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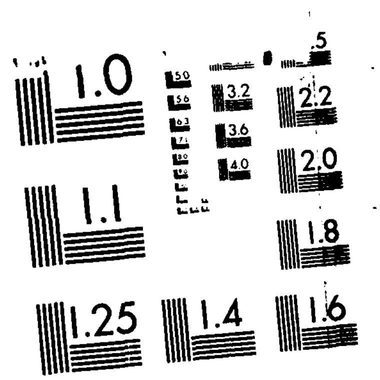

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ON THE SET COVERING POLYTOPE:
I. ALL THE FACETS WITH COEFFICIENTS IN {0,1,2}

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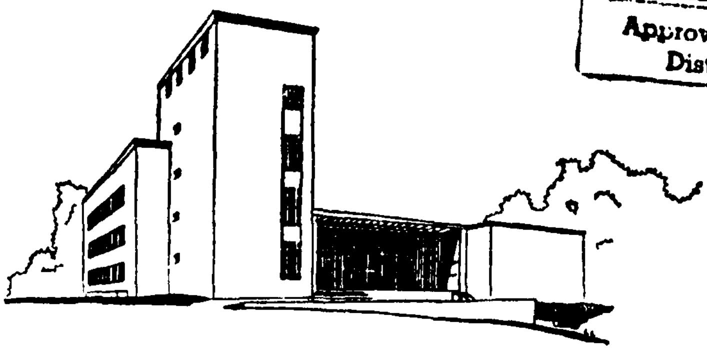

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Abstract

While the set packing polytope, through its connection with vertex packing, has lent itself to fruitful investigations, little is known about the set covering polytope. We characterize the class of valid inequalities for the set covering polytope with coefficients equal to 0, 1 or 2, and give necessary and sufficient conditions for such an inequality to be minimal and to be facet defining. We show that all inequalities in the above class are contained in the elementary closure of the constraint set, and that 2 is the largest value of k such that all valid inequalities for the set covering polytope with integer coefficients no greater than k are contained in the elementary closure. We point out a connection between minimal inequalities in the class investigated and certain circulant submatrices of the coefficient matrix. Finally, we discuss a procedure for generating all the inequalities in the above class, as well as inequalities that cut off a fractional solution to the linear programming relaxation of the set covering problem, and inequalities whose addition to the constraint set improves the lower bound given by a feasible solution to the dual of the linear programming relaxation.

1. Introduction

The set covering problem can be stated as

$$\text{(SC)} \qquad \min\{cx \mid Ax \geq 1, \ x \in \{0,1\}^n\},$$

where $A = (a_{ij})$ is an $m \times n$ matrix with $a_{ij} \in \{0,1\}$, $\forall\ i,j$, and 1 is the m-vector of 1's. If $Ax \geq 1$ is replaced by $Ax = 1$, the problem is called set partitioning. Both models have applications to crew scheduling, facility location, vehicle routing and a host of other areas (see Appendix to Balas and Padberg [3] for a bibliography of applications).

If we reverse the inequality in the definition of (SC), we obtain the set packing problem

$$\text{(SP)} \qquad \max\{cx \mid Ax \leq 1, \ x \in \{0,1\}^n\}$$

which is known to be equivalent to the vertex packing problem on the intersection graph G_A of A. Both (SC) and (SP) are NP-complete problems for a general 0-1 matrix A. As far as structural properties go, because of the connection between (SP) and vertex packing, the properties of the set packing polytope (the convex hull of points satisfying the constraints of (SP)) have been thoroughly studied. In particular, many classes of facets of this polytope have been identified (see, for instance, [6], [7]), as well as families of matrices A for which the corresponding polytope is given by the linear inequalities $Ax \leq 1$, $0 \leq x \leq 1$. The same cannot be said about the set covering polytope

$$P_I(A) := \text{conv}\{x \in R^n \mid Ax \geq 1, \ 0 \leq x \leq 1, \ x \text{ integer}\},$$

or its more relaxed relative, the polyhedron

$$P_I^*(A) := \text{conv}\{x \in R^n \mid Ax \geq 1, \ x \geq 0, \ x \text{ integer}),$$

about which much less is known. In the following, we will denote

$$P(A) := \{x \in R^n \mid Ax \geq 1, \ 0 \leq x \leq 1\}, \ P^*(A) := \{x \in R^n \mid Ax \geq 1, \ x \geq 0\}.$$

Let M and N be the row and column index sets, respectively, of A, and let

$\bar{a}^i$ and a_j denote the i^{th} row and the j^{th} column of A, respectively. For any $R \subseteq M$ and $S \subseteq N$, we will write A^R_S for the submatrix of A whose rows and columns are indexed by R and S, respectively. Also, we will denote $A^R = A^R_N$ and $A_S = A^M_S$. Finally, for $i \in M$, we will denote $N^i := \{j \in N \mid a_{ij} = 1\}$.

A polyhedron is the intersection of a finite number of halfspaces. A polytope is a bounded polyhedron. A face of a polyhedron is the intersection of the polyhedron with some of its boundary planes. For an n-dimensional polyhedron, the 0-dimensional faces are its vertices, the (n-1)-dimensional faces are its facets. An inequality is valid for a polyhedron P if it is satisfied by all $x \in P$. An inequality $\alpha x \geq \alpha_0$ is dominated by, or is a weakening of, the inequality $\beta x \geq \alpha_0$, if $\alpha \geq \beta$. If in addition $\alpha_j > \beta_j$ for some j, then $\alpha x \geq \alpha_0$ is strictly dominated by $\beta x \geq \alpha_0$. A coefficient α_j of a valid inequality $\alpha x \geq \alpha_0$ is minimal if $\alpha x \geq \alpha_0$ becomes invalid when α_j is decreased (without changing other coefficients). A valid inequality whose coefficients are all minimal is called minimal. Thus a minimal inequality is one not strictly dominated by any valid inequality. An inequality $\alpha x \geq \alpha_0$, valid for a polyhedron P, defines (or induces) a facet of P if and only if $\alpha x = \alpha_0$ for n (= dim P) affinely independent points $x \in P$. Valid inequalities that are facet defining are minimal, but the converse is not true.

Among the few facts known about the polyhedra $P^*_I(A)$ and $P_I(A)$ are the following: We assume throughout that A has no zero columns or zero rows.

1. Every vertex of $P^*_I(A)$ is a vertex of $P_I(A)$; hence any inequality that defines a bounded facet of $P^*_I(A)$ is also facet defining for $P_I(A)$; and any facet defining inequality for $P_I(A)$ that is satisfied by all $x \in P^*_I(A)$ is also facet defining for $P^*_I(A)$.

2. *A valid inequality for $P_I(A)$ (for $P_I^*(A)$) cuts off a vertex of $P(A)$ (of $P^*(A)$) if and only if it is not a positive linear combination of inequalities of the system $Ax \geq 1$, $0 \leq x \leq 1$ (of the system $Ax \geq 1$, $x \geq 0$).*

3. *$P_I^*(A)$ is full dimensional. $P_I(A)$ is full dimensional if and only if $|N^i| \geq 2$ for all $i \in M$.*

In the following we assume that $P_I(A)$ is full dimensional.

4. *The inequality $x_j \geq 0$ defines a facet of $P_I^*(A)$ if and only if $|N^i \setminus \{j\}| \geq 1$ for all $i \in M$. It defines a facet of $P_I(A)$ if and only if $|N^i \setminus \{j\}| \geq 2$ for all $i \in M$.*

5. *The inequality $x_j \leq 1$ defines a facet of $P_I(A)$.*

6. *All facet defining inequalities $\alpha x \geq \alpha_0$ for $P_I^*(A)$ (for $P_I(A)$) have $\alpha \geq 0$ if $\alpha_0 > 0$.*

7. *The inequality*

$$\Sigma(x_j : j \in N^i) \geq 1$$

defines a facet of $P_I^*(A)$ if and only if there exists no $k \in M$ with $N^k \subsetneq N^i$. It defines a facet of $P_I(A)$ if and only if (i) there exists no $k \in M$ with with $N^k \subsetneq N^i$; and (ii) there exists no $j \in N \setminus N^i$ such that $A_{N^i \cup \{j\}}$ contains the circulant of order $|N^i| + 1$ with exactly one 0 in every row and column.

8. *The only minimal valid inequalities (hence the only facet defining inequalities) for $P_I^*(A)$ (for $P_I(A)$) with integer coefficients and righthand side equal to 1 are those of the system $Ax \geq 1$.*

Statements 1 through 6 are easily seen to be true. A proof of statement 7 for $P_I^*(A)$ is to be found in [1], and for $P_I(A)$ in [5].

Proof of 8. Let $\pi x \geq 1$ be any inequality with $\pi \geq 0$, and let $S := \{j \in N \mid \pi_j > 0\}$. If there exists $i \in M$ with $N^i \subseteq S$, then $\pi x \geq 1$ is either not minimal, or identical to some inequality of $Ax \geq 1$. Otherwise $N^i \setminus S \neq \phi$, $i \in M$, and hence $\bar{x}$ defined by $\bar{x}_j = 0$, $j \in S$, $\bar{x} = 1$, $j \in N \setminus S$, satisfies

$-$ $A\bar{x} \geq 1$, but $\pi\bar{x} = 0 < -1$; thus $\pi x \geq -1$ is not a valid inequality for $P_I^*(A)$ or $P_I(A)$. ▌

Valid inequalities for a polyhedron related to $P_I^*(A)$, namely the convex hull of those $x \in P_I^*(A)$ that satisfy a given inequality $cx \leq z_U - 1$, have been studied in [1]. The inequalities derived there have been successfully used as cutting planes, as reported in [2].

In this paper we characterize a class of facets of the polytope $P_I(A)$. It is a well-known fact (see Chvatal [4]) that if one forms a positively weighted sum of the inequalities of the system $Ax \geq 1$, $0 \leq x \leq 1$, and rounds up every coefficient on both sides, one gets a valid inequality. Such inequalities are called rank 1 Chvatal inequalities. It is not easy, however, to identify the conditions under which such an inequality is facet inducing. To see the nature of the problem, consider the following

Example 1.1. Let

$$A = \begin{vmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Adding the rows of A, dividing by 2 and rounding up yields the inequality

$$x_1 + x_2 + x_3 + x_5 + x_6 + x_7 + x_8 \geq 2$$

which is easily seen to define a facet of $P(A)$. Now let A' be the matrix obtained by adding to the three rows of A a fourth one,

$$(0\ 0\ 1\ 0\ 0\ 0\ 1\ 1).$$

The above inequality remains of course valid, but it does not define a facet of $P_I(A')$. In fact, it becomes redundant, since the last inequality of the system $A'x \geq 1$ cuts off the unique fractional vertex $x = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0, 0)$ of $P(A)$, and $P(A')$ has only integer vertices: thus $P_I(A') = P(A')$. ▌

2. A Class of Facets of $P_I(A)$

We will be studying inequalities of the form $\alpha x \geq 2$, with $\alpha_j = 0$, 1 or 2, $j \varepsilon N$. As before, we let M and N be the row and column index sets, respectively, of A. For each such inequality, we denote

$$J_t(\alpha) = \{j \varepsilon N \mid \alpha_j = t\}, \quad t = 0, 1, 2$$

and simply write J_t for $J_t(\alpha)$ whenever the meaning is clear from the context.

With each nonempty subset $S \subseteq M$ we associate the inequality $\alpha^S x \geq 2$, where

$$(2.1) \qquad \alpha_j^S = \begin{cases} 0 & \text{if } a_{ij} = 0 \text{ for all } i \varepsilon S \\ 2 & \text{if } a_{ij} = 1 \text{ for all } i \varepsilon S \\ 1 & \text{otherwise} \end{cases}$$

Notice that if $|S| = 1$, say $S = \{i\}$, the inequality $\alpha^S x \geq 2$ is just $a^i x \geq 1$ multiplied by 2.

Let C denote the class of inequalities $\alpha^S x \geq 2$ for all $S \subseteq M$. It is easy to see that C is in fact the class of inequalities obtainable from the system $Ax \geq 1$ by the following procedure, which we will also call C:

Procedure C

(i) Add the inequalities $a^i x \geq 1$, $i \varepsilon S$;

(ii) divide the resulting inequality by $|S| - \varepsilon$, $0.5 < \varepsilon < 1$; and

(iii) round up all coefficients to the nearest integer.

Thus for any $S \subseteq M$, $\alpha^S x \geq 2$ is a valid inequality for $P_I(A)$. To show that the converse is also true, we define for every $Q \subseteq N$,

$$M(Q) := \{i \varepsilon M \mid a_{ij} = 0, \ \forall j \varepsilon Q\},$$

with $M(\phi) := M$.

Theorem 2.1. *- Every valid-inequality $\beta x \geq 2$ for $P_I(A)$, with β_j integer, $j \in N$, is dominated by the inequality $\alpha^S x \geq 2$, where $S = M(J_0(\beta))$.*

Proof. By contradiction. W.l.o.g., we may assume that $\beta_j \in \{0,1,2\}$. The inequality $\alpha^S x \geq 2$ is well defined; for otherwise, i.e., if $M(J_0(\beta)) = \phi$, then $\bar{x}$ defined by $\bar{x}_j = 1$, $j \in J_0(\beta)$, $\bar{x}_j = 0$ otherwise, satisfies $Ax \geq 1$ and violates $\beta x \geq 2$, a contradiction.

If $\beta x \geq 2$ is not dominated by $\alpha^S x \geq 2$, then $\beta_{j_*} < \alpha_{j_*}$ for some $j_* \in N$. From the definition of α^S with $S = M(J_0(\beta))$, $\beta_j = 0$ implies $\alpha^S_j = 0$; hence $\beta_{j_*} = 1$ and $\alpha^S_{j_*} = 2$. This in turn implies that $a_{ij} = 1$ for all $i \in M(J_0(\beta))$. Therefore $\bar{x}$ defined by $\bar{x}_j = 1$, for $j \in J_0 \cup \{j_*\}$, $\bar{x}_j = 0$ otherwise, satisfies $Ax \geq 1$, $0 \leq x \leq 1$, but violates $\beta x \geq 2$, a contradiction. ▌

Thus every valid inequality for $P_I(A)$ with coefficients equal to 0, 1 or 2 is dominated by some inequality in the class C. From now on we therefore restrict our attention to this family.

Next we identify those members of the class C that are not strictly dominated by other members. Given any pair of inequalities $\alpha^S x \geq 2$ and $\alpha^T x \geq 2$ in C, such that $J_0(\alpha^S) = J_0(\alpha^T)$ and $T \subset S$, it is clear from the definitions that $\alpha^S x \geq 2$ dominates $\alpha^T x \geq 2$. Hence among all inequalities $\alpha^S x \geq 2$ with a fixed J_0, it is sufficient to consider those with $S = M(J_0)$.

Further, given any inequality $\alpha^S x \geq 2$ with $S = M(J_0)$, we will say that the set J_0 is ***maximal*** if for every $j \in J_1$ there exists $k \in J_1 \setminus \{j\}$ such that $a_{ik} = 1$ for all $i \in M(J_0 \cup \{j\})$. In other words, J_0 is maximal if transferring any column from J_1 to J_0 requires the transfer of some column from J_1 to J_2. This concept plays an important role in the sequel.

Theorem 2.2. The inequality $\alpha^S x \geq 2$, where $S = M(J_0)$, is minimal if and only if J_0 is maximal.

Proof. Necessity. If J_0 is not maximal, then there exists $j \in J_1$ such that for every $k \in J_1 \setminus \{j\}$, $a_{ik} = 0$ for some $i \in M(J_0 \cup \{j\})$. But then the inequality $\alpha^T x \geq 2$, where $T = M(J_0 \cup \{j\})$, strictly dominates $\alpha^S x \geq 2$.

Sufficiency. Suppose $\alpha^S x \geq 2$ is not minimal. Then there exists $\alpha^T x \geq 2$ in C such that $\alpha^T \leq \alpha^S$ and $\alpha^T_{j*} < \alpha^S_{j*}$ for some $j_* \in N \setminus J_0(\alpha^S)$. If $j_* \in J_2(\alpha^S)$, then $T \supset S$; but since $S = M(J_0(\alpha^S))$, this implies $J_0(\alpha^T) \subsetneq J_0(\alpha^S)$, contrary to $\alpha^T \leq \alpha^S$. Hence $j_* \in J_1(\alpha^S)$. But then $J_0(\alpha^S) \cup \{j_*\} \subseteq J_0(\alpha^T)$ and for all $k \in J_1(\alpha^S) \setminus \{j_*\}$, there exists some $i \in M(J_0(\alpha^S) \cup \{j_*\})$ such that $a_{ik} = 0$ (since $k \in J_1(\alpha^T)$); i.e., $J_0(\alpha^S)$ is not maximal.▌

Example 2.1. Consider the set covering polytope defined by the matrix

$$A = \begin{vmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \end{vmatrix}.$$

The inequality $\Sigma(x_j : j=1,\ldots,6) \geq 2$, obtained by applying Procedure C to the subsystem consisting of rows 1, 2, 3, 4, 6, is not minimal, since the set $J_0 = \{7\}$ associated with it is not maximal: column 6 can be added to J_0 without having to transfer any column from J_1 to J_2, since each of the remaining columns of J_1, namely 1, 2, 3, 4, 5, contains at least one 0 in some row i such that $a_{i6} = a_{i7} = 0$. Thus $\Sigma(x_j : j=1,\ldots,5) \geq 2$ is a valid inequality. It is also minimal, since the associated set $J = \{6,7\}$ is maximal. Another valid inequality in class C is $2x_1 + x_2 + x_3 + x_4 \geq 2$. The associated set $J_0 = \{5,6,7\}$ is maximal, so this inequality is also minimal.▌

Next we address the question as to which inequalities of the class C are facet inducing for $P_I(A)$. One obvious necessary condition for this is that the inequality be minimal. In stating the conditions for an inequality to be facet defining, we will assume that $P_I(A)$ is full dimensional. This is the case if and only if

(2.2) $\Sigma(a_{ij} : j \in N) \geq 2$, $i \in M$.

Assuming that (2.2) holds involves no loss of generality; for if not, then either $P_I(A) = \phi$, or else there exists some $F \subset N$, $F \neq 0$, such that $x \in P_I(A)$ implies $x_j = 1$ for all $j \in F$. In the former case the inequality is obviously not facet defining; whereas in the latter case setting $x_j = 1$, $j \in F$, and removing the inequalities satisfied by this assignment, produces a set covering polytope for which (2.2) is satisfied.

For any valid inequality $\alpha^S x \geq 2$ in C, consider the set of pairs $j, h \in J_1$ such that

(2.3) $a_{ij} + a_{ih} \geq 1$ for all $i \in M(J_0)$.

We will call these pairs the *2-covers* of $A^{J_1}_{M(J_0)}$, the submatrix of A whose rows and columns are indexed by $M(J_0)$ and J_1, respectively. We define the *2-cover graph of* $A^{J_1}_{M(J_0)}$ as the graph G that has a vertex for every $j \in J_1$ and an edge for every 2-cover of $A^{J_1}_{M(J_0)}$.

Further, for every $k \in J_0$, we define $T(k)$ as the set of rows such that k is the only column in J_0 to cover $T(k)$; i.e.,

$$T(k) = \{i \in M \mid a_{ik} = 1, a_{ij} = 0 \text{ for all } j \in J_0 \setminus \{k\}\}.$$

Theorem 2.3. *Let $P_I(A)$ be full dimensional and let $\alpha^S x \geq 2$ be a minimal valid inequality for $P_I(A)$, with $S = M(J_0)$. Then $\alpha^S x \geq 2$ defines a facet of $P_I(A)$ if and only if*

(i) every component of the 2-cover graph of $A^{J_1}_{M(J_0)}$ has an odd cycle;

(ii) for every $k \in J_0$ such that $T(k) \neq \phi$ there exists either

(a) some $j(k) \in J_2$ such that $a_{ij(k)} = 1$ for all $i \in T(k)$; or

(b) some pair $j(k)$, $h(k) \in J_1$ such that $a_{ij(k)} + a_{ih(k)} \geq 1$ for all $i \in T(k) \cup M(J_0)$.

Proof. Necessity. Suppose $\alpha^S x \geq 2$ defines a facet of $P_I(A)$. Then there exists a collection of n affinely independent points $x^i \in P_I(A)$, such that $\alpha^S x^i = 2$ for $i = 1,\ldots,n$. Let X be the $n \times n$ matrix whose rows are the vectors x^i; then X is of the form (modulo row and column permutations)

$$X = \begin{pmatrix} X_1 & 0 & X_3 \\ 0 & X_2 & X_4 \end{pmatrix},$$

where the columns of X_1, X_2 and X_3 are indexed by J_1, J_2 and J_0 respectively, X_2 is the identity matrix of order $|J_2|$, and every row of X_1 is a row of the edge-vertex incidence matrix of the 2-cover graph G of $A^{J_1}_{M(J_0)}$. Since X is nonsingular, X_1 is of full column rank, and hence every component of G is nonbipartite. Thus (i) holds.

To show that (ii) also holds, suppose there exists $k \in J_0$ and $T(k)$ for which neither (a) nor (b) is satisfied. Then $x_k = 1$ for every $x \in P_I(A)$ such that $\alpha^S x = 2$, which in turn implies (since $P_I(A)$ is full-dimensional) that the inequalities $\alpha^S x \geq 2$ and $x_k \leq 1$ are the same, a contradiction.

Sufficiency. Suppose conditions (i) and (ii) are satisfied. We exhibit a set of n affinely independent points $x^k \in P_I(A)$ such that $\alpha^S x^k = 2$, $k = 1,\ldots,n$.

For $t = 0, 1, 2$, let e^t and 0^t denote the $|J_t|$-vector whose components are all 1 and all 0, respectively. For $t = 0, 1, 2$, let e^t_j be the j^{th} unit vector with $|J_t|$ components.

Our first $|J_0|$ vectors x^k, $k \in J_0$, are defined as

$$x^k = \begin{cases} (e^0 - e^0_k, \quad 0^1 \quad , \; e^2_j) & \text{for some } j \in J_2, \text{ if } T(k) = \phi \text{ and } J_2 \neq \phi \\ (e^0 - e^0_k, \quad e^1_j + e^1_h \quad , \; 0^2) & \text{for some 2-cover } (j,h) \text{ if } T(k) = \phi = J_2 \\ (e^0 - e^0_k, \quad 0^1 \quad , \; e^2_{j(k)}) & \text{if } T(k) \neq \phi \text{ and (a) holds (with } j(k) \text{ as in (a))} \\ (e^0 - e^0_k, \; e^1_{j(k)} + e^1_{h(k)} \; , \quad 0^2) & \text{if } (k) \neq \phi \text{ and not (a) but (b) holds (with } j(k), h(k) \text{ as in (b))} \end{cases}$$

By property (ii), these vectors exist and belong to $P_I(A)$.

Our next $|J_1|$ vectors are of the form

$$x^k = (e^0, e^1_{j(k)} + e^1_{h(k)}, 0^2) , \; k \in J_1.$$

where the pair $j(k), h(k) \in J_1$ satisfies (2.3), and the vectors $e^1_{j(k)} + e^1_{h(k)}$ are linearly independent. By property (i), there exists a set of $|J_1|$ vectors $x^k \in P_I(A)$ satisfying these conditions.

Finally, the last $|J_2|$ vectors are of the form

$$x^k = (e^0, 0^1, e^2_k), \qquad k \in J_2.$$

Here the vectors e^2_k form the identity matrix of order $|J_2|$. The existence of these vectors $x^k \in P_I(A)$ follows from the definition of J_2.

It is now easy to see that the matrix X whose rows are the n vectors x^k, $k \in J_0 \cup J_1 \cup J_2$, is nonsingular. Also, every x^k satisfies $\alpha^S x^k = 2$. Hence $\alpha^S x \geq 2$ defines a facet of $P_I(A)$.∎

Example 2.2. Consider the matrix A of example 2.1 and the valid inequality $\Sigma(x_j : j = 1,\ldots,5) \geq 2$ for $P_I(A)$, which was shown to be minimal. We have $J_1 = \{1,\ldots,5\}$, $J_0 = \{6,7\}$ and $M(J_0) = \{1,2,3,4\}$. The two-cover graph of $A^{J_1}_{M(J_0)}$, shown in Fig. 1, is connected and has odd cycles; thus condition

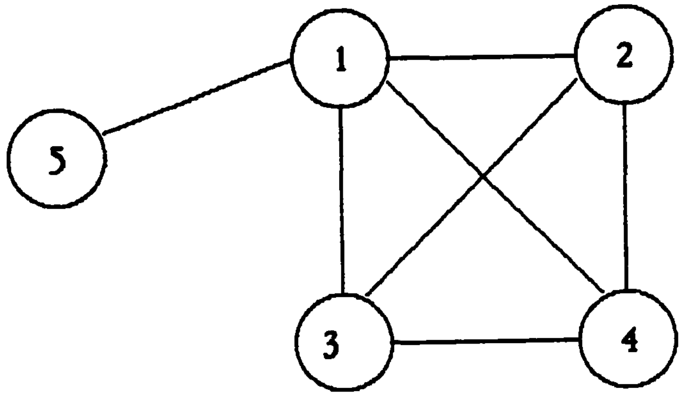

Fig. 1

(i) of Theorem 2.3 is satisfied. The only $k \in J_0$ such that $T(k) \neq \phi$ is 6, with $T(6) = \{6\}$; and any of the pairs $h, j \in J_1$ satisfies $a_{ih} + a_{ij} \geq 1$ for all $i \in M(J_0) \cup T(6) = \{1,2,3,4,6\}$. Hence $\Sigma(x_j : j = 1,\ldots,5) \geq 2$ induces a facet of $P_I(A)$. ▌

3. The Class C and Chvatal's Procedure

We have shown in section 2 that every valid inequality for $P_I(A)$ with coefficients equal to 0, 1 or 2 is dominated by some inequality in the class C, hence is obtainable by applying to the system $Ax \geq 1$ our Procedure C. The latter is easily seen to be a specialized version of Chvatal's procedure [4], which consists of recursively performing the following operations on the constraint set of an integer program:

0. Let the current set of constraints be

(I) $\quad \Sigma(a_{ij}x_j : j \in N) \geq b_i, \quad i \in M$, with all coefficients integer.

At the start, this is the set of inequalities which, together with the integrality conditions, defines feasibility.

1. Generate all distinct inequalities of the form $\gamma(\lambda)x \geq \gamma_0(\lambda)$, where

$$(3.1)\qquad \begin{aligned} \gamma_j(\lambda) &= \lceil \Sigma(\lambda_i a_{ij} : i \varepsilon M) \rceil, \quad j \varepsilon N \\ \gamma_0(\lambda) &= \lceil \Sigma(\lambda_i b_i : i \varepsilon M) \rceil \end{aligned}$$

with $\lambda_i \geq 0$, $i \varepsilon M$, where $\lceil a \rceil$ is the smallest integer greater than or equal to a, and where $\gamma_j(\lambda)$, $j \varepsilon N$, and $\gamma_0(\lambda)$, are relatively prime.

2. Redefine the system (I) by adding to it all the inequalities generated under 1, and go to 1.

Stop when no new inequalities can be obtained.

This procedure is known to yield the convex hull of the integer points satisfying the initial set (I) after a finite number of applications of the recursive steps 1, 2. The number of times the recursion needs to be applied to obtain a particular inequality is called the *rank* of the inequality. The original system together with the rank 1 inequalities forms the *elementary closure* of the system. Thus from Theorem 2.1 we have the following

Corollary 3.1. Every minimal valid inequality for $P_I(A)$ with coefficients equal to 0, 1 or 2 belongs to the elementary closure of the system $Ax \geq 1$, $0 \leq x \leq 1$.

Procedure C of section 2 is a specialized version of Step 1 above, with the multipliers λ_i defined by

$$(3.2)\qquad \lambda_i = \begin{cases} (|S|-\varepsilon)^{-1} & \text{if } i \varepsilon S \\ 0 & \text{otherwise} \end{cases}$$

for some $S \subseteq M$ and $0.5 < \varepsilon < 1$.

It turns out, however, that this choice of multipliers is much less special than it seems. In particular, again from Theorem 2.1, we have the following

Corollary 3.2. Let $\gamma(\lambda)x \geq \gamma_0(\lambda)$ be any rank 1 Chvatal inequality obtained from the system $Ax \geq 1$, $0 \leq x \leq 1$, with $\gamma_0(\lambda) = 2$. Then $\gamma(\lambda) \geq \gamma(\lambda^*)$, where $\lambda^* = (\lambda_i^*)$ is defined by (3.2), with $S = M(J_0(\lambda))$.

In view of our findings that all minimal valid inequalities for $P_I(A)$ with coefficients equal to 0, 1 or 2 belong to the elementary closure of $Ax \geq 1$, $0 \leq x \leq 1$, the question arises whether this property extends to some larger class of valid inequalities. Our next theorem answers this question in the negative.

Theorem 3.3. *For every $k \geq 3$, there exists a 0-1 matrix A and a minimal valid inequality $\beta x \geq k$ for $P_I(A)$ that is not contained in the elementary closure of the system $Ax \geq 1$, $0 \leq x \leq 1$.*

Proof. Let $k \geq 3$ and let A be the edge-vertex incidence matrix of K_{k+1}, the complete graph on $k + 1$ vertices. Then the inequality

$$\Sigma(x_j : j=1,\ldots,k+1) \geq k \tag{3.3}$$

is satisfied by every $x \in P_I(A)$, since any vertex cover of K_{k+1} contains at least k vertices. Also, (3.3) is minimal.

Now if the inequality (3.3) belongs to the elementary closure of the system $Ax \geq 1$, $0 \leq x \leq 1$, then there exists a set of multipliers $\lambda_{ij} \geq 0$, $i = 1,\ldots,k$, $j = i+1,\ldots,k+1$; μ_h and γ_h, $h = 1,\ldots,k+1$, satisfying the relations

$$\begin{array}{l} \lambda_{12} + \lambda_{13} + \lambda_{14} + \cdots + \lambda_{1,k+1} + \mu_1 - \gamma_1 \leq 1 \\ \lambda_{12} + \lambda_{23} + \lambda_{24} + \cdots + \lambda_{2,k+1} + \mu_2 - \gamma_2 \leq 1 \\ \lambda_{13} + \lambda_{23} + \lambda_{34} + \cdots + \lambda_{3,k+1} + \mu_3 - \gamma_3 \leq 1 \\ \text{-------------------------------------} \\ \lambda_{1,k+1} + \lambda_{2,k+1} + \cdots + \lambda_{k,k+1} + \mu_{k+1} - \gamma_{k+1} \leq 1 \end{array} \tag{3.4}$$

and

$$\Sigma(\lambda_{ij} : i=1,\ldots,k; j=i+1,\ldots,k+1) - \Sigma(\gamma_h : h=1,\ldots,k+1) > k - 1. \tag{3.5}$$

But adding the inequalities (3.4) yields

$$2\Sigma(\lambda_{ij} : i=1,\ldots,k; j=i+1,\ldots,k+1) + \Sigma(\mu_h - \gamma_h : h=1,\ldots,k+1) \leq k + 1,$$

or

$$\begin{aligned} &\Sigma(\lambda_{ij} : i=1,\ldots,k; j=i+1,\ldots,k+1) - \Sigma(\gamma_h : h=1,\ldots,k+1) \\ \leq\ &\Sigma(\lambda_{ij} : i=1,\ldots,k; j=i+1,\ldots,k+1) + \tfrac{1}{2}\Sigma(\mu_h - \gamma_h : h=1,\ldots,k+1) \\ \leq\ &\tfrac{1}{2}k + \tfrac{1}{2} \end{aligned}$$

which for $k \geq 3$ contradicts (3.5).▌

4. The Class C and Full Circulant Submatrices

In this section we examine the relationship between minimal valid inequalities for $P_I(A)$ with coefficients equal to 0, 1 or 2, and circulant submatrices of A with exactly one zero in every row and column. Such a matrix, if of order k, will be denoted C_k^{k-1}. This is the $k \times k$ matrix with exactly $k - 1$ ones and one zero in every row and column. It will be called the *full circulant* of order k.

Consider the inequality $\alpha^S x \geq 2$, where $S = M(J_0)$. Often there exist proper subsets $T \subset S$ such that $\alpha^T x \geq 2$ is the same inequality as $\alpha^S x \geq 2$. Such a subset $T \subset S$ will be called *C-equivalent to S.* If T is C-equivalent to S and no proper subset of T has this property, we say that T is a ***minimal C-equivalent subset*** of S. A set $S = M(J_0)$ may have several minimal C-equivalent subsets.

Now let the inequality $\alpha^S x \geq 2$ be minimal, i.e., let J_0 be maximal. If $M(J_0)$ has a C-equivalent subset of cardinality ≤ 2, then the inequality $\alpha^S x \geq 2$ is of course dominated by the sum of at most two inequalities of the system $Ax \geq 1$. Suppose this is not the case, i.e., every minimal C-equivalent subset of $M(J_0)$ has cardinality ≥ 3. Then we have the following

Theorem 4.1. *For every minimal C-equivalent subset T of $M(J_0)$, the matrix $A_T^{J_1}$ contains as a submatrix C_t^{t-1}, the full circulant of order $t = |T|$.*

Proof. Since T is minimal with respect to the property that $\alpha^T = \alpha^S$, and $\alpha^S x \geq 2$ is a minimal inequality, it follows that for any $Q \subsetneq T$, $\alpha^Q_{j_*} > \alpha^S_{j_*}$ for some $j_* \in N$. Since $J_0(\alpha^S) \subset J_0(\alpha^Q)$, $j_* \notin J_0(\alpha^S)$. Also, $j_* \notin J_2(\alpha^S)$. Thus $j_* \in J_1(\alpha^S)$, and $a_{ij_*} = 1$ for all $i \in Q$, $a_{ij_*} = 0$ for some $i \in T \setminus Q$. Since this is true for any proper subset of T and in particular for every subset of the form $Q = T \setminus \{i\}$ for some $i \in T$, it follows that for every row i $\in$ T there exists a column $j(i) \in J_1$ such that $a_{hj(i)} = 0$ for h = i and $a_{hj(i)} = 1$ for all $h \in T \setminus \{i\}$. Clearly, the t columns j(i), $i \in T$, must be distinct since every column has exactly one zero in position i. But the submatrix of $A_T^{J_1}$ consisting of these t columns is precisely C_t^{t-1}.∎

Example 4.1. Consider the matrix

$$A = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{vmatrix}$$

The inequality of C associated with the row set S = {2,3,4,5} is

$$x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \geq 2,$$

and it is minimal, since $J_0 = \{1\}$ is maximal: any attempt to extend J_0 results in the transfer of some column from J_1 to J_2. (For instance, if column 2 is appended to J_0, the set $M(J_0)$ shrinks to rows 3, 4 and column 5 is transferred to J_2).

However, the set $M(J_0) = S$ is not minimal for the inequality $\alpha^S x \geq 2$. Removing any one of the four rows of S produces a minimal C-equivalent subset. If T_1 denotes the subset {2,3,4}, A_{T_1} contains C_3^2 in columns 3, 5, 6. For $T_2 = \{2,3,5\}$, C_3^2 consists of columns 2, 3, 4 of A_{T_2}. Similarly, for

$T_3 = \{2,4,5\}$ and $T_4^2 = \{3,4,5\}$, C_3 consists of columns 2, 6, 7 of A_{T_3} and columns 4, 5, 7 of A_{T_4}, respectively.∎

Theorem 4.1 establishes a correspondence between the minimal inequalities of the class C and the full circulant submatrices of A. The correspondence is not one to one, since for any given minimal inequality $\alpha^S x \geq 2$ of C, there may be several minimal sets T_i C-equivalent to S, each one containing one or several full circulants of order $|T_i|$. Nevertheless the full circulants of A can be used to list all the minimal inequalities in the class C, as will presently be shown.

For convenience, we will adopt the notation

$$J_0(S) := \{j \in N \mid a_{ij} = 0 \text{ for all } i \in S\}, \quad S \subseteq M.$$

Clearly, $J_0(S) = J_0(\alpha^S)$, and also $J_0(T) = J_0(\alpha^S)$ for any C-equivalent subset T of S.

Corollary 4.2. Every minimal inequality in C can be obtained by using Procedure C with S restricted to subsets of M such that

(i) $J_0(S)$ is maximal;

(ii) A_S contains a full circulant of order $|S|$;

(iii) There exists no $T \supsetneq S$ satisfying (i) and (ii).

Proof. From Theorem 4.1, Procedure C can be restricted to sets S satisfying (i) and (ii). Further, Procedure C used with a set S that satisfies (i), (ii) but not (iii) yields the same inequality as when used with any $S' \supset S$ such that $J_0(S') = J_0(S)$ and S' satisfies (ii), (iii) (with $S = S'$).∎

The correspondence between full circulants and minimal members of the class C is also helpful in counting the latter, viz., in bounding their number.

Corollary 4.3. The number of minimal inequalities in C is $O(m^k)$, where $m = |M|$ and k is the cardinality of the largest full circulant submatrix of A.

Proof. From the previous Corollary, the number of minimal members of C is bounded by the number of row sets T such that A_T contains C_t^{t-1} with $t = |T|$. Since A has $\binom{m}{i}$ row sets of size i, if k is the order of the largest full circulant submatrix of A, then the number of row sets T such that A_T contains a full circulant of order $|T|$ is bounded by

$$\sum_{i=3}^{k} \binom{m}{i} < m^k. \blacksquare$$

It is a well known result in polyhedral combinatorics (see, for instance, [6], [7], [8]) that minimal inequalities for a polyhedron $P_I(A)$ can often be obtained by lifting minimal inequalities for a polyhedron $P_I(A^V)$ for some $V \subset N$. The above Theorem and its Corollaries suggest that the minimal inequalities of the class C for $P_I(A)$ might be obtainable by lifting the minimal inequalities of C for some polyhedra of the form $P_I(A^K)$, where K is the column index set of a full circulant. This, however, is not true, since in most cases where $\alpha^S x \geq 2$ is a minimal inequality for $P_I(A)$, the corresponding inequality

$$\Sigma(x_j : j \in K) \geq 2$$

is not minimal for $P_I(A^K)$. This is illustrated by the following

Example 4.2. Let

$$A = \begin{vmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{vmatrix}.$$

The inequality

$$x_1 + \ldots + x_6 \geq 2$$

is minimal for $P_I(A)$, but

$$x_1 + x_2 + x_3 \geq 2$$

is not minimal for $P_I(A^K)$, where $K = \{1,2,3\}$, although A^K contains the full circulant C_3^2. ∎

Thus restricting ourselves to the lifting of minimal inequalities for polyhedra of the form $P_I(A^K)$, with K defined as above, would make us miss many, if not most, minimal inequalities for $P_I(A)$. The situation changes, however, if instead of A^K we consider submatrices of the form A^L with $L = K \cup J_0$, as shown by our next result.

Theorem 4.4. *Let $\alpha^S x \geq 2$ be a minimal valid inequality for $P_I(A)$, let T be a minimal C-equivalent subset of $M(J_0)$, with $|T| \geq 3$, and let K be the column index set of a full circulant C_t^{t-1} contained in $A_T^{J_1}$, where $t = |T|$. Then*

(4.1) $$\Sigma(x_j : j \in K) \geq 2$$

is a minimal valid inequality for $P_I(A^L)$, where $L = K \cup J_0$.

Proof. If Procedure C is applied to $A^L x \geq 1$ with $S = M(J_0)$, we obtain the inequality (4.1), which is thus valid for $P_I(A^L)$. Also, J_0 is maximal for A^L, since it is by definition maximal for A. Thus (4.1) is minimal for $P_I(A^L)$. ∎

In the above Example, $K = \{1,2,3\}$, $J_0 = \{7,8\}$, and

$$x_1 + x_2 + x_3 \geq 2$$

is a minimal valid inequality for $P_I(A^L)$, where $L = \{1,2,3,7,8\}$.

The converse of Theorem 4.4 is not true in general; i.e., if A contains the full circulant C_t^{t-1} with row and column sets R and K, respectively, and (4.1) is a minimal valid inequality for $P_I(A^L)$, where $L = K \cup J_0(R)$, the corresponding inequality $\alpha^R x \geq 2$ obtained by applying Procedure C to the system $Ax \geq 1$ with $S = R$ is not necessarily minimal. To see this it is sufficient to notice that although $J_0(R)$ is maximal with respect to A^L, it need not be maximal with respect to A.

Example 4.3. Consider the matrix

$$A = \begin{vmatrix} 1&1&0&0&0&1&0&1 \\ 0&1&1&0&0&1&1&0 \\ 1&0&1&0&0&0&1&0 \\ 1&1&0&0&0&0&0&1 \\ 0&1&1&0&0&0&1&0 \\ 0&1&0&1&0&1&0&1 \\ 0&0&1&0&1&0&0&0 \end{vmatrix}$$

The circulant C_3^2 in the upper lefthand corner of A has row and column sets R = {1,2,3} and K = {1,2,3}, respectively, and $J_0(R) = \{4,5\}$.

The inequality

$$x_1 + x_2 + x_3 \geq 2$$

is minimal for $P_I(A^L)$, where $L = K \cup J_0(R) = \{1,\ldots,5\}$, since $J_0(R)$ is is maximal with respect to A^L. On the other hand, the inequality

$$x_1 + x_2 + x_3 + x_6 + x_7 + x_8 \geq 2,$$

obtained by applying Procedure C to $Ax \geq 1$ with S = R, is not minimal, since $J_0(R)$ is not maximal with respect to A: it is possible to add column 6 to J_0 without transferring any column from J_1 to J_2. If J_0' denotes the augmented set {4,5,6}, $M(J_0') = \{3,4,5\}$, and the inequality obtained by applying Procedure C with $S = M(J_0')$ is

$$x_1 + x_2 + x_3 + x_7 + x_8 \geq 2,$$

which is minimal and strictly dominates the inequality obtained by using S = R. ▌

Further connections between minimal and facet defining inequalities for polytopes of the form $P_I(A^L)$ and $P_I(A)$ can be established by using the theory of inequality lifting, but this is left to another paper.

5. Generating Minimal Inequalities

We first discuss systematic ways of generating all minimal inequalities in C, then we address the issue of generating certain subsets of inequalities.

The set of minimal inequalities in C can be partially ordered by inclusion applied to the corresponding sets J_0. In other words, if $\alpha x \geq 2$ and $\beta x \geq 2$ are minimal inequalities in C, one can say that $\alpha x \geq 2$ precedes $\beta x \geq 2$ if $J_0(\alpha) \subset J_0(\beta)$. Thus one can define a directed graph G with a node for every minimal inequality of C, and an arc for every pair of minimal inequalities such that one member of the pair is an immediate successor of the other.

To generate all minimal inequalities without predecessors, one can use Theorem 4.1 and list all maximal sets $T \subseteq M$ such that A_T contains a full circulant submatrix. For each such T, if $J_0(T)$ is maximal, then $\alpha^T x \geq 2$ is one of the minimal inequalities without predecessors; and conversely, all such inequalities can be obtained this way.

Given a minimal inequality $\alpha^S x \geq 2$, its immediate successors in G are those inequalities $\alpha^T x \geq 2$ such that $T \subset S$, $J_0(\alpha^T)$ is maximal, and there exists no W, with $T \subset W \subset S$ and $J_0(T) \neq J_0(W) \neq J_0(S)$, such that $J_0(W)$ is maximal. Since there is no reason to be interested in generating inequalities that are the sum of two inequalities of $Ax \geq 1$, this rule can be amended by requiring that $|T| \geq 3$.

These two rules, one for generating all the minimal inequalities without predecessors and the other for generating all immediate successors of a given inequality, suffice for generating the whole family of minimal inequalities in C. Note, however, that this procedure is not free of redundancy, in that a minimal inequality can have more than one predecessor. Thus some checking is required to exclude repetition.

Example 5.1. Consider the matrix:

$$A = \begin{vmatrix} 0&1&1&1&1&1&0&0&0&0&1&0&0&0&1 \\ 1&0&1&1&1&0&1&0&0&0&1&1&0&0&0 \\ 1&1&0&1&1&0&0&1&0&0&0&1&1&0&0 \\ 1&1&1&0&1&0&0&0&1&0&0&0&1&1&0 \\ 1&1&1&1&0&0&0&0&0&1&0&0&0&1&1 \end{vmatrix}$$

The following tableau shows the minimal inequalities in C, all of which happen to define facets of $P_I(A)$.

Ineq. No.	$M(J_0)$	J_0	$\alpha_j : j =$ 1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1,2,3,4,5	∅	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1,2,3,4	10	1	1	1	1	2	1	1	1	1	0	1	1	1	1	1
3	1,2,3,5	9	1	1	1	2	1	1	1	1	0	1	1	1	1	1	1
4	1,2,4,5	8	1	1	2	1	1	1	1	0	1	1	1	1	1	1	1
5	1,3,4,5	7	1	2	1	1	1	1	0	1	1	1	1	1	1	1	1
6	2,3,4,5	6	2	1	1	1	1	0	1	1	1	1	1	1	1	1	1
7	1,2,3	9,10,14	1	1	1	2	2	1	1	1	0	0	1	1	1	0	1
8	1,2,4	8,10	1	1	2	1	2	1	1	0	1	0	1	1	1	1	1
9	1,2,5	8,9,13	1	1	2	2	1	1	1	0	0	1	1	1	0	1	1
10	1,3,4	7,10	1	2	1	1	2	1	0	1	1	0	1	1	1	1	1
11	1,3,5	7,9	1	2	1	1	2	1	0	1	0	1	1	1	1	1	1
12	1,4,5	7,8,12	1	2	2	1	1	1	0	0	1	1	1	0	1	1	1
13	2,3,4	6,10,15	2	1	1	1	2	0	1	1	1	0	1	1	1	1	0
14	2,3,5	6,9	2	1	1	2	1	0	1	1	0	1	1	1	1	1	1
15	2,4,5	6,8	2	1	2	1	1	0	1	0	1	1	1	1	1	1	1
16	3,4,5	6,7,11	2	2	1	1	1	0	0	1	1	1	0	1	1	1	1

Consider the first inequality with $S = M = \{1,\ldots,5\}$. S contains the circulant C_5^4 whose columns are $1,\ldots,5$. We have $J_0(S) = \phi$, $\alpha^S = (1,\ldots,1)$ and $J_0(S)$ is maximal. Thus $\alpha^S x \geq 2$ is a minimal inequality.

Consider the second inquality with $T = \{1,2,3,4\}$ and $J_0(T) = \{10\}$, and $\alpha^T = (1,1,1,1,2,1,1,1,1,0,1,1,1,1,1)$. Since $J_0(T)$ is maximal, $\alpha^T x \geq 2$ is a minimal inequality. Since there exists no set W such that $T \subset W \subset S$ and $J_0(S) \neq J_0(W) \neq J_0(T)$, $\alpha^T x \geq 2$ is an immediate successor of $\alpha^S x \geq 2$.

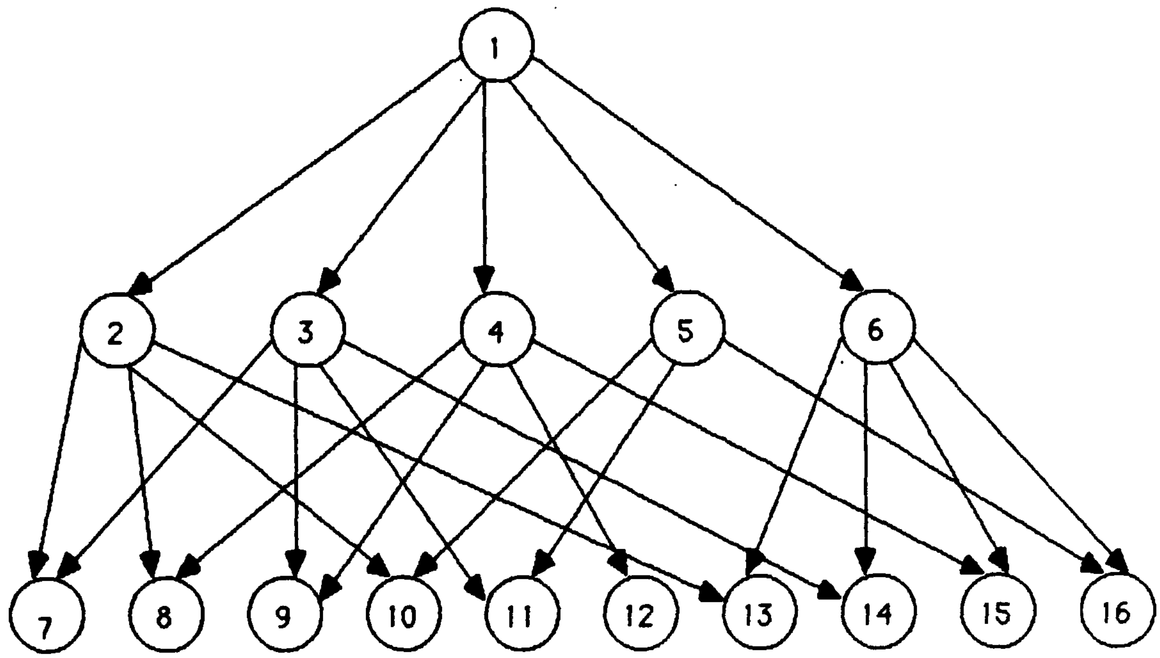

Fig. 2.

The other relations can be obtained similiarly. Fig. 2 shows the directed graph representing the precedence relation between the inequalities in the tableau. In this particular case, the graph has a single source, due to the fact that $J_0(S)$ is maximal for $S = M$.

An alternative way of generating all the minimal inequalities of C that are not the sum of two inequalities of $Ax \geq 1$, is to use the above defined ordering but work backwards, from larger to smaller sets J_0. To generate all minimal inequalities without successors, one can list all triplets T of pairwise nonorthogonal rows such that there exists no other triplet W with $J_0(T) \subsetneq J_0(W)$. For each such T, $J_0(T)$ is easily seen to be maximal, hence $\alpha^T x \geq 2$ is a minimal inequality without successors; and conversely, each such inequality can be obtained this way.

Given a minimal inequality $\alpha^S x \geq 2$, its immediate predecessors in G are those inequalities $\alpha^T x \geq 2$ such that $S \subset T$, $J_0(T)$ is maximal, and there exists no W with $S \subset W \subset T$ and $J_0(S) \neq J_0(W) \neq J_0(T)$ such that $J_0(W)$ is maximal. It follows from these conditions that $T \setminus S$ must have the property that $a_{ij} = 0$ for at least one $i \in T \setminus S$, $j \in J_2(S)$. Thus when no such T exists, the minimal inequality $\alpha^S x \geq 2$ has no predecessors.

Again, this procedure has redundancies, since a minimal inequality may have more than one successor; therefore checks are required to avoid repetition.

A frequent situation encountered in practice is the one where a fractional solution to the current problem is available, and one is interested in generating an inequality in C that cuts it off. Let $\bar{x}$ be a fractional solution to $Ax \geq 1$, $0 \leq x \leq 1$, with

$$I = \{j \in N \mid \bar{x}_j = 1\}, \quad F = \{j \in N \mid 0 < \bar{x}_j < 1\},$$

and let

$$Q = \{i \in M \mid \bar{x}_j = 1 \text{ implies } a_{ij} = 0\}.$$

Theorem 5.1. *Let*

$$(5.1) \qquad \sum(\alpha_j x_j : j \in F) \geq 2$$

be an inequality obtained by applying Procedure C to the system $A^F_Q x_F \geq 1$, $0 \leq x_F \leq 1$, such that

$$(5.2) \qquad \sum(\alpha_j \bar{x}_j : j \in F) < 2,$$

and let T be any C-equivalent subset of $M(J_0(\alpha_F))$ (=Q). Then the inequality $\alpha^S x \geq 2$ obtained by applying Procedure C to the system $Ax \geq 1$, $0 \leq x \leq 1$, with $S = T$, cuts off $\bar{x}$.

Conversely, if $\alpha x \geq 2$ is a valid inequality for $P_I(A)$ that cuts off $\bar{x}$, then $\alpha_j = 0$, $j \in I$, and (5.1) is a valid inequality for $P_I(A^F_Q)$ that cuts off $\bar{x}_F$.

Proof. In proving both statements we will make use of the following fact that we claim to be true. If $\alpha x \geq 2$ is a valid inequality for $P_I(A)$

that cuts off $\bar{x}$, then $\alpha_j = 0$ for all $j \varepsilon I$. For suppose that $\alpha_{j_*} \geq 1$ for some $j_* \varepsilon I$. If $\alpha_{j_*} = 2$, then $\alpha\bar{x} \geq \alpha_{j_*}\bar{x}_{j_*} = 2$, contrary to the assumption that $\alpha x \geq 2$ cuts off $\bar{x}$. Thus $\alpha_{j_*} = 1$. Then there exists $i_* \varepsilon M(J_0(\alpha))$ such that $a_{i_*j_*} = 0$, or else $\alpha x \geq 2$ would not be valid for $P_I(A)$. But then substituting $\bar{x}_{j_*} = 1$ into the inequalities $a^{i_*}x \geq 1$ and $\alpha x \geq 2$ yields

$$\Sigma(a_{i_*j}\bar{x}_j : j\varepsilon N\backslash\{j_*\}) \geq 1$$

and

$$\Sigma(\alpha_j\bar{x}_j : j\varepsilon N\backslash\{j_*\}) \geq 1,$$

respectively, with $a_{i_*j} \leq \alpha_j$ for all $j \varepsilon N \setminus \{j_*\}$. Since $\bar{x}$ satisfies the first of these inequalities, it cannot violate the second one. This proves the claim.

Now let (5.1) and T be as stipulated, and suppose $\alpha^S x \geq 2$, where $S \subseteq T$, does *not* cut *off* $\bar{x}$. *Since* $\bar{x}$ satisfies (5.2), it must be the case that $\alpha_j^S > 0$ for some $j \varepsilon I$, contrary to what we have just proved. Thus $\alpha x \geq 2$ cuts off $\bar{x}$.

Conversely, if $\alpha x \geq 2$ is any valid inequality for $P_I(A)$ such that $\alpha\bar{x} < 2$ then $\alpha_j = 0$ for $j \varepsilon I$ and hence (5.2) holds. Also, the same instance of Procedure C that yields $\alpha x \geq 2$ when applied to the system $Ax \geq 1$, $0 \leq x \leq 1$, yields the inequality (5.1) when applied to the system $A_Q^F x_F \geq 1$, $0 \leq x_F \leq 1$, hence (5.1) is valid for $P_I(A_F^F)$. ∎

Example 5.2. Consider the problem of minimizing $x_1 + x_2 + x_3 + 3x_4 + 3x_5 + 3x_7$ subject to $Ax \geq 1$, $x \varepsilon \{0,1\}^7$, where A is obtained from the matrix of Example 2.1 by adding the row (0,0,0,1,1,1,1). The (unique) optimal solution is $\bar{x} = (\frac{1}{2},\frac{1}{2},\frac{1}{2},0,0,1,0)$. We have $F = \{1,2,3\}$, $I = \{6\}$ and $Q = \{1,2,3,4\}$, and $\bar{x}_1 + \bar{x}_2 + \bar{x}_3 = \frac{3}{2} < 2$, i.e., condition (5.2) holds.

Applying Procedure C to $Ax \geq 1$, $0 \leq x \leq 1$, with $S = Q$, we obtain the inequality $x_1 + x_2 + x_3 + x_4 + x_5 \geq 2$, which cuts off $\bar{x}$.▌

Thus in order to generate a minimal inequality in C that cuts off a given fractional solution $\bar{x}$, one can apply either one of the two procedures discussed at the beginning of this section to the subsystem $A_S^F x_F \geq 1$, $0 \leq x_F \leq 1$, up to the point where an inequality for $P_I(A_S^F)$ is generated that cuts off $\bar{x}_F$; the corresponding inequality for $P_I(A)$ that cuts off $\bar{x}$ is then easily identified. If no inequality for $P_I(A_S^F)$ is found that cuts off $\bar{x}_F$, then there is no inequality for $P_I(A)$ in C that cuts off $\bar{x}$.

Some of the more recent methods for solving set covering problems never solve the linear programming relaxation of the problem and thus never generate fractional solutions to be cut off. These methods (see, for example, [3]) use instead subgradient optimization or other techniques to find an approximate (feasible) solution to the dual of the linear programming relaxation, whose objective function value provides a lower bound on the value of an optimal cover. To use the inequalities of the class C in this context, one has to be able to answer the following question: given a feasible solution u to the dual of the linear relaxation of the set covering problem, is there an inequality in C whose addition to the constraint set would make it possible to strengthen the lower bound associated with u? Our next theorem addresses this question.

Theorem 5.2. *Let $\alpha x \geq 2$ be a minimal valid inequality for $P_I(A)$, and let T be any C-equivalent subset of $M(J_0(\alpha))$. Further, let $u \in \mathbb{R}^m$ satisfy $u \geq 0$, $uA \leq c$, and define*

$$\delta(T)_k := \min\{c_j - \Sigma(u_i a_{ij} : i \in M\backslash T) : j \in J_k(\alpha)\},\ k = 1,2,$$

with

$$\delta(T) := \min\{\delta(T)_1, \tfrac{1}{2}\delta(T)_2\}$$

Then

$$\Sigma(u_i : i\varepsilon M\setminus T) + 2\delta(T) \leq cx$$

for all $x \varepsilon \{0,1\}^n$ *satisfying* $Ax \geq 1$.

Proof. Define $\bar{u} \varepsilon \mathbb{R}^{m+1}$ by

$$\bar{u}_i = \begin{cases} 0 & i \varepsilon T \\ u_i & i \varepsilon M \setminus T \\ \delta(T) & i = |M| + 1. \end{cases}$$

Then $\bar{u} \geq 0$ and

$$c_j - \Sigma(\bar{u}_i a_{ij} : i\varepsilon M) - \bar{u}_{|M|+1}\alpha_j$$

$$= c_j - \Sigma(u_i a_{ij} : i\varepsilon M\setminus T) - \delta(T)\alpha_j \qquad j \varepsilon N$$

$$\geq c_j - \Sigma(u_i a_{ij} : i\varepsilon M\setminus T) - \begin{cases} \delta(T)_1 \geq 0 & j \varepsilon J_1(\alpha) \\ \delta(T)_2 \geq 0 & j \varepsilon J_2(\alpha) \\ 0 \geq 0 & j \varepsilon J_0(\alpha), \end{cases}$$

i.e., $\bar{u}$ is a feasible solution to the linear program dual to

(5.3) $\quad \min\{cx : Ax \geq 1, \alpha x \geq 2, x \geq 0\}$.

Therefore

$$\Sigma(\bar{u}_i : i=1,\ldots,|M|+1) = \Sigma(u_i : i\varepsilon M\setminus T) + 2\delta(T)$$
$$\leq cx$$

for any x satisfying $Ax \geq 1$, $\alpha x \geq 2$, $x \geq 0$, hence for any $x \varepsilon \{0,1\}^n$ satisfying $Ax \geq 1$. ▌

Corollary 5.3. *Adding* $\alpha x \geq 2$ *to the constraint set* $Ax \geq 1$ *strengthens the lower bound on* cx *provided by* u *if and only if*

(5.4) $\quad \delta(T) > \frac{1}{2}\Sigma(u_i : i\varepsilon T)$.

If (5.4) holds and, in addition, u *is an optimal solution to the dual of*

$$(5.5)\quad \min\{cx : Ax \geq 1,\quad x \geq 0\},$$

then the inequality $\alpha x \geq 2$ *cuts off all optimal solutions to (5.5).*

Proof. The difference between the lower bounds provided by $\bar{u}$ and u (i.e., the difference due to $\alpha x \geq 2$) is

$$\begin{aligned} &\Sigma(u_i : i \varepsilon M\backslash T) + 2\delta(T) - \Sigma(u_i : i \varepsilon M) \\ &= 2\delta(T) - \Sigma(u_i : i \varepsilon T), \end{aligned}$$

which proves the first statement.

If this difference is positive and u is an optimal solution to the dual of (5.5), then for any optimal solution $\hat{x}$ to (5.5),

$$\begin{aligned} c\hat{x} &= \Sigma(u_i : i \varepsilon M) \\ &< \Sigma(\bar{u}_i : i \varepsilon M\backslash T) + 2\delta(T) \leq cx \end{aligned}$$

for any x satisfying $Ax \geq 1$, $\alpha x \geq 2$, $x \geq 0$. Hence the inequality $\alpha x \geq 2$ cuts off $\hat{x}$. ▌

Note that a straightforward modification of Theorem 5.2 and Corollary 5.3 holds for the case when the constraint set $Ax \geq 1$ is amended by $\alpha^i x \geq 2$, $i \varepsilon M'$, i.e., the dual constraint set $uA \leq c$ is replaced by

$$uA + \Sigma(u_i \alpha^i : i \varepsilon M') \leq c.$$

In other words, inequalities in C that improve the lower bound can be generated recursively.

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