Description of Modal Logics Inheriting Admissible Rules for $S4$

V.V. Rybakov, *Mathematics Department, Krasnoyarsk University pr. Svobodnyi 79, 660 062, Krasnoyarsk, Russia.*
E-mail: rybakov@math.kgu.krasnoyarsk.su

C. Gencer, *Mathematics Dept., Science Faculty, Ege University, Bornova-Izmir, 35100, Turkey. E-mail: gencer@fenfak.ege.edu.tr*

T. Oner, *Mathematics Dept., Science Faculty, Ege University, Bornova-Izmir, 35100, Turkey. E-mail: oner@fenfak.ege.edu.tr*

Abstract

We give a necessary and sufficient condition for any modal logic with fmp to inherit all inference rules admissible in $S4$. Using this condition we describe all tabular modal logics inheriting inference rules admissible for $S4$.

Keywords: Modal logic, Kripke Model, Inference Rule, Admissible Rule

1 Introduction, Motivation

Investigation of inference rules for non-standard logics has nowadays many aspects. One of them is exploration of admissible inference rules - those which do not enlarge the set of provable theorems being adjoined to arbitrary axiomatic system of a given logic (cf. Lorenzen ([6], 1955). Thus these rules are invariant for any given logic λ - they do not depend on the choice of axiomatic systems for λ, and they are the greatest class of rules which are compatible with the provability in λ.

At the beginning the attention was focused primary to algorithms and tools for recognizing admissibility of inference rules. It turns out that the majority of superintuitionistic logics, as well as IPC itself, and most part of transitive modal logics with fmp, which have been under consideration, have algorithms determining admissibility (cf. [10, 12]). A pure algebraic counterpart for these results is the decidability of quasi-equational theories of free algebras from corresponding varieties of algebras. Therefore it was of interest to consider elementary theories of corresponding free algebras, and it turns out that they are very often undecidable (see [8]). Some pure algebraic results about structure of these free algebras were obtained in [2], [3] and [1]. The questions about finite bases for admissible rules were investigated (cf, for example [12]) also. Structurally complete logics - those for which classes of admissible and derivable rules coincide - were also an active area for research. For instance, hereditarily structurally complete superintuitionistic logics were described by A.Citkin [4], and a similar description lately was given for transitive modal logics [10]. Next question for research was study the preservation of admissible rules. The matter is the logics generally

speaking do not inherit admissible rules while extending logics themselves. And in [7] a complete description of all superintuitionistic logics with fmp which preserve all rules admissible in IPC was found. The description employs the so-called co-cover property - the property to preserve the truth of theorems of given logic λ w.r.t. adjoining co-covers for antichains of clusters of any finite rooted λ-frame. Then similar description for modal logics inheriting admissible rules of $S4$ was given in [9] (and was reproduced in [12]). In this paper the idea of co-cover posters from [7] was an initial point for the technique employed. But it turns out that using this approach has a small gap since the direct transforming the notion of the co-covers posters for the case of superintuitionistic logics to the case of modal logics is not correct (because, for any given single-element cluster C, no p-morphisms from C onto any proper cluster). It leads us to recover the employed technique and we are doing this in the present paper.

First we give a correct for our aims notion of co-covers poster and the corresponding notion of co-cover property for modal logics. Then we establish, for any modal logic with fmp, a necessary and sufficient condition to inherit all inference rules admissible in $S4$ (recall, by the *inheriting* we mean that any rule admissible in $S4$ is also admissible in this logic). This condition is given in terms of the co-cover property. Finally, by using this criterion we describe tabular modal logics inheriting all rules admissible for $S4$.

2 Denotation, Preliminary Facts

We are assuming reader to be aware of algebraic and Kripke semantics for modal logics and also assume a certain knowledge of basic facts concerning inference rules and their admissibility, though we recall below briefly certain necessary facts and notation (for deep acquaintances confer [12] for advanced technique concerning inference rules, and [5] for general technique concerning modal logics). For any frame $\mathcal{F} := \langle F, R\rangle$, and any element $a \in F$, $C(a)$ will denote the cluster containing a. For any subset $\mathcal{X}$ of F $\mathcal{X}^R := \{a \mid \exists b \in \mathcal{X}(bRa)\}$, i.e. $\mathcal{X}^R$ is the upwards cone generated by $\mathcal{X}$, and $\mathcal{X}^{R+} := \{a \mid \exists b \in \mathcal{X}(bRa) \& \forall c \in \mathcal{X}(\neg(aRc))\}$. For any antichain $\mathcal{Y}$ of clusters from $\mathcal{F}$, a cluster C is a **co-cover** for $\mathcal{Y}$ if and only if

$$C^{R+} = \bigcup_{C_1 \in \mathcal{Y}} (C_1^R \cup C_1);$$

an element is a co-cover if the cluster containing this element is a co-cover. We say an element a (or cluster C) R-sees an element b (or a cluster C_1) if aRb (CRC_1, respectively). Also we will say b is R-visible from a, or R-seen from a. An element a (cluster C) is an **immediate R-successor** of an element b (cluster C_1) if aRb (CRC_1) and no elements c such that aRc, cRb and $\neg(cRa)$, $\neg(bRc)$ (CRc, cRC_1 and $\neg(cRC)$, $\neg(C_1Rc)$). Similarly we define **immediate R-predecessor**. For a pair of frames $\mathcal{F}_1, \mathcal{F}_2$, $\mathcal{F}_1 \sqsubseteq \mathcal{F}_2$ is the abbreviation for $\mathcal{F}_1$ is an open subframe of $\mathcal{F}_2$.

We are saying a frame $\mathcal{F}$ is a λ-frame for a logic λ if all theorems of λ are valid at $\mathcal{F}$, and $\lambda(\mathcal{F})$ is the logic generated by the frame $\mathcal{F}$. A frame $\mathcal{F}$ is **rooted, or sharp** if $\exists a \in \mathcal{F}$ such that $\forall b \in \mathcal{F}$, aRb, then we say $C(a)$ is the root of $\mathcal{F}$ and denote that cluster by $root(\mathcal{F})$. $S_m(\mathcal{F})$ denotes the set of all elements of $\mathcal{F}$ of depth not exceeding m, and $Sl_m(\mathcal{F})$ is the set of all elements of depth m from $\mathcal{F}$.

A rule $r := \alpha_1, ..., \alpha_n/\beta$ is admissible for a given logic λ iff, for any substitution ε of arbitrary formulas in place of variables,

$$[\forall j, 1 \leq j \leq n, (\alpha_j^{\varepsilon} \in \lambda)] \Rightarrow [\beta^{\varepsilon} \in \lambda].$$

The book [12] contains a construction of n-characterizing Kripke models $Ch_\lambda(n)$ and criteria for recognizing admissibility, which are important for understanding our paper. Therefore we recall briefly the construction of $Ch_\lambda(n)$. Given a modal logic λ with finite model property (fmp) which extends $K4$ and a set $P_n := \{p_1, ..., p_n\}$ of propositional letters. The first slice $S_1(Ch_\lambda(n))$ consists of the collection of all clusters with all possible valuations V of letters from P_n which do not have doubling - elements with the same valuation, and no clusters which are identical as Kripke models. Assuming $S_m(Ch_\lambda(m))$ to be constructed, we put in $Sl_{m+1}(Ch_\lambda(m))$ the clusters as follows. We chose any antichain $\mathcal{Y}$ of clusters from $S_m(Ch_\lambda(m))$ having at least one cluster of depth m and put in $Sl_{m+1}(Ch_\lambda(n))$ any cluster C from $S_1(Ch_\lambda(n))$ assuming C to be immediate predecessor for clusters from $\mathcal{Y}$ but only provided that

(i) the frame C^R is a λ-frame and
(ii) if $\mathcal{Y} := \{C_1\}$ then C is not a Kripke submodel of C_1.

Iterating this procedure we get as the result the model $Ch_\lambda(n)$. Recall, we are saying a model $\mathcal{M}$ is n-**characterizing** for a logic λ if, for any formula α, which is built up out of letters from P_n, $\alpha \in \lambda$ iff $\mathcal{M} \Vdash \alpha$. Recall certain necessary results first one is

Theorem 2.1 *[12] For any modal logic λ with fmp extending logic $K4$, the model $Ch_\lambda(n)$ is n-characterizing for λ.*

An element a from a Kripke model $\mathcal{M}$ is **definable** in $\mathcal{M}$ if there is a formula α such that $\forall x \in \mathcal{M}(x \Vdash_V \alpha) \Leftrightarrow x = a$, where V is the valuation from $\mathcal{M}$. A subset $\mathcal{X}$ of $\mathcal{M}$ is definable if there is a formula α such that $\forall x \in \mathcal{M}(x \Vdash_V \alpha) \Leftrightarrow x \in \mathcal{X}$. And the valuation V is definable in $\mathcal{M}$ if, for any letter p from the domain of V, $V(p)$ is definable.

Theorem 2.2 *[12] For any modal logic λ having fmp and extending $K4$, any element of the model $Ch_\lambda(n)$ is definable in $Ch_\lambda(n)$.*

We need a description of admissible rules by n-characterizing models. For a given frame $\mathcal{F}$, a given valuation V and a given inference rule $r := \alpha_1, ..., \alpha_n/\beta$, we say r is valid at $\mathcal{F}$ under V, and write $\mathcal{F} \Vdash_V r$, if as soon as $\forall x \in \mathcal{F}, \forall i (x \Vdash_V \alpha_i)$, we have $\forall x \in \mathcal{F}(x \Vdash_V \beta)$. A rule r is valid at a frame $\mathcal{F}$ if r is valid at $\mathcal{F}$ under any valuation, we write then $\mathcal{F} \Vdash r$. Lemma 3.4.2, Theorem 3.5.1. and Corollary 3.5.9 from [12] immediately imply

Theorem 2.3 *[12] (Semantic Criterion) An inference rule r is admissible in $S4$ iff r is valid at the frame $Ch_{S4}(n)$ for any n under any valuation.*

Theorem 2.4 *([12], Theorem 3.3.3) For any modal logic λ having fmp and extending $K4$ and for any inference rule r, r is admissible in λ iff r is valid in the frame of $Ch_\lambda(n)$ for any n under any definable valuation.*

Now define the co-cover property for the case of modal logics extending $S4$. First we need to recover the notion of the co-covers poster for the case of modal logic which was incorrectly given in [9].

Given an arbitrary finite frame $\mathcal{F} := \langle F, R\rangle$. A **co-covers poster** of $\mathcal{F}$ is the frame constructed out of the frame $\mathcal{F}$ by a finite number of the following steps. First, 0-step, if $\mathcal{F}$ does not have a single-element R-maximal cluster we add to $\mathcal{F}$ an R-maximal single-element cluster and we get the frame $\mathcal{F}_1$. Otherwise let $\mathcal{F}_1 := \mathcal{F}$. Then in any next step we choose (the choice is random, not deterministic) some non-trivial anti-chain $\mathcal{X}$ of clusters from the frame $\mathcal{Q}$ which we got at the previous step (initially $\mathcal{Q}$ is $\mathcal{F}_1$). If $\mathcal{Q}$ does not have a co-cover for $\mathcal{X}$ we *adjoin* to $\mathcal{Q}$ a new element which is a co-cover for $\mathcal{X}$. Otherwise we do not change $\mathcal{Q}$.

Definition 2.5 *We say that a logic λ extending $S4$ has the* **co-cover property** *if, for every rooted finite λ-frame $\mathcal{F}$, any co-covers poster of $\mathcal{F}$ is a λ-frame also.*

Thus, comparing with [9], the definition of the co-covers-poster (and consequently the definition of the co-cover property) is changed only in one respect: if the initial rooted frame does not have R-maximal single-element cluster then we adjoin such a cluster to the frame. This alteration is only we need in order to make our technique working. But this alteration involves necessity to revise the proof from [9], and this revised proof is presented below.

Definition 2.6 *We say a logic λ_1* **inherits (or preserves)** *all rules admissible in a logic λ if any rule r admissible for λ is admissible for λ_1 also.*

It is evident that if λ_1 inherits all rules admissible for λ then $\lambda \subseteq \lambda_1$, the converse does not hold generally speaking, which can be simply demonstrated with examples.

3 Preservation of Admissibility

Lemma 3.1 *If a modal logic λ extending $S4$ has the co-cover property and the finite model property then λ inherits all rules admissible in $S4$.*

Proof. We describe only steps of proof which are distinct from the proof of the similar result in [9]. Again we consider only rules with a single premise. Assume α/β is a rule which is admissible for $S4$ but is not admissible for λ. Then there are formulas γ_i such that $\alpha(\gamma_i) \in \lambda$ but $\beta(\gamma_i) \notin \lambda$. Formulas γ_i contain a finite number of propositional letters, say n letters. Then, for the n-characterizing model $Ch_\lambda(n)$ of λ (cf. Theorem 3.3.6 from [12]), the following holds: $Ch_\lambda(n) \Vdash_V \alpha(\gamma_i)$, $Ch_\lambda(n) \nVdash_V \beta(\gamma_i)$. Thus the valuation $S(x_i) := V(\gamma_i)$ invalidates α/β in a subframe a^R of $Ch_\lambda(n)$. Let $\mathcal{F}_0 := a^{R\leq}$, and let $\mathcal{F}(1)$ be a single-element reflexive frame. The frame $\mathcal{F}_1$ is defined as follows.

$$\mathcal{F}_1 \;=\; \begin{cases} \mathcal{F}_0 & \text{if } \exists \text{ cluster } C \in Sl_1(a^R)(||C|| = 1) \\ \mathcal{F}_0 \sqcup \mathcal{F}(1) & \text{if } \forall \text{ clusters } C \in Sl_1(a^R)(||C|| > 1) \end{cases}$$

$\mathcal{F}_1$ is a co-cover poster of $\mathcal{F}_0$ (0 -steps are needed). Since λ has the co-cover property, $\mathcal{F}_1$ is an open submodel of $Ch_\lambda(n)$. Assume the frame $\mathcal{F}_m$ is constructed. The frame $\mathcal{F}_{m+1}$ is obtained from $\mathcal{F}_m$ as follows. We take every non-trivial anti-chain $\mathcal{X}$ of clusters of $\mathcal{F}_m$ which has no co-covers in $\mathcal{F}_m$ and add to $\mathcal{F}_m$ a single reflexive element which is the co-cover for $\mathcal{X}$. By our construction $\mathcal{F}_{m+1}$ is a co-covers poster of $\mathcal{F}_m$ and,

since λ has the co-cover property, each $\mathcal{F}_m$ can be considered as an open subframe of $Ch_\lambda(n)$ and also $\mathcal{F}_m$ is an open subframe of $\mathcal{F}_{m+1}$. Thus $\mathcal{F}_\infty := \bigcup_{m<\infty} \mathcal{F}_m$ is an open subframe of $Ch_\lambda(n)$, i.e. $\mathcal{F}_\infty \sqsubseteq Ch_\lambda(n)$ in our denotation.

Since the valuation S invalidates the rule α/β in a subframe a^R of $\mathcal{F}_\infty$, the rule α/β is false in $\mathcal{F}_\infty$ w. r. t. the valuation S. Moreover, according to its construction, the model $Ch_{S4}(n)$ has co-covers for any non-trivial anti-chain of clusters, and, as it can be easily seen, $Ch_\lambda(n) \sqsubseteq Ch_{S4}(n)$.

From this together with the property of $\mathcal{F}_\infty$ to have co-covers for all non-trivial finite anti-chains of clusters, we can derive that there exists a p-morphism f of the frame $Ch_{S4}(n)$ onto $\mathcal{F}_\infty$. The distinction from the proof in [9] is that $\mathcal{F}_\infty$ has now single-element clusters which are maximal by accessibility relation, and this allows us to construct mentioned above p-morphism.

We omit details of this construction which are the same as in [9]. Using this p-morphism f and Theorem 2.3 we can conclude that α/β is not admissible in $S4$ by the same argument as in [9]. Thus λ inherits all admissible rules of $S4$. ■

Lemma 3.2 *If a modal logic λ extending $S4$ inherits all admissible rules of $S4$ and has the finite model property then λ has the co-cover property.*

Proof. Again we will concentrate only on the steps of the proof which are distinct from given ones in [9]. Let $\lambda \supseteq S4$ be a logic with the fmp. Assume λ does not have the co-cover property, i.e. there is an open subframe $\mathcal{F} := a^R$ of $Ch_\lambda(n)$ generated by an element a and there exists a sequence $\mathcal{F}_0, ..., \mathcal{F}_k$ of open finite subframes of $Ch_\lambda(n)$ which have the following properties: (i): $\mathcal{F}_0 := \mathcal{F}$ and $\mathcal{F}_1$ is either $\mathcal{F}$ if $\mathcal{F}$ has single-element R-maximal cluster or $\mathcal{F}_0; = \mathcal{F} \sqcup \{u\}$, where $\{u\}$ is a certain R-maximal single-element cluster. (ii): All antichains of clusters from $\mathcal{F}_i$ of depth less than i have in $\mathcal{F}_i$ some co-covers. (iii): The frame $\mathcal{F}_{i+1}$ is obtained from the frame $\mathcal{F}_i$ when $i < k$ as follows: We consider the set A_i consisting of all non-trivial anti-chains of clusters from $\mathcal{F}_i$ having at least one cluster of depth i. If all antichains from A_i have co-covers in $\mathcal{F}_i$ then we set $\mathcal{F}_{i+1} := \mathcal{F}_i$. Otherwise, for every anti-chain X from A_i which has no co-cover, we add to $\mathcal{F}_i$ a single reflexive co-cover for X. (iv): There exists an antichain Δ of clusters from $\mathcal{F}_k$ which has no co-cover in $Ch_\lambda(n)$ and has a cluster with depth k. Then, in particular, all antichains of $\mathcal{F}_k$ with depth of elements not exceeding $k-1$ have co-covers in $\mathcal{F}_k$.

From this point the proof is the same as in [9]. We only give below its scheme and the sequence of steps. First we introduce special modal formulas. Let $\mathcal{M} := S_k(Ch_\lambda(n)) \cup \mathcal{F}_k$ and $\mathcal{M}_1 := Sl_{k+1}(Ch_\lambda(n)) \cup \mathcal{M}$. Any cluster from $Ch_\lambda(n)$ is a subset of $\mathcal{M}_1$ or $\mathcal{M}$ or has empty intersection with $\mathcal{M}_1$ and $\mathcal{M}$. For each cluster $i \subseteq \mathcal{M}_1$ (now it will be more convenient to denote clusters by small letters like indexes), we introduce a new propositional variable p_i (new means that p_i is not a variable from the domain of the valuation from $Ch_\lambda(n)$). The special formulas $f(i)$ for clusters i of $\mathcal{M}_1$ are introduced by induction on the depth of i as follows (if i and j are clusters then iRj is denotation for $j \subseteq i^R$). If a cluster i belongs to $S_1(\mathcal{M}_1)$ then $f(i) := \Box p_i$.

If the formulas $f(i)$ are already introduced for all clusters i from $S_t(\mathcal{M}_1)$, and j is a cluster from $Sl_{t+1}(\mathcal{M}_1)$, then we put

$f(j) := \varphi(j) \wedge \varphi_1(j)$ where

$\varphi(j) := \Box p_j \wedge \bigwedge\{\neg\Box p_i | i \neq j, i \subseteq Sl_{t+1}(\mathcal{M}_1)\}$ and

$$\varphi_1(j) := \bigwedge_{jRi,\neg(iRj)} \Diamond f(i) \wedge \bigwedge\{\neg\Diamond f(i) | \neg(jRi), i \subseteq S_t(\mathcal{M}_1)\} \wedge \varphi_2(j) \text{ where}$$

$$\varphi_2(j) := \bigwedge_{\neg(iRj)} \neg\Box p_i \wedge u(j) \text{ and}$$

$$u(j) := \Box[\varphi(j) \wedge \bigwedge_{jRi,\neg(iRj)} \Diamond f(i) \wedge \bigwedge_{jRi,\neg(iRj)} \neg f(i)] \vee \bigvee_{jRi,\neg(iRj)} f(i).$$

Applying these denotations we introduce the inference rule r as follows. Let $g := \bigwedge_{i\in\mathcal{M}_1} \neg f(i) \wedge \bigvee_{i\in Sl_{k+1}(\mathcal{M}_1)} \Diamond f(i)$, and $r := \bigvee\{f(i) | i \subseteq \mathcal{M}_1\} \vee g/\neg f(a)$, where a is the root of the frame $\mathcal{F}_0$.

Lemma 3.3 *The rule r is not admissible in λ.*

Proof. Direct, as in [9] ■

Lemma 3.4 *The rule r is admissible in $S4$.*

Proof. Assume not. Then by Theorem 2.4 there is a definable valuation $\mathcal{W}$ in some n-characterizing model $Ch_{S4}(n)$ disproving r, i.e.

$$\mathcal{W}(\bigvee_{i\in\mathcal{M}_1} f(i) \vee g)) = |Ch_{S4}(n)|, \text{ and } \mathcal{W}(\neg f(a)) \neq |Ch_\lambda(n)| \tag{3.1}$$

It entails $\exists b_a \in |Ch_{S4}(n)|$: $b_a \Vdash_{\mathcal{W}} f(a)$. Applying definition of $f(i)$ it follows, $\forall b_i \in |Ch_{S4}(n)|$,

$$[(b_i \Vdash_{\mathcal{W}} f(i)) \& (j \subseteq \mathcal{M}_1) \& (iRj) \& \neg(jRi)] \Rightarrow \exists b_j [b_i R b_j \Vdash_{\mathcal{W}} f(j)] \tag{3.2}$$

Assume $b_i \Vdash_{\mathcal{W}} f(i)$, $b_j \Vdash_{\mathcal{W}} f(j)$ and $b_i R b_j$ hold. If $\neg(iRj)$ then $b_j \Vdash_{\mathcal{W}} \neg\Box p_i$ and also $b_i \Vdash_{\mathcal{W}} f(i)$, therefore $b_i \Vdash_{\mathcal{W}} \Box p_i$. This and $b_i R b_j$ imply that $b_j \Vdash_{\mathcal{W}} \Box p_i$, a contradiction. Using the same argument as in [9] we can show that $\forall b_i, b_j \in Ch_{S4}(n)$

$$[(b_i \Vdash_{\mathcal{W}} f(i)) \& (b_j \Vdash_{\mathcal{W}} f(j)) \& \neg(iRj) \Rightarrow \neg(b_i R b_j)] \text{ and by (3.2)} \tag{3.3}$$

$$b_i \Vdash_{\mathcal{W}} f(i) \& (iRj) \& \neg(jRi) \Rightarrow \exists b_j [(b_i R b_j) \& \neg(b_j R b_i) \& (b_j \Vdash_{\mathcal{W}} f(j))] \tag{3.4}$$

$$b_i \Vdash_{\mathcal{W}} f(i) \& (b_j \Vdash_{\mathcal{W}} f(j) \& (b_i R b_j) \Rightarrow iRj \tag{3.5}$$

$$[b_i \Vdash_{\mathcal{W}} f(i)] \& [b_j \Vdash_{\mathcal{W}} f(j)] \& \neg(jRi) \Rightarrow \neg(b_i R b_j) \& \neg(b_j R b_i) \tag{3.6}$$

$$(b_i \Vdash_{\mathcal{W}} f(i)) \& (b_i R d) \Rightarrow \exists j [(iRj) \& (d \Vdash_{\mathcal{W}} f(j))] \tag{3.7}$$

$$\forall j (aRj) \Rightarrow \exists b_j [(b_a R b_j) \wedge (b_j \Vdash_{\mathcal{W}} f(j))]. \tag{3.8}$$

Using all these properties and reasoning further by the same scheme as in [9] we derive a contradiction. ■

Lemmas 3.3 and 3.4 conclude the proof of Lemma 3.2. And from Lemmas 3.1 and 3.2 we immediately derive

Theorem 3.5 *In order for a modal logic λ extending $S4$ and possessing fmp to inherit all rules admissible in $S4$ it is necessary and sufficient that λ to have the co-cover property.*

It is possible to show that there are continuously many logics with fmp over $S4$ which preserve all rules admissible in $S4$. We can prove this exactly as in [9] since in this paper only posets were employed in the proof and, for posets, the definitions of co-cover property from mentioned paper and our new one coincide.

4 Tabular Logics Inheriting Admissible Rules of $S4$

In the light of Theorem 3.5 in order to give description of these logics it is sufficient to classify precisely which tabular logics have the co-cover property. Since this property defers now from given one in [9] we have to clarify how it influences the inheritance of admissible rules for tabular logics.

Recall that any tabular modal logic λ has a representation: $\lambda := \lambda(\mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_n)$, where $\mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_n$ is the disjoint union of certain rooted finite frames, where logics $\lambda(\mathcal{F}_i)$ are incomparable. We fix some such kind representation for any given tabular logic. First we recall a simple general fact:

Lemma 4.1 *A tabular modal logic $\lambda := \lambda(\mathcal{F}_1 \sqcup \cdots \sqcup \mathcal{F}_n)$ extending $S4$, preserves all inference rules admissible for $S4$ iff each logic $\lambda(\mathcal{F}_i)$ does.*

Proof. Necessity. Assume the rule $r := \Box\alpha(x_k)/\Box\beta(x_k)$ is admissible in $S4$ but is not admissible in $\lambda(\mathcal{F}_j)$. Then there exist formulas γ_k such that $\Box\alpha(\gamma_k) \in \lambda(\mathcal{F}_j)$ but $\Box\beta(\gamma_k) \notin \lambda(\mathcal{F}_j)$. Consider the rule $\Box\alpha \vee \Box z/\Box\beta \vee \Box z$, where z is a new variable not occurring in r. This rule is admissible in $S4$ due to the disjunction property of $S4$. Let

$$\theta := \bigvee_{i \neq j} \Box Ax(\lambda(\mathcal{F}_j)),$$

where $Ax(\lambda(\mathcal{F}_j))$ is a formula axiomatizing the logic $\lambda(\mathcal{F}_j)$ and all formulas from the disjunction do not have common variables. Consider the rule r_1 obtained from r by substitution of θ in place of z. We have $\mathcal{F}_j \Vdash \Box\alpha(\gamma_k)$, therefore the premise of r_1 is a theorem of λ. Since $\mathcal{F}_j \nVdash \Box\beta(\gamma_k)$ and logics $\lambda(\mathcal{F}_j)$ are incomparable, i.e. $\lambda(\mathcal{F}_{i_1}) \not\subseteq \lambda(\mathcal{F}_{i_2})$, for every $i_1 \neq i_2$, $\mathcal{F}_j \nVdash \theta$, and $\mathcal{F}_j \nVdash \Box\beta(\gamma_k)$. Thus the conclusion of r_1 is not a theorem of λ and r_1 is not admissible in λ.

Sufficiency. Assume the rule $\Box\alpha/\Box\beta$ is admissible in $S4$ but is not admissible for λ. Then there exist formulas γ_k such that $\Box\alpha(\gamma_k) \in \lambda$ but $\Box\beta(\gamma_k) \notin \lambda$. Since, for any j, $\lambda \subseteq \lambda(\mathcal{F}_j)$ we have $\Box\alpha(\gamma_k) \in \lambda(\mathcal{F}_j)$. And since $\Box\beta(\gamma_k) \notin \lambda$, for a certain j, $\Box\beta(\gamma_k) \notin \lambda(\mathcal{F}_j)$, i.e. the rule is not admissible in $\lambda(\mathcal{F}_j)$. ■

Lemma 4.2 *If a tabular logic λ has the co-cover property and a finite frame $\mathcal{F}$ is a co-covers poster for a rooted λ-frame then the width of $\mathcal{F}$ does not exceed 2.*

Proof. In principle the same reasoning as in [9] despite of distinction of the co-cover property: assume $\mathcal{F}$ has a 3-elements antichain of clusters. Using the co-cover property, taking consequently co-covers posters of $\mathcal{F}$, we obtain that, for any $n \in N$,

there is a frame $\mathcal{F}_n$ with depth n such that all theorems of λ are valid in $\mathcal{F}_n$. But λ is tabular, i.e., $\lambda = \lambda(\mathcal{Q})$ for some finite frame $\mathcal{Q}$ which has, say, a depth k. Then $\varphi(k) \in \lambda$, where $\varphi(k)$ is the formula expressing the property of frames to be of depth not more than k. This contradicts the validness of λ in the above mentioned frames $\mathcal{F}_n$ for each n. ■

For our description below we need the following rooted finite frames.

$\mathcal{M}_1 := \langle \{a, a_1, a_2, b_1, c_1\}, R\rangle$, where $a = root(\mathcal{M}_1), aRa_1, aRa_2,$

$a_1Rb_1, b_1Rc_1, a_2Rc_1$;

$\mathcal{M}_2 := \langle \{a, a_1, a_2, b_1\}, R\}, R\rangle$, where $a = root(\mathcal{M}_2), aRa_1, aRa_2, a_1Rb_1$;

$\mathcal{M}_3 := \langle \{a, a_1, a_2, b_1, b_2\}, R\rangle$, where $a = root(\mathcal{M}_3), aRa_1, aRa_2,$

$a_1Rb_1, a_1Rb_2, a_2Rb_2$;

$\mathcal{M}_4 := \langle \{a, a_1, a_2, b_1, b_2, c\}, R\rangle$, where $a = root(\mathcal{M}_4), aRa_1, aRa_2,$

$a_1Ra_2, a_2Ra_1, a_1Rb_1, a_1Rb_2, a_2Rb_1, a_2Rb_2, b_1Rc, b_2Rc$;

$\mathcal{M}_5 := \langle \{a, a_1, a_2, b_1, b_2\}, R\rangle$, where $a = root(\mathcal{M}_5), aRa_1, aRa_2,$

$a_1Ra_2, a_2Ra_1, a_1Rb_1, a_1Rb_2, a_2Rb_1, a_2Rb_2$.

For any rooted frame $\mathcal{F}$ we say $\mathcal{F}$ is **properly rooted** iff the root of $\mathcal{F}$ is single-element or if there is only one immediate successor cluster for the root.

Theorem 4.3 *A tabular logic $\lambda := \lambda(\mathcal{F}_1 \cup \cdots \cup \mathcal{F}_n)$ inherits all inference rules admissible in S4 iff (i) each $\mathcal{F}_j$ is a properly rooted frame with width not exceeding 2 and (ii) there is a single-element R-maximal cluster in $\mathcal{F}_j$, and (iii), for all i, $1 \leq m \leq 5$, $\lambda \not\subseteq \lambda(\mathcal{M}_i)$.*

Proof. First we verify **necessity**. Using Theorem 3.5 and Lemma 4.1 it follows that any $\lambda(\mathcal{F}_j)$ has the co-cover property, and by Lemma 4.2 we derive that any $\mathcal{F}_j$ is of width not exceeding 2.

Lemma 4.4 *λ is not included in $\lambda(\mathcal{M}_i)$, where $1 \leq i \leq 3$.*

Proof. Assume not and fix i. The logic λ is tabular and $\mathcal{M}_i^+ \in Var(\lambda)$, therefore applying $Var(\lambda) = HSP(\mathcal{F}_1 \sqcup, ..., \sqcup \mathcal{F}_n)$ and Stone duality theorems we get, for some j, there exists an onto p-morphism

$$f_j : \mathcal{Q}_j = C^R \sqsubseteq \mathcal{F}_j \to \mathcal{M}_i,$$

where C is a cluster of $\mathcal{F}_j$. And we may assume C is an R-maximal cluster with that property. Reasoning further as in [9] we derive a contradiction. ■

Assume now every $\mathcal{F}_j$ is properly rooted and not all R-maximal clusters of $\mathcal{F}_j$ are proper and that, for $i \in 1, 2, 3$, λ is not included in $\lambda(\mathcal{M}_i)$. Then the following holds.

Lemma 4.5 *$\lambda \not\subseteq \lambda(\mathcal{M}_i)$, where $i = 4, 5$.*

Proof. The same as in [9]. ■

Now suppose some $\mathcal{F}_j$ is not properly rooted. Then this frame has the root cluster with at least two elements and with two immediate successor clusters C_1 and C_2. In this case we immediately obtain a contradiction by adding a co-cover c for C_1 and C_2 and then by adding a co-cover for c and the root.

Finally, if a certain $\mathcal{F}_j$ has only proper R-maximal clusters then we generate the co-covers poster for $\mathcal{F}_j$ by adjoining new single-element R-maximal cluster c and then add a co-cover u for this c and the root of $\mathcal{F}_j$, i.e.

$$\mathcal{Q} := \{u\} \cup [\mathcal{F}_j \sqcup \{c\}],$$

where u is the root of $\mathcal{Q}$. The resulting frame $\mathcal{Q}$ is $\lambda(\mathcal{F}_j)$-frame according to the co-cover property for $\lambda(\mathcal{F}_j)$, but this frame is deeper than $\mathcal{F}_j$ itself - a contradiction. Thus the necessity of the condition in our theorem is proved.

Sufficiency. We have that the width of frames $\mathcal{F}_j$ is 2 or smaller and all of them are properly rooted and have single-element R-maximal clusters. Also we have $\lambda \not\subseteq \lambda(\mathcal{M}_i)$, $1 \leq i \leq 5$. By Lemma 4.1 and Theorem 3.5 it is sufficient to show that each $\lambda(\mathcal{F}_j)$ has the co-cover property.

Lemma 4.6 *$\lambda(\mathcal{F}_j)$ has the co-cover property.*

Proof. The same as in [9]. This lemma concludes the proof of our theorem. ■

To conclude this paper we would like to set several open questions. It is interesting to consider logics inheriting admissible rules of the smallest transitive normal modal logic $K4$ and to describe tabular logics preserving admissible rules of $K4$. This is important since (i) $K4$ is a basic, in a sense, transitive modal logic; (ii) these tabular logics are still inside of the proposed technique. Though, it seems, the presence of irreflexive elements in Kripke frames of these logics will make all considerations more complicated. But, anyway, there is a chance that the presence of irreflexivity could also reduce drastically the family of such logics and, quite to the contrary, the proofs could be even shorter. Another portion of open questions concerns study of the structure and properties of all modal logics inheriting all admissible rules of a given modal logic λ with fmp, it is of interest even for particular important modal logics. For instance: whether these classes are closed w.r.t. lattices operations; whether they contain logics with particular properties (without fmp, Kripke incomplete, for instance). Another open question is: whether the property to inherit all admissible rules of $S4$ is decidable for finitely axiomatizable logics with fmp.

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