq c_2 x^{-a_2}$, $x > x_0$, where $x_0, c_2, a_2 > 0$, and $a_2\beta > 1 + \sigma$, then the solution w_t is global and satisfies $0 \leq w_t(x) \leq Ct^{-a_2}$, $x \geq 0$, for some constant $C > 0$. This extends the results previously obtained in the case of α-stable generators. Systems of semilinear PDE's with gamma generators are also considered.

Key words: Semilinear partial differential equations, Feynman-Kac representation, blow-up of semilinear systems, gamma processes.
Mathematics Subject Classification: 60H30, 35K57, 35B35, 60J57, 60E07, 60J75.

1 Introduction

Critical exponents for blowup of semilinear Cauchy problems of the prototype

$$\frac{\partial w_t}{\partial t} = L w_t + w_t^{1+\beta}, \qquad w_0 = \varphi, \tag{1}$$

where L is a Lévy generator, $\beta > 0$ is constant and $\varphi \geq 0$, have been studied by many authors during the last years. The case of d-dimensional Laplacian $L = \Delta$ has been thoroughly investigated (see e.g. [7] and [4] for surveys), and has originated many techniques that are now standard tools in the theory of semilinear problems. When L is the fractional power $\Delta_\alpha = -(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leq 2$, it was shown in a series of papers [1, 8, 9, 12, 13] that the critical parameter for blow-up of (1) is $d_c := \alpha/\beta$, meaning that if $d \le d_c$ then (1) possesses no global nontrivial solutions, and if $d > d_c$, then (1) admits a nontrivial global solution for all sufficiently small initial values. The approaches developed in those works use subtle comparison arguments [13], or probabilistic representations of solutions (in terms of branching particle systems [8, 9], or by means of the Feynman-Kac formula [1, 12]). A feature common to these methods is that they rely significantly on the symmetry and scaling properties of stable distributions.

In this paper we investigate finite-time blow-up and existence of non-trivial global solutions of the semilinear equation

$$\frac{\partial w_t}{\partial t} = \varGamma w_t + \nu t^{\sigma} w_t^{1+\beta}, \qquad w_0(x) = \varphi(x), \qquad x \in \mathbb{R}_+, \tag{2}$$

where φ is a nonnegative function, ν, σ and β are positive constants, and $\varGamma$ is the pseudo-differential operator

$$\varGamma f(x) = \int_0^{\infty} (f(x+y) - f(x)) \frac{e^{-y}}{y}\, dy,$$

i.e. the generator of the standard gamma process. In the linear case, such equations are of interest in reliability models based on the gamma process [16]. The symmetrized generator

$$\tilde{\varGamma} f(x) = \int_{-\infty}^{\infty} (f(x+y) - f(x)) \frac{e^{-|y|}}{|y|} dy$$

has symbol

$$\log(1 + |\xi|) = \lim_{\alpha \to 0} \alpha^{-1}((1 + |\xi|)^{\alpha} - 1), \qquad \xi \in \mathbb{R},$$

and can be viewed as the weak limit of $\alpha^{-1}((I - \Delta_{1/2})^{\alpha} - I)$ as α goes to 0. Similarly, the one-sided stable process can be renormalized to converge in distribution to a gamma process, cf. [3], [14]. Thus, another motivation for studying (2) is that it constitutes a natural follow-up to the previous investigations, as it can be considered in a sense as a "limiting case" $\alpha \to 0$, although, unlike in the α-stable case, the gamma process enjoys no scaling or symmetry property, or dimensional-dependent behavior. However, its density function is explicitly known and this allows us to follow closely the approaches in [1] and [9] to make work the probabilistic representations of (2) for our purposes.

Our solutions will be understood in the mild sense (see e.g. [11]), and therefore we can consider bounded, measurable initial values $\varphi \ge 0$. We will show as a consequence of Corollary 4.2 and Theorem 5.1 that any initial value satisfying

$$c_1 x^{-a_1} \le \varphi(x), \qquad x > x_0,$$

for some positive constants x_0, c_1, a_1, yields a non-global solution of (2) if $a_1\beta < 1+\sigma$, or if $a_1\beta = 1+\sigma$ and $\beta > 1$. Similarly, if the initial value of (2) satisfies

$$\varphi(x) \leq c_2 x^{-a_2}, \qquad x > x_0,$$

where x_0, c_2, a_2 are positive numbers and $a_2\beta > 1+\sigma$, then the solution u_t is global and satisfies $0 \leq u_t(x) \leq Ct^{-a_2}$, $x \geq 0$, for some constant $C > 0$. For the particular case $\sigma = 0$, if $\varphi(x) \sim_{x\to\infty} cx^{-a}$ for some $c > 0$ and $a > 0$, then blow-up of (2) occurs if $a\beta \leq 1$ or if $\beta = a^{-1} > 1$, and a global solution exists if $a\beta > 1$. Hence, if $\sigma = 0$ and for some $\varepsilon > 0$

$$\liminf_{x\to\infty} x^{-\varepsilon+1/\beta}\varphi(x) > 0,$$

then the solution of (2) blows-up, whereas if

$$\limsup_{x\to\infty} x^{\varepsilon+1/\beta}\varphi(x) = 0,$$

then the solution of (2) exists globally.

Note that without additional difficulty we may replace the operator Γ in (2) with the generator Γ_λ given by

$$\Gamma_\lambda f(x) = \int_0^\infty (f(x+y) - f(x))\frac{e^{-\lambda y}}{y}\,dy, \qquad x \in \mathbb{R}_+,$$

where λ is a strictly positive parameter. Indeed, for $f \in \mathrm{Dom}(\Gamma_\lambda)$ we have the relation $\Gamma_\lambda f(x) = \Gamma f_\lambda(\lambda x)$, where $f_\lambda(x) = f(x/\lambda)$. This means that f_λ is solution of (2) if and only if f is solution of (2) with Γ_λ in place of Γ.

In the case of systems of equations of the form

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma_\lambda u_t + \nu u_t^{1+\beta_1} v_t^{\beta_2}, \qquad u_0 = \varphi_1, \\[2ex] \dfrac{\partial v_t}{\partial t} = \Gamma_\mu v_t + F_t(u_t, v_t), \qquad v_0 = \varphi_2, \end{cases}$$

with $\lambda \neq \mu$, the solution cannot be constructed directly from the case $\lambda = \mu = 1$, nevertheless the existence and blow-up criteria for solutions are independent of the values of $\lambda, \mu > 0$. In this case we show that if $\varphi_1(x) \geq cx^{-a_1}$ and $\varphi_2(x) \geq cx^{-a_2}$, for x large enough, then blow-up occurs provided $a_1\beta_1 + a_2\beta_2 < 1$. We also study the semilinear system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma_{\lambda_1} u_t + \nu_1 u_t^{\beta_{11}} v_t^{\beta_{12}}, \qquad u_0 = \varphi_1, \\[2ex] \dfrac{\partial v_t}{\partial t} = \Gamma_{\lambda_2} v_t + \nu_2 u_t^{\beta_{21}} v_t^{\beta_{22}}, \qquad v_0 = \varphi_2, \end{cases}$$

$\nu_1, \nu_2 > 0$, with integer exponents $\beta_{ij} \geq 1$ and initial values satisfying $\varphi_1(x) \leq c_1 x^{-a_1}$ and $\varphi_2(x) \leq c_2 x^{-a_2}$ for x large enough, where $a_1, a_2 \in (1, \infty)$. We show that this system admits a global solution provided $(a_1 \wedge a_2)[(\beta_{11} + \beta_{12}) \wedge (\beta_{11} + \beta_{12}) - 1] > 1$ and the constants $c_1, c_2 > 0$ are sufficiently small. In particular, the solution of the system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma u_t + u_t v_t \\ \\ \dfrac{\partial v_t}{\partial t} = \Gamma v_t + u_t v_t, \end{cases}$$

with $u_0(x) \sim c x^{-a_1}$ and $v_0(x) \sim c x^{-a_2}$ for x large enough, is global if $\min(a_2, a_1) > 1$ and c is sufficiently small. We also show that blow-up occurs if $\min(a_2, a_1) < 1$, and deal under additional assumptions with critical cases with time-dependent non-linearities.

Our methods of proof are inspired in the approaches developed in [1] and [9]. To prove explosion of semilinear equations we use the Feynman-Kac representation as well as estimates of probability transition densities, analogously to the α-stable case as treated in [1]. Existence of global solutions is deduced using a general criterion, originally obtained in [9].

The paper is organized as follows. In Section 2 we recall some basic facts about the gamma process and its infinitesimal generator, and obtain bounds for the gamma semigroup that will be useful in the sequel. In Section 3 we recall the Feynman-Kac representation of (2), and derive from this representation a criterion for blow-up of semilinear PDE's. Using a general argument deduced from [15], we show existence of global solutions in Section 4. Blow-up of solutions of (2) is dealt with in Section 5, and systems of semilinear PDE's with gamma generators are considered in Section 6.

2 Estimates of the gamma semigroup

Let

$$G(t) = \int_0^\infty x^{t-1} e^{-x} \, dx, \qquad t > 0,$$

denote the gamma function, and let $(X_t^\Gamma)_{t \in \mathbb{R}_+}$ denote the standard gamma process with densities

$$\gamma_t(x) = \frac{x^{t-1}}{G(t)} e^{-x} 1_{[0,\infty)}(x), \qquad x \in \mathbb{R}, \quad t > 0,$$

and generator

$$\Gamma f(x) = \int_0^\infty (f(x+y) - f(x)) \frac{e^{-y}}{y}\, dy.$$

Let $\{T_t^\Gamma,\ t \geq 0\}$ denote the operator semigroup generated by Γ, which is given by

$$T_t^\Gamma \varphi(y) = E[\varphi(X_t^\Gamma + y)] = \int_0^\infty \varphi(x+y)\gamma_t(x)dx = \int_y^\infty \varphi(x)\gamma_t(x-y)dx, \qquad (3)$$

$y \in \mathbb{R}_+$. In the next lemma we prove asymptotic estimates for the semigroup $\{T_t^\Gamma,\ t \geq 0\}$, using results of [2] on the median of the gamma density. Recall that for $t > 1$, γ_t is increasing on $[0, t-1]$ and decreasing on $[t-1, \infty)$.

Lemma 2.1 *Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be bounded and measurable. Assume that there exist $c_1 \in [0, \infty)$, $c_2 \in (0, \infty]$, and $a_1 \geq a_2 > 0$ such that for all x large enough,*

$$c_1 x^{-a_1} \leq \varphi(x) \leq c_2 x^{-a_2}. \qquad (4)$$

Then, for all $\eta \geq 0$ and $0 < \varepsilon \leq 1$ there exists $t_0 = t_0(\varepsilon, \eta) > 0$ such that

1. *For all $t > t_0$ and all $y \geq 0$,*

$$\left(\frac{1-\varepsilon}{3}\right)^{a_1} \frac{c_1}{2} t^{-a_1} 1_{[0,t+\eta]}(y) \leq T_t^\Gamma \varphi(y) \leq c_2(1+\varepsilon)t^{-a_2}. \qquad (5)$$

2. *For all $t > t_0$ and any $0 \leq y \leq \eta + t/2$,*

$$(1-\varepsilon)\frac{c_1}{2^{1+a_1}} t^{-a_1} 1_{[0,\eta+t/2]}(y) \leq T_t^\Gamma (1_{[t-1/3,2t]}\varphi)(y) \leq c_2(1+\varepsilon)t^{-a_2}. \qquad (6)$$

3. *For all $t > t_0$ and any $0 \leq y \leq \eta \leq 1$,*

$$(1-\varepsilon)\frac{\eta c_1}{\sqrt{2\pi}} t^{-a_1-1/2} 1_{[0,\eta]}(y) \leq T_t^\Gamma (1_{[t-\eta,t]}\varphi)(y) \leq (1+\varepsilon)\frac{\eta c_2}{\sqrt{2\pi}} t^{-a_2-1/2}. \qquad (7)$$

Proof. There exists $x_0 > 0$ such that for all $0 < y < t + \eta$,

$$\begin{aligned}
T_t^\Gamma \varphi(y) &= \int_0^\infty \varphi(x+y)\gamma_t(x)dx \\
&\geq c_1 \int_{x_0}^\infty (x+y)^{-a_1}\gamma_t(x)dx \\
&\geq c_1 \int_{x_0}^\infty (x+t+\eta)^{-a_1}\gamma_t(x)dx \\
&\geq c_1 \frac{G(t-a_1)}{G(t)} \int_{x_0}^\infty (1+(t+\eta)/x)^{-a_1}\gamma_{t-a_1}(x)dx \\
&\geq c_1 \frac{G(t-a_1)}{G(t)} \int_{t-a_1-1/3}^\infty (1+(t+\eta)/x)^{-a_1}\gamma_{t-a_1}(x)dx \\
&\geq c_1 \frac{G(t-a_1)}{G(t)} \int_{t-a_1-1/3}^\infty \left(1+\frac{t+\eta}{t-a_1-1/3}\right)^{-a_1}\gamma_{t-a_1}(x)dx \\
&\geq c_1 \frac{G(t-a_1)}{G(t)} \frac{(1-\varepsilon)^{a_1}}{2^{a_1}} \int_{t-a_1-1/3}^\infty \gamma_{t-a_1}(x)dx \\
&\geq \frac{c_1}{2} \frac{(1-\varepsilon)^{a_1}}{(3-\varepsilon)^{a_1}} t^{-a_1},
\end{aligned}$$

for all sufficiently large t, provided $(a_1 + 1/3)/t < \varepsilon$ and $\eta/t < \varepsilon$. Here we used the equivalence $G(t-a)/G(t) \sim t^{-a}$ as $t \to \infty$ which follows from Stirling's formula $G(t) \sim \sqrt{2\pi} t^{t-1/2} e^{-t}$, and the fact that the median of the gamma distribution with parameter $t - a_1$ is greater than $t - a_1 - 1/3$, see Theorem 2 of [2]. Similarly we have for all $y > 0$ and t big enough:

$$\begin{aligned}
T_t^\Gamma \varphi(y) &= \int_0^\infty \varphi(x+y)\gamma_t(x)dx \\
&\leq c_2 \int_0^\infty (x+y)^{-a_2}\gamma_t(x)dx \\
&\leq c_2 \int_0^\infty x^{-a_2}\gamma_t(x)dx \\
&\leq c_2 \frac{G(t-a_2)}{G(t)} \int_0^\infty \gamma_{t-a_2}(x)dx \\
&\leq c_2(1+\varepsilon)t^{-a_2},
\end{aligned}$$

which proves (4). Concerning (6) we have for $0 < y \le \eta + t/2$ and t sufficiently large:

$$
\begin{aligned}
\int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)dx &\ge c_1 \int_{t-1/3}^{2t} x^{-a_1}\gamma_t(x-y)dx \\
&\ge c_1(2t)^{-a_1}\int_{t-1/3}^{2t} \gamma_t(x-y)dx \\
&\ge c_1(2t)^{-a_1}\int_{t-1/3}^{-\eta+3t/2} \gamma_t(x)dx \\
&\ge c_1(2t)^{-a}\left(\frac{1}{2} - \int_{-\eta+3t/2}^{\infty} \gamma_t(x)dx\right) \\
&\ge (1-\varepsilon)\frac{c_1}{2}(2t)^{-a},
\end{aligned}
$$

since $\int_{t-1/3}^{\infty}\gamma_t(x)dx \ge 1/2$ and $\int_{-\eta+3t/2}^{\infty}\gamma_t(x)dx = P(X_t^{\Gamma} \ge -\eta + \frac{3t}{2}) \to 0$ as $t \to \infty$ by the law of large numbers. Similarly we have for t large enough:

$$
\begin{aligned}
\int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)dx &\le c_2 \int_{t-1/3}^{2t} x^{-a_2}\gamma_t(x-y)dx \\
&\le c_2(t-1/3)^{-a_2}\int_{t-1/3}^{2t} \gamma_t(x-y)dx \\
&\le c_2(t-1/3)^{-a_2} \\
&\le (1+\varepsilon)c_2 t^{-a_2}.
\end{aligned}
$$

Concerning (7) we have, for $0 < y \le \eta \le 1$ and $t > 2$:

$$
\gamma_t(t-1)\int_{t-\eta}^{t}\varphi(x)dx \ge \int_{t-\eta}^{t}\varphi(x)\gamma_t(x-y)dx \ge (\gamma_t(t) \wedge \gamma_t(t-2))\int_{t-\eta}^{t}\varphi(x)dx.
$$

Since for any $l \ge 0$,

$$
\gamma_t(t-l) = \frac{(t-l)^{t-1}}{G(t)}e^{-t+l} \sim \frac{(t-l)^{t-1}e^{l}}{\sqrt{2\pi}t^{t-1/2}} \sim \frac{t^{-1/2}}{\sqrt{2\pi}}, \quad t \to \infty,
$$

it follows that for any $0 < \varepsilon < 1$ and for all sufficiently large t,

$$
(1+\varepsilon)\frac{t^{-1/2}}{\sqrt{2\pi}}\int_{t-\eta}^{t}\varphi(x)dx \ge \int_{t-\eta}^{t}\varphi(x)\gamma_t(x-y)dx \ge (1-\varepsilon)\frac{t^{-1/2}}{\sqrt{2\pi}}\int_{t-\eta}^{t}\varphi(x)dx.
$$

It remains to note that

$$
\int_{t-\eta}^{t} x^{-a}dx = \frac{t^{-a}}{1-a}(1-(1-\eta/t)^{1-a}) \sim \eta t^{-a}
$$

for all $a \ge 0$ as t goes to infinity, and to use (4). $\square$

Remark 2.1 *Let $\{T^\lambda_t,\ t \geq 0\}$ be the operator semigroup having generator Γ_λ. From the relation $T^\lambda_t \varphi(x) = [T^\Gamma_t \varphi_\lambda](\lambda x)$ we get for $t > t_0$ and $y \geq 0$:*

$$\frac{c_1}{2}\left(\frac{1-\varepsilon}{3}\right)^{a_1}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,t+\eta]}(y) \leq T^\lambda_t \varphi(y) \leq c_2(1+\varepsilon)\left(\frac{t}{\lambda}\right)^{-a_2},$$

$$(1-\varepsilon)\frac{c_1}{2^{1+a_1}}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,\eta+t/2]}(y) \leq T^\Gamma_t(1_{[t-1/3,2t]}\varphi)(y) \leq c_2(1+\varepsilon)\left(\frac{t}{\lambda}\right)^{-a_2},$$

$$(1-\varepsilon)\frac{\eta c_1}{\sqrt{2\pi}}\left(\frac{t}{\lambda}\right)^{-a_1} 1_{[0,\eta]}(y) \leq T^\Gamma_t(1_{[t-\eta,t]}\varphi)(y) \leq (1+\varepsilon)\frac{\eta c_2}{\sqrt{2\pi}}\left(\frac{t}{\lambda}\right)^{-a_2}.$$

Recall that for $0 \leq s < t$ and $x > 0$, the conditional law of X^Γ_s given $X^\Gamma_t = x$ is the beta distribution with density

$$\beta_{s,t}(z,x) := \frac{\gamma_s(z)\gamma_{t-s}(x-z)}{\gamma_t(x)} = \frac{1}{x}\frac{G(t)}{G(s)G(t-s)}\left(\frac{z}{x}\right)^{s-1}\left(1-\frac{z}{x}\right)^{t-s-1}, \qquad z \in [0,x]. \tag{8}$$

Using the result of [10] on the median of the beta distribution we obtain the following estimates.

Lemma 2.2 *Let $\eta > 0$. We have*

$$P_y(0 < X^\Gamma_s < s+\eta | X^\Gamma_t = x) \geq 1/2 \tag{9}$$

for all $0 < s < t/2$, $0 < y < \eta$, $0 < t-2\eta < t-\eta < x < t$, and

$$P_y(0 < X^\Gamma_s < 2s+t/2 | X^\Gamma_t = x) \geq 1/2 \tag{10}$$

for all $0 < s < t/2$, $0 < y < t/2$ and $0 < t/2 < x < 2t$.

Proof. We have

$$\begin{aligned}
P_y(0 < X^\Gamma_s < s+\eta | X^\Gamma_t = x) &= P(0 < y + X^\Gamma_s < s+\eta | X^\Gamma_t = x-y) \\
&\geq P(0 < X^\Gamma_s < s | X^\Gamma_t = x-y) \\
&= \int_0^s \beta_{s,t}(z, x-y)dz \\
&= \frac{G(t)}{G(s)G(t-s)}\int_0^{s/(x-y)} z^{s-1}(1-z)^{t-s-1}dz \\
&\geq \frac{G(t)}{G(s)G(t-s)}\int_0^{s/t} z^{s-1}(1-z)^{t-s-1}dz \\
&= \int_0^{s/t} \beta_{s,t}(z,1)dz \\
&\geq 1/2,
\end{aligned}$$

since from Theorem 1 of [10], the median $m_{s,t}$ of the standard beta density $\beta_{s,t}(\cdot, 1)$ with mean s/t satisfies

$$0 < m_{s,t} < \frac{s}{t} < m_{s,t} + \frac{t-2s}{(t-2)t},$$

provided $s < t/2$. Similarly we have

$$\begin{aligned} P_y(0 < X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x) &= P(0 < y + X_s^\Gamma < 2s + t/2 | X_t^\Gamma = x - y) \\ &\geq P(0 < X_s^\Gamma < 2s | X_t^\Gamma = x - y) \\ &= \int_0^{2s} \beta_{s,t}(z, x-y) dz \\ &= \frac{G(t)}{G(s)G(t-s)} \int_0^{2s/(x-y)} z^{s-1}(1-z)^{t-s-1} dz \\ &\geq \frac{G(t)}{G(s)G(t-s)} \int_0^{s/t} z^{s-1}(1-z)^{t-s-1} dz \\ &\geq 1/2. \end{aligned}$$

□

3 Feynman-Kac representation and subsolutions

Let $(X_t)_{t \in \mathbb{R}_+}$ be a Lévy process in $\mathbb{R}_+$ having generator L and operator semigroup $\{T_t,\ t \geq 0\}$. We assume that the transition densities p_t, $t > 0$ of $(X_t)_{t \in \mathbb{R}_+}$ satisfy $p_t(x, y) = p_t(y - x)$ for all $x, y \in \mathbb{R}_+$, and that $p_t(x, y) = 0$ if $y < x$. Recall (see e.g. [5]) that the mild solution of

$$\frac{\partial w_t}{\partial t}(y) = L w_t(y) + \zeta_t(y) w_t(y), \qquad w_0 = \varphi, \tag{11}$$

admits the Feynman-Kac representation

$$w_t(y) = E\left[\varphi(y + X_t) \exp \int_0^t \zeta_{t-s}(y + X_s) ds\right], \qquad t \geq 0, \quad y \geq 0. \tag{12}$$

If ζ_t is positive (12) implies

$$w_t(y) \geq E\left[\varphi(y + X_t)\right] = T_t \varphi(y), \qquad y \in \mathbb{R}_+, \quad t \geq 0.$$

Thus, the solution of

$$\frac{\partial w_t}{\partial t} = L w_t, \qquad w_0 = \varphi \geq 0,$$

is also a subsolution of (11) provided $\zeta_t \geq 0$. By linearity this implies the following lemma.

Lemma 3.1 *Let $\varphi \geq 0$ be bounded and measurable. If u_t, v_t respectively solve*

$$\frac{\partial u_t}{\partial t}(y) = Lu_t(y) + \zeta_t(y)u_t(y), \qquad \frac{\partial v_t}{\partial t}(y) = Lv_t(y) + \xi_t(y)v_t(y),$$

with $u_0 \geq v_0$ and $\zeta_t \geq \xi_t$, then $u_t \geq v_t$.

We will use the fact (which follows from Lemma 3.1) that if u_t is a subsolution of

$$\frac{\partial w_t}{\partial t}(y) = Lw_t(y) + \nu w_t^{1+\beta}(y), \qquad w_0 = \varphi, \tag{13}$$

where $\nu, \beta > 0$, then any solution of

$$\frac{\partial v_t}{\partial t}(y) = Lv_t(y) + \nu u_t^{\beta}(y)v_t(y), \qquad v_0 = \varphi,$$

remains a subsolution of (13). Notice that from the Feynman-Kac representation,

$$\begin{aligned} w_t(y) &= \int_0^\infty \varphi(y+x)E\left[\exp\int_0^t \zeta_{t-s}(y+X_s)\,ds\Big|X_t = x\right]p_t(x)dx \\ &= \int_y^\infty \varphi(x)p_t(x-y)E\left[\exp\int_0^t \zeta_{t-s}(y+X_s)ds\Big|y+X_t = x\right]dx \\ &= \int_y^\infty \varphi(x)p_t(x-y)E_y\left[\exp\int_0^t \zeta_{t-s}(X_s)ds\Big|X_t = x\right]dx \\ &\geq \int_y^\infty \varphi(x)p_t(x-y)\exp\left(E_y\left[\int_0^t \zeta_{t-s}(X_s)ds\Big|X_t = x\right]\right)dx, \end{aligned} \tag{14}$$

where on the last line we used Jensen's inequality. Hence, when $L = \Gamma$, (14) reads

$$w_t(y) \geq \int_y^\infty \varphi(x)\gamma_t(x-y)\exp\left(\int_0^t\int_y^x \beta_{s,t}(z-y,x-y)\zeta_{t-s}(z)dzds\right)dx,$$

where $\beta_{s,t}(z-y,x-y)$ is given by (8). We close this section with a lemma that will be helpful in the proof of explosion, see §4 of [6] for the case $L = \Delta$.

Lemma 3.2 *Let $\sigma \in \mathbb{R}$ and $\nu > 0$. Assume that the solution u_t of*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^\sigma v_t(y)w_t(y), \qquad w_0 = \varphi, \tag{15}$$

satisfies

$$\lim_{t\to\infty}\inf_{0\leq x\leq 1} u_t(x) = \infty,$$

where $v : \mathbb{R}_+^2 \to \mathbb{R}_+$ is a measurable function such that $u_t^\beta \leq v_t$ for all $t \geq 0$. Then u_t blows-up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x)\,dx = \infty.$$

In particular, explosion in $L^p(\mathbb{R}_+)$-norm occurs for all $p \in [1,\infty]$.

Proof. Given $t_0 > 0$, let $u_t = w_{t_0+t}$ and $K(t_0) = \inf_{0\le y\le 1} w_{t_0}(y)$. The mild solution of (15) is given by

$$u_t(x) = \int_0^\infty \gamma_t(y-x)u_0(y)\,dy + \nu \int_0^t s^\sigma \int_0^\infty \gamma_{t-s}(y-x)u_s(y)v_{s+t_0}(y)\,dy\,ds.$$

Thus, for any $\varepsilon \in (0,1)$ and $t < (1-\varepsilon)\beta \wedge 1$,

$$\begin{aligned}
&\int_0^1 u_t(x)\,dx \\
&\ge \int_0^1 \int_0^\infty \gamma_t(y-x)u_0(y)dydx + \nu \int_0^t s^\sigma \int_0^1 \int_0^\infty \gamma_{t-s}(y-x)u_s^{1+\beta}(y)dydxds \\
&\ge \int_0^1 \int_x^1 \gamma_t(y-x)u_0(y)dydx + \nu \int_0^t s^\sigma \int_0^1 \int_x^1 \gamma_{t-s}(y-x)u_s^{1+\beta}(y)dydxds \\
&\ge K(t_0)\int_0^1 \int_0^y \gamma_t(x-y)dxdy + \nu \int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)\int_0^y \gamma_{t-s}(x-y)dxdyds \\
&\ge K(t_0)\int_0^1 \int_0^y \gamma_t(x)dxdy + \nu \int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)\int_0^y \gamma_{t-s}(x)dxdyds \\
&\ge \frac{1}{4}K(t_0)\int_0^1 \int_0^y \frac{x^{t-1}}{G(t)}dxdy + \frac{\nu}{4}\int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)\int_0^y \frac{x^{t-s-1}}{G(t-s)}dxdyds \\
&\ge \frac{1}{4}\frac{K(t_0)}{tG(t)}\int_0^1 y^t dy + \frac{\nu}{4}\int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)\frac{y^{t-s}}{(t-s)\gamma(t-s)}dyds \\
&\ge \frac{1}{4}K(t_0)\int_0^1 y^\beta dy + \frac{\nu}{4}\int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dyds \\
&\ge \frac{K(t_0)}{4(1+\beta)} + \frac{\nu}{4}\int_0^t s^\sigma \int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dyds,
\end{aligned}$$

where we used the inequalities $0 \le t-s \le t < (1-\varepsilon)\beta$ and $0 \le tG(t) \le 1$, $0 \le t \le 1$. Hölder's inequality yields

$$\begin{aligned}
\left(\int_0^1 u_s(y)dy\right)^{1+\beta} &\le \left(\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dy\right)\left(\int_0^1 y^{-(1-\varepsilon)}dy\right)^\beta \\
&= \varepsilon^{-\beta}\int_0^1 u_s^{1+\beta}(y)y^{(1-\varepsilon)\beta}dy,
\end{aligned}$$

hence letting $\tilde{u}(t) = \int_0^1 u_t(x)dx$ we get

$$\tilde{u}(t) \ge \frac{K(t_0)}{4(1+\beta)} + \frac{\nu\varepsilon^\beta}{4}\int_0^t s^\sigma \tilde{u}^{1+\beta}(s)ds, \qquad t < (1-\varepsilon)\beta \wedge 1.$$

It remains to choose t_0 such that the blow-up time of the equation

$$\tilde{u}(t) = \frac{K(t_0)}{4(1+\beta)} + \frac{\nu\varepsilon^\beta}{4}\int_0^t s^\sigma \tilde{u}^{1+\beta}(s)ds, \qquad t < (1-\varepsilon)\beta \wedge 1,$$

is smaller than $(1-\varepsilon)\beta \wedge 1$. $\square$

Choosing $v_t = u_t^\beta$ in Lemma 3.2 yields immediately:

Corollary 3.1 *Let $\sigma \in \mathbb{R}$ and $\nu > 0$. If the solution u_t of*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^\sigma w_t^{1+\beta}(y), \qquad w_0 = \varphi,$$

satisfies

$$\lim_{t\to\infty} \inf_{0\le x\le 1} u_t(x) = \infty,$$

then u_t blows-up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x)\, dx = \infty.$$

□

4 Existence of global solutions

We have the following non-explosion result, obtained originally by Nagasawa and Sirao [9] for integer $\beta \ge 1$.

Theorem 4.1 *Let $\sigma \in \mathbb{R}$ and $\beta, \nu > 0$. Assume that*

$$\int_0^\infty r^\sigma \|T_r^\Gamma \varphi\|_\infty^\beta \, dr < \frac{b}{\nu\beta}$$

for some $b > 0$. Then the equation

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi, \tag{16}$$

admits a global solution $u_t(x)$ which satisfies

$$0 \le u_t(x) \le \frac{b^{1/\beta} T_t^\Gamma \varphi(x)}{\left(b - \nu\beta \int_0^t r^\sigma \|T_r^\Gamma \varphi\|_\infty^\beta dr\right)^{1/\beta}}, \qquad x \in \mathbb{R}_+, \quad t \ge 0.$$

Proof. This is an adaptation of the proof of Theorem 3 in [15] to our context of time-dependent non-linearities. Recall that the mild solution of (16) is given by

$$u_t(x) = T_t^\Gamma \varphi(x) + \nu \int_0^t r^\sigma T_{t-r}^\Gamma u_r^{1+\beta}(x)\, dr. \tag{17}$$

Defining

$$B(t) = \left(b - \beta\nu \int_0^t r^\sigma \|T_r^\Gamma \varphi\|_\infty^\beta dr\right)^{-1/\beta}, \qquad t \ge 0,$$

we have $B(0)=b^{-1/\beta}$ and

$$\frac{d}{dt}B(t)=\nu t^\sigma\|T_t^\Gamma\varphi\|_\infty^\beta\left(b-\beta\nu\int_0^t r^\sigma\|T_r^\Gamma\varphi\|_\infty^\beta dr\right)^{-1-1/\beta}=\nu t^\sigma\|T_t^\Gamma\varphi\|_\infty^\beta B^{1+\beta}(t),$$

hence

$$B(t)=b^{-1/\beta}+\nu\int_0^t r^\sigma\|T_r^\Gamma\varphi\|_\infty^\beta B^{1+\beta}(r)\,dr.$$

Let $(t,x)\mapsto v_t(x)$ be a continuous function such that $v_t(\cdot)\in C_0(\mathbb{R}_+)$, $t\geq 0$, and

$$T_t^\Gamma\varphi(x)\leq v_t(x)\leq b^{-1/\beta}B(t)T_t^\Gamma\varphi(x),\qquad t\geq 0,\ x\in\mathbb{R}_+.$$

Let now

$$R(v)(t,x)=T_t^\Gamma\varphi(x)+\nu\int_0^t r^\sigma T_{t-r}^\Gamma v_r^{1+\beta}(x)dr.$$

We have

$$\begin{aligned}R(v)(t,x) &\leq T_t^\Gamma\varphi(x)+\nu b^{-1/\beta}\int_0^t r^\sigma B^{1+\beta}(r)T_{t-r}^\Gamma(T_r^\Gamma\varphi(x))^{1+\beta}dr\\ &\leq T_t^\Gamma\varphi(x)+\nu b^{-1/\beta}\int_0^t r^\sigma B^{1+\beta}(r)T_{t-r}^\Gamma T_r^\Gamma\varphi(x)\|T_r^\Gamma\varphi\|_\infty^\beta dr\\ &= b^{1/\beta}T_t^\Gamma\varphi(x)\left(b^{-1/\beta}+\nu\int_0^t r^\sigma B^{1+\beta}(r)\|T_r^\Gamma\varphi\|_\infty^\beta dr\right),\end{aligned}$$

hence

$$T_t^\Gamma\varphi(x)\leq R(v)(t,x)\leq b^{1/\beta}B(t)T_t^\Gamma\varphi(x),\qquad t\geq 0,\ x\in\mathbb{R}_+.$$

Let

$$u_t^0(x)=T_t^\Gamma\varphi(x),\quad\text{and}\quad u_t^{n+1}(x)=R(u^n)(t,x),\quad n\in\mathbb{N}.$$

Then $u_t^0(x)\leq u_t^1(x)$, $t\geq 0$, $x\in\mathbb{R}_+$. Since T_t^Γ is non-negative, using induction we obtain

$$0\leq u_t^n(x)\leq u_t^{n+1}(x),\qquad n\geq 0.$$

Letting $n\to\infty$ yields, for $t\geq 0$ and $x\in\mathbb{R}_+$,

$$0\leq u_t(x)=\lim_{n\to\infty}u_t^n(x)\leq b^{1/\beta}B(t)T_t^\Gamma\varphi(x)\leq\frac{b^{1/\beta}T_t^\Gamma\varphi(x)}{\left(b-\nu\beta\int_0^t r^\sigma\|T_r^\Gamma\varphi\|_\infty^\beta dr\right)^{1/\beta}}<\infty.$$

Consequently, u_t is a global solution of (17) due to the monotone convergence theorem.

□

As a consequence, an existence result can be obtained under an integrability condition on φ.

Corollary 4.1 *Let $1 \le q < \infty$, $\sigma > -1$ and $\nu > 0$. If $\varphi \in L^q(\mathbb{R}_+)$ is non-negative and $\beta > 2q(1+\sigma)$, then the solution u_t of*

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^\sigma w_t^{1+\beta}, \qquad w_0 = \varphi,$$

is global and satisfies, for some $c > 0$,

$$0 \le u_t(x) \le c t^{-1/(2q)}, \qquad x \in \mathbb{R}_+,$$

for all t large enough.

Proof. From Hölder's inequality and (3) we have

$$|T_t^\Gamma \varphi(y)| \le \|\varphi\|_q \|\gamma_t\|_p, \qquad 1/p = 1 - 1/q,$$

where

$$\begin{aligned}
\|\gamma_t\|_p &= \left(\int_0^\infty \frac{x^{p(t-1)}}{G(t)^p} e^{-px} dx \right)^{1/p} \\
&= \frac{G(p(t-1)+1)^{1/p}}{p^{t-1} G(t)} \left(\int_0^\infty \frac{(px)^{p(t-1)}}{G(p(t-1)+1)} e^{-px} dx \right)^{1/p} \\
&= \frac{G(p(t-1)+1)^{1/p}}{p^{t-1+1/p} G(t)} \\
&\sim \frac{(p(t-1)+1)^{(t-1)+1/(2p)} e^{1-1/p}}{p^{t-1+1/p} t^{t-1/2}} (2\pi)^{-1/2+1/(2p)} \\
&\sim t^{1/2} \frac{(1 - 1/t + 1/(pt))^t (p(t-1)+1)^{1/(2p)} e^{1-1/p}}{(t-1+1/p) p^{1/p}} (2\pi)^{-1/(2q)} \\
&\sim t^{1/2} \frac{(p(t-1)+1)^{1/(2p)}}{(t-1+1/p) p^{1/p}} (2\pi)^{-1/(2q)} \\
&\sim t^{-1/2} t^{1/(2p)} p^{-1/(2p)} (2\pi)^{-1/(2q)} \\
&\sim (2\pi t)^{-1/(2q)} p^{-1/(2p)},
\end{aligned}$$

as $t \to \infty$. Hence for some $t_0 > 0$ and $c > 0$,

$$\int_0^\infty t^\sigma \|T_t^\Gamma \varphi\|_\infty^\beta dt \le \|\varphi\|_\infty^\beta \int_0^{t_0} t^\sigma dt + c \|\varphi\|_q^\beta \int_{t_0}^\infty t^\sigma \|\gamma_t\|_p^\beta dt < \infty$$

provided $\beta > 2q(1+\sigma)$, and the conclusion follows from Theorem 4.1. □

Under a polynomial growth assumption on φ we get the following more precise result as another corollary of Theorem 4.1.

Corollary 4.2 *Let $\sigma \in \mathbb{R}$ and assume that there exist $c \geq 0$, $a \geq 0$ and $x_0 \geq 0$ such that*

$$\varphi(x) \leq cx^{-a}, \qquad x > x_0.$$

If $a\beta > 1 + \sigma$ then the solution u_t of

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^{\sigma} w_t^{1+\beta}, \qquad w_0 = \varphi$$

is global, and there exists $C > 0$ such that

$$0 \leq u_t(x) \leq Ct^{-a}, \qquad x \in \mathbb{R}_+,$$

for all t large enough.

Proof. Apply Theorem 4.1 and (5) of Lemma 2.1. □

5 Blow-up of solutions

In this section we obtain a partial converse to Corollary 4.2.

Theorem 5.1 *Assume that $\varphi \geq 0$ satisfies $\varphi(x) \geq cx^{-a}$ for all x large enough, where $a, c \geq 0$. Let $\nu > 0$, $\beta > 0$ and $a\beta < 1 + \sigma$. Then the equation*

$$\frac{\partial w_t}{\partial t} = \Gamma w_t + \nu t^{\sigma} w_t^{1+\beta}, \qquad w_0 = \varphi,$$

blows up in finite time. In the critical case $a\beta = 1 + \sigma$, $a \neq 0$, finite-time blow-up occurs under the additional assumption $\beta > 1$, i.e. $a < 1 + \sigma$.

This result is a consequence of the lemmas 3.1 and 3.2 above, and of the lemmas 5.1 and 5.2 below.

Lemma 5.1 *Assume that $\varphi \geq 0$ is such that $\varphi(x) \geq cx^{-a}$ for all x large enough, where $a, c \geq 0$. Let $\nu > 0$, $\beta > 0$ and $a\beta < 1 + \sigma$. Let g_t be the solution of*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^{\sigma} (T_t^{\Gamma} \varphi)^{\beta}(y) w_t(y), \qquad w_0 = \varphi. \tag{18}$$

Then

$$\lim_{t \to \infty} \inf_{0 \leq x \leq 1} g_t(x) = \infty.$$

Proof. Let $0 < \eta < 1$. The Feynman-Kac representation and (5) yield, for $0 < y < \eta + t/2$, $t > 6t_0$ (where t_0 is defined in Lemma 2.1), and some $c_0 > 0$:

$$\begin{aligned}
g_t(y) &= \int_y^\infty \varphi(x)\gamma_t(x-y)E_y\left[\exp\left(\nu\int_0^t (t-s)^\sigma (T^\Gamma_{t-s}\varphi(X^\Gamma_s))^\beta ds\right)\Big|X^\Gamma_t = x\right]dx \\
&\geq \int_y^\infty \varphi(x)\gamma_t(x-y)E_y\left[\exp\left(c_0\nu\int_{t_0}^{t/2} 1_{[0,\eta+t-s]}(X^\Gamma_s)(t-s)^{\sigma-a\beta}ds\right)\Big|X^\Gamma_t = x\right]dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)\exp\left(c_0\nu\int_{t_0}^{t/2}(t-s)^{\sigma-a\beta}P_y(0<X^\Gamma_s<\eta+t-s|X^\Gamma_t=x)ds\right)dx \\
&\geq \int_{t-1/3}^{2t} \varphi(x)\gamma_t(x-y)\exp\left(c_0\nu\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta}P_y(0<X^\Gamma_s<2s+t/2|X^\Gamma_t=x)ds\right)dx \\
&\geq c_1 1_{[0,\eta+t/2]}(y)t^{-a}\exp\left(\frac{c_0\nu}{2}\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta}ds\right),
\end{aligned}$$

where we used (6) and (10) to obtain the last inequality. Hence

$$\begin{aligned}
g_t(y) &\geq 1_{[0,\eta+t/2]}(y)c_1t^{-a}\exp\left(\frac{c_0\nu}{2}\int_{t_0}^{t/6}(t-s)^{\sigma-a\beta}ds\right) \qquad (19)\\
&= 1_{[0,\eta+t/2]}(y)c_1t^{-a}\exp\left(\frac{c_0\nu}{2(1+\sigma-a\beta)}\left((t-t_0)^{1+\sigma-a\beta}-\left(\frac{5t}{6}\right)^{1+\sigma-a\beta}\right)\right),
\end{aligned}$$

and it suffices that $a\beta < 1+\sigma$ in order to get $\inf_{0<y<1} g_t(y) \to \infty$ as $t \to \infty$. □

Notice that the criteria for blow-up of Lemma 5.1 can easily be adapted to other time-dependent non-linearities. We now turn to the critical case $a\beta = 1+\sigma$.

Lemma 5.2 *Let $\sigma > -1$, $\nu > 0$, and assume that $\varphi \geq 0$ is such that $\varphi(x) \geq cx^{-(1+\sigma)/\beta}$ for all x large enough, where $\beta > 1$. Then the solution h_t of the equation*

$$\frac{\partial w_t}{\partial t}(y) = \Gamma w_t(y) + \nu t^\sigma w_t(y) g_t^\beta(y), \qquad w_0 = \varphi,$$

where g_t solves (18)*, satisfies* $\lim_{t\to\infty}\inf_{0\leq x\leq 1} h_t(x) = \infty$.

Proof. Let $0 < \eta < 1$. Since $a\beta = 1+\sigma$, with $a > 0$, we see from (19) that there exists $t_0 > 0$ such that for all $t > 3t_0$,

$$g_t(y) \geq ct^{-a}1_{[0,\eta+t/2]}(y). \qquad (20)$$

Jensen's inequality, (7) and (9) yield, for $t > t_0$ and $0 < y < \eta$,

$$\begin{aligned}
h_t(y) &= \int_y^\infty \varphi(x)\gamma_t(x-y)E_y\left[\exp\left(\nu\int_0^t (t-s)^\sigma g_{t-s}^\beta(X_s^\Gamma)ds\right)\Big|X_t^\Gamma = x\right]dx \\
&\geq \int_y^\infty \varphi(x)\gamma_t(x-y)\exp\left(\nu\int_0^t E_y\left[(t-s)^\sigma g_{t-s}^\beta(X_s^\Gamma)\Big|X_t^\Gamma = x\right]ds\right)dx \\
&\geq \int_{t-\eta}^t \varphi(x)\gamma_t(x-y)\exp\left(c\nu\int_{t_0}^{t/3}(t-s)^{\sigma-a}P_y(0 < X_s^\Gamma < \eta + (t-s)/2|X_t^\Gamma = x)ds\right)dx \\
&\geq \int_{t-\eta}^t \varphi(x)\gamma_t(x-y)\exp\left(c\nu\int_{t_0}^{t/3}(t-s)^{\sigma-a}P_y(0 < X_s^\Gamma < \eta + s|X_t^\Gamma = x)ds\right)dx \\
&\geq \int_{t-\eta}^t \varphi(x)\gamma_t(x-y)\exp\left(\frac{c\nu}{2}\int_{t_0}^{t/3}(t-s)^{\sigma-a}ds\right)dx \\
&= c_3 t^{-(1+\sigma)/\beta-1/2}\exp\left(\frac{c\nu}{2(1+\sigma-a)}\left((t-t_0)^{\sigma-a+1} - \left(\frac{2t}{3}\right)^{\sigma-a+1}\right)\right).
\end{aligned}$$

Hence the conclusion holds provided $\sigma - a + 1 > 0$, i.e. $\beta > 1$. $\square$

6 Systems of semilinear equations

First we consider the following system of semilinear equations

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \mathit{\Gamma}_{\lambda_1} u_t + \nu_1 u_t^{\beta_{11}} v_t^{\beta_{12}} \\ \\ \dfrac{\partial v_t}{\partial t} = \mathit{\Gamma}_{\lambda_2} v_t + \nu_2 u_t^{\beta_{21}} v_t^{\beta_{22}}, \end{cases} \tag{21}$$

where $u_0 = \varphi_1$ and $v_0 = \varphi_2$ are nonnegative bounded measurable functions, $\nu_1, \nu_2 > 0$, and $\beta_{ij} \in \{1, 2, \ldots\}$, $i, j = 1, 2$. The solution of this system can be expressed in terms of a continuous-time, two-type branching process evolving in the following way. The particles of type $i = 1, 2$ live independent exponential lifetimes of mean $1/\nu_i$. During its lifetime a type-i particle develops an independent Markov motion of generator $\mathit{\Gamma}_{\lambda_i}$ and, at the end of its life, it branches, leaving behind β_{i1} individuals of type 1 and β_{i2} individuals of type 2 that appear where the parent particle died, and evolve independently under the same rules. The state space of such branching process is the space $\mathcal{N}_f(S)$ of finite counting measures on $S := \mathbb{R}_+ \times \{1, 2\}$, where a measure

$$\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$$

represents a population consisting of n individuals of type 1 at positions $x_1, \ldots, x_n$, and m individuals of type 2 at positions $y_1, \ldots, y_m$. Let X_t^μ be the random element of $\mathcal{N}_f(S)$ representing the population configuration at time $t \geq 0$, starting from a given $\mu \in \mathcal{N}_f(S)$. For any bounded measurable $f : S \to [0, \infty)$ we define

$$w_t(\mu) = E_\mu \left[e^{S_t} \prod_{z \in \mathrm{supp}(X_t^\mu)} f(z) \right], \qquad \mu \in \mathcal{N}_f(S), \quad t \geq 0,$$

where E_μ denotes expectation with respect to $P(\,\cdot\,|X_0 = \mu)$, and $S_t = \nu_1 \int_0^t N_{s,1}\, ds + \nu_2 \int_0^t N_{s,2}\, ds$, where $N_{s,i}$ is the number of particles of type i in the population at time s. Choosing f so that $f(\cdot, i) = \varphi_i$ for $i = 1, 2$, one can show [8] that the solution of (21) is given by $u_t = w_t(\cdot, 1)$ and $v_t = w_t(\cdot, 2)$, where for shortness of notation we write $w_t(x, i)$ when $\mu = \delta_{(x,i)}$. We now prove the following theorem.

Theorem 6.1 *Let the initial values φ_1, φ_2 of (21) be bounded measurable functions such that $0 \leq \varphi_1(x) \leq c_1 x^{-a_1}$ and $0 \leq \varphi_2(x) \leq c_2 x^{-a_2}$ for x large enough and some constants $c_1, c_2 > 0$, where $a_1, a_2 \in (1, \infty)$. If $(a_1 \wedge a_2)[(\beta_{11} + \beta_{12}) \wedge (\beta_{11} + \beta_{12}) - 1] > 1$ and c_1, c_2 are sufficiently small, then the solution of (21) is global.*

Proof. Without loss of generality we assume that $f(x, i) := \varphi_i(x) \leq c_i(x^{-a_i} \wedge 1)$ for all $x > 0$ and $i = 1, 2$. Let $\kappa = \kappa(t)$ denote the number of branchings occurring in the interval $[0, t]$, and let $w_t^{(k)}(\mu) = E_\mu \left[e^{S_t} \prod_{z \in \mathrm{supp}(X_t^\mu)} f(z);\ \kappa = k \right]$, $\mu \in \mathcal{N}_f(S)$, $k \in \mathbb{N}$. Therefore,

$$w_t(\mu) = \sum_{k=0}^{\infty} w_t^{(k)}(\mu), \qquad \mu \in \mathcal{N}_f(S), \quad t \geq 0.$$

Writing $\gamma_t^{\lambda_i}$ for the transition densities of the gamma process of parameter λ_i, $i = 1, 2$, and defining

$$\pi_t f(x, i) := \int_{\mathbb{R}} f(y, i) \gamma_t^{\lambda_i}(y - x)\, dy, \qquad (x, i) \in S, \quad t \geq 0,$$

we see that, for $\mu = \sum_{i=1}^n \delta_{(x_i,1)} + \sum_{j=1}^m \delta_{(y_j,2)}$,

$$w_t^{(0)}(\mu) = \left(\prod_{i=1}^n \pi_t f(x_i, 1) \right) \left(\prod_{j=1}^m \pi_t f(y_j, 2) \right)$$

and

$$
\begin{aligned}
w_t^{(1)}(\mu) &= 1_{\{n\neq 0\}}\nu_1 \sum_{i=1}^{n}\int_0^t \int_{\mathbb{R}} p_s^{\lambda_1}(z-x_i)\,(\pi_{t-s}f(z,1))^{\beta_{11}}\,(\pi_{t-s}f(z,2))^{\beta_{12}}\,dz \\
&\quad \times \prod_{\substack{l=1\\ l\neq i}}^{n} \pi_s w_{t-s}^{(0)}(x_l,1) \prod_{h=1}^{m} \pi_s w_{t-s}^{(0)}(y_h,2)\,ds \\
&+ 1_{\{m\neq 0\}}\nu_2 \sum_{j=1}^{m}\int_0^t \int_{\mathbb{R}} p_s^{\lambda_2}(z-y_j)\,(\pi_{t-s}f(z,1))^{\beta_{21}}\,(\pi_{t-s}f(z,2))^{\beta_{22}}\,dz \\
&\quad \times \prod_{l=1}^{n} \pi_s w_{t-s}^{(0)}(x_l,1) \prod_{\substack{h=1\\ h\neq j}}^{m} \pi_s w_{t-s}^{(0)}(y_h,2)\,ds \\
&\leq \nu_1 n \prod_{l=1}^{n} \pi_t f(x_l,1) \prod_{h=1}^{m} \pi_t f(y_h,2) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_{11}+\beta_{12}-1} ds \\
&+ \nu_2 m \prod_{l=1}^{n} \pi_t f(x_l,1) \prod_{h=1}^{m} \pi_t f(y_h,2) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_{21}+\beta_{22}-1} ds,
\end{aligned}
$$

where we used $\|f\|_\infty \leq 1$ and $\pi_s w_{t-s}^{(0)}(z,i) = \pi_t f(z,i)$, $(z,i) \in S$, $t \geq 0$. Hence,

$$
\begin{aligned}
w_t^{(1)}(\mu) &\leq (\nu_1 \vee \nu_2)(n+m) w_t^{(0)}(\mu) \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{[(\beta_{11}+\beta_{12})\wedge(\beta_{21}+\beta_{22})]-1} ds, \\
&\qquad \mu = \sum_{i=1}^{n} \delta_{(x_i,1)} + \sum_{j=1}^{m} \delta_{(y_j,2)}, \quad t \geq 0.
\end{aligned}
$$

By induction on k one can prove that for $t \geq 0$, $\mu = \sum_{i=1}^{n} \delta_{(x_i,1)} + \sum_{j=1}^{m} \delta_{(y_j,2)}$ and $k \geq 1$,

$$
w_t^{(k)}(\mu) \leq \frac{\nu^k}{k!} \prod_{i=0}^{k-1} (n+m+i(\beta^*-1)) \left(\int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_*-1} ds \right)^k w_t^{(0)}(\mu), \tag{22}
$$

where $\nu = \nu_1 \vee \nu_2$, $\beta_* = (\beta_{11}+\beta_{12}) \wedge (\beta_{21}+\beta_{22})$ and $\beta^* = (\beta_{11}+\beta_{12}) \vee (\beta_{21}+\beta_{22})$. Setting $X_0 = \mu = \delta_{(z,i)}$ in (22) yields

$$
w_t(z,i) \leq \pi_t f(z,i) \left(1 + \sum_{k=1}^{\infty} v_k(t)\right), \qquad t \geq 0, \tag{23}
$$

where

$$
v_k(t) = \frac{1}{k!} \prod_{i=0}^{k-1} (1+i(\beta^*-1)) \left(\nu \int_0^t \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_*-1} ds \right)^k .
$$

Taking $M > 0$ large enough we obtain from Remark 2.1 that

$$
v_k(t) \leq \left(\beta^* \nu \left(\int_0^M \left(\sup_{z\in S} \pi_s f(z)\right)^{\beta_*-1} ds + \text{Const.} \int_M^\infty \left((c_1 \vee c_2) s^{-a_1\wedge a_2}\right)^{\beta_*-1} ds \right)\right)^k .
$$

If c_1, c_2 are so small that $v_k(t) < 1$ uniformly in t for all k, then the solution of (21) is global. □

Next, consider the nonlinear system of equations:

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma_\lambda u_t + \nu t^\sigma u_t^{1+\beta_1} v_t^{\beta_2} \\ \\ \dfrac{\partial v_t}{\partial t} = \Gamma_\mu v_t + F_t(u_t, v_t), \end{cases} \tag{24}$$

$u_0 = \varphi_1$, $v_0 = \varphi_2$, $\lambda, \mu, \nu > 0$, where F_t is a positive and measurable function.

Proposition 6.1 *Assume that $\varphi_1(x) \geq cx^{-a_1}$ and $\varphi_2(x) \geq cx^{-a_2}$ for x large enough, with $a_1, a_2 \geq 0$. Then (24) blows-up if $a_1\beta_1 + a_2\beta_2 < 1 + \sigma$, and also if $a_1\beta_1 + a_2\beta_2 = 1 + \sigma$ under the additional assumption $\beta_1 > 1$.*

Proof. From Lemma 3.1 and Lemma 2.1 we have $T_t^\Gamma \varphi_2(y) \geq c_2 \mu^{a_2} t^{-a_2} 1_{[0,t]}(y)$, and

$$v_t^{\beta_2}(y) \geq (T_t^\Gamma \varphi_2(y))^{\beta_2} \geq c_2^{\beta_2} \mu^{a_2\beta_2} t^{-a_2\beta_2} 1_{[0,t]}(y).$$

We conclude by an application of Theorem 5.1 and Lemma 3.1. □

In the remaining part of this section we obtain conditions for explosion in finite time of the system

$$\begin{cases} \dfrac{\partial u_t}{\partial t} = \Gamma u_t + t^{\sigma_1} u_t v_t \\ \\ \dfrac{\partial v_t}{\partial t} = \Gamma v_t + (1 \vee t)^{\sigma_2} u_t v_t, \end{cases} \tag{25}$$

with $u_0 = \varphi_1$, $v_0 = \varphi_2$, and $\sigma_1, \sigma_2 \in \mathbb{R}$.

Lemma 6.1 *Assume that $\sigma_2 \geq \sigma_1$ and that for some initial conditions $\varphi_1 \leq \varphi_2$, the solution u_t of (25) satisfies*

$$\inf_{0 \leq x \leq 1} u_t(x) \to \infty$$

as $t \to \infty$. Then u_t blows-up in finite time, in the sense that there exists $t > 0$ such that

$$\int_0^1 u_t(x)dx = \infty.$$

Proof. By linearity, $u_t - v_t$ is solution of

$$\frac{\partial}{\partial t}(u_t - v_t) = \Gamma(u_t - v_t) + u_t v_t (t^{\sigma_1} - (1 \vee t)^{\sigma_2}), \tag{26}$$

with $u_0 - v_0 = \varphi_1 - \varphi_2 \leq 0$, hence from the integral form of (26):

$$(u_t - v_t)(x) = T_t^\Gamma(u_0 - v_0) + \int_0^t (s^{\sigma_1} - (1 \vee s)^{\sigma_2}) T_{t-s}^\Gamma (u_s v_s)(x) ds,$$

we have $u_t - v_t \leq 0$, $t \geq 0$. It remains to apply Lemma 3.2 to the equation

$$\frac{\partial u_t}{\partial t}(y) = \Gamma u_t(y) + t^{\sigma_1} v_t(y) u_t(y),$$

with $\beta = 1$, $\nu = 1$, and to use the inequality $v_t \geq u_t$. □

The above explosion criterion also implies blow-up in all L^p norms, $p \in [1,\infty]$, and is used in the next proposition.

Proposition 6.2 *Assume that* $\sigma_2 \geq \sigma_1$ *and* $\varphi_1(x) \geq cx^{-a_1}$, $\varphi_2(x) \geq cx^{-a_2}$, *for* x *large enough. Then* (25) *blows-up if* $\min(a_1,a_2) < 1+\sigma_1$. *In the critical case* $\min(a_1,a_2) = 1+\sigma_1$, *blow-up occurs if* $\max(a_1,a_2) < 1+\sigma_2$.

Proof. It suffices to prove blow-up for any pair of functions φ_1, φ_2 such that $\varphi_1(x) = cx^{-a_1}$ and $\varphi_2(x) = cx^{-a_2}$ for x large enough. Moreover, without loss of generality we may assume that $a_1 \geq a_2$ and $\varphi_1 \leq \varphi_2$. From (5) of Lemma 2.1, there exists $t_0 > 0$ such that for all $t \geq t_0$ and $y \in \mathbb{R}_+$,

$$u_t(y) \geq T_t^\Gamma \varphi_1(y) \geq ct^{-a_1} 1_{[0,t+\eta]}(y)$$

and

$$v_t(y) \geq T_t^\Gamma \varphi_2(y) \geq ct^{-a_2} 1_{[0,t+\eta]}(y).$$

The Feynman-Kac formula, (6) and (10) yield, for $0 \leq y \leq \eta + t/2$ and $t > 2 \vee t_0$,

$$\begin{aligned}
u_t(y) = & \int_{-\infty}^{\infty} \varphi_1(x)\gamma_t(x-y) E_y \left[\exp \int_0^t v_{t-s}(X_s^\Gamma) ds \Big| X_t^\Gamma = x \right] dx \\
\geq & \int_y^{\infty} \varphi_1(x)\gamma_t(x-y) E_y \left[\exp \left(c \int_{t_0}^{t/6} (t-s)^{-a_2+\sigma_1} 1_{[0,\eta+t-s]}(X_s^\Gamma) ds \right) \Big| X_t^\Gamma = x \right] dx \\
\geq & \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y) \\
& \times \exp \left(c \int_{t_0}^{t/6} (t-s)^{-a_2+\sigma_1} P_y(0 < X_{t-s}^\Gamma < \eta + t - s | X_t^\Gamma = x) ds \right) dx \\
\geq & \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y) \\
& \times \exp \left(c \int_{t_0}^{t/6} (t-s)^{-a_2+\sigma_1} P_y(0 < X_{t-s}^\Gamma < 2s + t/2 | X_t^\Gamma = x) ds \right) dx \\
\geq & \int_{t-1/3}^{2t} \varphi_1(x)\gamma_t(x-y) \exp \left(\frac{c}{2} \int_{t_0}^{t/6} (t-s)^{-a_2+\sigma_1} ds \right) dx \\
\geq & \; c_2 t^{-a_1} \exp \left(\frac{1}{2} \int_{t_0}^{t/6} (t-s)^{-a_2+\sigma_1} ds \right) \\
\geq & \; c_2 t^{-a_1} \exp \left(\frac{c}{2(1+\sigma_1-a_2)} \left((t-t_0)^{\sigma_1-a_2+1} - \left(\frac{5t}{6} \right)^{\sigma_1-a_2+1} \right) \right).
\end{aligned}$$

Hence, with $\eta = 1$, we infer blow-up from Lemma 6.1 if $a_2 < 1+\sigma_1$. Turning to the critical case, if $a_2 = 1+\sigma_1$ the above estimate yields $u_t(y) \geq c_2 1_{[0,\eta+t/2]}(y) t^{-a_1}$, and from (9) and (7) we have, for all $0 \le y \le \eta$,

$$
\begin{aligned}
v_t(y) &= \int_{-\infty}^{\infty} \varphi_2(x)\gamma_t(x-y)E_y\left[\exp\int_0^t u_{t-s}(X_s^\Gamma)ds\Big|X_t^\Gamma = x\right]dx \\
&\ge \int_{t-\eta}^{t} \varphi_2(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c_2\int_{t_0}^{t}(t-s)^{-a_1+\sigma_2}P_y(0<X_s^\Gamma<\eta+(t-s)/2|X_t^\Gamma = x)ds\right)dx \\
&\ge \int_{t-\eta}^{t} \varphi_2(x)\gamma_t(x-y) \\
&\quad \times \exp\left(c_2\int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2}P_y(0<X_s^\Gamma<\eta+s|X_t^\Gamma = x)ds\right)dx \\
&\ge c_2\int_{t-\eta}^{t}\varphi_2(x)\,dx\,t^{-1/2}\exp\left(\frac{c_2}{2}\int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2}ds\right) \\
&\ge c_2 t^{-a_2-1/2}\exp\left(\frac{c_2}{2}\int_{t_0}^{t/3}(t-s)^{-a_1+\sigma_2}ds\right).
\end{aligned}
$$

Hence, Lemma 6.1 implies blow-up provided $a_1 < 1+\sigma_2$. □

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