JNM: Journal of Natural Sciences and Mathematics
Vol. 1, pp. 15-21 (December 2007/Thu Al-Hijjah 1428$_{\mathrm{H}}$)

Two Basic Inequalities for Shape Operator of Hypersurfaces

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Abstract: Two basic inequalities for shape operator of Riemannian hypersurfaces are established.

Keywords: Ricci curvature, k-Ricci curvature, scalar curvature, shape operator.

2000 Mathematics Subject Classification: 53C40.

Received for JNM on May 6, 2007.

1 Introduction

In [1] B.-Y. Chen recalled that one of the basic interests of submanifold theory is to establish simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold. Many famous results in differential geometry can be regarded as results in this respect. The main extrinsic invariant is the squared mean curvature and the main intrinsic invariants include the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature. In the literature, we find several work done in establishing basic inequalities involving the squared mean curvature and one of the classical curvature invariants namely the scalar curvature, the sectional curvature and the Ricci curvature for different kind of submanifolds of real space forms and complex space forms. First results in these directions were given by B.-Y. Chen in [1], [2] and [3]. Motivated by a result of B.-Y. Chen [3], a basic inequality, involving the Ricci curvature and the squared mean curvature of the submanifold of any Riemannian manifolds, was proved recently [5]. The goal was achieved by use of the concept of k-Ricci curvature [3]. Then some basic inequalities were obtained [4] for a submanifold of any Riemannian manifold involving the squared mean curvature and one of the intrinsic invariants namely the scalar curvature and the sectional curvature of the submanifold.

In this paper we apply these results to find two basic inequalities for shape operator of Riemannian hypersurfaces. In section 2, we recall the definitions of Ricci curvature, k-Ricci curvature and scalar curvature. Then we list required known results for later use. In section 3, we give main results.

2 Some known results

Let M be an n-dimensional Riemannian manifold equipped with a Riemannian metric g. Let Π_k be a k-plane section of T_pM and X a unit vector in Π_k. We choose an orthonormal basis $\{e_1, \ldots, e_k\}$ of Π_k such that $e_1 = X$. The Ricci curvature Ric_{Π_k} of Π_k at X is defined by [3]

$$\mathrm{Ric}_{\Pi_k}(X) = K_{12} + K_{13} + \cdots + K_{1k}, \tag{2.1}$$

where K_{ij} is the *sectional curvature* of the plane section spanned by e_i and e_j at $p \in M$. $\mathrm{Ric}_{\Pi_k}(X)$ is called a *k-Ricci curvature*. The scalar curvature $\tau\left(\Pi_k\right)$ of the k-plane section Π_k is given by [3]

$$\tau\left(\Pi_k\right) = \sum_{1 \le i < j \le k} K_{ij}, \tag{2.2}$$

where $\{e_1, \ldots, e_k\}$ is any orthonormal basis of the k-plane section Π_k. Geometrically, $\tau(\Pi_k)$ is the scalar curvature of the image $\exp_p(\Pi_k)$ of Π_k at p under the exponential map at p. The *Ricci curvature* of X now becomes $\mathrm{Ric}\,(X) = \mathrm{Ric}_{T_pM}(X)$. The scalar curvature $\tau(p)$ of M at p is identical with the scalar curvature of the tangent space T_pM of M at p, that is, $\tau\,(p) = \tau\,(T_pM)$. If Π is a plane section, $\tau(\Pi)$ is simply the sectional curvature $K(\Pi)$ of Π.

Now, let M be an n-dimensional Riemannian submanifold of an m-dimensional Riemannian manifold $(\tilde{M}, g)$. The induced metric on M will also be denoted by g. The equation of Gauss is given by

$$R(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) + g\left(\sigma(X,W), \sigma(Y,Z)\right) - g\left(\sigma(X,Z), \sigma(Y,W)\right) \tag{2.3}$$

for all $X, Y, Z, W \in TM$, where $\tilde{R}$ and R are the Riemann curvature tensors of $\tilde{M}$ and M respectively; and σ is the second fundamental form. The mean curvature vector $H(p)$ is given by $nH(p) = \mathrm{Tr}\,(\sigma)$. The submanifold M is *totally geodesic* in $\tilde{M}$ if $\sigma = 0$, and *minimal* if $H = 0$. If $\sigma\,(X,Y) = g\,(X,Y)\,H$ for all $X, Y \in TM$, then M is *totally umbilical*. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM and e_r belongs to an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of the normal space $T_p^{\perp}M$. We put

$$\sigma_{ij}^r = \left\langle \sigma\left(e_i, e_j\right), e_r \right\rangle \quad \text{and} \quad \left\|\sigma\right\|^2 = \sum_{i,j=1}^{n} \left\langle \sigma\left(e_i, e_j\right), \sigma\left(e_i, e_j\right) \right\rangle .$$

Let K_{ij} and $\tilde{K}_{ij}$ denote the sectional curvature of the plane section spanned by e_i and e_j at p in the submanifold M and in the ambient manifold $\tilde{M}$ respectively. In view of the equation of Gauss (2.3), we have

$$K_{ij} = \tilde{K}_{ij} + \sum_{r=n+1}^{m} \left(\sigma_{ii}^{r} \sigma_{jj}^{r} - (\sigma_{ij}^{r})^2 \right). \tag{2.4}$$

From (2.4) it follows that

$$2\tau(p) = 2\tilde{\tau}(T_pM) + n^2 \|H\|^2 - \|\sigma\|^2, \tag{2.5}$$

where $\tilde{\tau}(T_pM) = \sum_{1 \leq i < j \leq n} \tilde{K}_{ij}$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\tilde{M}$.

We recall the following three results, in which basic inequalities involving intrinsic invariants namely the Ricci curvature, the scalar curvature and the sectional curvature and the extrinsic invariant namely the squared mean curvature of any submanifold of a Riemannian manifold are established.

Theorem 2.1 ([5, Theorem 3.1]) *Let M be an n-dimensional submanifold of a Riemannian manifold. For a unit vector $X \in T_pM$, it follows that*

$$\frac{1}{4} n^2 \|H\|^2 \geq \mathrm{Ric}(X) - \widetilde{\mathrm{Ric}}_{(T_pM)}(X), \tag{2.6}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}(X)$ is the n-Ricci curvature of T_pM at X with respect to the ambient manifold $\tilde{M}$. The equality case of (2.6) is satisfied by the unit vector $X \in T_pM$ if and only if

$$\sigma(X, X) = \frac{n}{2} H(p) \quad \text{and} \quad \sigma(X, Y) = 0 \tag{2.7}$$

for all $Y \in T_pM$ such that $g(X, Y) = 0$. The equality case of (2.6) holds for all unit vectors $X \in T_pM$ and for all $p \in M$ if and only if either M is totally geodesic or M is a totally umbilical surface.

Proof. Here we give a very simple proof, which is taken from [4]. We put

$$2\sigma'(X, Y) = \sigma(X, Y) - \frac{n}{2} g(X, Y) H, \qquad X, Y \in T_pM.$$

Then for a unit vector $X \in T_pM$, we obtain

$$\begin{aligned} 0 &\leq \sum_{i=1}^{n} g(\sigma'(X, e_i), \sigma'(X, e_i)) \\ &= \sum_{i=1}^{n} g\left(\sigma(X, e_i), \sigma(X, e_i)\right) - ng\left(H, \sigma(X, X)\right) + \frac{n^2}{4} \|H\|^2. \end{aligned}$$

According to the Gauss equation (2.3) and the above inequality, we can easily get our theorem. ■

Theorem 2.2 *([4, Theorem 4.2]) For an n-dimensional submanifold M in a Riemannian manifold, at each point $p \in M$, we have*

$$\frac{n(n-1)}{2}\,\|H\|^2 \geq \tau(p) - \tilde{\tau}(T_pM), \tag{2.8}$$

where $\tilde{\tau}(T_pM) = \sum_{1\leq i<j\leq n} \tilde{K}_{ij}$ denotes the scalar curvature of the n-plane section T_pM in the ambient manifold $\tilde{M}$. The equality holds at each point $p \in M$ if and only if M is totally umbilical.

Theorem 2.3 *[4, Theorem 5.2] Let M be an n-dimensional $(n \geq 3)$ submanifold of an m-dimensional Riemannian manifold $\tilde{M}$. Then, for each point $p \in M$ and each plane section $\Pi \subset T_pM$, we have*

$$\tau - K(\Pi) \leq \frac{n^2(n-2)}{2(n-1)}\,\|H\|^2 + \tilde{\tau}(T_pM) - \tilde{K}(\Pi). \tag{2.9}$$

*The equality in (2.9) holds at $p \in M$ if and only if there exist an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM and an orthonormal basis $\{e_{n+1}, \ldots, e_m\}$ of $T_p^{\perp}M$ such that **(a)** $\Pi = \mathrm{Span}\{e_1, e_2\}$ and **(b)** the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+1, \ldots, m$, become*

$$A_{n+1} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)\,I_{n-2} \end{pmatrix}, \tag{2.10}$$

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-2} \end{pmatrix}, \qquad r \in \{n+2, \ldots, m\}. \tag{2.11}$$

3 Main results

Let M be a orientable hypersurface in an m-dimensional Riemannian manifold $\tilde{M}$. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal field of M and A the shape operator of M with respect to N. Then Gauss and Weingarten formulae are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \text{and} \qquad \tilde{\nabla}_X N = -AX.$$

The equation of Gauss is given by

$$R(X,Y)Z = \tilde{R}(X,Y)Z + g(AY,Z)AX - g(AX,Z)AY. \tag{3.1}$$

We shall need the following:

Lemma 3.1 *Let M be hypersurface in an m-dimensional Riemannian manifold $\tilde{M}$. Let Π be a plane section in T_pM and X be a unit vector in T_pM. Then*

$$K(\Pi) = \tilde{K}(\Pi) + \det(A|_{\Pi}), \tag{3.2}$$

$$\mathrm{Ric}(X) = \widetilde{\mathrm{Ric}}_{(T_pM)}(X) + \mathrm{Tr}(A)\, g(AX, X) - g\left(A^2X, X\right), \tag{3.3}$$

$$2\tau(p) = 2\tilde{\tau}(T_pM) + (\mathrm{Tr}(A))^2 - \mathrm{Tr}\left(A^2\right), \tag{3.4}$$

where $\widetilde{\mathrm{Ric}}_{(T_pM)}(X)$ is the $(m-1)$-Ricci curvature of T_pM at X with respect to the ambient manifold $\tilde{M}$ and $\tilde{\tau}(T_pM)$ be the scalar curvature of the $(m-1)$-plane section T_pM in the ambient manifold $\tilde{M}$.

Proof. Let M be a hypersurface in an m-dimensional Riemannian manifold $\tilde{M}$. Now, let $\{e_1, \ldots, e_{m-1}\}$ be an orthonormal basis of the tangent space T_pM. Then from (3.1) we get

$$K_{ij} = \tilde{K}_{ij} + g(Ae_i, e_i)g(Ae_j, e_j) - g(Ae_i, e_j)g(Ae_j, e_i), \tag{3.5}$$

where K_{ij} and $\tilde{K}_{ij}$ are the sectional curvature of the plane section spanned by e_i and e_j at p in the hypersurface M and in the ambient manifold $\tilde{M}$ respectively. Then (3.2) is implied by (3.5). From (3.5) for a fixed $i \in \{1, \ldots, m-1\}$ we have

$$\begin{aligned}
\sum_{j\neq i}^{m-1} K_{ij} &= \sum_{j\neq i}^{m-1} \tilde{K}_{ij} + g(Ae_i, e_i)\sum_{j\neq i}^{m-1} g(Ae_j, e_j) - \sum_{j\neq i}^{m-1} g(Ae_i, e_j)g(Ae_i, e_j) \\
&= \sum_{j\neq i}^{m-1} \tilde{K}_{ij} + g(Ae_i, e_i)\left(\sum_{j=1}^{m-1} g(Ae_j, e_j) - g(Ae_i, e_i)\right) \\
&\quad - \left(\sum_{j=1}^{m-1} g(Ae_i, e_j)g(Ae_i, e_j) - g(Ae_i, e_i)g(Ae_i, e_i)\right) \\
&= \sum_{j\neq i}^{m-1} \tilde{K}_{ij} + g(Ae_i, e_i)\sum_{j=1}^{m-1} g(Ae_j, e_j) - \sum_{j=1}^{m-1} g(Ae_i, e_j)g(Ae_i, e_j),
\end{aligned}$$

which gives

$$\mathrm{Ric}(e_i) = \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) + \mathrm{Tr}(A)\, g(Ae_i, e_i) - g(Ae_i, Ae_i). \tag{3.6}$$

From (3.6) for a unit vector $X \in T_pM$, we immediately get (3.3). Also from (3.6) we get

$$\sum_{i=1}^{m-1} \mathrm{Ric}(e_i) = \sum_{i=1}^{m-1} \widetilde{\mathrm{Ric}}_{(T_pM)}(e_i) + \mathrm{Tr}(A)\sum_{i=1}^{m-1} g(Ae_i, e_i) - \sum_{i=1}^{m-1} g\left(A^2e_i, e_i\right),$$

which gives (3.4). ■

Now, we prove two basic inequalities for shape operator of a hypersurface in the following two Theorems:

Theorem 3.2 *Let M be a hypersurface of an m-dimensional Riemannian manifold. Then for each unit vector $X \in T_pM$, it follows that*

$$\frac{1}{4}\left(\operatorname{Tr}(A)\right)^2 \geq \operatorname{Tr}(A)\, g(AX,X) - g\left(A^2X,X\right). \tag{3.7}$$

The equality case of (3.7) is true for all unit vectors in T_pM, $p \in M$ if and only if either M is totally geodesic or M is a totally umbilical surface.

Proof. From $(m-1)\,H = \sum_{i=1}^{m-1} g(Ae_i,e_i)\,N$, we get

$$(m-1)^2 \|H\|^2 \;=\; \left(\operatorname{Tr}(A)\right)^2. \tag{3.8}$$

Now, using (3.3) and (3.8) in (2.6), we find the inequality (3.7). Rest of the proof is straightforward. ■

Theorem 3.3 *Let M be a hypersurface of an m-dimensional $(m > 3)$ Riemannian manifold. Then, for each point $p \in M$ and each plane section $\Pi \subset T_pM$, we have*

$$\frac{1}{2(m-2)}\left(\operatorname{Tr}(A)\right)^2 \leq \frac{1}{2}\operatorname{Tr}\left(A^2\right) + \det(A|_\Pi). \tag{3.9}$$

The equality in (3.9) holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1,\ldots,e_{m-1}\}$ of T_pM such that $\Pi = \operatorname{Span}\{e_1,e_2\}$ and the shape operator A becomes of the form

$$A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (a+b)\, I_{m-3} \end{pmatrix}. \tag{3.10}$$

Proof. If M is a hypersurface of an m-dimensional $(m > 3)$ Riemannian manifold, then from (3.2) and (3.4) we get

$$\tau(p) - K(\Pi) = \tilde{\tau}(T_pM) - \tilde{K}(\Pi) - \det(A|_\Pi) + \frac{1}{2}\left(\operatorname{Tr}(A)\right)^2 - \frac{1}{2}\operatorname{Tr}\left(A^2\right).$$

Using the Schwarz inequality $(m-1)\operatorname{Tr.} A^2 \geq (\operatorname{Tr.} A)^2$ and the above equation in (2.9) we get (3.9). ■

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