5

A First Survey of Gabor Multipliers

Hans G. Feichtinger and Krzysztof Nowak

ABSTRACT
We describe various basic facts about *Gabor multipliers* and their continuous analogue which we will call *STFT-multipliers*. These operators are obtained by going from the signal domain to some transform domain, and applying a pointwise multiplication operator before resynthesis. Although such operators have been in use implicitly for quite some time this paper appears to be the first systematic mathematical treatment of Gabor multipliers. Indeed, typical time-frequency localization operators, or thresholding algorithms involve simple 0/1-multipliers.
The main results of this chapter are of qualitative nature and describe how the properties of the Gabor multiplier depend on the decay of the multiplier sequence, the time-frequency (TF) concentration properties of the Gabor atom in use, and the time-frequency-lattice. These properties will be described in terms of the mapping properties of the corresponding Gabor multiplier between modulation spaces, or membership in some operator ideal (such as trace class or Hilbert Schmidt operator). It is also possible to give relatively precise estimates on behaviour of the sequence of eigenvalues of such operators, especially for the case of tight Gabor frames.
We shall also discuss the problem of injectivity of the linear mapping from the multiplier symbol to the operator, recovery of Gabor multipliers from lower symbols, and the related question, concerning best approximation of operators (e.g., from the Hilbert-Schmidt class) by Gabor multipliers of a certain type.

5.1 Introduction

For many years it has been considered as the key problem in Gabor analysis to exactly describe those windows or Gabor atoms g and time-frequency lattices Λ, for which the corresponding Weyl-Heisenberg (=WH) families, obtained by moving some Gabor atom g along a given TF-lattice Λ, form a Gabor frame. In the positive case standard frame theory allows to show that arbitrary functions f can be expanded into a Gabor series, where the building blocks are time-frequency shifted copies of g, and suitable coefficients (for L^2 functions they are the minimal norm coefficients) for

these series can be obtained as samples of the short-time Fourier transform of f with respect to some *dual Gabor atom*, such as the canonical dual atom $\tilde{g} = S^{-1}g$, i.e., by applying the inverse frame operator to g.

The standard results of Gabor analysis show that for a *sufficiently nice* atom $g \in L^2$ the following is true: For all TF-lattices which are sufficiently dense (e.g. with lattice constants (a, b) small enough) the WH-family generated from (g, Λ) is a Gabor frame with an equally *nice* dual atom $\tilde{g}$ (cf. [26], Thm.6.1., [7], Chap.3, or [76]). In this situation there exists also a (canonical) tight Gabor atom $h = S^{-1/2}g$, associated with the pair (g, Λ), which can be used in many ways like orthonormal systems (cf. [58] for details). Very recently Leinert and Gröchenig (see [45]) have shown that the L^2-frame property together with good time-frequency concentration of g (in the sense of a weighted L^1-condition over the TF-plane) ensure that $\tilde{g}$ has the same TF-concentration.

While dense TF-lattices yield Gabor frames, WH-families generated from sufficiently coarse TF-lattices Λ° will generate Riesz-bases for their closed linear span and even Riesz-projection bases for a family of modulation spaces (cf. [32] for this concept). Indeed, by moving from the lattice Λ generating a Gabor family to its adjoint Λ° , consisting of all TF-shifts which commute with those from Λ is equivalent to switch from Gabor frames to Riesz bases (for their closed linear span), according to the Ron-Shen principle (cf. [13]), and vice versa.

In the present paper *we are taking a step from function space theory towards operator theory*, by switching our attention now to (certain families of) operators based on these non-orthogonal expansions, instead of just studying the properties of the expansions themselves.

It is the purpose of this chapter to describe *the foundations of a theory of Gabor multipliers*, i.e., of those linear operators which arise from pointwise multiplication of Gabor coefficients or the Short-Time Fourier transform. Among others we shall address the basic questions: What are the *properties of such operators* (e.g., boundedness properties between various function spaces)? As expected, decay properties of the multiplier sequence as well as the time-frequency concentration of the building blocks in use play a role. When will we get a Hilbert-Schmidt or trace class operator on L^2? More quantitatively, can we predict the decay of eigenvalues of an operator from the global behaviour of the multiplier?

We will also study the mapping from the pointwise multiplier to the operator, and ask when it is *injective*. Under which conditions is there a best approximation of a general linear mapping (say a Hilbert Schmidt operator) by a Gabor multiplier using a given atom g and TF-lattice Λ. Furthermore we address the question to which extent the operators depend continuously on the atoms g resp. TF-lattices Λ used to generate the Gabor multiplier, with the full TF-plane being a natural limiting case.

5.2 Notations and Conventions

Throughout this paper we shall assume some familiarity with the theory of Gabor expansions and modulation spaces, as for example, explained in detail in K.Gröchenig's book [13], Chapters 11 and 12.

The ordinary Lebesgue spaces on $\mathbb{R}^d$ are denoted by $(L^p(\mathbb{R}^d), \|\cdot\|_p)$, for the range $1 \leq p \leq \infty$. We write $M(\mathbb{R}^d)$ for the space of bounded (regular Borel) measures on $\mathbb{R}^d$. It is considered as the dual space of $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$. Consequently a bounded sequence is w^*-convergent in $M(\mathbb{R}^d)$ if and only if it is vaguely convergent, i.e., $\mu_n(k) \to \mu_0(k)$ for arbitrary continuous functions k with compact support. As usual $L^1(\mathbb{R}^d))$ is viewed as a closed subspace of $M(\mathbb{R}^d)$ (consisting of those measures which are absolutely continuous with respect to the Lebesgue measure).

As usual in Gabor analysis we use the symbols T_x for the translation and M_ω for the modulation operator, i.e., pointwise multiplication with $\chi_\omega(x) = e^{2\pi i \omega x}$. The combined TF-shift (time-frequency shift) for $\lambda = (x, \omega)$ is $\pi(\lambda) = M_\omega T_x$. Moreover, $\pi : \lambda \mapsto \pi(\lambda)$ is (only) a *projective representation* of $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d)$, since the product of two TF-shifts is the TF-shift of the sum only up to phase factors. With any (test) function g and a signal or distribution f satisfying suitable (integrability) criteria, e.g. $f, g \in L^2(\mathbb{R}^d)$, one can associate the STFT (Short-Time Fourier Transform), which we like to define as follows:

$$STFT_g(f) = V_g(f) : \lambda = (t, \omega) \mapsto \langle f, \pi(\lambda) g \rangle \tag{5.2.1}$$

For every non-zero $g \in L^2(\mathbb{R}^d)$ is (up to the normalization factor $\|g\|_2$) isometric from $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$, and as suitable inverse STFT (ISTFT) on the range one can use the adjoint mapping V_g^*. Hence (in a weak sense), the reconstruction formula reads for $g \in L^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$ as follows:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} V_g(f)\ \pi(\lambda) g\ d\lambda, \tag{5.2.2}$$

showing that f is described as a (smeared) sum of building blocks $\pi(\lambda)g, \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}^d}$, with amplitudes being given by the STFT. This makes it of course plausible that a change of the amplitude allows to *operate* on the signal f and making changes in certain areas of the TF-plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$, before resynthesis. Thus TF-localization operators are obtained by multiplying the STFT $V_g(f)$ with some 0/1-function describing an area of interest. The operators obtained in such a way will be called STFT-multipliers, and are a very special case of the type of operators which are in the central topic of this paper.

Indeed, an important underlying idea of the theory of Gabor multipliers is the idea, that the evident redundancy in the STFT-representation should be used in order to work only with samples of $V_g(f)$, e.g. samples taken over a sufficiently fine TF-lattice $\Lambda \lhd L^2(\mathbb{R}^d)$, such as $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$.

The discrete analogue of (5.2.2) is then a representation of the form

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \ \pi(\lambda) g, \tag{5.2.3}$$

valid for all $f \in L^2(\mathbb{R}^d)$ or even tempered distribution, depending on the decay and smoothness properties of the pair (g, γ) of dual windows. Usually it is assumed that for a given g the TF-lattice is chosen to be sufficiently dense, such that the family $(g_\lambda)_{\lambda \in \Lambda}$ is a Gabor frame for the Hilbert space $L^2(\mathbb{R}^d)$, or equivalently $(\sum_{\lambda \in \Lambda} |V_g(f)|^2)^{1/2}$ defines an equivalent norm on $L^2(\mathbb{R}^d)$. This is the case if and only if the standard frame operator $S_g(f) = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda) g$ is bounded and invertible on $L^2(\mathbb{R}^d)$. It is a standard fact of Gabor analysis that one can achieve validity of (5.2.3) for all $f \in L^2(\mathbb{R}^d)$ by choosing as γ the so-called canonical *dual Gabor atom*, which we will denote (for fixed Λ) simply by $\tilde{g}$, but usually there are many other choices for γ, such that (5.2.3) is valid.

Another option is to chose $g = \gamma$ in such a way that (5.2.3) holds true despite this extra coupling. Corresponding building blocks g generate *tight* Gabor frames, and are therefore called *tight Gabor atoms*. Such tight Gabor atoms can be obtained in the following way: assume that (g, Λ), generates a Gabor frame, i.e., that $(g_\lambda)_{\lambda \in \Lambda}$ is a frame. Then the family $(\pi(\lambda)h)_{\lambda \in \Lambda}$, with $h = S^{-1/2}g$ is a tight Gabor frame. Again this particular tight Gabor window is called the *canonical tight Gabor window* associated with the pair (g, Λ). As shown recently (cf. [58]) it is the L^2 function which minimizes the distance $\|g - h\|_2$, among all other L^2-functions for which (h, Λ) generates a tight Gabor frame.

Since the description of operators *acting somehow on the TF-plane* requires a detailed description of the TF-behaviour of functions or (tempered) distributions it is clear, that corresponding function spaces have to be used extensively. As it is well-known by now the appropriate family for this purpose are the so-called **modulation spaces** $M^{p,q}_m(\mathbb{R}^d)$, among them the classical spaces $M^s_{p,q}(\mathbb{R}^d)$, using weights of the form $w_s(t, \omega) = (1 + |\omega|^2)^{s/2}$. One finds in this family also the classical Sobolev spaces $H^s(\mathbb{R}^d)$, (by the choice $p = q = 2$) or the Segal algebra $S_0(\mathbb{R}^d) = M^0_{1,1}(\mathbb{R}^d)$ and its dual $S_0'(\mathbb{R}^d) = M^0_{\infty,\infty}(\mathbb{R}^d)$, which will take a special role within Gabor analysis. The last two spaces are in fact invariant under the Fourier transform (as well as metaplectic transformations), in the same way as the space $M^{p,q}_{v_s}(\mathbb{R}^d)$, which are obtained by using radial symmetric weights $v_s(t, \omega) = (1 + |t|^2 + |\omega|^2)^{1/2}$ for the case $p = q$. Background information is also found in [32] and [31]. To fix symbols we also recall that we use the symbol $\mathcal{L}$ to denote the space of all bounded linear operators from one Banach space to another, such as $\mathcal{L}(S_0, S_0')$ or $\mathcal{L}(S_0, S_0)$.

Besides standard facts from functional analysis which we take for granted we want to specifically recall the concept of so-called *Gelfand triples*, as described in [31]. A *Gelfand triple* consists of some Banach space $(B, \|\cdot\|_B)$,

continuously and densely embedded into some Hilbert space $\mathcal{H}$, such that consequently $\mathcal{H}$ itself is embedded (w^*-densely) into the dual Banach space $(B', \|\cdot\|_B')$. We shall use the symbol $(B, \mathcal{H}, B')$ for such a triple. The (sesquilinear) inner product on $\mathcal{H}$ then extends in a natural way to a natural pairing between B' and B (again expressed as $\langle \sigma, f\rangle$). As we shall see examples of such Gelfand triples arise in abundance within Gabor analysis.

Typical examples of relevance for us are the Banach sequence spaces $\big(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\big)$, or the Gelfand triple arising from $S_0(\mathbb{R}^d)$, with $\mathcal{H} = L^2(\mathbb{R}^d)$. It is denoted by $\big(S_0(\mathbb{R}^d), L^2(\mathbb{R}^d), S_0'(\mathbb{R}^d)\big)$. The kernel theorem for $S_0(\mathbb{R}^d)$ implies that the operators from $S_0(\mathbb{R}^d)$ into $S_0'(\mathbb{R}^d)$ are identified with their distributional kernels from $S_0'(\mathbb{R}^{2d})$. Consequently it is the dual space to the space of operators with kernels in $S_0(\mathbb{R}^{2d})$, which is essentially the same as $\mathcal{L}(S_0', S_0)$. The Hilbert space in this context is the class of Hilbert Schmidt operators $\mathcal{HS}$ on $L^2(\mathbb{R}^d)$. The Gelfand triple of Banach spaces of operators arising in this way will be denoted by $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$.

If we talk about a bounded *linear mapping between Gelfand triples* this means that the boundedness is valid at each level, the Banach space, the Hilbert space, and the dual Banach space (usually also preserving w^*-convergence of sequences in B').

5.3 Basic Theory of Gabor Multipliers

This section starts with a rather general definition of *Gabor multipliers*. A number of basic results can be obtained as a combination of known facts about both the *analysis* and the *synthesis* mapping associated with a Gabor or Weyl-Heisenberg family, and the standard properties of multiplication operators, acting between Banach sequence spaces, based for example, on Hölder's inequality.

Since the atoms which will be involved should generate Bessel families with respect to general TF-lattices Λ windows g will be most often taken from the Segal algebra $S_0(\mathbb{R}^d)$, as the corresponding Gabor families are Bessel family for *arbitrary* TF-lattices Λ, in particular for those of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d \lhd \mathbb{R}^d \times \widehat{\mathbb{R}^d}$.

Definition 5.3.1. *Let g_1, g_2 be two L^2-functions, Λ a TF-lattice for $\mathbb{R}^d$, i.e., a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}^d}$. Furthermore let $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the* **Gabor multiplier** *associated to the triple (g_1, g_2, Λ) with (strong or)* **upper symbol m** *is given as*

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle\, \pi(\lambda)g_2. \qquad (5.3.1)$$

Convention: If in some discussion certain parameters, e.g., both the atoms and the TF-lattice Λ are fixed they may be simply omitted as *lower*

indices. For example, we may discuss the convergence of G_{m_n}, for a sequence of multipliers $\mathbf{m}_n$, while fixing the atoms and the TF-lattice. We simply write $G_{g,\Lambda,\mathbf{m}}$ for the case $g_1 = g = g_2$.

It is obvious from this definition that Gabor multipliers are essentially (infinite) linear combinations of rank one operators $f \mapsto \langle f, \pi(\lambda)g_1\rangle\, \pi(\lambda)g_2$, with coefficients m_λ. Whenever $g_1 = g = g_2$ and $\|g\|_2 = 1$ these building blocks are just the orthogonal projections onto the $1D$-subspaces of L^2 generated by the elements of the WH-family $(\pi(\lambda)g)_{\lambda\in\Lambda}$. We denote this family of projection operators in the rest of this paper by $(P_\lambda)_{\lambda\in\Lambda}$. Depending on the properties of the *analysis window* g_1, the *synthesis window* g_2 and the *multiplier sequence* $\mathbf{m} = (m_\lambda)_{\lambda\in\Lambda}$ the overall operator $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is bounded between various spaces. Typically one would require that both g_1 and g_2 are *Bessel atoms* with respect to the given lattice Λ, and that $\mathbf{m}$ is bounded. In this case the coefficient mapping using g_1, mapping f to the sequence of sampling values of $STFT_g f$ over Λ maps $L^2(G)$ into $\ell^2(\Lambda)$ (by definition), and also the synthesis mapping $\mathbf{c} \mapsto \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda)g_2$ is bounded from $\ell^2(\Lambda)$ to $L^2(G)$, and thus the overall operator is bounded on $L^2(G)$.

There are many good reasons to assume that the windows g_1 and g_2 should be chosen from Feichtinger's Segal algebra $S_0(\mathbb{R}^d)$ (cf. [32] for many properties relevant in the context of Gabor analysis). Furthermore, $S_0(\mathbb{R}^d)$ is much larger than the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, used often in such a context *just for convenience*. On the other hand $L^2(\mathbb{R}^d)$ is too large as a reservoir of windows and does not allow any more the most interesting results to be described below.

There are many other sufficient conditions which guarantee that $f \in S_0(\mathbb{R}^d)$. For example, any band-limited L^1-function is in $S_0(\mathbb{R}^d)$, but also compactly supported functions with an integrable Fourier transform. Generally speaking a mild combination of decay and smoothness assumption on f guarantee that $f \in S_0(\mathbb{R}^d)$ (cf.[Grö96] or [13], Prop.12.1.6, or [32], Thm.3.2.17). For $d = 1$ it is enough to know that $f, f', f" \in L^1(\mathbb{R})$ ([43]).

Among others it is well established (cf. again [32]) that S_0-atoms will automatically be Bessel atoms for arbitrary lattices Λ), while for general L^2-functions the Bessel property may depend in a drastic way on the corresponding TF-lattice in use (cf.[33]). Consequently the restriction to S_0 atoms will allow a discussion of the effect resulting from modifications of the lattice constants describing the TF-lattices involved in the construction. Furthermore this assumption will allow us to derive that $G_{\mathbf{m}}$ is Hilbert-Schmidt if $\mathbf{m} \in \ell^2(\Lambda)$, and a trace class operator on $L^2(\mathbb{R}^d)$, mapping $S_0'(\mathbb{R}^d)$ into $S_0(\mathbb{R}^d)$ (cf. Cor.3.3.5 in [32], p. 144), if $\mathbf{m} \in \ell^1(\Lambda)$.

In order to concentrate on the essential properties we shall state some of our results only for the case $g_1 = g_2 = g$, assuming that (g, Λ) generates a *tight* Gabor frame. In particular in such a case a minimal *symbolic calculus* is valid, in the sense that the constant multiplier $\mathbf{m} \equiv 1$ yields a multiple of the identity operator. Summarizing these basic facts we have:

Theorem 5.3.2. *Assume that $g \in S_0(\mathbb{R}^d)$. Then one has:*
(i) If $\mathbf{m} \in \ell^\infty(\Lambda)$, then $G_{\mathbf{m}} = G_{g,\Lambda,\mathbf{m}}$ defines a bounded operator on (S_0, L^2, S_0'), and the operator norm of $G_{\mathbf{m}}$ can be estimated (up to some constant) by $\|\mathbf{m}\|_\infty$.
(ii) The Gabor multiplier generated by $\mathbf{m}(\lambda) \equiv 1$ is a multiple of the identity operator if and only if (g, Λ) generates a tight Gabor frame.
(iii) $G_{\mathbf{m}}$ is a compact operator on $L^2(\mathbb{R}^d)$ and on $S_0(\mathbb{R}^d)$, if $\mathbf{m} \in c_o(\Lambda)$, i.e., if $m(\lambda) \to 0$ for $\lambda \to \infty$ (in the sense of Λ)
(iv) If $\mathbf{m} \in \ell^2(\Lambda)$, then $G_{\mathbf{m}}$: $S_0'(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d) \to S_0(\mathbb{R}^d)$.
(v) For $\mathbf{m} \in \ell^1(\Lambda)$ the operator $G_{\mathbf{m}}$ maps $S_0'(\mathbb{R}^d)$ into $S_0(\mathbb{R}^d)$.

Proof: These statements follow from the boundedness properties of the coefficient resp. synthesis mappings (for fixed lattice Λ), as described in some detail in Section 3.3.3 of [32]. □

Of course it would be possible to make similar statements for other classes of windows. For example, any $g \in S_0'(\mathbb{R}^d)$ in combination with an ℓ^1 multiplier sequence yields still a (compact) linear operator from $S_0(\mathbb{R}^d)$ into $S_0'(\mathbb{R}^d)$, to mention a rather extreme possible variant. A more traditional approach to TF-analysis making use of Schwartz functions an tempered distributions would probably make use of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)'$ (instead of $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$) in the above context.

For general pairs (g_1, g_2) from $S_0(\mathbb{R}^d)$ an even more compact formulation of the above theorem using the terminology of *Gelfand triples* can be given:

Theorem 5.3.3. *For every pair (g_1, g_2) in $S_0(\mathbb{R}^d)$, and any TF-lattice Λ the mapping from the strong symbol (multiplier) $(m_\lambda)_{\lambda\in\Lambda}$ to the corresponding Gabor multiplier $G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps the Gelfand triple $\left(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\right)$ into the bounded operators with kernel in the corresponding Gelfand triple $\left(S_0(\mathbb{R}^d \times \widehat{\mathbb{R}^d}), L^2(\mathbb{R}^d \times \widehat{\mathbb{R}^d}), S_0'(\mathbb{R}^d \times \widehat{\mathbb{R}^d})\right)$, i.e., into $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$.*

As already indicated in the introduction, modulation spaces are the right family of function spaces to describe distributions by means of the behaviour of their Gabor coefficients. The established atomic theory of modulation spaces makes some of the basic results concerning Gabor multipliers quite easy to prove. The following proposition describes a typical example. showing that modulation spaces are highly suitable to describe the mapping properties of Gabor multipliers when the upper symbol has some decay or growth rate. It is valid for arbitrary Weyl-Heisenberg systems (not necessarily Gabor frames).

Proposition 5.3.4. *[Gabor multipliers on modulation spaces,I]*
Define for $r \in \mathbb{R}$ the symbol $\mathbf{m}_r$ by $\mathbf{m}_r(\lambda) := (1 + |\omega|^2)^{r/2}$, if $\lambda = (x, \omega)$. Then for any pair of Schwartz atoms g_1, g_2 the Gabor multiplier $G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps $M^{s_1}_{p,q}(\mathbb{R}^d)$ to $M^{s_2}_{p,q}(\mathbb{R}^d)$, with $s_2 = s_1 + r$.

Proof: Since it is well known (cf. [13]) that Schwartz atoms induce both continuous coefficient as well as synthesis mappings between modulation

spaces and the corresponding (natural) solid sequence spaces over $\mathbf{Z}^{2d}$ the proof is a simple consequence of the mapping properties of a multiplication operator of the given kind at the sequence space level, i.e., on mixed-norm sequence spaces with polynomial weights. □

Remark 5.3.5. *A possible choice in the above theorem would be to start from a Schwartz atom g_1, such that $(g_1, a\mathbb{Z}^d \times b\mathbb{Z}^d)$ induces a Gabor frame, and to choose $g_2 = \tilde{g}$, the canonical dual Gabor atom. Indeed in this case one has automatically $g_2 = \tilde{g} \in \mathcal{S}(\mathbb{R}^d)$ as well. For the case of the standard Gaussian this situation occurs if (and only if) $ab < 1$.*

At a finer level summability properties of the Gabor multiplier $(m_\lambda)_{\lambda\in\Lambda}$ are sufficient to operate on the parameters (p, q). For convenience we assume again that the atoms are Schwartz, but membership in suitable spaces $M^s_{1,1}$ is indeed sufficient.

Proposition 5.3.6. *[Gabor multipliers on modulation spaces, II] Assume that g_1 and g_2 are in $\mathcal{S}(\mathbb{R}^d)$, and that $(m_\lambda)_{\lambda\in\Lambda} \in \ell^r(\Lambda)$. Then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps $M^s_{p_1,q_1}$ into $M^s_{p_2,q_2}$ with $p_1, q_1, p_2, q_2 \in [1, \infty]$ whenever $1/p_2 = 1/p_1 + 1/r \geq 0$ and $1/q_2 = 1/q_1 + 1/r \geq 0$, for any $s \in \mathbb{R}$.*

5.4 From Upper Symbol to Operator Ideal

In this section we summarize the mapping properties between the space of symbols and the membership of the resulting Gabor multiplier in one of the typical operator ideals within the bounded operators on the Hilbert space $L^2(\mathbb{R}^d)$. Again we fix a pair (g_1, g_2) in $S_0(\mathbb{R}^d)$, and the TF-lattice Λ.

Theorem 5.4.1. *Assume that g, g_1, g_2 are in $S_0(\mathbb{R}^d)$. Then one has:*
(i) If $\mathbf{m}$ is bounded, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a bounded operator on $L^2(\mathbb{R}^d)$.
(ii) If $\mathbf{m}$ is real-valued, then $G_{g,\Lambda,\mathbf{m}}$ is a self-adjoint operator on $L^2(\mathbb{R}^d)$.
(iii) If $\mathbf{m} \in c_o(\Lambda)$, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a compact operator on $L^2(\mathbb{R}^d)$.
(iv) If $\mathbf{m} \in \ell^2(\Lambda)$, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a Hilbert-Schmidt operator on $L^2(\mathbb{R}^d)$.
(vi) If $\mathbf{m} \in \ell^1(\Lambda)$, then $G_{g_1,g_2,\Lambda,\mathbf{m}}$ is a trace class operator on $L^2(\mathbb{R}^d)$.

Proof: Most of these statements follow from general facts about operator ideal properties of linear operators on $L^2(\mathbb{R}^d)$ with kernels in the Gelfand triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$. Obviously L^2-kernels correspond (exactly) to Hilbert-Schmidt operators. On the other hand the operators in $\mathcal{B}$, i.e., with S_0-kernels are absolutely convergent sums of rank 1 operators, and hence they are trace class. Since the sequences with finitely non-zero coefficients give finite rank operators their density in $c_o(\Lambda)$ implies (iii). Relation (ii) is easily verified directly and actually the main reason to make use of the symmetric case $g_1 = g = g_2$ for the discussion of eigenvalue behaviour or operators with real symbols. □

Remark 5.4.2. *The main statements of the above theorem can be summarized in the terminology of Gelfand triples by saying that for atoms $g_1, g_2 \in S_0(\mathbb{R}^d)$ the mapping $(m_\lambda)_{\lambda\in\Lambda} \mapsto G_{g_1,g_2,\Lambda,\mathbf{m}}$ maps the Gelfand triple of sequence spaces $\left(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\right)$ into the Gelfand triple of operator ideals, consisting of trace class operators, $\mathcal{HS}$ and the class of all bounded linear operators on $L^2(\mathbb{R}^d)$.*

Remark 5.4.3. *Obviously one can obtain by means of complex interpolation corresponding statements for Schatten S^p-classes from the above results, whenever the upper symbol is in $\ell^p(\Lambda)$, for $1 \leq p < \infty$.*

Remark 5.4.4. *Note that Thm. 5.3.2 iv) in conjunction with Thm. 5.4.1 ii) implies that the eigenvectors of $G_{g,\Lambda,\mathbf{m}}$ belong to $S_0(\mathbb{R}^d)$ if $(m_\lambda)_{\lambda\in\Lambda}$ is a real-valued sequence in $\ell^2(\Lambda)$.*

5.5 Eigenvalue Behavior of Gabor Multipliers

In this section we will assume that $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a tight Gabor frame, which is normalized in such a way that $f = \sum_{\lambda\in\Lambda} \langle f, \pi(\lambda)g\rangle\, \pi(\lambda)g$. [1] Our assumption implies that $G_{\mathbf{m}}$ is the identity operator for $\mathbf{m}_\lambda \equiv 1$. We are interested in the behaviour of the sequence of eigenvalues of Gabor multipliers based on such tight frames which are using atoms with a certain amount of TF-concentration. Their existence for sufficiently dense TF-lattices Λ can be taken as granted.

As a consequence of our normalization the spectrum of $G_{\mathbf{m}}$ is contained in the interval $[c, d]$ whenever the multiplier sequence $(m_\lambda)_{\lambda\in\Lambda}$ takes values in $[c, d]$. This normalization also allows one to calculate the L^2-norm of any $f \in L^2(\mathbb{R}^d)$ directly from Gabor coefficients (this would not be possible for non-tight atoms). Further comments on the relevance of tight Gabor systems are given in [73] and [58], where also iterative methods to determine tight Gabor atoms are described.

In this section we shall present results concerning the eigenvalue distribution of Gabor multiplier operators which resemble similar results for the continuous case that can be found in the literature. Recall that it happens often that discrete models involving a high degree of oversampling is to consider the analogous continuous situation and draw information from such a limiting case, either directly or by analogy. As a matter of fact, continuous models often allow a larger variety of methods and their treatment may be technically simpler. In many situations, it occurs that the results and methods of the continuous model may be transferred back to the discrete setting. This happens in particular in the theory of multipliers based on

[1] Recall that this normalization of the Gabor atom will *not* allow to assume at the same time that $\|g\|_2 = 1$, as we will be in the non-critical, hence redundant regime

redundant Gabor frames.

The results presented in this section are discrete analogs of the statements obtained in [16] and [38] for STFT-multipliers, continuous analogues of Gabor multipliers. Technically these proofs (which will appear elsewhere) are obtained by appropriate modifications of the proofs that have been given for the continuous case.

Time-frequency localization operators in the Gabor context (with 0/1 symbols) or even general Gabor multipliers share many features with other situations where a symbolic calculus of operators exists. It occurs that in a well-defined sense they are modeled on Toeplitz operators acting on the Fock space (see [10]). The principal class of symbols, which is crucial for applications, consists of characteristic functions of finite sets. Gabor multipliers with such symbols are called time-variant filters. Characteristic functions take only two values 0, 1 and they are idempotent with respect to pointwise multiplication. General principles of symbolic calculi assert that operators inherit algebraic features of their symbols. This is also true for Gabor multipliers. One of the important tasks is to understand the relationship between symbols and corresponding operators.

On the first level of accuracy Gabor multiplier with symbol χ_Ω, the characteristic function of a finite set Ω, resembles the orthogonal projection onto the linear space spanned by the functions $(\pi(\lambda)g)_{\lambda\in\Omega}$. The second level of accuracy is related to the boundary of Ω. The number of the eigenvalues which are contained inside the interval (δ_1, δ_2), where $0 < \delta_1 < \delta_2 < 1$, is comparable to the size of a fixed strip around the boundary of Ω.

Hankel operators and commutators of pointwise multiplications and projections onto the space of STFT transforms are the main tools in the STFT context. Now we discuss how to transfer them to the present context of tight Gabor frames. The mapping $W_g : L^2(\mathbb{R}^d) \to \ell^2(\Lambda)$ given by the formula

$$W_g f(\lambda) = (\langle f, \pi(\lambda)g\rangle)_{\lambda\in\Lambda}$$

is an isometry, and the operator $P_g : \ell^2(\Lambda) \to \ell^2(\Lambda)$ given as

$$P_g H(\lambda) = \sum_{\rho\in\Lambda} H(\rho)\langle \pi(\rho)g, \pi(\lambda)g\rangle$$

is the orthogonal projection onto its range, i.e., $W_g(L^2(\mathbb{R}^d))$. The operator $P_g M_{\mathbf{m}} P_g$, where $M_{\mathbf{m}}$ denotes the operator of pointwise multiplication, has the following matrix representation

$$\begin{bmatrix} W_g G_{\mathbf{m}} W_g^* & 0 \\ 0 & 0 \end{bmatrix}$$

with respect to the decomposition $\ell^2(\Lambda) = W_g(L^2(\mathbb{R}^d))) \oplus W_g(L^2(\mathbb{R}^d))^\perp$. The above representation shows that as far as non-zero eigenvalues are concerned we may exchange operators $G_{\mathbf{m}}$ and $P_g M_{\mathbf{m}} P_g$. We define *Hankel*

operator $H_{\mathbf{m}}$ with the symbol $\mathbf{m}$ by the formula $H_{\mathbf{m}} = (I - P_g)M_{\mathbf{m}}P_g$. Hankel operators are closely related to commutators $C_{\mathbf{m}} = M_{\mathbf{m}}P_g - P_gM_{\mathbf{m}}$ of operators of pointwise multiplication $M_{\mathbf{m}}$ and projections P_g. Indeed, one may express commutators $C_{\mathbf{m}}$ in terms of Hankel operators $H_{\mathbf{m}}$ and vice versa,

$$C_{\mathbf{m}} = H_{\mathbf{m}} - H_{\mathbf{m}}^*,$$
$$H_{\mathbf{m}} = C_{\mathbf{m}}P_g.$$

In many contexts, including ours, Hankel operators $H_{\mathbf{m}}$ and commutators $C_{\mathbf{m}}$ are interchangeable. Hankel operators measure the degree by which the mapping

$$m \to P_gM_{\mathbf{m}}P_g$$

fails to be a homomorphism. One may directly verify that

$$P_gM_{\mathbf{m_1}\mathbf{m_2}}P_g - P_gM_{\mathbf{m_1}}P_gM_{\mathbf{m_2}}P_g = H_{\overline{\mathbf{m_1}}}^*H_{\mathbf{m_2}}.$$

The above formula is the main algebraic ingredient of our proofs. It also stresses the role of Hankel operators in the study of functional calculus of Gabor multipliers.

Distribution of the eigenvalues is one of the fundamental objects describing Gabor multipliers. In the context of the STFT transform the eigenvalues were investigated by Daubechies and Heil-Ramanathan-Topiwala (see [11], [47] and the references given there). Our first result is a formula of *Szegö type*. It describes the asymptotic behavior of traces of functions of Gabor multipliers. The symbol function is subjected to dilations and the dilation factor tends to infinity. As a consequence we deduce first order asymptotics of the eigenvalue distribution (compare [38]).

Theorem 5.5.1. *Let us assume that a tight Gabor frame* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ *and a Riemann integrable function* $\mathbf{m}$ *with compact support defined on* $\mathbb{R}^{2d}$ *are given. Let us also assume that* $0 \leq \mathbf{m}(\eta) \leq 1$. *Then, for any continuous function* h *defined on the interval* $[0,1]$, *the following asymptotic formula holds:*

$$\lim_{R\to\infty} \frac{tr(G_{\mathbf{m}_R}h(G_{\mathbf{m}_R}))}{R^{2d}} = \frac{||g||^2}{V_\Lambda}\int_{\mathbb{R}^{2d}} \mathbf{m}(\eta)h(\mathbf{m}(\eta))d\eta, \tag{5.5.1}$$

where $\mathbf{m}_R(\eta) = \mathbf{m}(\eta/R)$ *and* V_Λ *is the volume of the fundamental domain corresponding to the lattice* Λ.

Let us denote the sequence of the eigenvalues of $G_{\mathbf{m}_R}$ by $\sigma_i(G_{\mathbf{m}_R})$. The following corollary shows that the asymptotic behavior of the eigenvalues is very closely related to the behavior of the symbol function.

Corollary 5.5.2. *Under the assumptions of Theorem 5.5.1 we have.*
(i) Let $0 < \delta < 1$. *If* $|\{\eta : \mathbf{m}(\eta) = \delta\}| = 0$, *then*

$$\lim_{R\to\infty} \frac{\#\{i : \sigma_i(G_{\mathbf{m}_R}) > \delta\}}{R^{2d}} = \frac{||g||^2}{V_\Lambda}|\{\eta : \mathbf{m}(\eta) > \delta\}|. \tag{5.5.2}$$

In particular for any bounded, Riemann measurable set Ω

$$\lim_{R\to\infty} \frac{\#\{i : \sigma_i(G_{\chi_{R\Omega}}) > \delta\}}{R^{2d}} = \frac{||g||^2}{V_\Lambda}|\Omega|. \tag{5.5.3}$$

(ii) If $0 < \delta_1 < \delta_2 < 1$ *and* $|\{\eta : \mathbf{m}(\eta) = \delta_i\}| = 0$, *for* $i = 1, 2$, *then*

$$\lim_{R\to\infty} \frac{\#\{i : \delta_1 < \sigma_i(G_{\mathbf{m}_R}) < \delta_2\}}{R^{2n}} = \frac{||g||^2}{V_\Lambda}|\{\eta : \delta_1 < \mathbf{m}(\eta) < \delta_2\}|. \tag{5.5.4}$$

Our second result is a refinement and a generalization of statement (ii) of Corollary 5.5.2 for time-variant filters G_{χ_Ω}. It deals with general families of localization domains Ω and it identifies the geometric features corresponding to the second order behavior of the eigenvalue distribution. We show that under suitable assumptions the size of the eigenvalue plunge region $\#\{i : \delta_1 < \sigma_i(G_{\chi_\Omega}) < \delta_2\}$ is comparable to the size of a fixed strip around the domain of localization. In what follows we shall write Ω^c for $\Lambda \setminus \Omega$.

- Let us assume that there is a positive number r such that the ball $B_r(0)$, with radius r and center 0, generates Λ and

  $$\langle \pi(\lambda)g, \pi(\lambda')g\rangle \neq 0, \tag{5.5.5}$$

  for all λ, λ' such that $|\lambda - \lambda'| \leq r$.

- Let us also assume that the reproducing kernel $V_g g$ has sufficient decay, namely

  $$|\langle \pi(\lambda)g, \pi(\lambda')g\rangle| = V_g(\lambda - \lambda') \leq c(1 + |\lambda - \lambda'|)^{-s} \tag{5.5.6}$$

  for some constant $c > 0$, for some $s > 2d + 1$.

- Let $\mathcal{G}$ be a family of finite subsets $\Omega \subset \Lambda$ satisfying

  $$|S^k_\Omega| \leq c|\partial^r \Omega|, \tag{5.5.7}$$

  where

  $$\partial^r \Omega = \{\lambda \in \Omega : B_r(\lambda) \cap \Omega^c \neq \emptyset\} \cup \{\lambda \in \Omega^c : B_r(\lambda) \cap \Omega \neq \emptyset\}$$

  and

  $$S^k_\Omega = \{\lambda \in \Omega : k \leq d(\lambda, \Omega^c) < k + 1\}.$$

Theorem 5.5.3. *Let us assume that a tight Gabor frame* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ *and a family* $\mathcal{G}$ *of finite subsets of* Λ *satisfying the above conditions are given. Then for any* $\delta_1 > 0$ *sufficiently close to* 0 *and* $\delta_2 < 1$ *sufficiently close to* 1 *there are positive constants* c_1, c_2 *such that for all* $\Omega \in \mathcal{G}$

$$c_1|\partial^r \Omega| \leq \#\{k : \delta_1 < \sigma_k(G_{\chi_\Omega}) < \delta_2\} \leq c_2|\partial^r \Omega|.$$

Remark 5.5.4. *(i) It is clear that conditions (5.5.5), (5.5.6) are satisfied for any Schwartz function and any sufficiently dense lattice Λ. Condition (5.5.6) holds whenever $g \in M^\infty_{v_s}$, in the terminology of modulation spaces (cf. [13], Chap. 12), where v_s is a radial symmetric weight over the TF-plane with growth order s.*

(ii) If D is a bounded domain with smooth boundary, then the family $\mathcal{G} = \{RD \cap \Lambda : R \geq 1\}$ satisfies (5.5.7).

Remark 5.5.5. *Numerical experiments have confirmed the above statements consistently. Indeed, even for relatively low redundancy one can find tight Gabor atoms g such that for most reasonable sets (which are only supposed to have no rough boundary) Ω the eigenvalue distribution of the corresponding TF-localization operator is characterized by a plateau, whose size corresponds the volume of a blurred version of Ω, while the size of the plunge region is determined by the length of the boundary.*

5.6 Changing the Ingredients

So far we have been concentrating on the properties of Gabor multipliers obtained from a fixed pair (g, Λ) generating a (tight) Gabor frame. It is an immediate consequence of the above considerations that small *changes* of the *multiplier sequence* (in the appropriate norm) will cause only small changes of the resulting Gabor multiplier (now measured in the corresponding space of operators, or a suitable operator ideal over $L^2(\mathbb{R}^d)$). The continuous dependence of properties of $G_{g_1,g_2,\Lambda,\mathbf{m}}$ on the two windows, g_1 used for analysis and g_2 for the synthesis operator are well established and thus we do not have to discuss them in detail here (cf. [32], section 3.3).

In contrast the influence of the *TF-lattice parameters* on the operator is much more delicate and interesting. For example, the change of lattice constants usually will effect the corresponding Gabor operator in a way which is not continuous anymore in any the standard operator norms, but only in the sense of the strong operator topology (cf. again [32], section 3.6 for a discussion for the case of small lattice constants, and the recent paper [37] for a detailed discussion of the general case).

There is one more difficulty. As we are now going to compare multiplier sequences which are well defined (and say square summable) over *different* TF-lattices the statements require the use of *continuous* multipliers defined over the full TF-plane, so that their samples are suitable multiplier sequences for whatever TF-lattice Λ is chosen. The appropriate description in full generality would require the use of so-called Wiener amalgam spaces $W(C, \ell^p)$, which consist of functions which are locally continuous but show a global ℓ^p-behaviour. In order to avoid additional terminology in this note we will use continuous S_0-multipliers or multipliers in the Sobolev space $H^s(\mathbb{R}^{2d})$, with $s > d$ (which consist of continuous functions, due to the

Sobolev embedding theorem). The results described in this way are only slightly weaker than optimal results formulated amalgam spaces.

A typical result of the desired type is the following one:

Theorem 5.6.1. *Let g_1, g_2 be atoms in $S_0(\mathbb{R}^d)$, and for some $s > d$ let $\mathbf{m} \in H^s(\mathbb{R}^{2d})$ be given, which is then both a continuous and square integrable function. Furthermore let (a_k, b_k) be a sequence of lattice constants satisfying $(a_k, b_k) \to (a_0, b_0)$ for $k \to \infty$, for some pair (a_0, b_0) of positive lattice constants. Write G_k for the Gabor multipliers, with windows g_1 and g_2, the TF-lattices $a_k\mathbb{Z}^d \times b_k\mathbb{Z}^d$, and corresponding multiplier sequences $\mathbf{m}_k = (\mathbf{m}(a_k n, b_k l))_{n,l\in\mathbb{Z}^d}$. Then the operators G_k converge to G_0 in the $\mathcal{HS}$-norm.*

We have to skip the proof of this statement here, as it is lengthy and will require the use of Kohn-Nirenberg symbols of the corresponding operators (cf. [34]). An alternative variant with stronger assumptions on the symbol, but also with a stronger conclusion, follows. Its proof (or rather an outline of arguments) can be well described in the present context.

Theorem 5.6.2. *Let $g_1, g_2 \in S_0(\mathbb{R}^d)$, $(a_k, b_k) \to (a_0, b_0)$ and G_k as in Thm. 5.6.1, but now with $\mathbf{m} \in S_0(\mathbb{R}^{2d})$, hence continuous and integrable over the TF-plane. Then $G_k \to G_0$ in the trace class operator norm over $L^2(\mathbb{R}^d)$.*

Proof: The first key argument of the proof is the fact that the sampling values of $\mathbf{m}$ are essentially concentrated over a bounded domain in the TF-plane and uniformly small in the $\ell^1(\Lambda)$-norm on its complement, for all the (convergent) lattices simultaneously. Therefore the problem can be reduced (up to some ε) to the case of finite sums within the index set $\mathbb{Z}^d \times \mathbb{Z}^d$. For those finite sums it is sufficient to make use of the fact that TF-shifts act (strongly) continuous on the atoms $g_i \in L^2(\mathbb{R}^d)$, for $i = 1, 2$, which in turn implies that the rank one operators $f \mapsto \langle f, \pi(a_k n, b_k l)g_1\rangle \pi(a_k n, b_k l)g_2$ corresponding to a fixed label $(n, l) \in \mathbb{Z}^{2d}$ get arbitrarily close to each other in the trace class norm. Since functions in $S_0(\mathbb{R}^{2d})$ have the property that their restrictions to discrete subgroups Λ are in $\ell^1(\Lambda)$, one only has to take care that the sampling mappings

$$f \to R_k(f) = (f(a_k n, b_k l))_{n,l\in\mathbb{Z}^d}$$

from $S_0(\mathbb{R}^{2d})$ into $\ell^1(\mathbb{Z}^d \times \mathbb{Z}^d)$ are strongly convergent to R_0 for $k \to \infty$. These technical details (in a more general form) are the content of Lemma 2.1. in [37]. □

Remark 5.6.3. *As the atoms in Thm. 5.6.2 are in $S_0(\mathbb{R}^d)$, one has essentially by the same argument as above also convergence in the $\mathcal{B}$-norm, resp. in the norm of $\mathcal{L}(S_0', S_0)$.*

Of course the norm in $\mathcal{L}(S_0', S_0)$ will grow if - for example - a uniformly bounded sequence of such multipliers tends (uniformly over compact sub-

sets of the TF-plane) to the constant **1**. Furthermore, such a family of operators is bounded in $\mathcal{L}(S_0', S_0')$. The corresponding operators then yield approximations to the identity operator on $S_0'(\mathbb{R}^d)$, by means of regularizing operators from $S_0'(\mathbb{R}^d)$ to $S_0(\mathbb{R}^d)$.

If we are starting from a tight atom the change of the lattice will usually destroy the tightness condition. Therefore it seems reasonable to repair this defect by replacing the synthesis atom by the dual atom, in order to compensate for this effect, at least for the case of constant multipliers. Recent numerical tests carried out by M. Dörfler indicate that it is indeed true that this modification improves the approximation quality, as long as the multiplier symbol consists of samples of a rather smooth function **m** on the TF-plane. A theoretical justification of this phenomenon has not been obtained so far.

Another aspect of this problem is the continuous dependence of dual atoms on the lattice parameters. Such a discussion does not make sense for arbitrary L^2 atoms g, as the property of generating a Gabor frame may depend in a chaotic way on the lattice constants in this general case (cf. [33]). However, mild additional conditions (like membership of g to $S_0(\mathbb{R}^d)$) are sufficient to ensure stability with respect to a change of lattice parameters, see recent results by Feichtinger/Kaiblinger (cf. [37]). For simplicity we consider again time-frequency lattices of the form $a\mathbf{Z}^d \times b\mathbf{Z}^d$. One of the main results of [37] (Thm. 3.9) then specializes to:

Theorem 5.6.4. *Assume that $g \in S_0(\mathbb{R}^d)$. Then the set*

$$GF(g) := \{(a,b) \mid a>0, b>0, (g,a,b) \text{ generates a Gabor frame}\}$$

is open in $\mathbb{R}^+ \times \mathbb{R}^+$. Furthermore, the dual atom depends continuously in the S_0-norm on the parameters (a,b) over GF(g).

Remark 5.6.5. *Writing $\widetilde{g}_{a,b}$ for the dual Gabor atoms in g with respect to the TF-lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$ the statement can be spelled out as follows: Given (a_0,b_0) such that (g,a_0,b_0) generates a Gabor frame. Then for $\varepsilon > 0$ given there exists $\delta > 0$ such that (g,a,b) generates a Gabor frame, with dual atom in S_0, for any pair (a,b) satisfying $|(a,b)-(a_0,b_0)| < \delta$, and*

$$\|\widetilde{g}_{a,b} - \widetilde{g}_{a_0,b_0}\|_{S_0} < \varepsilon.$$

We mention here that the above result is based on the important recent result by Gröchenig and Leinert ([45]), showing that the dual atom is automatically in $S_0(\mathbb{R}^d)$ whenever an $S_0(\mathbb{R}^d)$-atom g generates a Gabor frame (just with respect to the Hilbert space $L^2(\mathbb{R}^d)$). Under slightly stronger conditions on g (which apply, for example, to all Schwartz atoms), and for TF-lattice constants which are *sufficiently small* the result has been already known from [32].

Remark 5.6.6. *It can even be shown (cf. [37]) that the set of all triples (g,a,b) in $S_0(\mathbb{R}^d) \times \mathbb{R}^+ \times \mathbb{R}^+$ for which (g,a,b) generates a Gabor frame*

is open (with respect to the natural topology on those triples). Based on the Ron-Shen principle a corresponding statement holds for the triples of the form $(g, 1/b, 1/a)$. They form Riesz projection bases for the same open set (cf. [32], Prop. 3.5.13, or [37], Thm. 4.2).

With these results in the back of our minds we can prove the following:

Theorem 5.6.7. *Assume that $g \in S_0(\mathbb{R}^d)$ generates a Gabor frame with respect to some TF-lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$, let $\mathbf{m} \in S_0(\mathbb{R}^{2d})$ be a continuous and integrable function on the TF-plane, and let (a_k, b_k) be any sequence satisfying $(a_k, b_k) \to (a_0, b_0)$ for $k \to \infty$. Write G_k for the Gabor multiplier with fixed window $g = g_1$ and $g_2 = \tilde{g}_k$ adapted to the TF-lattice, and multiplier sequence $\mathbf{m}_k = (\mathbf{m}(a_k n, b_k l))_{n,l \in \mathbb{Z}^d}$. Then one has for $k \to \infty$:*

$$G_k \to G_0 \quad \textit{in the trace class norm over} \quad L^2(\mathbb{R}^d).$$

Proof: The result is essentially a combination of the two previous results. □

While at the qualitative level the result stated in the previous theorem is obtained by combining those for fixed windows with convergence results of canonical dual windows in practice those new operators appear to have better approximation properties, at least for smooth symbols. This is certainly plausible, because the choice of canonical dual pairs (or tight Gabor atoms) assures exact reproduction in the case of constant multipliers. Thinking of "good Gabor expansions" as a local (in the TF-sense) procedure we may well expect to have a very good approximation behaviour for symbols which behave locally like constants (i.e., for very smooth symbols).

5.7 From Gabor Multipliers to their Upper Symbol

Having seen that good (decay) properties of $(m_\lambda)_{\lambda \in \Lambda}$ imply corresponding good properties of the resulting Gabor multiplier we are interested to find out whether the converse is true as well. To ask this, however, brings us to a more fundamental question, which we see as the main question of this section: Under which conditions can one *reconstruct the multiplier sequence* $(m_\lambda)_{\lambda \in \Lambda}$ *from the operator* $G_{g,\Lambda,\mathbf{m}}$ (knowing of course which Gabor system (g, Λ) has been used). Again it is useful to consider the Hilbert case first, i.e., to ask under which condition the mapping from $\ell^2(\Lambda)$ to $\mathcal{HS}$: $(m_\lambda)_{\lambda \in \Lambda} \mapsto G_{g_1,g_2,\Lambda,\mathbf{m}}$ has a bounded inverse. Most of the results stated in this section are discussed in some detail in [34]. Our main result reads as follows [2]:

Theorem 5.7.1. *Assume that (g, Λ) generates an S_0-Gabor frame for $L^2(\mathbb{R}^d)$, with $\|g\|_2 = 1$, and write P_λ for the projection $f \mapsto \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$.*

[2] Note that in this context a different normalization of g is preferred

i) Then the family $(P_\lambda)_{\lambda\in\Lambda}$ *is a Riesz basis for its closed linear span within the Hilbert space* $\mathcal{HS}$ *of all Hilbert-Schmidt operators on* $L^2(\mathbb{R}^d)$ *if and only if the function* $H(s)$*, defined as the* Λ*-Fourier transform of* $\left(|STFT_g(g)(\lambda)|^2\right)_{\lambda\in\Lambda}$ *is does not have zeros.*
ii) An operator T *belongs to the closed linear span of this Riesz basis if and only if it belongs to* $\mathcal{GM}_2$*, the space of Gabor multiplier with* $\ell^2(\Lambda)$*-symbol.*
iii) The canonical biorthogonal family to $(P_\lambda)_{\lambda\in\Lambda}$ *is of the form* $(Q_\lambda)_{\lambda\in\Lambda}$*,*

$$Q_\lambda = \pi(\lambda)\circ Q\circ\pi^{-1}(\lambda) \text{ for } \lambda\in\Lambda,$$

for a uniquely determined Gabor multiplier $Q\in\mathcal{B}$*.*
iv) The best approximation of $T\in\mathcal{HS}$ *by Gabor multipliers based on the pair* (g,Λ) *is of the form*

$$PG(T) := \sum_{\lambda\in\Lambda} \langle T, Q_\lambda\rangle_{\mathcal{HS}}\ P_\lambda. \tag{5.7.1}$$

Hence PG *describes the orthogonal projection from* $\mathcal{HS}$ *onto* $\mathcal{GM}_2(g,\Lambda)$*.*

Corollary 5.7.2. *The restriction of the mapping* PG *defined above to the class* $\mathcal{B}$ *of all operators with kernels in* $S_0(\mathbb{R}^{2d})$ *is continuous on that class, while on the other hand* PG *is well defined and bounded on* $\mathcal{L}(S_0,S_0')$*. In particular,* $PG(T)$ *is a well defined operator from* $S_0(\mathbb{R}^d)$ *to* $S_0'(\mathbb{R}^d)$ *for every bounded linear operator* T *on* $L^2(\mathbb{R}^d)$*.*

Corollary 5.7.3. *A Gabor multiplier associated with the pair* (g,Λ)*, with* $g\in S_0(\mathbb{R}^d)$*, belongs to* $\mathcal{B}$ *if and only if it is of the form* $\sum_{\lambda\in\Lambda} c_\lambda P_\lambda$*, with an absolutely summable sequence* $(c_\lambda)_{\lambda\in\Lambda}\in\ell^1(\Lambda)$*.*

On the other hand any w^**-convergent sequence of Gabor multipliers which is bounded in the sense of* $\mathcal{L}(S_0,S_0')$ *converges to a Gabor multiplier, which is of the form* $\sum_{\lambda\in\Lambda} d_\lambda P_\lambda$*, for some bounded sequence* $(d_\lambda)_{\lambda\in\Lambda}\in\ell^\infty(\Lambda)$*.*

Proof: (Sketch of the argument for Thm. 5.7.1) The main statement can be derived (see [79] for the characterization of Riesz bases) from the fact that the entries of the Gram matrix for the system $(P_\lambda)_{\lambda\in\Lambda}$ are the Hilbert-Schmidt scalar products

$$\langle P_\lambda, P_{\lambda'}\rangle_{\mathcal{HS}} = |\langle\pi(\lambda)g,\pi(\lambda')g\rangle|^2 = |\langle g,\pi(\lambda'-\lambda)g\rangle|^2 = |STFT_g g(\lambda'-\lambda)|^2,$$

i.e., the Gram matrix is circulant with respect to the discrete Abelian group Λ. It is clear that this matrix is boundedly invertible if and only if the transfer function corresponding to the convolution kernel defined by $\left(|STFT_g(g)(\lambda)|^2\right)_{\lambda\in\Lambda}$ is bounded away from zero, as stated in the theorem. In the positive case there exists a biorthogonal Riesz basis$(Q_\lambda)_{\lambda\in\Lambda}$ within $\mathcal{HS}$, of a similar structure, i.e., whose elements are obtained from an operator Q_0 by conjugation with $\pi(\lambda)$, i.e., $Q_\lambda = \pi^{-1}(\lambda)\circ Q_0\circ\pi(\lambda)$, for all $\lambda\in\Lambda$. Indeed, if $g\in S_0(\mathbb{R}^d)$ one can derive from Wiener's inversion

theorem that $Q_0 = \sum_{\lambda\in\Lambda} d_\lambda P_\lambda$, for some $\ell^1(\Lambda)$-sequence $\mathbf{d} = (d_\lambda)$. Hence $Q_\lambda \in \mathcal{B}$ is a bounded family of (trace class) operators in $\mathcal{B}$. In [34] one can find details on this argument, also shedding some light on the connection to the case of spline-type spaces. Using this pair of biorthogonal Riesz bases within $\mathcal{HS}$ it is clear that the projection onto the closed linear span (within $\mathcal{HS}$) takes the form described by formula (5.7.1). □

Theorem 5.7.1 can also be used to characterize those Gabor multipliers which belong to the Banach spaces $\mathcal{B}, \mathcal{HS}$ or $\mathcal{L}(S_0, S_0')$, respectively. In order to give have a unified terminology we define for any $p \in [1, \infty]$ the space

$$\mathcal{GM}_p := \{\sum_{\lambda\in\Lambda} c_\lambda P_\lambda, \text{with} (c_\lambda)_{\lambda\in\Lambda} \in \ell^p(\Lambda)\}. \tag{5.7.2}$$

Corollary 5.7.4. *Assume that the situation described in Thm. 5.7.1 is given. Then the range of the linear projection PG within $\mathcal{B}$ is just $\mathcal{GM}_1$, while the range on all of $\mathcal{L}(S_0, S_0')$ is just $\mathcal{GM}_\infty$.*

Equivalently, the mapping on the Gelfand triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$, which maps the operator T to the upper symbol of its best approximating Gabor multiplier is a surjective Gelfand-triple mapping onto $\big(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\big)$.

Proof: We only have to observe that the fact that $(Q_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{B}$ implies that the mapping $T \to (\langle T, Q_\lambda\rangle)_{\lambda\in\Lambda}$ maps back from $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$ into $\big(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\big)$. That the mapping is surjective stems simply from the fact that the projection mapping PG coincides with the identity when applied to the space $\mathcal{GM}_p$, $p = 1, 2, \infty$. □

5.8 Best Approximation by Gabor Multipliers

The most interesting case described above is the following: For given $g \in S_0(\mathbb{R}^d)$ we are interested in TF-lattices Λ such that the family $(P_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span in $\mathcal{HS}$, with biorthogonal family $(Q_\lambda)_{\lambda\in\Lambda}$ in $\mathcal{B}$. Then it is not difficult to describe the best approximation of operators by Gabor multipliers as follows:

Theorem 5.8.1. *For a given Hilbert Schmidt operator $T \in \mathcal{HS}$ the best approximation by Gabor multipliers associated to the pair (g, Λ) is given by*

$$PGM_{g,\Lambda}(T) = \sum_{\lambda\in\Lambda} \langle T, Q_\lambda\rangle_{\mathcal{HS}} P_\lambda = \sum_{\lambda\in\Lambda} \langle T, P_\lambda\rangle_{\mathcal{HS}} Q_\lambda. \tag{5.8.1}$$

For fixed (g, Λ) the mapping $T \mapsto PGM_{g,\Lambda}(T)$ is the orthogonal projection from the Hilbert space $\mathcal{HS}$ onto the closed subspace of Gabor multipliers with $\ell^2(\Lambda)$-symbols.

Proof: The above result is simply a reformulation of the general fact that $(P_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis for the space of all Gabor multipliers, and that $(Q_\lambda)_{\lambda\in\Lambda}$ is its biorthogonal family. □

In the above equation (5.8.1) the role of the families $(P_\lambda)_{\lambda\in\Lambda}$ and $(Q_\lambda)_{\lambda\in\Lambda}$ may of course be interchanged. Since the scalar product of an operator T with respect to P_λ in the $\mathcal{HS}$-sense can be replaced by a simple scalar product in $L^2(\mathbb{R}^d)$, according to

$$\langle\, T, P_\lambda \,\rangle_{\mathcal{HS}} = \langle\, T\,(\pi(\lambda)g)\,, \pi(\lambda)g \,\rangle_{L^2}, \tag{5.8.2}$$

this alternative viewpoint has some interesting consequences. Note that the mapping $\lambda \mapsto \langle\, T\,(\pi(\lambda)g)\,, \pi(\lambda)g \,\rangle_{L^2}$, is known as the *lower symbol* of the operator T. If $g \in S_0(\mathbb{R}^d)$ it is well defined for any $T \in \mathcal{L}(S_0, S_0')$. Inserting this term into the projection formula shows that the best approximation of $T \in \mathcal{HS}$ by Gabor multipliers can be calculated already from the lower symbol of the operator:

Corollary 5.8.2.

$$PGM_{g,\Lambda}(T) = \sum_{\lambda\in\Lambda} \langle\, T[\pi(\lambda)g], \pi(\lambda)g \,\rangle\; Q_\lambda. \tag{5.8.3}$$

The fact that the kernels of the operators P_λ and $Q_\lambda, \lambda \in \Lambda$ are uniformly bounded in the S_0-sense implies that the mapping $T \mapsto PGM_{g,\Lambda}(T)$ is not only well defined in $\mathcal{HS}$, but on all of $\mathcal{L}(S_0, S_0')$, i.e., the space of all linear operators with kernels in $S_0'(\mathbb{R}^{2d})$. Their lower symbol is a sequence is only in $\ell^\infty(\Lambda)$, but the series 5.8.3 is still well defined (as a w^*-convergent series) in $\mathcal{L}(S_0, S_0')$. On the other hand its restriction to $\mathcal{B}$ yields Gabor multipliers with $\ell^1(\Lambda)$-coefficients, i.e., the projection mapping is also continuous in the more sensitive norm defined on $\mathcal{B}$ (the S_0-norm of their kernels).

We may summarize these findings in the following form:

Theorem 5.8.3. *For (g, Λ) as described above one has:*
i) The mapping from operators to lower symbols, i.e.,

$$T \mapsto\; (\langle T(\pi(\lambda)g), \pi(\lambda)g\rangle)_{\lambda\in\Lambda} \tag{5.8.4}$$

maps the Gelfand triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$ into $\big(\ell^1(\Lambda), \ell^2(\Lambda), \ell^\infty(\Lambda)\big)$.
ii) The mapping $T \mapsto PGM_{g,\Lambda}(T)$ is continuous on the Gelfand triple $(\mathcal{B}, \mathcal{HS}, \mathcal{B}')$.

Detailed proofs of these facts are given in [34], where the analogy to the situation of spline type spaces is described. The prototype of such spaces are (e.g. cubic) splines, for which the integer translates of a B-spline form a Riesz basis for their closed linear span. In this case a cubic spline function is in $L^2(\mathbb{R}^d)$ if and only it has a representation with ℓ^2 coefficients with respect to the standard spline basis. In this particular case, as well as more generally for the case that the generating atom belongs to $S_0(\mathbb{R}^d)$, more can be said with respect to L^p-norms: the corresponding projection mapping is also (uniformly) bounded on all the L^p-spaces, for $1 \le p \le \infty$, and maps general L^p-functions on spline-type functions with ℓ^p-coefficients,

which form a closed subspace of $L^p(\mathbb{R}^d)$. The situation described above for operators and their projections onto spaces of Gabor multipliers is thus completely parallel to the situation encountered for the case of spline spaces (for $p = 1, 2, \infty$).

5.9 STFT-multipliers and Gabor Multipliers

In this section we want to describe that under very natural conditions on the atom g and the multiplier function m the corresponding STFT-multiplier can be approximated (in various relatively strong operator norms) by the Gabor multipliers whose symbols are sampled version of the continuous multipliers. Indeed, in some sense they can be seen as Riemann-type approximations to the continuous STFT-multiplier.

Again we are trying to give the reader a good idea about the kind of results which are possible in this context, by providing a few typical examples in the context of Hilbert Schmidt operators, and not the most general and technically more involved consequences of the underlying principles.

First let us make sure that $STFT$-multipliers $SM_{\mathbf{m}}$ with (upper) symbol (and atom g) are well defined.

Definition 5.9.1. *For $g \in S_0(\mathbb{R}^d)$ the STFT-multiplier with symbol $\mathbf{m}$ is given in a formal way by the formula*

$$SM_{\mathbf{m}}(f) = \mathtt{ISTFT}(\mathbf{m} \cdot \mathtt{STFT}(f)), \tag{5.9.1}$$

or more practical (in the weak sense on $L^2(\mathbb{R}^d)$) by

$$SM_{\mathbf{m}}(f) = \int\int_{TF} m(x,\omega)\langle f, \pi(x,\omega)g\rangle \ \pi(\lambda) \ g \ dx \ d\omega \tag{5.9.2}$$

Proposition 5.9.2. *If $g \in S_0(\mathbb{R}^d)$ and $\mathbf{m} \in L^2(\mathbb{R}^{2d})$ then the resulting operator $SM_{\mathbf{m}}$ is a Hilbert Schmidt operator on the Hilbert space $L^2(\mathbb{R}^d)$. On the other hand, if the multiplier is in $L^1(\mathbb{R}^{2d})$ then the corresponding operator is a regularizing integral operator with S_0-kernel, i.e., $SM_{\mathbf{m}} \in \mathcal{B}$. In particular, it is trace class on $L^2(\mathbb{R}^d)$, and maps $S_0'(\mathbb{R}^d)$ into $S_0(\mathbb{R}^d)$.*

Remark 5.9.3. *In the above proposition it is even possible to replace the L^1-condition by some bounded measure over $\mathbb{R}^{2d}$ and one still obtains a true statement. In short, for $\mu \in M(\mathbb{R}^{2d})$ one can define SM_μ by replacing the measure $m(x,\omega)dxd\omega$ by the bounded measure μ in equation (5.9.2).*

As the full TF-plane can of course be seen as the limiting case of TF-lattices with lattice constants $(a, b) \to (0, 0)$ the following result should not come as a complete surprise. We spell out the L^1-version of this result as it appears to us to be the most interesting (and perhaps surprising) one. The analogue of Remark 5.6.5 in this context is then

Remark 5.9.4. *In the limiting case $(a,b) \to (0,0)$ the shape of the dual atoms $\widetilde{g}_{a,b}$ is known to approximate that of the original atom g. Indeed, suitable normalized versions converge to g in the S_0-norm:*

$$\|(ab)^{-d}\widetilde{g}_{a,b} - g\|_{S_0} \to 0 \text{ for } (a,b) \to (0,0).$$

This fact has been shown in [32],as Cor. 3.6.12. Before coming to the statement on the approximation of STFT-multipliers we have to recall a definition, which also plays a role in connection with the characterization of relatively compact subsets in Banach spaces of functions (cf. [15]).

Definition 5.9.5. *A bounded subset S of $M(\mathbb{R}^{2d})$ is called tight if for every $\varepsilon > 0$ there exists a compact subset $K \subseteq \mathbb{R}^{2d}$ such that $|\nu|(\mathbb{R}^{2d} \setminus K) < \varepsilon$ for all $\nu \in S$.*

Theorem 5.9.6. *Let $\mu_0 \in M(\mathbb{R}^{2d})$ be a bounded measure, and $(\mu_m)_{m\geq 1}$ be a bounded, tight and w^*-convergent sequence in $M(\mathbb{R}^{2d})$, with limit μ_0. For a fixed pair of S_0-atoms g_1 and g_2 we denote the corresponding STFT multiplier operators by SM_{μ_k} and SM_{μ_0}. Then $SM_k \to SM_0$, for $k \to \infty$, in the trace class norm over $L^2(\mathbb{R}^d)$.*

Proof: We only indicate the strategy of proof, which is based on two main arguments. First of all the problem is translated into a result about the convergence of ordinary convolution products, by applying the Kohn-Nirenberg transform $T \mapsto \sigma(T)$ (cf. [31]) to the operators under consideration. Then the convergence statement is a consequence of the fact that for any $h \in S_0(\mathbb{R}^{2d})$ one has $\|\mu_n * h - \mu_0 * h\|_{S_0} \to 0$ for $k \to \infty$ (cf. [23]) under the given circumstances, because $S_0(\mathbb{R}^{2d})$ is a so-called homogeneous Banach space, i.e., $||T_z f||_{S_0} = ||f||_{S_0}$ for all $z \in M(\mathbb{R}^{2d})$ and $||T_z f - f||_{S_0} \to 0$ for $z \to 0$, for all $f \in S_0$. □

Remark 5.9.7. *The proof actually implies that the norm convergence of these operators takes place in the class $\mathcal{B}$. Consequently we may claim convergence in $\mathcal{L}(S_0', S_0)$ as well.*

The next theorem can be seen as an immediate consequence of the general principle above. Again only a prototypical case is described.

Theorem 5.9.8. *Let $g \in S_0(\mathbb{R}^d)$, and $\mathbf{m} \in S_0(\mathbb{R}^{2d})$ by a (continuous and integrable) multiplier function be given. Furthermore let (a_k, b_k) be a sequence of lattice constants satisfying $(a_k, b_k) \to (0,0)$ for $k \to \infty$. Write $G^k_{\mathbf{m}}$ for the corresponding discrete Gabor multipliers, with fixed window g, TF-lattices $a_k\mathbf{Z}\times b_k\mathbf{Z}$, and multiplier sequence $\left((a_k b_k)^{-1}\mathbf{m}(a_k n, b_k l)\right)_{n,l\in\mathbf{Z}^d}$. Then $G^k_{\mathbf{m}}$ converges to $SM_{\mathbf{m}}$ in the trace class operator norm over $L^2(\mathbb{R}^d)$.*

Proof: It is not hard to verify that the bounded, discrete measures

$$\mu_k = \frac{1}{a_k b_k} \sum_{n,l} \mathbf{m}(na_k, lb_k)\delta_{(na_k, lb_k)}$$

form a bounded, tight and $w*$-convergent sequence of bounded measures over $\mathbb{R}^{2d}$, with limit μ (just as Riemannian sums approximate the integral), and consequently Theorem 5.9.6 applies. □

An L^2-$\mathcal{HS}$ variant of Thm. 5.9.6 is the following result, which makes use of the concept of tight (and bounded) subsets of $L^2(\mathbb{R}^{2d})$.

Definition 5.9.9. *A bounded subset $S \subseteq L^2(\mathbb{R}^d)$ is called* **tight** *whenever for every $\varepsilon > 0$ there exists some compact subset $K \subseteq \mathbb{R}^{2d}$ such that*

$$\|f.(1 - \mathbf{1}_K)\|_2 < \varepsilon \qquad \text{for any } f \in S.$$

Corollary 5.9.10. *Assume that $g \in S_0(\mathbb{R}^d)$ is given. Let $\mathbf{m}_k$ be a tight sequence in $L^2(\mathbb{R}^{2d})$, with weak limit $\mathbf{m}_0$. Then it is true that $SM_{\mathbf{m}_k} \to SM_{\mathbf{m}_0}$ for $k \to \infty$ in the $\mathcal{HS}$-norm over $L^2(\mathbb{R}^d)$.*

It appears to us that again the best way to verify this result is to translate it via the Kohn-Nirenberg transform into an equivalent question about the convolution between L^2 and S_0-functions. Details will be given elsewhere.

5.10 Compactness in Function Spaces

This short section is strongly inspired by the recent paper [15], characterizing the relatively compact subsets in coorbit spaces (such as the modulation spaces). We present a kind of discrete variant of their result, in the context of modulation spaces.

Theorem 5.10.1. *Let $M^s_{p,q}(\mathbb{R}^d)$ be a modulation space, with $1 \leq p, q < \infty$, and assume that $g \in \mathcal{S}(\mathbb{R}^d)$ generates a Gabor frame with respect to some TF-lattice $a\mathbb{Z}^d \times b\mathbb{Z}^d$.*

Let $\mathcal{FIN}$ denote the collection of all finite subsets $F \subseteq a\mathbb{Z}^d \times b\mathbb{Z}^d$, which is a directed set by the rule $F_1 \preceq F_2$ if $F_1 \subseteq F_2$. Write G_F for the corresponding net of partial sums of the canonical Gabor expansion, i.e.,

$$G_F(f) = \sum_{(n,l)\in F} \langle f, \pi(an, bl)\tilde{g} \rangle \ \pi(an, bl)\ g \tag{5.10.1}$$

or, more precisely,

$$G_F(f) = \sum_{(n,l)\in F} \langle f, M_{bl}T_{an}\tilde{g} \rangle \ M_{bl}T_{an}\ g.$$

Then $(G_F(f))_{\mathcal{FIN}}$ converges to f for each $f \in M^s_{p,q}(\mathbb{R}^d)$ (i.e., we have $G_F \to Id$ in the strong operator topology).

Moreover, a bounded subset $S \subseteq M^s_{p,q}(\mathbb{R}^d)$ is relatively compact if and only if $G_F(f) \to f$ uniformly on S.

Proof: The first part of the theorem is more or less a reformulation of the main result on atomic decompositions as developed in [26], applied to modulation spaces. We only had to restrict our attention to those coorbit spaces which are derived from solid BF-spaces (in this case weighted mixed norm spaces over the TF-plane) which contain the compactly supported functions as a dense subspace). We also use the fact that under the assumption $g \; in \mathcal{S}(\mathbb{R}^d)$ the canonical dual is also automatically in $\mathcal{S}(\mathbb{R}^d)$ (cf. [45]), hence both the analysis and the synthesis operator enjoy the natural boundedness conditions between the modulation spaces under consideration and the corresponding weighted mixed norm sequences spaces over $\mathbb{Z}^{2d}$.

The second part of the theorem then follows easily from two general observations. Since the operators G_F are uniformly bounded over arbitrary modulation spaces it is clear that they are uniformly convergent over relatively compact subsets $S \subseteq M^s_{p,q}(\mathbb{R}^d)$, once they are strongly convergent. On the other hand it is clear that each of these operators is a finite rank operator. Consequently one can argue that a set S on which the net G_F is uniformly convergent can be approximated arbitrary well by relatively compact subsets, hence it must be relatively compact itself. □

5.11 Gabor Multipliers and Time-Variant Filters

The possibility of introducing so-called *time-frequency localization operators* has been among the driving forces to study the continuous wavelet transform, but also in parallel to it the STFT (cf. [14, 11, 6] for prominent examples, or section 2.8 in [7]). For early occurrences of discrete variants of Gabor multipliers (usually with 0/1-symbols, describing a certain region of interest) we refer to [21, 22, 61, 74], and in particular Chap. 9 on time-variant filtering in the book of Qian-Chen ([68]). Filtering the time-frequency content of a signals is indeed one of the main applications of Gabor multipliers.

Generally speaking, there are at least three traditional types of time-frequency filters:
(i) compositions of timepass and bandpass filters and their generalizations (see [62], [78], [7] and the references therein)
(ii) restrictions of the reproducing formulas based on coherent state expansions (see [11], [7], [72], [69, 71], [47] and the references therein)
(iii) Weyl pseudo-differential operators with symbols with compact support (see [39],[60], [59], [HRT97]) .
Other useful approaches are however also possible (see for example [52]). All three types of filters are related to specific operator calculi. It occurs that good understanding of operator calculus leads to precise knowledge of eigenvalue behavior of the corresponding filter. This is the essence of our

method behind the estimates presented in this section. At this occasion we would like to mention that a new interesting approach to the calculus of Gabor multipliers based on Gaussian windows was taken recently by Coburn in [7], [8].

When it comes to digital signal processing actual data to be handled are vectors of finite length (or mathematically equivalent) discrete *and* periodic measures. The same is true for digital image processing. The need to perform time-variant filters is of course coupled with the wish to carry out such operations with minimal computational costs. Therefore also in this setting there is a need to understand the approximation of linear mappings, given in whatever form or representation, be it the matrix kernel or the Kohn-Nirenberg symbol of the operator.

Let us mention here that in practice these operations have to be applied to signals of finite length. Hence we are dealing essentially with problems in linear algebra. Although in this finite-dimensional setting certain critical limiting cases which make the continuous situation often very delicate cannot occur, one is faced with numerical issues of speed and computational complexity. We only mention here that such questions are treated in [36], where a matrix version of the approximation results described in section 5.8 are given.

Let us conclude this section with an alternative view on Gabor multipliers from a *filter bank point of view*. The equivalence between Gabor frame expansions and certain filter bank systems with perfect reconstruction is well established (cf. [3, 4, 5, 10, 9]). Given this connection it is easy to see that the very definition of Gabor multipliers implies that they have a very nice and simple interpretation (hence also a *cheap* implementation) in this context. Indeed, in a filter bank description the Gabor coefficient mapping turns into a splitting of the information contained in the signal into channels (each of which has to carry only a fraction of the information contained in the original signal). The *perfect reconstruction property* of such a system ensures that by using another (appropriate) synthesis window the original input signal is exactly recovered, if *no manipulation* takes place within the channels.

Applying now simple multiplications on these coefficients (before applying the synthesis mapping) is just a reinterpretation of what a Gabor multiplier should be. Altogether this means of course that Gabor multipliers are very important operators for the purpose of (time-variant) digital filtering, and their efficient use will hopefully benefit from the theoretical studies given in the present note.

References

[1] F. Berezin. Wick and anti-Wick operator symbols. *Math. USSR*, Sb. 15, pages 577–606, 1971.

[2] P. Boggiatto. Multi-quasi-elliptic operators and anti-Wick symbols. *Ann.Univ.Ferrara*, Nouva Ser., Sez. VII 41, Suppl., pages 139–149, 1996.

[3] H. Bölcskei, F. Hlawatsch, and H.G. Feichtinger. Equivalence of DFT filter banks and Gabor expansions. In *SPIE Proc., Wavelet Applications in Signal and Image Processing III*, volume 2569, pages 128–139, San Diego (CA), July 1995.

[4] H. Bölcskei, F. Hlawatsch, and H.G. Feichtinger. Oversampled FIR and IIR DFT filter banks and Weyl–Heisenberg frames. In *Proc. IEEE ICASSP-96*, volume 3, pages 1391–1394, Atlanta(GA), 1996.

[5] H. Bölcskei, and F. Hlawatsch. Oversampled modulated filter banks. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 295–322. Birkhäuser, Boston, 1998. Chap. 9.

[6] O. Christensen and C. Heil. Perturbations of Banach frames and atomic decompositions. *Math. Nach.*, 185:33–47, 1997.

[7] L.A. Coburn, The Measure Algebra of the Heisenberg Group, *Journal of Functional Analysis* 161, 509–525, (1999).

[8] L.A. Coburn, On the Berezin-Toeplitz calculus, *ESI-preprint* 834, (2000).

[9] Z. Cvetkovic. Oversampled modulated filter banks and tight Gabor frames in $\ell^2(\mathbb{Z})$. In *Proc. ICASSP-95*, 1456–1459, Detroit (MI), 1995.

[10] Z. Cvetković and M. Vetterli. Oversampled filter banks. *IEEE Trans. Signal Processing*, 46(5):1245–1255, May 1998.

[11] I. Daubechies. Time-frequency localization operators: a geometric phase space approach. *IEEE Trans. Inform. Theory*, 34/4, July 1988.

[12] I. Daubechies. The wavelet transform, time-frequency localization and signal analysis. *IEEE Trans. Info. Theory*, 36:961–1005, 1990.

[13] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Reg. Conf. Series in Applied Math. SIAM, 1992.

[14] I. Daubechies and T. Paul. Time–frequency localization operators: A geometric phase space approach II. The use of dilations and translations. *Inverse Problems*, 4:661–680, 1988.

[15] M. Dörfler, H.G. Feichtinger, and K. Gröchenig: Compactness criteria in function spaces, preprint 2001.

[16] F.DeMari, H.G. Feichtinger and K. Nowak. Uniform eigenvalue estimates for time-frequency localization operators. *to appear in the Journal of London Mathematical Society.*

[17] J. Du and M.W. Wong. A product formula for localization operators. *Bull. Korean Math. Soc.*, 37(1):77–84, 2000.

[18] J. Du and M.W. Wong. Gaussian functions and Daubechies operators. *Integral Equations Oper. Theory*, 38(1):1–8, 2000.

[19] J. Du and M.W. Wong. Gaussian series and Daubechies operators. *Appl. Anal.*, 76(1-2):83–91, 2000.

[20] J. Du and M.W. Wong. Traces of localization operators. *C. R. Math. Acad. Sci., Soc. R. Can.*, 22(2):92–96, 2000.

[21] S. Farkash and S. Raz. Time variant filtering via the Gabor representation. In *Signal Processing V*, pages 509–512, 1990.

[22] S. Farkash and S. Raz. Linear systems in Gabor time–frequency space. *Trans. IEEE Sign.Proc.*, 42:611–617, 1994.

[23] H. G. Feichtinger. Compactness in translation invariant Banach spaces of distributions and compact multipliers. *J.Math.Anal.Appl.*, 102:289–327, 1984.

[24] H. G. Feichtinger and K. Gröchenig. A unified approach to atomic characterizations via integrable group representations. *Lect.Notes in Math.*, 1302:52–73, 1988.

[25] H.G. Feichtinger. Atomic characterizations of modulation spaces through Gabor-type representations. In *Proc. Conf. Constructive Function Theory*, *Rocky Mountain J. Math.* 19:113–126, 1989.

[26] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J.Funct.Anal.*, 86:307–340, 1989.

[27] H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, II. *Monatsh. f. Math.*, 108:129–148, 1989.

[28] H. G. Feichtinger, K. Gröchenig, and D. Walnut. Wilson bases and modulation spaces. *Math. Nachr.*, 155:7–17, 1992.

[29] H.G. Feichtinger. An elementary approach to the generalized Fourier transform. In T. Rassias, editor, *Topics in Mathematical Analysis*, pages 246–272. World Sci.Publ., 1988. volume in the honour of Cauchy, 200^{th} anniv.

[30] H. G. Feichtinger and T. Strohmer, editors. *Gabor Analysis and Algorithms: Theory and Applications.* Birkhäuser, Boston, 1998.

[31] H. G. Feichtinger and W. Kozek. Quantization of TF–Lattice invariant operators on elementary LCA groups. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pg. 233–266. Birkhäuser, Boston, 1998. Chap. 7.

[32] H. G. Feichtinger and G. Zimmermann. A space of test functions for Gabor analysis. In H.G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, pages 123–170. Birkhäuser, Boston, 1998. Chap. 3.

[33] H.G. Feichtinger and A.J.E.M. Janssen. Validity of WH-frame bound conditions depends on lattice parameters. *Appl. Comput. Harm. Anal.*, 8(1):104–112, 2000.

[34] H. G. Feichtinger. Gabor multipliers and spline-type spaces over LCA groups, In "Wavelet Analysis: Twenty Years' Developments", (D. X. Zhou, ed.) World Scientific Press, Singapore, 2002, to appear.

[35] H. G. Feichtinger and M. Dörfler. Eigenvalue analysis of time-frequency localization operators, in preparation, 2002.

[36] H. G. Feichtinger and M. Hampejs. Best approximation of matrices by Gabor multipliers, preprint, 2002.

[37] H. G. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of Gabor frames. 2002, submitted.

[38] H. G. Feichtinger and K. Nowak. A Szegö-type theorem for Gabor-Toeplitz localization operators. *Michigan Math. J.*,49:13–21,2001.

[39] P. Flandrin, *Maximal energy concentration in a time-frequency domain*, ICASSP 88, 2176–2179, (1988)

[40] G.B. Folland. *Harmonic Analysis in Phase Space.* Annals of Math. Studies. Princeton Univ. Press, Princeton (NJ), 1989.

[41] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Func. Anal.*, 93:34–170, 1990.

[42] K. Gröchenig. An uncertainty principle related to the Poisson summation formula. Studia Math. 121(1):87–104, 1996.

[43] K. Gröchenig. Personal communication.

[44] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Birkhäuser, Boston, 2001.

[45] K. Gröchenig and M. Leinert: Wiener's Lemma for twisted convolution and Gabor frames, 2001, submitted.

[46] Z. He and M.W. Wong. Localization operators associated to square integrable group representations. *Panam. Math. J.*, 6(1):93–104, 1996.

[47] C. Heil, J. Ramanathan, and P. Topiwala : Asymptotic singular value decay of time-frequency localization operators. In *Mathematical Imaging: Wavelet Applications in Signal and Image Processing II*, Bellingham, WA, to appear. SPIE Proceedings Series Volume 2303.

[48] C. Heil, J. Ramanathan, and P. Topiwala. Singular values of compact pseudodifferential operators. J. Funct. Anal. 150 (1997), 426-452.

[49] C. Heil, J. Ramanathan, and P. Topiwala. Linear independence of time-frequency translates. *Proc. Amer. Math. Soc.*, 124:2787–2795, 1996.

[50] F. Hlawatsch and W. Kozek. Time–frequency weighting and displacement effects in linear time–varying systems. In *Proc. IEEE ISCAS-92*, pages 1455–1458, 1992.

[51] F. Hlawatsch and W. Kozek. Time–frequency projection filters and TF signal expansions. *IEEE Trans. Signal Proc.*, 42(12): 3321–3334, 1994.

[52] F. Hlawatsch, W. Krattenthaler, and W. Kozek. Time–frequency subspaces and their applications to time–varying filtering. In *Proc. IEEE ICASSP–90*, pages 1607–1610, Albuquerque (NM), Apr. 1990.

[53] F. Hlawatsch, G. Matz, H. Kirchauer, and W. Kozek. Time-frequency formulation, design, and implementation of time-varying optimal filters for signal estimation. *IEEE Trans. Signal Processing*, vol. 48, pp. 1417–1432, May 2000.

[54] F. Hlawatsch and G. Matz, "Linear time-frequency filters," in *Time-Frequency Signal Analysis and Processing* (B. Boashash, ed.), Englewood Cliffs (NJ): Prentice Hall, 2002.

[55] R.B. Holmes. Mathematical foundations of signal processing. *SIAM Rev.*, 21(3):361–388, 1979.

[56] A.J.E.M. Janssen. Duality and biorthogonality for Weyl-Heisenberg frames. *J. Four. Anal. Appl.*, 1(4):403–436, 1995.

[57] A.J.E.M. Janssen. From continuous to discrete Weyl-Heisenberg frames through sampling. *J. Fourier Anal. Appl.*, 3(5):583–596, 1997.

[58] A. J. E. M. Janssen and T. Strohmer Characterization and computation of canonical tight windows for Gabor frames. *J. Fourier Anal. Appl.* , 8:1, 1–28, 2002.

[59] W. Kozek. On the generalized weyl correspondence and its application to time–frequency analysis of linear systems. In *IEEE Int. Symp. on Time–Frequency and Time–Scale Analysis*, pg. 167–170, Victoria, Canada, Oct. 1992.

[60] W. Kozek. Time–frequency signal processing based on the wigner–weyl–framework. *Signal Processing*, **29**, 77–92, 1992.

[61] M.L. Kramer. Improved time–frequency filtering using an STFT analysis–modification–synthesis method. In *Proc. IEEE-SP Internat. Symp. on Time-Frequency and Time-Scale Analysis*, pages 264–267, Philadelphia/PA, October 1994.

[62] H.J. Landau, H. Widom, *Eigenvalue distribution of time and frequency limiting*, J. Math. Anal. Appl. **77**, 469–481, (1980).

[63] G.F. Margrave and M.P. Lamoureux. Gabor Deconvolutoin: The Annual Research Report of the Consortium for Research in Elastic Wave Exploration Seismology, CREWES Res. Report, Volume 13 (2001).

[64] G. Matz and F. Hlawatsch. Time-frequency transfer function calculus (symbolic calculus) of linear time-varying systems (linear operators) based on a generalized underspread theory. *J. Math. Phys., Special Issue on Wavelet and Time-Frequency Analysis*, vol. 39, pp. 4041–4071, Aug. 1998.

[65] G. Matz and F. Hlawatsch, Linear Time-Frequency Filters: Online Algorithms and Applications. In: Applications in Time-Frequency Signal Processing, A. Papandreou-Suppappola, Ed., Boca Raton (FL): CRC Press, 2002.

[66] M. Niedzwiecki. Identification of time–varying systems usings combined parameter estimation and filtering. *IEEE Trans. Acoust. Speech Sig. Proc.*, 38:679–686, 1990.

[67] A. Pietsch. Eigenvalues of Integral Operators II, Math. Ann. 262 (1983), 343-376.

[68] S. Qian and D. Chen. *Joint Time-Frequency Analysis: Method and Application*. Prentice Hall, Englewood Cliffs, NJ, 1996.

[69] J. Ramanathan and P. Topiwala. Time-frequency localization operators of Cohen's class. In J. Byrnes et. al., editor, *Wavelets and their applications (Il Ciocco, 1992)*, pages 313–324. Kluwer Acad. Publ., Dordrecht, 1994.

[70] J. Ramanathan and P. Topiwala. Time-frequency localization via the Weyl correspondence. *SIAM J. Math. Anal.*, 24/5:1378–1393, 1993.

[71] J. Ramanathan and P. Topiwala. Time-frequency localization and the spectrogram. *Appl. Comp. Harm. Anal.*, 1(2):209–215, 1994.

[72] R. Rochberg, A correspondence principle for Toeplitz and Calderón-Toeplitz operators, Israel Math. Conf. Proc. 5, 229–243, (1992).

[73] T. Strohmer. Approximation of dual Gabor frames, window decay, and wireless communications. *Appl. Comput. Harm. Anal.* , 11(2):243-262,2001.

[74] A. Teuner and B.J. Hosticka. Adaptive filter for two-dimensional Gabor transformation and its implementation. *IEE Proceedings-1*, 140/1:2–6, 1993.

[75] R. Vio and W. Wamsteker. Joint Time-Frequency Analysis: a tool for exploratory signal analysis and filtering of non-stationary time series, Astronomy and Astrophysics, 2002, submitted.

[76] D.F. Walnut. Continuity properties of the Gabor frame operator. *J. Math. Anal. Appl.*, 165(2):479–504, 1992.

[77] M. Wang and Z. Bao. Modified short–time Fourier transform. *Opt. Engrg.*, 34(5):1333–1337, 1995.

[78] H. Widom *Asymptotic expansions for pseudodifferential operators on bounded domains*, Lecture Notes in Math., 1152, Springer-Verlag, Berlin, 1985

[79] R. Young. *An Introduction to Non-harmonic Fourier*. Series. Acad. Press, New York, 1980.