Using discrete invariants for fault detection of hybrid systems

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Abstract. The paper is concerned with diagnosis of higher order hybrid systems by means of discrete models. If discrete variables are given or can be found that are invariant in time for the normal operation, the fault detection problem can be solved by means of checking these invariants and neglecting other dynamical restrictions. As this is computationally simple, it can make real time applications possible which are out of reach nowadays. It will be shown that the existence of invariants for hybrid systems is not so rare as for pure continuous systems, where often only physical constraints like energy or momentum are preserved for closed systems. Hybrid systems even can have discrete invariants while the included continuous system is not hamiltonian.

1 INTRODUCTION

The fault detection problem of hybrid systems can be solved by the use of discrete models representing the qualitative dynamics of the system. Their application for higher order systems often is restricted due to complexity problems. The reason for these problems is the large number of discrete states that increases exponentially with the dynamical order of the continuous variable system and also with the number of quantization levels. The paper proposes a way to simplify these discrete models for special cases of hybrid systems and gives methods to use these simplified models for fault detection.

Physical invariants like energy, momentum, impulse or mass are well known in continuous systems analysis, but only seldom used for the description of discrete systems. Once an invariant for the normal behavior is found and formally described by a function of the quantized measurement variables, the fault detection program can make use of this. At each instant of time, the value of the invariant function is computed. If it happens to be not constant, the system shows no normal behavior and a fault has been detected.

Discrete invariants in principle can exist for any hybrid system. Some interesting questions in that area are

- Which class of hybrid systems has discrete invariants?
- How can these invariants be represented?
- Can invariants be computed from a model of the system?
- How can hybrid systems be designed that have invariants?

Not all of these questions will be answered in this paper, but results that do exemplarily show some properties are derived. Next, a short overview of the literature in discrete modelling and diagnosis of quantized systems is given.

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Nondeterministic automata are models that are appropriate for qualitative modelling of hybrid systems, [9]. The parallel observer based approach to process diagnosis [8] proposed in the middle of the last decade is based on automata models which have been proven to be applicable in a lot of industrial fields, [3, 7]. This approach as well has been intensively studied theoretically, where it turns out, that the qualitative model is a discrete approximation of the Frobenius-Perron operator, [12].

One problem of the automata approach to diagnosis is due to the very large number of discrete states that models for higher order plants generically have. Attempts to decompose a complex hybrid system may help in this case, leading to automata networks as models, [6]. But this is only a good choice if the subsystems are not strongly coupled. It may also be possible that the diagnostic task is separable and the subtasks can be solved independently with the help of parallel processing, [4].

Nowadays, methods for finding correct and quick solutions of diagnosis problems are essential for higher tasks, like reconfiguration of a faulty plant which only is possible if the diagnosis problem is solved first, [1].

Once the automaton as a qualitative model is found, discrete invariants can be found by applying graph partitioning techniques to the corresponding automaton, [2]. But these methods do not take into account the special structure of hybrid systems. In the design phase of these systems, the quantizer has to be chosen. In practice, this can either mean the physical positioning of threshold sensors or the selection of virtual thresholds in the case that continuous measurements are available. In many real world engineering problems, the latter is the case.

In the basic physics literature, invariants play a major role to classify systems, to find first integrals of differential equations of motion and much research has been done to relate invariants and symmetries of systems.

Invariant based approaches to controller design have been adopted to hybrid systems recently, [14]. But this is not the focus of this paper, because it is not posing a controller design problem. It is also known that invariants can be used within simulation algorithms, [5], but in this paper we will use invariants for diagnostic problems.

The paper is organized as follows. In Section 2, the class of hybrid systems will be introduced. In Section 3, the diagnostic problem will be formulated. The used class of qualitative models will be described in Section 4. Thereafter, discrete invariants will be defined in Section 5. With the help of two examples given in Section 6, the basic principle of fault detection with invariants is discussed in Section 7. The paper closes with the Conclusions and an Outlook.

2 HYBRID SYSTEMS

In Figure 1 the block diagram of a hybrid system under consideration is given. A *continuous system* with the vector $\mathbf{x}$ of states, the vector $\mathbf{u}$ of inputs and the vector $\mathbf{y}$ of outputs is started with initial state $\mathbf{x}_0$. The outputs are mapped by a *Quantiser* to values $[\mathbf{y}]$ of a discrete set. Thus, for each continuous output variable, a unique value of a discrete set is defined by the quantiser.

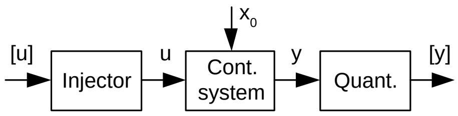

Figure 1. Block diagram of a hybrid system

For this paper, it is assumed that the qualitative input $[\mathbf{u}]$ stays constant over time, thus for each value $[u_0]$ the setup is given in Figure 2. These hybrid systems now can also be specified as *autonomous quantized systems*. Additionally, a possible fault is assumed to influence the continuous part of the overall system and thus, is an additional qualitative input to the hybrid system.

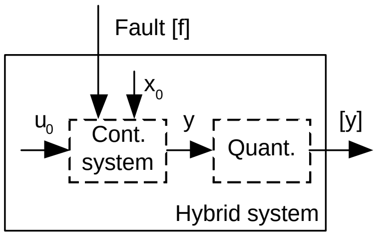

Figure 2. Autonomous quantised system with fault input

This is a standard setup that has been investigated many times before. Autonomous linear continuous systems can be represented by a state space model

$$\dot{\boldsymbol{x}} = \boldsymbol{A}_c\boldsymbol{x} \ , \tag{1}$$

$$\boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \ , \tag{2}$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0 \ . \tag{3}$$

The quantizer can be described by a mapping

$$q : \mathrm{IR}^r \to \mathcal{Y} \quad \text{with} \quad q(\boldsymbol{y}) = [\boldsymbol{y}] \tag{4}$$

that maps each vector $\boldsymbol{y}$ of the continuous output space IR^r uniquely to a qualitative value $[\boldsymbol{y}] \in \mathcal{Y}$. The set $\mathcal{Y}$ of qualitative output values is finite and thus countable.

The output signal can be given as a discrete event sequence

$$(([\boldsymbol{y}(t_0)], t_o), ([\boldsymbol{y}(t_1)], t_1), \ldots) \tag{5}$$

holding the exact continuous event times $t_0, t_1, \ldots$ where the quantised output signal value changes, [10, 11]. If the system is part of a automatized process, usually measurements are gathered at discrete time points with a fixed sampling time T_a. Thus, from eqn. (1) the discrete time equation

$$\boldsymbol{x}(k+1) = \boldsymbol{A}_d\boldsymbol{x}(k) \tag{6}$$

can be derived and the quantized measurement can be given by a discrete time sequence

$$[\boldsymbol{Y}] = ([\boldsymbol{y}(0)], [\boldsymbol{y}(1)], \ldots) \ . \tag{7}$$

3 FAULT DETECTION PROBLEM

The problem of fault detection considered here is restricted to the fact, that the only information given about the system are in the qualitative measurement sequences $[\boldsymbol{Y}]$. Neither an initial condition $\boldsymbol{x}_0$ nor an output $\boldsymbol{y}$ of the continuous part of the hybrid system should be used.

The hybrid system has a nominal behavior, given in principle by (1–4). Discrepancies from this nominal behavior which are the results of faults should be detected. To determine where a fault is located (isolation problem) or which fault has occurred (identification problem) is not in the scope of this paper, although extensions could be made which also give solutions to these problems.

The fault detection problem often is the first step in a diagnostic procedure, followed by isolation and identification routines. It has to be solved most frequently, reliable and with the least possible effort.

4 QUALITATIVE MODELS

A nondeterministic automaton, [13]

$$\boldsymbol{N} = (\mathcal{X}, \mathcal{Y}, L) \tag{8}$$

having the set of states $\mathcal{X}$, the set of inputs $\mathcal{Y}$ and the behavior relation L can be used as a qualitative model of a quantized autonomous systems (6–4) if the behavior relation is chosen to cover the hybrid systems dynamics. That can be ensured by the relation

$$L(z', w|z) = \begin{cases} 1 & \Longleftrightarrow \quad \begin{array}{l} \exists\ (\boldsymbol{x}', \boldsymbol{x}, \boldsymbol{y}) \ \text{ with} \\ \boldsymbol{x}' = \boldsymbol{A}_d\boldsymbol{x}, \boldsymbol{y} = \boldsymbol{C}\boldsymbol{x} \\ [\boldsymbol{x}'] = z', [\boldsymbol{x}] = z, [\boldsymbol{y}] = w \end{array} \\ 0 & \Longleftrightarrow \quad \text{else} \ . \end{cases} \tag{9}$$

An automaton is called *separable* if there exists at least two sets $\mathcal{M}_i$ and $\mathcal{M}_j$ of automaton states for which no transition between any states of different sets can take place, i.e. for all $i \neq j$

$$L(z', w|z) = 0 \quad \forall \ \ (z', z) \in \mathcal{M}_i \times \mathcal{M}_j \ \text{ or } \ \mathcal{M}_j \times \mathcal{M}_i \tag{10}$$

holds. A *partition* p of the discrete output space $\mathcal{Y}$ is a mapping

$$p([\boldsymbol{y}]) : \mathcal{Y} \to \bar{\mathcal{Y}} \ , \tag{11}$$

that assigns a discrete value $\bar{y} \in \bar{\mathcal{Y}}$ to every quantized output variable $[\boldsymbol{y}] \in \mathcal{Y}$. A partition is called *trivial* if the cardinality of $\bar{\mathcal{Y}}$ is one.

5 DISCRETE INVARIANTS

We directly give the definition of a discrete invariant which is the basis of the latter investigations.

Definition: The hybrid system (1–4) has a *discrete invariant* if a non trivial partition p of the discrete output space exists that has the property that

$$p([y(k)]) = p_0 \ \forall \ k \tag{12}$$

holds.

The initial condition (3) of the continuous system gives the value of a discrete invariant

$$p_0 = p([y(0)]) = p(q(\boldsymbol{C}\boldsymbol{x}(0))) \ . \tag{13}$$

6 SYSTEMS WITH DISCRETE INVARIANTS

With the help of two simple systems, the principle possibilities will be illustrated. Both examples are second order systems, but with different eigenvalues. The state variables are also given as outputs, i.e. $\boldsymbol{C} = \boldsymbol{I}$ holds.

Two Tanks System

The Tanks System presented in Figure 3 is autonomous. The liquid levels x_1 and x_2 change accordingly, because of the connecting pipe. There is a continuous invariant, because the volume does not change in time. Thus the sum $x_1 + x_2$ is an invariant of the continuous system.

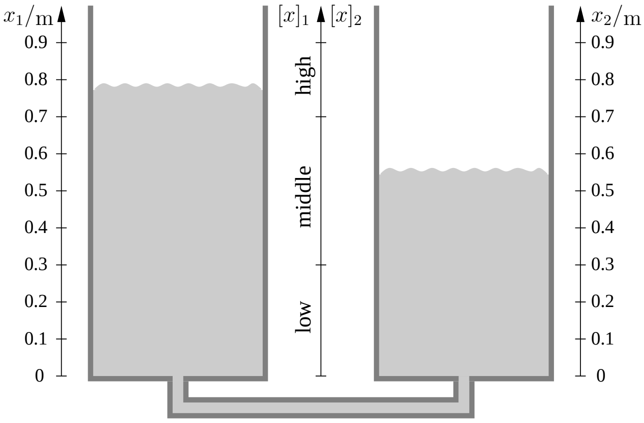

Figure 3. Two Tanks System

Each of the liquid levels is quantized in 3 intervals, denoted *low, middle, high*. The partition of the state space thus is rectangular and the transitions displayed in Figure 4 can occure. The qualitative dynamics is given by an automaton shown in Figure 5.

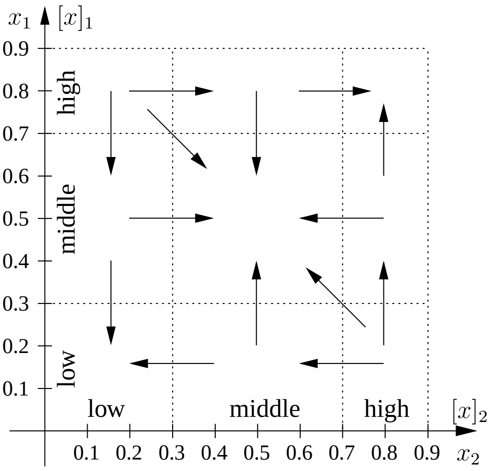

Figure 4. Rectangular quantization

This is the generic situation in qualitative modelling. The discrete model is nondeterministic and the tree of all possible qualitative state trajectories does connected all states of the model. The known continuous invariant $x_1 + x_2$ is not retrieved by this qualitative model. This situation changes if a quantization of the state space is chosen like displayed in Figure 6.

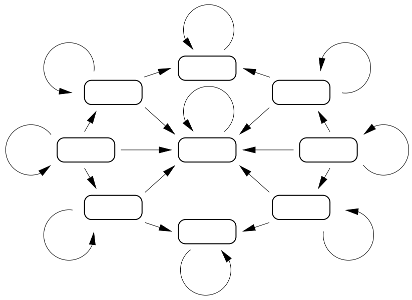

Figure 5. Qualitative model – rectangular quantization

While the interval borders going from low left to up right are crossed by some trajectories, the borders going from low right to up left are not. There are no arrows displayed and the border lines do show a constant value for the known invariant, the sum of both state variables.

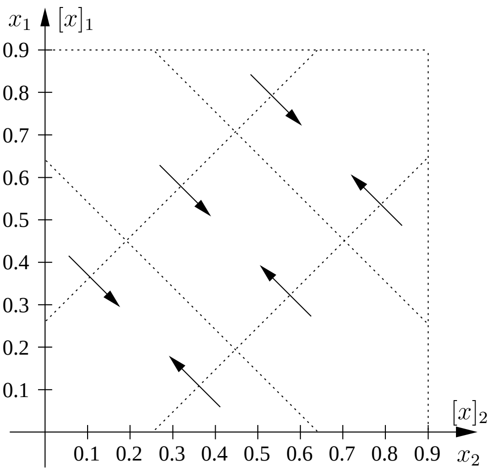

Figure 6. Invariance quantization

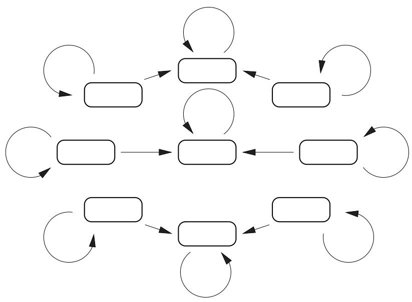

Figure 7. Qualitative model – invariance quantization

Of course, the automaton graph of the qualitative model looks different, see Figure 7. Still there is nondeterminism, but the automaton graph is separable into three subgraphs. These correspond to different intervals for the total liquid volume in the two Tanks.

If the states of all connected subgraphs are cumulated, the result is a trivial graph with no connectivity and three vertices. The states of this automaton also represent the continuously invariant variable and the dynamics is trivial, because each state is deterministically mapped to itself.

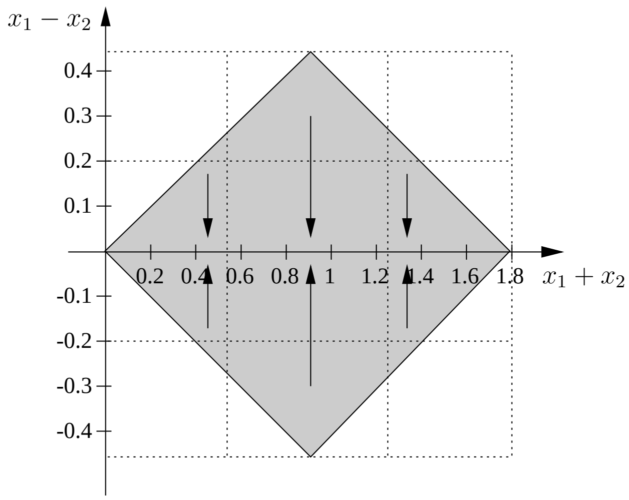

Figure 8. Invariance quantization – state trasformation

By coordinate transformations it is possible to find a rectangular partition that retains a separable qualitative dynamics. For the special case of the Two Tanks System, the transformation

$$\tilde{x}_1 = x_1 + x_2 \ , \tag{14}$$

$$\tilde{x}_2 = x_1 - x_2 \tag{15}$$

leads with a rectangular partition to the same automaton as shown in Figure 7. The partition is visualized in Figure 8. Remark that the limits of operation are now no longer parallel to the new coordinate axes but given by the shaded area. Thus, the shapes of the region in state space are the same like in Figure 6.

From this example, it could be seen, that the existence of an invariant of the continuous system is not sufficient for the hybrid system to have a discrete invariant. If transformations of the model are possible, where one new state variable $\tilde{x}_i$ is an invariant, i.e.

$$\tilde{x}_i(k+1) = \tilde{x}_i(k) \tag{16}$$

holds, each partition that is rectangular to this transformed state variable has the property that there are no transitions going over these borders.

Thus, we have the preliminary result that for each system with at least one integral action (i.e. one or more discrete eigenvalues of 1), a set of state quantizers can be found such that the corresponding qualitative model is separable. This was true for this very special example of the tank, which can also be fully described with a model of order one. But what about other continuous systems?

Spring-Mass-Damper System

A second example that shows oscillatory behavior is the spring-mass system given in Figure 9. Systems of this kind do occur in many mechanical applications, where often this system only is one subsystem of many coupled device that all can undergo oscillations.

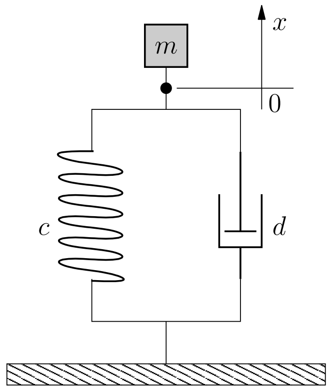

Figure 9. Spring-Mass-Damper-System

The system can be described by the state vector $(x, \dot{x})^T$ and the continuous time state space model

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{17}$$

which can be sampled with a period T an then resulting in a linear discrete time state space model of the form (6). We will not give it here explictly, because the principle parameters damping d, mass m and the spring constant c are not so easily retrievable in the discrete time model.

The continuous model (17) can be transformed to polar coordinates

$$r = \sqrt{x_1^2 + x_2^2} \tag{18}$$

$$\phi = \arctan \frac{x_2}{x_1} \tag{19}$$

resulting in the transformed state space model

$$\dot{r} = d\, r \tag{20}$$

$$\dot{\phi} = const. \tag{21}$$

The second equation gives hope for extending the discrete invariants approach by defining the quantization shown in Figure 10, where the states already have been transformed to its principal axes. This is a rather special quantization adopted to the system dynamics but nevertheless it can be chosen systematically. Its sectors are labelled by boolean tags, from which will be profitted later.

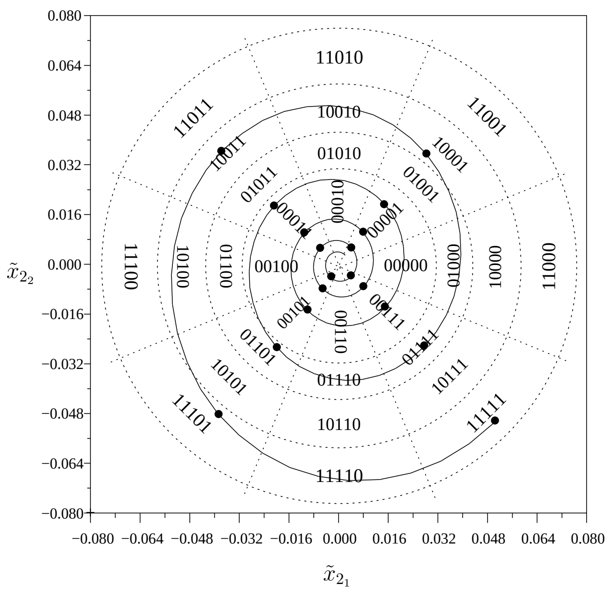

Figure 10. Radial state space quantization and trajectory – nominal

In Figure 10, one trajectory of the non faulty system also is displayed. The solid line shows the continuous trajectory while the dots mark the state at the sampling time intervals. The qualitative state sequence can then be given in terms of the boolean values $b_1b_2b_3b_4b_5$ of the sectors the continuous trajectory goes through.

Two possible faults are considered in the following:

- broken damper, which ideally leads to $d = 0$,
- weakened spring $c_{fault} < c_{normal}$.

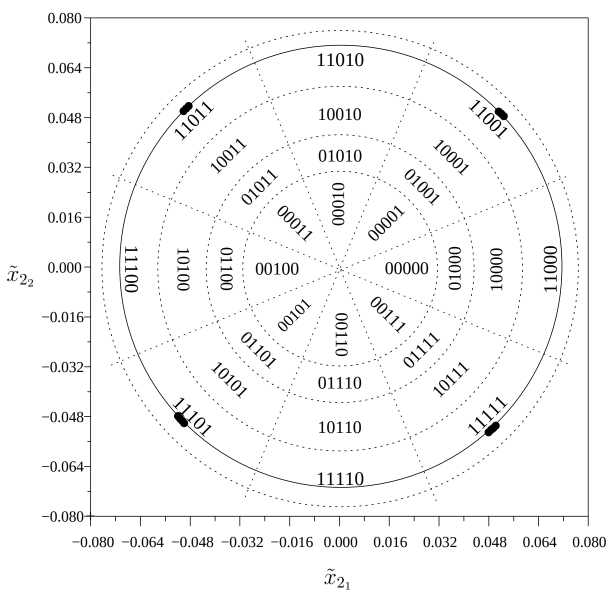

Figure 11. Trajectory with $d = 0$

Figure 11 shows a trajectory with broken damper. The system is theoretically infinitely oscillating with a certain frequency. If the spring constant becomes lower, the dynamics is not that clearly different from the nominal one, as seen by the example of Figure 12.

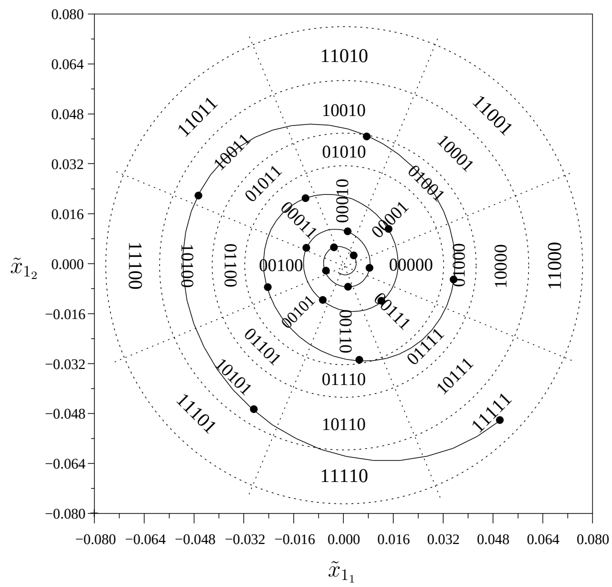

Figure 12. Trajectory with $c_{fault} < c_{normal}$

The measured quantized state sequences fort the normal and the two faulty cases are summarized in Table 1. Some interesting things can be derived from it. In the nominal case, the discrete variable b_5 is invariant ($b_5 = const.$), here $b_5 = 1$ holds. This results from the phase condition (21) and the fact that the sampling time is chosen such that the angle shows a 2π-symmetry.

k	normal	fault d	fault c
1	(11111)	(11111)	(11111)
2	(11101)	(11101)	(10101)
3	(10011)	(11011)	(10011)
4	(10001)	(11001)	(01001)
5	(01111)	(11111)	(00010)
6	(01101)	(11101)	(00110)
7	(00011)	(11011)	(00100)
8	(00001)	(11001)	(00011)

Table 1. Qualitative state sequences for normal and faulty cases

Thus, a faulty spring could be detected by only looking at the boolean variable in the last column. For the weakened spring (fault c), it starts for $k = 1, \ldots, 4$ with the value 1 and then changes to 0. At the time $k = 5$, the fault is detected by this violation of the invariant.

For the faulty case of the broken damper, the fault can not be detected by the same method, because the b_5 value for the whole trajectory is 1, the invariant is not violated. But in this special case, the faulty behavior itself has an invariant, namely $b_1(k) = 1$ that needs to be variant for a nominal behavior. It clearly corresponds to an energy level that the system can not leave, while the nominal system has to have a change.

7 FAULT DETECTION METHOD

The proposed fault detection method requires a basic setup of a hybrid system that is given in Figure 2, including a linear autonomous system which can be transformed to a block diagonal form and a quantizer. The sampling period as well as the quantization can be chosen freely. Further, in this paper we assume that there are no disturbances and the state variables can be measured directly. The following steps have to be done in order to set up a diagnostic unit.

- **Step 1 [Analyse non faulty system]:** Transform the system matrix $\boldsymbol{A}_c$ of the fault free system with a linear transformation matrix $\boldsymbol{T}$ such that the transformed system is block diagonal and has the form:

$$\begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_r \\ \dot{x}_{r+1} \\ \dot{x}_{r+2} \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} \lambda_1 \cdots 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots \ddots \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 \cdots \lambda_r & 0 & 0 & \cdots & 0 & 0 \\ 0 \cdots 0 & -\delta_1 & \omega_1 & \cdots & 0 & 0 \\ 0 \cdots 0 & -\omega_1 & -\delta_1 & \cdots & 0 & 0 \\ \vdots \ddots \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 \cdots 0 & 0 & 0 & \cdots & -\delta_c & \omega_c \\ 0 \cdots 0 & 0 & 0 & \cdots & -\omega_c & -\delta_c \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} \quad (22)$$

- **Step 2 [Choose sampling period]:** The sampling time T_a has to be chosen according to the phase conditions

$$T_a = \frac{2\pi}{\omega_1 \nu_1} = \cdots = \frac{2\pi}{\omega_c \nu_c}, \quad (23)$$

  with integer ν_i, i.e. $\exists\ (\nu_1, \cdots, \nu_c) \in \mathbb{N}^c$. If all frequencies ω_i are rational ($\forall i : \omega_i \in \mathbb{Q}$), it is always possible to find such a vector.
- **Step 3 [Generate quantization]:** Quantizations are done independently for the first r state variables corresponding to real eigenvalues (logarithmic spaced) and the remaining ones for the $2c$ complex eigenvalues (radial spaced, with ν_i intervals for the angle and logarithmic for the radius), like given in the spring-mass-damper example of Section 6.
- **Step 4 [Choose binary area coding]:** A coding scheme giving unique boolean vectors $\boldsymbol{b}$ for all quantized values has to be found. It can be in principle be arbitrary, but may be varied with respect to easy representations of the invariance functions, derived in the next step.
- **Step 5 [Determine partition function]:** The qualitative model (9) is derived for the chosen quantization and the non trivial partition p is determined, which can be represented by a binary function.
- **Step 6 [Implement diagnostic unit]:** The diagnostic unit only has to run the following procedure at each sampling time step. The state measurement is transformed with the matrix $\boldsymbol{T}^{-1}$ found in Step 1 and the result is quantized according to the quantization q found in Step 3 resulting in the value of the invariant at time k

$$p_{mes}(k) = p(q(\boldsymbol{T}^{-1}\boldsymbol{x}_{mes}(k))) . \quad (24)$$

  The algorithm works sequentially, is very fast and a fault is derived if $p_{mes}(k) \neq p_{mes}(k-1)$ holds for some time k.

It is clear that this method can result in wrong alarms in case of disturbances or modelling uncertainties. In some examples, we have seen, that the frequency of invariance violations can be used as a rough engineering measure for the reliability of the diagnostic result.

8 CONCLUSION AND OUTLOOK

The paper has shown that discrete invariants can be found for quantized systems with certain combinations of continuous systems dynamics and quantization. Further, problems of fault diagnosis can in principle be solved by regarding discrete invariants. As there is no need for computing trajectories of a dynamical model but only function evaluations are needed, fault detection gets extremely simple.

It has been shown, that discrete invariants do not only exist in trivial systems but there is chance to even design systems with respect to this property if there is freedom in choosing either the quantization or the continuous part of the hybrid system.

Extensions for the case of hybrid systems with qualitative input sequences will be made. The main issue that has to be regarded for this extension is the behavioral change of the continuous variable system by its own inputs. If the invariants are also changed by the inputs, it will be hard to find *one* quantization that ensures discrete invariants for all input values. But, if continuous measurement are available, the quantization or the transformation resp. can be changed according to the actual value of the input and the presented methods can be applied.

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