Efficient Computation of Equilibrium Prices for Markets with Leontief Utilities

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Abstract. We present a polynomial time algorithm for the computation of the market equilibrium in a version of Fisher's model, where the traders have Leontief utility functions. These functions describe a market characterized by strict complementarity. Our algorithm follows from a representation of the equilibrium problem as a concave maximization problem, which is of independent interest. We also show how to apply this representation to a more general market setting, where the traders have utility functions from a wide family which includes CES utilities.

1 Introduction

Back in 1891, Fisher [2, 17] introduced a market model given by a set of buyers and a set of divisible goods. Buyers have specified incomes, goods are available in given amounts, and the preferences of each buyer are expressed in terms of a concave utility function. The equilibrium problem consists of finding prices (of goods) and allocations (of goods to buyers) which *clear* the market and allow each buyer to maximize her utility function.

Devanur et al. [5] introduced a polynomial time algorithm for the *linear version* of Fisher's model, i.e., for the special case where the buyers have linear utility functions. Their approach is based on a primal-dual scheme, and boils down to a number of max-flow computations. A polynomial time algorithm for the linear case of Fisher's model was already implicit from the work of Gale ([10], pp. 281-287). Gale showed that the allocation which leads to equilibrium prices can be obtained by maximizing a concave function subject to linear constraints. The solution to this program can be approximated in polynomial time by using the ellipsoid algorithm. As pointed out in [5], since the equilibrium prices (in the linear case) are rational, the ellipsoid method indeed returns the exact solution.

The above results for the linear case are relevant because they are a valuable starting point for our understanding of computational and structural properties of equilibrium problems. However linear utility functions are realistic only for very particular markets or for the analysis of small price variations. In real world markets, utilities are typically concave functions.

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In this paper, we consider the equilibrium problem for a more realistic version of Fisher's model, where traders have utility functions which are known as *Leontief utility functions*, or *fixed proportions utility functions*. These functions have the form $u(z_1, \ldots, z_n) = \min\{\frac{z_1}{b_1}, \ldots, \frac{z_n}{b_n}\}$, where $b_j > 0$, and $z = (z_1, \ldots, z_n)$ represent a bundle (or basket) of goods. These utilities express strict complementarity in the preferences. Indeed a buyer with this utility wants to get a basket of goods proportional to $(b_1, \ldots, b_n)$ (see, e.g., [18], p. 1009).

Leontief utilities are an important special case of a rather general and widely used family of utility functions, known as *constant elasticity of substitution* (see next section for definitions and properties.)

Our result builds upon the construction of a constrained nonlinear maximization problem, where the constraints are linear and express the feasibility of the allocations. The function to be maximized is simply the product, over all buyers, of the individual utility of each buyer raised to her income. Using a duality-type argument, we prove that from the solution to this maximization problem we can derive equilibrium allocations, from which we can in turn compute equilibrium prices by Linear Programming.

We also show that, unlike in the linear case, the equilibrium prices need not be rational. Therefore there is no choice but to settle for an approximation.

From the above properties, we immediately derive a polynomial time approximation scheme for computing market clearing prices:

1. we first use the ellipsoid method to find the values of the individual utilities which maximize the product of the utilities;
2. from such values, we then compute the optimal allocations, by matrix by vector multiplication;
3. we finally use Linear Programming to compute the equilibrium prices by finding a nonnegative solution to a system of linear equations which relates equilibrium prices to optimal allocations.

This paper shows that there is a realistic market setting for which the equilibrium problem can be solved in polynomial time; it also shed some further light on the structure of the market equilibrium problem. Indeed, as it happens for Gale's construction for the linear case, our solution is based on the existence of a function which aggregates the preferences of the buyers in a simple way, and which leads, at equilibrium prices, to the optimization of the individual preferences, thus essentially translating an equilibrium problem into an optimization problem. More precisely, our work shows that the optimal allocations are given by the values which maximize the product of the buyers' utilities raised to their respective incomes, subject to constraints dictated by the data which describe the market problem.

We also show that our approach extends to a more general scenario when the utility functions can be chosen from a fairly general class, which includes constant elasticity of substitution utility functions. We sketch the ideas of this extension in this paper, and report the details in a joint paper with Jain and Vazirani [3].

Polynomial time algorithms for market equilibrium problems are known only in a few other instances. There is a polynomial time algorithm for markets with Cobb-Douglas utilities, which are concave functions maximized when the buyers spend a fixed fraction of their income on each of the goods (see next section for precise definitions). The algorithm has been introduced by Curtis Eaves [8], and is based on simple linear algebra considerations which make it possible to translate the equilibrium problem into that of finding the nonnegative solution to a particular linear system. Eaves shows how to compute such solution in a direct way, avoiding Linear Programming, and achieving a cubic time bound.

The rest of this paper is organized as follows.

In Section 2 we provide background on both existential and computational results concerning market equilibria.

In Section 3 we present our main results. We show that equilibrium prices for the Leontief setting can be computed from allocations which solve a convex optimization problem. We also show that the equilibrium prices can be irrational, thus pointing out the need of shooting for approximate solutions.

In Section 4 we sketch some more general results, which show that the approach of Section 3 extends to a fairly general class of utility functions, which include CES functions.

2 Background and some history

We now concisely describe the market model. Let us consider m economic agents which represent producers and/or consumers of n goods. Each agent has a utility function $u : \mathbf{R}^n \to \mathbf{R}$, which represents her preferences for the different baskets of goods, and an initial, typically suboptimal, endowment of goods $w \in \mathbf{R}^n$. At given prices $\pi \in \mathbf{R}^n$, each agent will sell her endowment, and get the basket of goods $z \in \mathbf{R}^n$ which maximizes u subject to her budget constraint $\pi^T z \leq \pi^T w$.

Arrow-Debreu celebrated Theorem [1] states that, under some quite mild assumptions on the utility functions, there is a price vector $\hat{\pi}$ such that the solution to the above maximization problem by each agent leads to an allocation $z(\hat{\pi})$ which clears the market. These prices are called *equilibrium prices*.

The proof of Arrow-Debreu Theorem follows from Kakutani's Fixpoint theorem, which is a generalization of Brouwer's Fixpoint theorem.

The above described market model is usually called *Arrow-Debreu model*. If we restrict the model by getting rid of the production component, we obtain a market where all the agents are traders which want to exchange their initial endowments in order to get a basket of goods which maximizes their utilities. This setting is called the *exchange model*. If we assume that the goods are initially available in the market, and that agents go to the market with fixed amounts of money (their income), then we get *Fisher's model*, where all the agents are buyers.

The market equilibrium problem, as well as the related problem of finding a Nash equilibrium in the mixed strategies for a two person nonzero sum game, has been analyzed from a computational viewpoint by Papadimitriou in [15]. Papadimitriou explores the nature of these problems, as revealed by their intimate connection with fixpoint theorems, pointing out that they are characterized by *inefficient proofs of existence*. Indeed the fixpoint theorems provide existential proofs which can be used a basis for (inefficient) computations. The actual computational complexity of this family of problems turns out to be wide open and of great relevance in computational complexity (see [16], Sections 2 and 5).

Many attempts have been made to find efficient algorithms for the solution of the market equilibrium problem, using the interplay with the computation of fixed points. In particular, it is worth mentioning the work of Scarf and some coauthors [9, 11, 17, 18]. For example, in [17] Scarf suggested an algorithm for the approximation of fixed points of continuous mappings of a simplex into itself. In [13], Kuhn showed the connection between Scarf's result and Sperner's lemma and proposed a technique for the subdivision of the simplex, which yields a simple algorithm for the simplicial approximation of fixed points.

Unfortunately, none of these results lead to algorithms with polynomial running time.

As already mentioned in the Introduction, efficient algorithms have been obtained for Fisher's model, when the utilities are linear [5]. The result in [5] has been extended in several directions. For instance, it has been used as the main ingredient in approximation schemes for the exchange model with linear utilities [12, 7]. It has also inspired the definition of a new model, the *spending constraint model* [6], to which the technique used in [5] can still be applied.

Another instance where price equilibria can be found efficiently arises when the utilities are Cobb-Douglas functions. A Cobb-Douglas utility function is a function of the form $u(\mathbf{z}) = \prod_{j=1}^{n} z_j^{\alpha_j}$ where $\sum \alpha_j = 1$. In this case, Eaves has shown a nice and simple cubic time algorithm which works for the exchange model [8]. It is not difficult to show that a trader with a Cobb-Douglas utility spends a fixed fraction of her income on each good. In what follows we will assume that there are m traders and n goods. Then the market can be described in terms of two $n \times m$ matrices, A and W, whose entries are the utility exponents and the initial endowments, respectively. More precisely, the i-th column of the matrix A (W, resp.) contains the vector of utilities (initial endowments, resp.) of player i. Using the special properties of Cobb-Douglas utility functions, Eaves has shown that the equilibrium prices can be obtained from the solution of a linear system associated with the matrix $E = WA^T$, which leads to an algorithm consisting of one matrix multiplication and one application of Gaussian elimination.

Leontief, Cobb-Douglas, and linear utilities are special cases of constant elasticity of substitution (CES, for short) utility functions. A CES function has the form

$$u(z_1, \ldots, z_n) = \left(\sum_{j=1}^{n} \alpha_j^{\frac{1}{\sigma}} z_j^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

where σ is the constant representing the given *elasticity of substitution*.

Leontief utilities are obtained in the limit as σ tends to zero, with zero elasticity of substitution, i.e., strict complementarity, while Cobb-Douglas utilities (obtained as σ tends to one) correspond to unitary elasticity of substitution. Conversely, note that the case of linear utilities (obtained as σ tends to infinity) represents a situation where goods are *perfect substitutes*. (For more precise definitions and properties of the most popular utility functions, see [19], Chapters 1 and 7.)

We conclude this section by mentioning a related stream of work, which has been devoted to analyzing the equilibrium in markets with indivisible commodities [4]. In such case, an equilibrium is not guaranteed to exist (due to indivisibility), and thus approximation algorithms have to be brought into the problem, which becomes that of finding market clearing prices for which the agents' utilities are *close* to their maximum value. In particular, in [4] the notion of *deficiency* (the smallest decrease in the agents' utilities for which the market still clears) has been isolated. Unfortunately, it has been proven that it is **NP**-hard to approximate the deficiency, even in the case of linear utilities. Efficient algorithms have instead been obtained under suitable restrictions on the number of players and/or of goods.

3 Main Results

3.1 Preliminaries

We consider a market with a set $B = \{1, \ldots, m\}$ of buyers and a set $G = \{1, \ldots, n\}$ of goods. For each $1 \le i \le m$, let the real number $e_i > 0$ denote the initial endowment (or money) of buyer i. Associated with buyer i, there is also a vector $a_i = (a_{1i}, \ldots, a_{ni})$, with $a_{ji} > 0$, that describes her utility function. We will use the variable $x_{ji} \ge 0$ to denote the amount of the j'th good in buyer i's basket, and the vector $x_i = (x_{1i}, \ldots, x_{ni}) \in \mathbb{R}^n$ to denote a basket of buyer i. The utility function $u_i(x_i)$ of buyer i is given by[1]

$$u_i(x_i) = \min_{1 \le j \le n} \frac{x_{ji}}{a_{ji}}.$$

Let $q_j > 0$ denote the amount of good j in the market, for $1 \le j \le n$.

Given a market with a set B of buyers, with a vector a_i and endowment e_i for buyer i, and a set G of goods, with an amount q_j for each good j, an *equilibrium* is given by a *price vector* $\bar{\pi} = (\bar{\pi}_1, \ldots, \bar{\pi}_n) \in \mathbb{R}^n$, where $\bar{\pi}_j \ge 0$ is called the *price* of good j, and a basket $\bar{x}_i = (\bar{x}_{1i}, \ldots, \bar{x}_{ni}) \in \mathbb{R}^n$ for each buyer i, where each $\bar{x}_{ji} \ge 0$, satisfying the following conditions:

1. For each buyer i, the basket $\bar{x}_i$ maximizes her utility given the price vector $\bar{\pi}$ and her endowment e_i. That is, $\bar{x}_i$ is the vector that maximizes $u_i(x_i)$ subject to the constraint that $\bar{\pi}^T x_i \le e_i$. Note that in our case this is equivalent to the requirement that $u_i(\bar{x}_i) = e_i / \left(\bar{\pi}^T a_i\right)$ and $\bar{\pi}^T \bar{x}_i = e_i$.

[1] This definition implies that buyer i has some interest in each good. Our approach readily generalizes to the scenario where there is a subset $G_i \subseteq G$ of goods and $u_i(x_i) = \min_{j \in G_i} x_{ji}/a_{ji}$.

2. Each good j is cleared, that is, $\sum_{i=1}^{m} \bar{x}_{ji} = q_j$ for each $1 \le j \le n$.

3.2 Computing the Equilibrium

Let M be a market with a set $B = \{1, \dots, m\}$ of buyers and a set $G = \{1, \dots, n\}$ of goods, with a vector $a_i \in \mathbb{R}^n$ and endowment $e_i > 0$ for each buyer i, and a set $G = \{1, \dots, n\}$ of goods, with an amount $q_j > 0$ for each good j. In this section we give a new proof of the existence of an equilibrium for such a market. This proof immediately implies an efficient algorithm for computing an equilibrium.

Let A be the $n \times m$ matrix whose entry in the j'th row and i'th column is a_{ji}. That is, the i'th column of A is the vector a_i. Let $q \in \mathbb{R}^n$ be the vector $(q_1, \dots, q_n)$. Let $\beta = (\beta_1, \dots, \beta_m)$ stand for a variable in $\mathbb{R}^m$. Consider the following optimization problem, which we call CP:

$$\begin{aligned} \text{maximize } & (\beta_1^{e_1} * \beta_2^{e_2} * \cdots * \beta_m^{e_m})^{1/\sum_i e_i} \\ \text{Subject to } & \sum_{1 \le i \le m} a_{ji}\beta_i \le q_j, \text{ for } 1 \le j \le n \\ & \beta_i \ge 0, \text{ for } 1 \le i \le m. \end{aligned}$$

Since each $a_{ji} > 0$, the set of feasible solutions for this problem is bounded, hence compact, and so the continuous objective function attains its maximum. The objective function is concave, and so CP is a convex optimization problem. CP can be concisely stated as:

$$\begin{aligned} \text{maximize } & (\beta_1^{e_1} * \beta_2^{e_2} * \cdots * \beta_m^{e_m})^{1/\sum_i e_i} \\ \text{Subject to } & A\beta \le q \\ & \beta_i \ge 0, \text{ for } 1 \le i \le m. \end{aligned}$$

Let $\hat{\beta} = (\hat{\beta}_1, \dots, \hat{\beta}_m)$ be an optimal solution to CP. We must have that $(\hat{\beta}_1^{e_1} * \hat{\beta}_2^{e_2} * \cdots * \hat{\beta}_m^{e_m})^{1/\sum_i e_i} > 0$. This is because for a sufficiently small $\delta > 0$, the point $(\delta, \dots, \delta) \in \mathbb{R}^m$ is feasible for CP (since each $q_j > 0$) and the value of the objective function at this point is $\delta > 0$.

There is one constraint in CP corresponding to each good j, namely the constraint $\sum_{1 \le i \le m} a_{ji}\beta_i \le q_j$. Let G^t be the subset of goods for which the corresponding constraint is tight at $\hat{\beta}$. That is, $G^t = \{j | \sum_{1 \le i \le m} a_{ji}\hat{\beta}_i = q_j\}$. Let G^l denote the remaining goods. G^t is non-empty because otherwise for a sufficiently small $\delta > 0$ the solution $(1+\delta)\hat{\beta}$ is feasible for CP and has an objective function value of

$$(1+\delta)(\hat{\beta}_1^{e_1} * \hat{\beta}_2^{e_2} * \cdots * \hat{\beta}_m^{e_m})^{1/\sum_i e_i} > (\hat{\beta}_1^{e_1} * \hat{\beta}_2^{e_2} * \cdots * \hat{\beta}_m^{e_m})^{1/\sum_i e_i}.$$

Let D^* denote the $m \times m$ diagonal matrix, with diagonal entries $(\hat{\beta}_1, \dots, \hat{\beta}_m)$. Then AD^* is the matrix whose (j, i)-th entry is $\hat{\beta}_i a_{ji}$. That is, the i'th column of AD^* is the vector $\hat{\beta}_i a_i$. We claim that the vector $(e_1, \ldots, e_m) \in \mathbb{R}^m$ is in the cone of the vectors $\{(\hat{\beta}_1 a_{j1}, \ldots, \hat{\beta}_m a_{jm}) |\ j \in G^t\}$ (the row vectors of AD^* that correspond to goods in G^t).

Claim 1. *For each good $j \in G^t$, there exists $\bar{\pi}_j \geq 0$ such that for each $1 \leq i \leq m$, we have $\sum_{j \in G^t} \bar{\pi}_j \hat{\beta}_i a_{ji} = e_i$.*

Proof. If the claim is false, then by Farkas Lemma [14] there is a vector $(t_1, \ldots, t_m) \in \mathbb{R}^m$ such that $e_1 t_1 + \cdots + e_m t_m > 0$ and for each $j \in G^t$, $\sum_{1 \leq i \leq m} t_i \hat{\beta}_i a_{ji} \leq 0$. We will argue that, for a sufficiently small $\delta > 0$, the vector $\bar{\beta} = (\hat{\beta}_1(1 + \delta t_1), \ldots, \hat{\beta}_m(1 + \delta t_m))$ is a feasible solution for CP and the value of the objective function at $\bar{\beta}$ is larger than at $\hat{\beta}$. This contradicts the fact that $\hat{\beta}$ is an optimal solution for CP.

For each $j \in G^t$, we have

$$\sum_{1 \leq i \leq m} \bar{\beta}_i a_{ji} = \sum_{1 \leq i \leq m} \hat{\beta}_i a_{ji} + \delta \sum_{1 \leq i \leq m} t_i \hat{\beta}_i a_{ji} \leq q_j + 0 = q_j,$$

so $\bar{\beta}$ satisfies the constraint in CP corresponding to j. For each $j \in G^l$, we have

$$\sum_{1 \leq i \leq m} \bar{\beta}_i a_{ji} = \sum_{1 \leq i \leq m} \hat{\beta}_i a_{ji} + \delta \sum_{1 \leq i \leq m} t_i \hat{\beta}_i a_{ji} < q_j + \delta \sum_{1 \leq i \leq m} t_i \hat{\beta}_i a_{ji},$$

so for a sufficiently small δ the point $\bar{\beta}$ satisfies the constraint in CP corresponding to j. Finally, we must have $\hat{\beta}_i > 0$ for otherwise the value of the objective function at $\hat{\beta}$ is 0. This implies that, for a sufficiently small δ, $\bar{\beta}_i = \hat{\beta}_i(1 + \delta t_i) > 0$ as well. So the point $\bar{\beta}$ is a feasible solution for CP. Now, using the Taylor expansion,

$$\begin{aligned}
&(\bar{\beta}_1^{e_1} * \cdots * \bar{\beta}_m^{e_m}) - (\hat{\beta}_1^{e_1} * \cdots * \hat{\beta}_m^{e_m}) \\
&= (\hat{\beta}_1^{e_1} * \cdots * \hat{\beta}_m^{e_m})((1 + \delta t_1)^{e_1} * \cdots * (1 + \delta t_m)^{e_m} - 1) \\
&= (\hat{\beta}_1^{e_1} * \cdots * \hat{\beta}_m^{e_m})(\delta(e_1 t_1 + \cdots + e_m t_m) + \delta^2(\cdots) + O(\delta^3)) \\
&> 0\,,
\end{aligned}$$

for sufficiently small δ, since $e_1 t_1 + \cdots + e_m t_m > 0$ and $\hat{\beta}_1^{e_1} * \cdots * \hat{\beta}_m^{e_m} > 0$. But this means that the value of the objective function at $\bar{\beta}$ is greater than that at $\hat{\beta}$. This finishes the proof of the claim.[2] □

Theorem 2. *Let β^* be an optimal solution to CP, and assume that there are values $\bar{\pi}_j$ for each $j \in G^t$ as in Claim 1. Let $\bar{\pi}_j = 0$ for each $j \in G^l$. Set $\bar{x}_{ji} = \hat{\beta}_i a_{ji}$, for each $j \in G^t$ and each $1 \leq i \leq m$. Also set $\bar{x}_{ji} = \hat{\beta}_i a_{ji}$, for each $j \in G^l$ and each $2 \leq i \leq m$. Set*

$$\bar{x}_{j1} = q_j - \sum_{2 \leq i \leq m} \hat{\beta}_i a_{ji} = q_j - \sum_{2 \leq i \leq m} \bar{x}_{ji},$$

[2] The proof can also be established using the Kuhn-Tucker stationary-point necessary optimality theorem ([14], page 105).

for each $j \in G^l$. Then the vector $\bar{\pi} \in \mathbb{R}^n$ and the vectors $\bar{x}_i = (\bar{x}_{1i}, \ldots, \bar{x}_{ni})$, for $1 \le i \le m$, are an equilibrium for the market M.

Proof. Note that $\bar{x}_{j1} \ge \hat{\beta}_1 a_{j1}$ because $\hat{\beta}$ is a feasible solution for CP. Each component of $\bar{\pi}$ is nonnegative by construction. We have $\bar{x}_{ji} \ge \hat{\beta}_i a_{ji} \ge 0$ for any $1 \le i \le m$ and $1 \le j \le n$, since each $\hat{\beta}_i \ge 0$ and each $a_{ji} > 0$. We now establish that the two conditions for equilibrium hold.

1. Since $\bar{x}_{ji} \ge \hat{\beta}_i a_{ji}$, we have

$$u_i(\bar{x}_i) = \min_{1 \le i \le n} \frac{\bar{x}_{ji}}{a_{ji}} \ge \hat{\beta}_i,$$

for each good i. In fact, $u_i(\bar{x}_i) = \hat{\beta}_i$ since $\bar{x}_{ji} = \hat{\beta}_i a_{ji}$ for each good $j \in G^t$. Now the price of the basket $\bar{x}_i$ is

$$\sum_{j \in G^t} \bar{\pi}_j \bar{x}_{ji} + \sum_{j \in G^l} \bar{\pi}_j \bar{x}_{ji} = \sum_{j \in G^t} \bar{\pi}_j \hat{\beta}_i a_{ji} + 0 = e_i.$$

We have, for each $1 \le i \le m$,

$$\sum_{1 \le j \le n} \bar{\pi}_j \hat{\beta}_i a_{ji} = \sum_{j \in G^t} \bar{\pi}_j \hat{\beta}_i a_{ji} + \sum_{j \in G^l} \bar{\pi}_j \hat{\beta}_i a_{ji} = e_i + 0 = e_i.$$

Thus $u_i(\bar{x}_i) = \hat{\beta}_i = e_i / \left(\bar{\pi}^T a_i\right)$, and so we have established that $\bar{x}_i$ maximizes u_i for the given prices $\bar{\pi}$ and the endowment.

2. For each $j \in G^t$, we have

$$\sum_{1 \le i \le m} \bar{x}_{ji} = \sum_{1 \le i \le m} \hat{\beta}_1 a_{ji} = q_j$$

by definition of G^t. For each $j \in G^l$, we have $\sum_{1 \le i \le m} \bar{x}_{ji} = q_j$ by construction of the $\bar{x}_{ji}$. Thus all goods are cleared. □

The Algorithm

The proof that an equilibrium exists yields the following algorithm for computing an equilibrium. We first solve the convex optimization problem CP to obtain $\hat{\beta}$. We then find the sets G^t and G^l by direct inspection. We then find values $\bar{\pi}_j \ge 0$ for each $j \in G^t$ such that, for each $1 \le i \le m$, $\sum_{j \in G^t} \bar{\pi}_j \hat{\beta}_i a_{ji} = e_i$. Note that this problem, which we denote by FP, involves finding a feasible solution for a system of linear inequalities and equalities. We set $\bar{\pi}_j = 0$ for each $j \in G^l$. We then use $\hat{\beta}$ to compute the baskets $\bar{x}_i$ for each buyer i as described in the proof. Since both CP and FP can be solved in polynomial time using the ellipsoid algorithm, we obtain a polynomial time algorithm to compute the equilibrium.

As we show below, the vector $\hat{\beta}$ can unfortunately consist of irrational numbers. This means that we have to settle for an approximation to $\hat{\beta}$, and therefore an approximate equilibrium, where the baskets and prices are such that the goods are almost cleared and the basket of each buyer almost optimizes her utility, given the prices and her endowment.

3.3 An Alternative Formulation

We can formulate the problem of computing an equilibrium in an alternative way as a feasibility problem with convex constraints. The problem FEAS is to find $\beta_1, \ldots, \beta_m$ and $\pi_1, \ldots, \pi_n$ satisfying

$$\begin{aligned}
\sum_{1 \leq i \leq m} \beta_i a_{ji} &\leq q_j \text{ for } 1 \leq j \leq n \\
\beta_i \sum_{1 \leq j \leq n} \pi_j a_{ji} &\geq e_i \text{ for } 1 \leq i \leq m \\
\sum_{1 \leq i \leq n} \pi_j q_j &= \sum_i e_i \\
\pi_j &\geq 0 \text{ for} 1 \leq j \leq n \\
\beta_i &\geq 0 \text{ for} 1 \leq i \leq m.
\end{aligned}$$

It is not hard to show that any solution $\hat{\beta}_1, \ldots, \hat{\beta}_m$ and $\bar{\pi}_1, \ldots, \bar{\pi}_n$ of FEAS yields an equilibrium for our setting, and the prices and utilities at any equilibrium are a solution to FEAS. Note that this formulation does not guarantee the existence of such a solution. Also note that the second constraint of FEAS defines a convex set for non-negative values of the β_i and the π_j, so this is indeed a convex feasibility problem.

3.4 Uniqueness of Utilities at Equilibrium

We now argue that the utilities at equilibrium are unique and can be irrational. We first establish the following theorem that, combined with the fact that CP is maximized at a unique point, implies that the utilities at equilibrium are unique. Note that our market is a classic case where the equilibrium prices need not be unique.

Theorem 3. *Let M be a market with a set $B = \{1, \ldots, m\}$ of buyers and a set $G = \{1, \ldots, n\}$ of goods, with a vector $a_i \in \mathbb{R}^n$ and endowment $e_i > 0$ for each buyer i, and an amount $q_j > 0$ for each good j. Let $\bar{\pi} \in \mathbb{R}^n$ be the vector of prices and $\bar{x}_i \in \mathbb{R}^n$ be a basket for buyer i, for each $1 \leq i \leq m$, so that the prices and the baskets constitute an equilibrium for the market M. Let $\bar{\beta}_i = \min_{1 \leq j \leq n} \bar{x}_{ji}/a_{ji}$ denote the utility of buyer i at equilibrium, and let $\bar{\beta} = (\bar{\beta}_1, \ldots, \bar{\beta}_m) \in \mathbb{R}^m$. Then $\bar{\beta}$ is an optimal solution to CP.*

Proof. We begin with some simple observations on the equilibrium. We must have $\bar{\pi}_j > 0$ for at least one good j and $\bar{\beta}_i > 0$ for each buyer i. By definition of our utility functions, we have $\bar{x}_{ji} \geq \bar{\beta}_i a_{ji}$ for each good j and buyer i. Let G^t denote the set of goods j for which $\bar{x}_{ji} = \bar{\beta}_i a_{ji}$ for every buyer i. Let G^l denote the remaining goods. For each good $j \in G^l$, there is a buyer i such that

$\bar{x}_{ji} > \bar{\beta}_i a_{ji}$; the fact that the basket x_i optimizes i's utility at prices $\bar{\pi}$ forces $\bar{\pi}_j = 0$. Thus $\bar{\pi}_j = 0$ for any $j \in G^l$. This also implies that for each buyer i,

$$\sum_{j \in G^t} \bar{\pi}_j \bar{\beta}_i a_{ji} = \sum_{j \in G^t} \bar{\pi}_j \bar{x}_{ji} = \sum_{j \in G} \bar{\pi}_j \bar{x}_{ji} = e_i,$$

where the last equality follows from the fact that for i's utility to be optimal at the given price $\bar{\pi}$, she must spend her entire endowment.

In other words, the vector $(e_1, \ldots, e_m) \in \mathbb{R}^m$ is in the cone of vectors

$$\{(\bar{\beta}_1 a_{j1}, \ldots, \bar{\beta}_m a_{jm}) |\ j \in G^t\}.$$

In the rest of this section we essentially reverse the direction of the argument in the previous section to show that $\bar{\beta}$ is an optimal solution to CP. We first check that $\bar{\beta}$ is a feasible solution for CP. For each good $j \in G^t$, we have

$$\sum_{1 \le i \le m} \bar{\beta}_i a_{ji} = \sum_{1 \le i \le m} \bar{x}_{ji} = q_j.$$

For each good $j \in G^l$, we have

$$\sum_{1 \le i \le m} \bar{\beta}_i a_{ji} < \sum_{1 \le i \le m} \bar{x}_{ji} = q_j.$$

Let us assume that $\bar{\beta}$ is not an optimal solution to CP and that $\hat{\beta}$ is an optimal solution. Let $\tau = \hat{\beta} - \bar{\beta}$. We express τ as $(\bar{\beta}_1 t_1, \ldots, \bar{\beta}_m t_m)$; this is possible since each $\bar{\beta}_i > 0$. Let $t = (t_1, \ldots, t_m)$.

We claim that the vector t satisfies two properties: (1) $\sum_{1 \le i \le m} t_i \bar{\beta}_i a_{ji} \le 0$ for each $j \in G^t$, and (2) $e_1 t_1 + \cdots + e_m t_m > 0$. The claim implies, by Farkas Lemma [14], that the vector $(e_1, \ldots, e_m) \in \mathbb{R}^m$ does not lie in the cone spanned by the vectors $\{(\bar{\beta}_1 a_{j1}, \ldots, \bar{\beta}_m a_{jm}) |\ j \in G^t\}$, which is a contradiction. We now show that the vector t does satisfy the two properties.

1. Since the feasible region of CP is convex, the vector $\bar{\beta} + \delta\tau$ must be feasible for CP for sufficiently small $\delta > 0$. So for any $j \in G^t$ and sufficiently small $\delta > 0$, we have

$$\begin{aligned}
&\sum_{1 \le i \le m} \bar{\beta}_i (1 + \delta t_i) a_{ji} \le q_j \\
\Rightarrow &\sum_{1 \le i \le m} \bar{\beta}_i a_{ji} + \delta \sum_{1 \le i \le m} t_i \bar{\beta}_i a_{ji} \le q_j \\
\Rightarrow\ & q_j + \delta \sum_{i \le i \le m} t_i \bar{\beta}_i a_{ji} \le q_j \\
\Rightarrow &\sum_{1 \le i \le m} t_i \bar{\beta}_i a_{ji} \le 0
\end{aligned}$$

2. Because the objective function of CP is concave, the objective function at $\bar{\beta}+\delta\tau$ must be strictly greater than the objective function at $\bar{\beta}$ for sufficiently small $\delta > 0$. This implies that

$$\prod_{1\leq i\leq m} \bar{\beta}_i^{e_i}(1+\delta t_i)^{e_i} - \prod_{1\leq i\leq m} \bar{\beta}_i^{e_i} > 0$$

for sufficiently small $\delta > 0$. Now, using the Taylor expansion,

$$\prod_{1\leq i\leq m} \bar{\beta}_i^{e_i}(1+\delta t_i)^{e_i} - \prod_{1\leq i\leq m} \bar{\beta}_i^{e_i} =$$
$$(\delta \sum_{1\leq i\leq m} e_i t_i + \delta^2 \frac{1}{2}((\sum_{1\leq i\leq m} e_i t_i)^2 - \sum_{1\leq i\leq m} e_i t_i^2) + O(\delta^3)) \prod_i \bar{\beta}_i^{e_i}$$

If $\sum_{1\leq i\leq m} e_i t_i < 0$, the coefficient of δ in the above expression is negative, so the expression is negative for sufficiently small $\delta > 0$. If $\sum_{1\leq i\leq m} e_i t_i = 0$, the coefficient of δ in the above expression is 0 but the coefficient of δ^2 is negative, so the expression is negative for sufficiently small $\delta > 0$. So we must have $\sum_{1\leq i\leq m} e_i t_i > 0$.

This finishes the proof of the theorem. □

Uniqueness and Irrationality

We have established that the utilities at equilibrium must be an optimal solution to CP. It is easily verified that the objective function of CP is strictly quasi-concave and therefore CP has a unique optimal solution. Thus the utilities at equilibrium are unique. We now present an example of a market with two goods and three buyers for which the utilities at equilibrium are irrational. We have $q_1 = q_2 = 3$, $e_1 = e_2 = e_3 = 1$, and $a_1 = (1, 1/2)$, $a_2 = (1/2, 1)$, and $a_3 = (1/4, 1/5)$. The utilities at equilibrium are the solution to the program:

$$\begin{aligned} \text{maximize } & (\beta_1 * \beta_2 * \beta_3)^{1/3} \\ \text{Subject to } & \beta_1 + \frac{\beta_2}{2} + \frac{\beta_3}{4} \leq 3 \\ & \frac{\beta_1}{2} + \beta_2 + \frac{\beta_3}{5} \leq 3 \\ & \beta_i \geq 0, \text{ for } 1 \leq i \leq 3. \end{aligned}$$

The solution to this program is $\beta_1 = 2/\sqrt{3}$, $\beta_2 = 1 + 1/\sqrt{3}$, and $\beta_3 = 10 - 10/\sqrt{3}$. So the utilities at equilibrium are irrational for this market. This implies that both the price vector and the baskets must contain irrational elements.

4 Generalizations

Let M be a market with a set $B = \{1, \ldots, m\}$ of buyers and a set $G = \{1, \ldots, n\}$ of goods, with a concave utility function $u_i : \mathbb{R}^n \to \mathbb{R}$ and endowment $e_i > 0$ for each buyer i, and an amount $q_j > 0$ for each good j. As before, we will denote by the variable x_{ji} the amount of good j in buyer i's basket, and by the vector $x_i = (x_{1i}, \ldots, x_{ni})$ the basket of buyer i.

Consider the following optimization problem CPG:

$$\begin{aligned}
&\text{maximize } \Big(\prod_{1\le i\le m} u_i(x_i)^{e_i}\Big)^{\frac{1}{e_1+\cdots+e_m}}\\
&\text{Subject to } \sum_{1\le i\le m} x_{ji} \le q_j, \text{ for } 1 \le j \le n\\
&\qquad\qquad x_{ji} \ge 0, \text{ for } 1 \le i \le m,\ 1 \le j \le n.
\end{aligned}$$

Since each u_i is a concave function, the objective function of CPG is concave, so this is a convex programming problem. Let $x \in \mathbb{R}^{mn}$ denote the vector $(x_{11}, \ldots, x_{n1}, x_{12}, \ldots, x_{n2}, \ldots, x_{1m}, \ldots, x_{nm})$. Let

$$\begin{aligned}
\theta(x) &= -\Big(\prod_{1\le i\le m} u_i(x_i)^{e_i}\Big)^{\frac{1}{e_1+\cdots+e_m}}\\
g_j(x) &= \sum_{1\le i\le m} x_{ji} - q_j, \text{ for } 1 \le j \le n\\
h_{ji}(x) &= -x_{ji}, \text{ for } 1 \le i \le m,\ 1 \le j \le n.
\end{aligned}$$

Then CPG can be restated as a minimization problem:

$$\begin{aligned}
&\text{minimize } \theta(x)\\
&\text{Subject to } g_j(x) \le 0, \text{ for } 1 \le j \le n\\
&\qquad\qquad h_{ji}(x) \le 0, \text{ for } 1 \le i \le m,\ 1 \le j \le n.
\end{aligned}$$

Let us assume that the utility functions u_i satisfy some fairly general differentiability conditions. A vector $\bar{x} \in \mathbb{R}^{mn}$ and real numbers

$$\bar{\pi}_1, \ldots, \bar{\pi}_n, \bar{\lambda}_{11}, \ldots, \bar{\lambda}_{n1}, \bar{\lambda}_{12}, \ldots, \bar{\lambda}_{n2}, \ldots, \bar{\lambda}_{1m}, \ldots, \bar{\lambda}_{nm},$$

if they exist, are said to solve the Kuhn-Tucker stationary-point problem (KTP) if they satisfy the following four conditions ([14], page 94):

1. $\nabla\theta(\bar{x}) + \sum_j \bar{\pi}_j \nabla g_j(\bar{x}) + \sum_{j,i} \bar{\lambda}_{ji} \nabla h_{ji}(\bar{x}) = 0$.
2. $\bar{x}$ is a feasible solution to CPG.
3. $\sum_j \bar{\pi}_j g_j(\bar{x}) + \sum_{j,i} \bar{\lambda}_{ji} h_{ji}(\bar{x}) = 0$.
4. $\bar{\pi}_j \ge 0$, for $1 \le j \le n$, and $\bar{\lambda}_{ji} \ge 0$, for $1 \le j \le n$ and $1 \le i \le m$.

The Kuhn-Tucker stationary-point necessary optimality theorem ([14], page 105) states that if $\bar{x}$ is an *optimal* solution to CPG, then there exist $\bar{\pi}_j$ and $\bar{\lambda}_{ji}$ so that $\bar{x}$, the $\bar{\pi}_j$, and the $\bar{\lambda}_{ji}$ are a solution to KTP. The four requirements of KTP then imply that the baskets corresponding to $\bar{x}$ and prices obtained by multiplying the vector $(\bar{\pi}_1, \ldots, \bar{\pi}_n)$ by a suitable number yield a market equilibrium for M, *provided* each utility function u_i satisfies certain additional conditions. One sufficient condition is that

$$\sum_{1 \leq j \leq n} x_{ji} \frac{\partial u_i}{\partial x_{ji}} = u_i(x_i).$$

Observe that this sufficient condition is satisfied by linear utility functions, Cobb-Douglas utility functions, CES utility functions (and indeed all differentiable scalable utility functions [19]). This yields a proof for the existence for an equilibrium when each utility function u_i is one of these types of functions. Also observe that the u_i are not required to be all of the same type.

This proof of existence also yields an efficient algorithm for computing an equilibrium: once we have solved the convex program CPG using the ellipsoid algorithm, we can immediately read off the prices from the four requirements of KTP. This becomes evident if we explicitly write out the four requirements in our case. (There is no need to solve a linear program as we did in the case of Leontief utilities.)

Leontief utilities, and in general utility functions defined as a minimum of a set of linear functions that have a value of 0 at the origin, are not differentiable and hence cannot be plugged directly into this paradigm but they can be dealt with using a slight variant.

Note that all the utility functions u we mentioned in this section are scalable, that is, they have the property that $u(\alpha z) = \alpha u(z)$ for any real $\alpha \geq 0$ and any basket $z \in \mathbb{R}^n$. That is, if we scale each component of a basket by a number, the utility scales by the same number. We point out that this is not as restrictive as it may seem at first since a utility function can be viewed as simply specifying a preference relation over all the baskets. Thus for example the function $\sqrt{u(z)}$ yields the same preference relation as $u(z)$. In closing, we wish to point out that it is unclear if our approach can be generalized even further without some kind of normalization like the sufficient condition described above.

5 Conclusions

In this paper we have investigated the problem of computing a market equilibrium in Fisher's model. In Section 3 we have shown that the equilibrium problem can be solved efficiently for markets with Leontief utility functions, and in Section 4 we have suggested significant extensions, which are reported in [3].

Acknowledgment. We wish to acknowledge fruitful discussions with Sriram Pemmaraju on the topics of this paper. The first author would like to thank Varsha Dani, for several conversations on market equilibria, in particular on Cobb-Douglas utility functions.

References

1. K.J. Arrow and G. Debreu, Existence of an Equilibrium for a Competitive Economy, Econometrica 22 (3), pp. 265–290 (1954).
2. W.C. Brainard and H. Scarf, How to Compute Equilibrium Prices in 1891. Cowles Foundation Discussion Paper 1270 (2000).
3. B. Codenotti, K. Jain, K. Varadarajan, V. V. Vazirani. Market Equilibrium for Scalable Utilities and Production Models via Variational Calculus, submitted (2004).
4. X. Deng, C. H. Papadimitriou, M. Safra, On the Complexity of Equilibria, STOC 02.
5. N. R. Devanur, C. H. Papadimitriou, A. Saberi, V. V. Vazirani, Market Equilibrium via a Primal-Dual-Type Algorithm. FOCS 2002, pp. 389-395. (Full version with revisions available on line.)
6. N. R. Devanur, V. V. Vazirani, Extensions of the spending constraint-model: existence and uniqueness of equilibria (extended abstract). ACM Conference on Electronic Commerce 2003, pp. 202-203 (2003).
7. N. R. Devanur, V. V. Vazirani, An Improved Approximation Scheme for Computing Arrow-Debreu Prices for the Linear Case. FSTTCS 2003, pp. 149-155 (2003).
8. B. C. Eaves, Finite Solution of Pure Trade Markets with Cobb-Douglas Utilities, Mathematical Programming Study 23, pp. 226-239 (1985).
9. B. C. Eaves and H. Scarf, The Solution of Systems of Piecewise Linear Equations, Mathematics of Operations Research, Vol. 1, No. 1, pp. 1-27 (1976).
10. D. Gale. The Theory of Linear Economic Models. McGraw Hill, N.Y. (1960).
11. T. Hansen and H. Scarf, The Computation of Economic Equilibrium, Cowles Foundation Monograph No. 24, New Haven: Yale University Press (1973).
12. K. Jain, M. Mahdian, and A. Saberi, Approximating Market Equilibria, Proc. APPROX 2003.
13. H.W. Kuhn, Simplicial Approximation of Fixed Points, Proc. National Academy of Sciences of the United States of America Vol. 61, n. 4, pp. 1238-1242 (1968).
14. O. L. Mangasarian. Nonlinear Programming, McGraw-Hill, 1969.
15. C. H. Papadimitriou, On the Complexity of the Parity Argument and other Inefficient Proofs of Existence, Journal of Computer and System Sciences 48, pp. 498-532 (1994).
16. C. H. Papadimitriou, Algorithms, Games, and the Internet, STOC 01, (2001).
17. H. Scarf, The Approximation of Fixed Points of a Continuous Mapping, SIAM J. Applied Math., 15, pp. 1328-1343 (1967).
18. H. Scarf, The Computation of Equilibrium Prices: An Exposition, in Arrow and Intriligator, editors, Handbook of Mathematical Economics, Volume II, pp. 1008-1061 (1982).
19. H. Varian, Microeconomic Analysis, New York: W.W. Norton, 1992.