Arbitrary Order and A-Stable Numerical Method for Solving Algebraic Ordinary Differential Equation by Power Series

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Abstract: The arithmetic operations and functions of power series can be defined by Fortran 95 or C++ language[2] easily. The functions represented by these language, which consist of arithmetic operations, pre-defined functions and conditional statements, can be expanded in power series.

Using this power series, the solution for an algebraic ordinary differential equation can be expanded up to arbitrary order. It is the arbitrary order calculation formula as Runge-Kutta formula for the ordinary differential equation. The power series can be used for the evaluation of the errors and determination of the optimal step size within given error allowance easily. In addition, we can transform Power series into Padé series, which give an arbitrary order and A-stable formula for solving algebraic ordinary differential equation numerically.

Key-Word : algebraic differential equation, A-stable, arbitrary order, power series, Padé series, C++ language

1. Introduction

A differential-algebraic equation (DAE) has the form

(1.1) $F(y, y', x) = 0$

with initial values

(1.2) $y(x_0) = y_0, \quad y'(x_0) = y_1$

where F and y is a vector function for which we assumed sufficient differentiablity. And the initial values to be consistent, i.e.

(1.3) $F(y_0, y_0', x_0) = 0$

The solutions of (1.1) can be assumed that

(1.4) $y = y_0 + y_1 x + e x^2$,

where e is a vector function which is the same size as y_0 and y_0'. Substitute (1.4) into (1.1) and neglect higher order terms, we have the linear equation of e in the form

(1.5) $Ae = B$

where A and B is a constant matrix, a constant vector respectively. Solve this equation of (1.5), the coefficients of x^2 in (1.4) can be determined. Repeating above procedure for higher order terms, we can get the arbitrary order power series of the solutions for (1.1). This method was probably known long before.

The power series given by above procedure can be transformed into Padé series.

According to E. Hairer and G. Wanner [3,p60], if $M \le L \le M + 2$, where M and L is the degree of numerator and denominator in Padé series respectively, then this Padé series gives an A-stable formula for

an ordinary differential equation.

2. Power Series

In this section, we explain basic idea to expand the functions into power series. Please refer to Rall[7] , Henrici[4] and Hirayama[5] in detail.

2.1 Arithmetic of Power Series

Arithmetic program for power series can be made easily. Power series can be defined in the form

$$(2.1)\quad \mathrm{f}(x) = \mathrm{f}_0 + \mathrm{f}_1 x + \mathrm{f}_2 x^2 + \mathrm{f}_3 x^3 + \cdots,$$

$$(2.2)\quad \mathrm{g}(x) = \mathrm{g}_0 + \mathrm{g}_1 x + \mathrm{g}_2 x^2 + \mathrm{g}_3 x^3 + \cdots,$$

$$(2.3)\quad \mathrm{h}(x) = \mathrm{h}_0 + \mathrm{h}_1 x + \mathrm{h}_2 x^2 + \mathrm{h}_3 x^3 + \cdots.$$

2.1.1 Addition and Subtraction

If $\mathrm{h}(x) = \mathrm{f}(x) \pm \mathrm{g}(x)$, the coefficients of f, g, h have the relation

$$(2.4)\quad \mathrm{h}_i = \mathrm{f}_i \pm \mathrm{g}_i$$

2.1.2 Multiplication

If $\mathrm{h}(x) = \mathrm{f}(x)\,\mathrm{g}(x)$, the coefficients of f, g, h have the relations

$$(2.5)\quad \mathrm{h}_n = \sum_{k=0}^{n} \mathrm{f}_k\, \mathrm{g}_{n-k}$$

2.1.3 Division

If $\mathrm{h}(x) = \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}$, the coefficients of f, g, h have the relations

$$(2.6)\quad h_0 = \frac{f_0}{g_0},$$

$$(2.7)\quad \mathrm{h}_n = \frac{1}{g_0}\left(\mathrm{f}_n - \sum_{k=0}^{n-1} \mathrm{h}_k \mathrm{g}_{n-k}\right) \quad (n \geq 1)$$

2.1.4 Exponential Function

If $\mathrm{h}(x) = e^{\mathrm{f}(x)}$, then $\dfrac{d\,\mathrm{h}}{dx} = \mathrm{h}\dfrac{d\,\mathrm{f}}{dx}$. Substitute (2.1) and (2.3) into this differential equation and compare coefficients of power series, we can get the following relations

$$(2.8)\quad h_0 = e^{f_0},$$

$$(2.9)\quad \mathrm{h}_n = \frac{1}{n}\sum_{k=1}^{n} k\, \mathrm{h}_{n-k} \mathrm{f}_k \quad (n \geq 1).$$

2.1.5 Power Function

If $\mathrm{h}(x) = \mathrm{f}(x)^a$ (a : constant), then $\mathrm{f}(x)\dfrac{d\,\mathrm{h}(x)}{dx} = a\,\mathrm{h}(x)$. Substitute (2.1) and (2.3) into this differential equation and compare coefficients of power series, we can get the following relations

$$(2.10)\quad \mathrm{h}_0 = \mathrm{f}_0{}^a,$$

$$(2.11)\quad \mathrm{h}_n = \frac{1}{n\,\mathrm{f}_0}\sum_{k=1}^{n}\{(a+1)k - n\}\mathrm{f}_k \mathrm{h}_{n-k} \quad (n \geq 1).$$

We can get same type differential equation and same type relation between coefficients of power series for another elementary transcendental functions.

Above relations is valid at not only origin but also anywhere.

2.2 Power Series of Function

We can expand x into power series at $x = a$ in the form

$$a + (x - a)$$

i.e.

$$f_0 = a, \quad f_1 = 1, \quad f_n = 0 \quad (n \geq 2) \text{ in (2.1)}$$

Substitute this power series for x into the function consist of x, we can get power series of this function.

2.3 Example of Power Series

The power series of the following function at $x = 1$ is calculated.

$$(2.12)\quad \mathrm{f}(x) = \sqrt{x}e^x + \sin x$$

The program for C++ language is

```
1 : #include "power.h" // define power series
2 : main(){
3 :       power f, x;       // declare f and x
4 :       x[0]=1 ; x[1]=1 ;   //   x=1+1(x-1)
5 :       f=sqrt(x)*exp(x)+sin(x); // compute
6 :       cout << "f=" << f << endl ;   // output
7 : }
```

output:

```
f=3.55975+4.61773*x+1.95776*x^2
+0.872674*x^3+0.268664*x^4
```

+0.0802463*x^5

where x means (x-1).

3.Power Series of Solution for DAE

In order to substitute (1.4) into (1.1), we define another type power series in the form

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + (f_n + p_1 e_1 + \cdots + p_m e_m) x^n \quad (3.1)$$

where $p_1, p_2, \ldots, p_m$ are constants, $e_1, e_2, \ldots, e_m$ are basis of vector e in (1.4), m is size of vector e.

Arithmetic and functions of this type power series can be defined easily for $n \geq 1$ in same manner as power series. .

y in (1.4) is a vector with m elements. Every elements can be represented by the power series in (3.1).

$$y_i = y_{i,0} + y_{i,1} x + y_{i,2} x^2 + \cdots + e_i x^n, \quad (3.2)$$

where y_i is i th element of y.

Substitute (3.2) into (1.1), we can get the following

$$f_i = (f_{i,n} + p_{i,1} e_1 + \cdots + p_{i,m} e_m) x^{n-j} + O(x^{n-j+1}) \quad (3.3)$$

where f_i is i th element of $f(y, y', x)$ in (1.1) and j is 0 if $f(y, y', x)$ have y', 1 if don't. From (3.3) and (1.5), we can determine the linear equation in (1.5) as follow

$$A_{i,j} = p_{i,j}, \quad (3.4)$$

$$B_i = -f_{i,n}. \quad (3.5)$$

Solve this linear equation, we have e_i ($i = 1, \cdots, m$). Substitute e_i into (3.2), we have y_i ($i = 1, \cdots, m$) which are polynomials of degree n.

Repeating this procedure from (3.2) to (3.5), we can get the arbitrary order power series of the solution for DAE in (1.1).

Let step size of x to be h and substitute it into the power series of y and derivative of y, we have y and y' at $x = x_0 + h$.

If we repeat above procedure, we have numerical solution of DAE in (1.1).

4. Padé Series

In above section the power series of solution for DAE can be given easily. Transform the power series into Padé Series, we have A-stable formula for DAE.

4.1 Calculation of Padé Series

The power series can be transformed into Padé Series easily. Padé Series is defined in the following

$$a_0 + a_1 x + a_2 x^2 + \cdots = \frac{p_0 + p_1 x + \cdots + p_M x^M}{1 + q_1 x + \cdots + q_L x^L} \quad (4.1)$$

Multiply the both side of (4.1) by the denominator of right side in (4.1) and compare the coefficients of both side in (4.1), we have

$$a_l + \sum_{k=1}^{m} a_{l-k} q_k = p_l \quad (l = 0, \cdots, M). \quad (4.2)$$

$$a_l + \sum_{k=1}^{L} a_{l-k} q_k = 0 \quad (l = M+1, \cdots, M+L). \quad (4.3)$$

Solve the linear equation in (4.3), we have q_k $(k = 1, \cdots, L)$. And substitute q_k into (4.2), we have p_k $(k = 0, \cdots, M)$.

The coefficients in (4.3) is Toeplitz matrix. Although a fast algorithm developed by Levinson for Toeplitz matrix[6,p43] is known, Gaussian elimination is used to solve above linear equations..

4.2 A-Stability

We consider the following test differential equation

$$\frac{dy}{dx} = ly \quad \text{with initial values} \quad y(0) = y_0 \quad (4.4)$$

As well known, the solution of this differential equation is

$$y = y_0 e^{lx} = y_0 \sum_{k=0}^{\infty} \frac{(lx)^k}{k!}. \quad (4.5)$$

For actual computation, we must truncate the infinite series in right side of (4.5) to the finite series in the form

(4.6) $y = y_0 \sum_{k=0}^{n} \frac{(lx)^k}{k!}$.

If the absolute values of the series parts in right side of (4.6) is less than 1 when $\mathrm{Re}\, lx < 0$, we call (4.6) A-stable formula.

There is not A-stable formula if we use the truncated power series shown in (4.6). If we use the infinite series in (4.5), it is A-stable formula. Therefore Corliss et al.[1] used very higher order power series for more stability.

According to E. Hairer and G. Wanner [3,p60], if $M \le L \le M+2$, where M and L is the degree of numerator and denominator in Padé series respectively, then this Padé series gives an A-stable formula for an ordinary differential equation.

An A-stable formula can be given to transform the power series into Padé series.

5. Numerical Examples

We consider the following differential algebraic equation [8]

$$\begin{aligned} v_1' + v_3 v_2' - (v_2 + 1)v_2' &= -v_1 + 1 + \sin x \\ (v_3 + 1)v_1' + v_1 v_2' &= -e^{-x} \\ 0 &= v_1 v_2 v_3 - 0.5e^{-x}\sin(2x) \end{aligned} \quad (5.1)$$

and initial values

$$v(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v'(0) = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}. \quad (5.2)$$

The exact solution is

$$\begin{aligned} v_1(x) &= e^{-x} \\ v_2(x) &= \sin x \\ v_3(x) &= \cos(x) \end{aligned} \quad (5.3)$$

From initial values, the solutions of (5.1) can be supposed as

$$\begin{aligned} v_1(x) &= 1 - x + e_1 x^2 \\ v_2(x) &= x + e_2 x^2 \\ v_3(x) &= 1 + e_3 x^2 \end{aligned} \quad (5.4)$$

Substitute (5.4) into (5.1) and neglect higher order terms, we have

$$\begin{matrix} e_2 x^2 + O(x^3) \\ (-2 + 4e_1 + 2e_2)x + O(x^2) \\ (-2 + 2e_1 + 2e_2 - 2e_3)x + O(x^2) \end{matrix} \quad (5.5)$$

These formula corresponds to (3.3). The linear equation that corresponds to (3.4) can be given in the following

$$Ae = B \quad (5.6)$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 4 & 2 & 0 \\ 2 & 3 & -2 \end{pmatrix}, \quad (5.7)$$

$$B = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}. \quad (5.8)$$

From equation (5.6), we have

$$e = \begin{pmatrix} 0.5 \\ 0 \\ -0.5 \end{pmatrix} \quad (5.9)$$

and

$$\begin{aligned} v_1(x) &= 1 - x + 0.5x^2 \\ v_2(x) &= x \\ v_3(x) &= 1 - 0.5x^2 \end{aligned} \quad (5.10)$$

From (5.10) the solutions of (5.1) can be supposed as

$$\begin{aligned} v_1(x) &= 1 - x + 0.5x^2 + e_1 x^3 \\ v_2(x) &= x + e_2 x^3 \\ v_3(x) &= 1 - 0.5x^2 + e_3 x^3 \end{aligned} \quad (5.11),$$

in like manner. Substitute (5.4) into (5.1) and neglect higher order terms, we have

$$\begin{matrix} (0.166667 + e_2)x^3 + O(x^4) \\ (-1.5 + 6e_1 + 3e_2)x^2 + O(x^3) \\ (1 + 3e_1 + 3e_2 - 3e_3)x^2 + O(x^3) \end{matrix} \quad (5.12).$$

From (5.12) we have the liner equation

$$\begin{pmatrix} 0 & 1 & 0 \\ 6 & 3 & 0 \\ 3 & 3 & -3 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} -0.166667 \\ -1.5 \\ -1 \end{pmatrix}. \quad (5.13)$$

Solve this linear equation, we have

$$(5.14)\quad e=\begin{pmatrix}-0.166667\\-0.666667\\0\end{pmatrix}$$

Therefore

$$\begin{aligned} v_1(x) &= 1-x+0.5x^2-0.166667x^3 \\ (5.15)\quad v_2(x) &= x-0.166667x^3 \\ v_3(x) &= 1-0.5x^2 \end{aligned}$$

Repeating above procedure we have

```
v1=1-x+0.5*x^2-0.166667*x^3
     +0.0416667*x^4-0.00833333*x^5
     +0.00138889*x^6-0.000198413*x^7
     +2.48016e-05*x^8-2.75573e-06*x^9
v2=x-0.166667*x^3+5.55112e-17*x^4
     +0.00833333*x^5+1.00614e-16*x^6
     -0.000198413*x^7+7.71952e-17*x^8
     +2.75573e-06*x^9
v3=1-0.5*x^2+1.38778e-17*x^3
   +0.0416667*x^4+4.59702e-17*x^5
   -0.00138889*x^6+4.07931e-18*x^7
   +2.48016e-05*x^8-2.849e-18*x^9
```

This is in the output form from the computer. The power series v1 can be transformed into Padé series.

v1:

```
pp=1-0.5*x+0.107143*x^2-0.0119048*x^3
     +0.000595238*x^4
qq=1+0.5*x+0.107143*x^2+0.0119048*x^3
    +0.000595238*x^4
```

where pp, qq is the numerator and the denominator of Padé series. We can also get Padé series for v2, v3.

Substitute step size h into v1, v2 and v3, we have $v_1(h)$, $v_2(h)$ and $v_3(h)$. We also have $v_1'(h)$, $v_2'(h)$ and $v_3'(h)$.

If the power series v1, v2, v3 converges very fast, the transformation of v1, v2, v3 into Padé series can be eliminated.

Consider $v_1(h)$, $v_2(h)$, $v_3(h)$, $v_1'(h)$, $v_2'(h)$ and $v_3'(h)$ as initial values at $x=h$ for differential algebraic equation (5.1), we can calculate $v_1(x)$, $v_2(x)$, $v_3(x)$, $v_1'(x)$, $v_2'(x)$ and $v_3'(x)$ at $x=2h$ by the above procedure.

Table 1 Numerical solution of (5.1).

x	v1	v2	v3
0.1	0.904837	0.0998334	0.995004
0.2	0.818731	0.1986690	0.980067
0.3	0.740818	0.2955200	0.955336
0.4	0.670320	0.3894180	0.921061
0.5	0.606531	0.4794260	0.877583
0.6	0.548812	0.5646420	0.825336
0.7	0.496585	0.6442180	0.764842
0.8	0.449329	0.7173560	0.696707
0.9	0.406570	0.7833270	0.621610
1.0	0.367879	0.8414710	0.540302

We show Table 1 for the solution of (5.1) by above numerical method. The problem of (5.1) is very difficult to solve by the numerical method proposed in [8]. But it is very easy to solve it by the present numerical methods. The numerical values on table 1 are coincide with the exact solutions of (5.1).

6. Conclusion

An arbitrary and A-stable formula for differential algebraic equation is given. The algorithm is very simple and effective for most of differential algebraic equations.

It is easy to convert above formula to adaptive one while we don't show above.

This algorithm is programmed by C++ language and Fortran 95. Although it can be programmed by the old program language which don't have the overload function like C language, FORTRAN 77 and Java, its program became very complex.

We can also use the present method to expand the implicit function into power series and Padé series.

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