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Exploring some alternatives to non-quadratic LMI conditions for analyzing nonlinear systems based on Takagi-Sugeno modelling

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Table of contents

List of Figures

General introduction

Introduction générale en Français:

L'étude des systèmes dynamiques est l'une des principales préoccupations de la communauté automaticienne. En effet, la maîtrise des comportements des systèmes dynamiques nécessite plusieurs étapes essentielles, allant de la modélisation à la spécification de systèmes de commande. Un modèle est une représentation mathématique d'un système. Qu'il soit par exemple physique ou encore biologique, l'étude d'un système nécessite de prendre en compte aussi précisément que possible sa dynamique [Murray & Åström, 2012]. De ce fait, le modèle d'un système joue un rôle important dans son automatisation, par exemple pour des simulations numériques, des prévisions, l'analyse de la stabilité, la synthèse d'un contrôleur ou encore d'un observateur. Par conséquent, à des fins le contrôle automatique d'un système dynamique, les modèles sont souvent décrits par des relations mathématiques liant leurs entrées et sorties. Qu'ils s'agissent de systèmes décrivant des applications de la vie quotidienne ou industrielles, la plupart de ceux-ci ont des dynamiques dont les comportements évoluent au cours du temps.

Un système physique peut donc être considéré comme ensemble de phénomènes corrélés et dont l'évolution temporelle peut-être modélisée d'un point de vue analytique par l'intermédiaire d'équations dynamiques, c'est-à-dire par un modèle de connaissance mathématique destiné à représenter mieux la réalité. Ces modèles mathématiques sont souvent représentés par des équations différentielles ordinaires (ODE) et, dans le cadre des techniques de l'automatique avancée, par des représentations dans l'espace d'état. Ces dernières permettent de relier les entrées et les sorties par un vecteur d'état qui décrit l'évolution du système. De nombreux exemples de modélisation dans l'espace d'état peuvent être ainsi considérés ; par exemple pour les systèmes mécaniques, électriques ou chimiques, etc. [Kumar & Daoutidis, 1995][T.Kulakowski et al, 2007][Li et al, 2007][Schulte & Guelton, 2006].

Sur la base de la variabilité de temps ou de l'évolution des paramètres, les modèles dynamiques peuvent être linéaires, non linéaires, stationnaires, variables dans le temps, décrits en temps continu ou discret… La synthèse de contrôleurs adéquats et les théories de l'automatique d'une manière générale tiennent compte de telles propriétés. De nos jours, la théorie de l'automatique linéaire est bien établie et les systèmes linéaires, représentés par les ODEs linéaires, satisfont généralement des propriétés de superposition ou d'homogénéité facilitant leur étude [Delarminat, 1993][Kuo, 1995]. Cependant, la plupart des systèmes réels sont non linéaires et, par conséquent, leurs études s'avèrent plus complexes. Par exemple, un système linéaire admet un point d'équilibre (généralement fixé à l'origine du modèle) alors que dans le cadre non linéaire plusieurs points d'équilibre ou d'attraction peuvent coexister. Pour traiter les systèmes non linéaires, selon les objectifs de contrôle, une première tentative peut être de considérer une approximation (linéarisation) autour d'un point particulier de l'espace opérationnel. Ainsi, autour de ce point particulier, les théories de l'automatique linéaire peuvent être en grande partie utilisées. Cependant, ces approches linéaires de l'automatique s'avèrent inadéquats lorsque l'objectif consiste à travailler dans un espace de fonctionnement plus large, s'éloignant du point de linéarisation [Khalil, 1996][Sastry, 1999]. Par exemple, un pendule est stable autour de son point d'équilibre inférieur et instable autour de son point d'équilibre supérieur (position érigée). De ce fait, à partir d'une position initiale donnée, hors de ses positions d'équilibres, il retourne de manière autonome à sa position d'équilibre stable. Dans ce contexte, la stabilité intrinsèque d'un système est une caractéristique importante à analyser. Dans le cas où le système est stable, un contrôleur peut ne pas être nécessaire à moins qu'une amélioration de performances soit exigée. Dans le cas contraire, si le système est instable, il est généralement nécessaire de concevoir un contrôleur adéquat pour le stabiliser. Par conséquent, un des défis principaux de l'automatique est la synthèse de contrôleur permettant de stabiliser un système dynamique et garantissant l'accomplissement d'une tâche désirée. Parmi les approches de l'automatique non linéaires, deux méthodologies de synthèse s'opposent :

- *Les approches ou les techniques non-formelles qui n'exigent pas la connaissance précise d'un modèle formalisé peuvent être employées pour concevoir des contrôleurs. Parmi celles-ci, les commande floue de type Mamdani [Mamdani, 1974], la commande par réseau de neurones [Ferrari & Stengel, 2002b][Zilkova et al, 2006], ou encore la commande par modes glissants [Edwards & Spurgeon, 1998][Perruqetti & Barbot, 2002], ont été largement discutées dans la littérature. Ces approches* *fournissent souvent des résultats satisfaisants en termes de performances mais leur inconvénient principal réside dans le fait qu'elles permettent difficilement de garantir (ou de prouver) la stabilité du système dynamique en boucle fermée sur l'ensemble de son espace opérationnel.*

- *Les approches formelles exigent a priori une connaissance plus précise du modèle du système à commander. Celui-ci est nécessaire pour synthétiser un contrôleur adéquat à l'aide de méthodologies analytiques telles que la théorie de Lyapunov. Dans ce cadre, ces approches sont utiles pour assurer la stabilité du système en boucle fermée en tous points ou dans un sous-espace compact de l'espace d'état. Néanmoins, lorsque la taille du système augmente (nombre de variables d'état) ou lorsque la dynamique du système est fortement non linéaire (fonctions de commutation, beaucoup de termes non linéaires…) l'obtention d'une garantie de stabilité s'avère difficile [Khalil, 1996][Liberzon, 2003]. En d'autres termes, ces approches souffrent d'un certain conservatisme qui peut-être considéré comme un frein à de nombreuses applications.*

Dans le cadre de ce travail de thèse, nous nous intéresserons principalement aux approches formelles dans le but d'explorer des pistes de réduction du conservatisme. Une manière élégante de représenter un système non linéaire est de le considérer par une représentation multi-modèle, c'est à dire une collection de systèmes linéaires interconnectés par des fonctions non linéaires. Plusieurs catégories des multi-modèles existent dans la littérature. Par exemples les systèmes linéaires à paramètres variables (LPV) [Henrion & Garulli, 2004] ou encore les systèmes quasi-LPV, également appelés systèmes flous de types Takagi-Sugeno (T-S) [Takagi & Sugeno, 1985][Zerar et al, 2009]. Notons que ces derniers possèdent une propriété intéressante d'approximation universelle des systèmes non linéaires, c'est-à-dire qu'ils permettent de représenter ceux-ci de manière exacte sur un espace compact de leurs variables d'état [Tanaka & Wang, 2001]. Ainsi, l'intérêt principal des modèles Quasi-LPV est qu'ils permettent d'étendre de nombreux concepts théoriques de l'automatique linéaire à la classe des systèmes non linéaires.

Dans la plupart des cas, la modélisation d'un système réel nécessitent des approximations ou hypothèses susceptibles de dégrader leurs performances en boucle fermée [Dubuisson, 1990][Oustaloup & Mathieu, 1999]. Dans ce contexte, il s'avère intéressant de considérer des incertitudes de modélisation qui permettront la synthèse des lois de commande robustes [Zhou & Doyle, 1998]. Ainsi, il est important de souligner le dilemme de modélisation auquel les automaticiens ont à faire face : un modèle comportant trop d'approximation peut décrire *un comportement erroné vis-à-vis du système étudié alors qu'un modèle trop détaillé peut être inutilisable, car trop complexe, avec les techniques d'analyse actuelles. Aussi, l'intérêt des modèles T-S a été démontré au travers de nombreuses études. Par exemple au sujet de l'analyse de leur stabilité ou encore la synthèse de lois de commande ou d'observateurs par compensation parallèle distribuée (PDC) [Tanaka & Wang, 2001][Lendek et al., 2010][Tanaka et al., 1999][Wang et al., 1996][Arino & Sala, 2007].*

La théorie de Lyapunov est l'une des techniques les plus plébiscité pour analyser la stabilité d'un système non linéaire et, en particulier pour les modèles T-S. L'idée principale est de trouver une fonction de Lyapunov, analogue à une fonction d'énergie, qui décroît strictement le long des trajectoires temporelles du système [Malisoff & Mazenc, 2009]. Dans ce contexte, de nombreux résultats sur l'analyse des modèles T-S sont basées sur une fonction candidate quadratique de Lyapunov (commune à l'ensemble des modèles locaux) [Tanaka & Wang, 1998][Johansson & Rantzer, 1998][Chadli et al., 2005]. Cette approche permet de réécrire de nombreux problèmes, allant de l'analyse de la stabilité à la synthèse de lois de commande ou d'observateurs, sous la forme d'inégalités linéaires matricielles (LMI). Ces dernières peuvent être résolues, lorsqu'une solution existe, par des algorithmes d'optimisation convexe tels que l'algorithme du point intérieur [Boyd & Vandenberghe, 2004]. Cependant, les conditions restent toutefois conservatives puisqu'il est souvent possible de trouver des solutions à un problème d'analyse d'un système non linéaire donné alors que des solutions au problème LMI associé ne peuvent-être atteintes. Ce conservatisme peut provenir de différentes sources : le modèle T-S choisi pour représenter un système non linéaire, la manière de s'affranchir des informations contenues dans les fonctions d'appartenance du modèle T-S pour obtenir des conditions LMI, ou encore le choix de la fonction candidate de Lyapunov [Sala, 2009]. Il est donc souvent nécessaire de rechercher à relâcher les conditions de stabilité et/ou de stabilisation LMI dévolues à l'analyse des modèles T-S [Liu & Zhang, 2002][Kim & Lee, 2000][Feng et al., 2006].

Les travaux présentés au sein de ce manuscrit concernent la relaxation des conditions de stabilité en termes de choix de la fonction candidate de Lyapunov appropriée à certaines classes de systèmes de T-S. De nombreux efforts de recherche ont été récemment réalisés dans le cadre non-quadratique. En effet, plusieurs travaux employant des fonctions candidates de Lyapunov non-quadratiques (NQLF) on été proposés [Blanco et al., 2001][Tanaka et al., 2003][Guerra & Vermeiren, 2004] (voir également [Feng, 2006] pour une synthèse bibliographique de ces techniques). En effet, les NQLF s'avèrent efficaces dans *la mesure où elles partagent les mêmes fonctions d'activation que les modèles T-S qu'elles permettent d'analyser. Cependant, celles-ci souffrent d'inconvénients dans le cas continu. En effet, les dérivés des fonctions d'appartenance apparaissent au sein des conditions de Lyapunov, ce qui rend difficile l'obtention de conditions LMI [Guerra et al., 2012].*

De nos jours, en dépit du succès et de la popularité des techniques LMI pour l'analyse des modèles T-S, les inconvénients décrits ci-dessus peuvent-être considérés comme une limite de ces approches, particulièrement dans le cadre non-quadratique. Par conséquent, un changement de perspective est peut-être nécessaire et les travaux de cette thèse tendent à explorer de nouvelles pistes dans ce sens. Très récemment, une alternative aux approches LMI est apparue au travers d'approches issue de l'optimisation polynômiale. En effet, certains problèmes de l'optimisation convexes, tels que des problèmes LMI, peuvent-être exprimé sous la forme de problème d'optimisation de polynômes décrits sous forme de « sommes de carrés » (SOS, pour Sum-Of-Squares). Par conséquent, avec les premiers travaux sur l'analyse de la stabilité des modèles T-S par des approches SOS, une étape importante vers des techniques alternatives aux LMIs a été franchie [Tanaka et al ., 2003][Sala & Arino, 2009][Tanaka et al., 2009][Naramani & Lam, 2010][Guelton et al., 2011] [Bernal et al., 2011].

Ces travaux de thèse tentent donc d'explorer des techniques alternatives aux LMIs dans le cadre non quadratique pour l'analyse des systèmes T-S. Ainsi, ce manuscrit est organisé comme suit :

Dans le ***premier chapitre****, les concepts principaux relatifs aux modèles Takagi-Sugeno sont présentés. Les résultats élémentaires, basés sur l'emploie de fonctions de Lyapunov quadratiques, seront présentés sous forme LMI et leurs sources de conservatisme seront discutées. Certaines des solutions existantes pour réduire ces sources de conservatisme seront alors présentées afin de positionner les contributions proposées dans les chapitres suivants.*

Comme première alternative aux conditions LMI dans le cadre non-quadratiques, le ***deuxième chapitre*** *traite de l'emploie d'une fonction candidate de Lyapunov à commutation. Dans ce contexte, sur la base de travaux récent sur la synthèse de lois de commande PDC à commutations [Othake & Tanaka, 2006], une méthodologie de synthèse de lois de commande robuste à commutation pour les systèmes T-S incertains et perturbés est proposée. Dans un premier temps, la stabilisation des modèles incertains est considérée sans perturbations*

externes. Puis, une extension aux systèmes sujets aux perturbations externes est fournie sur la base de l'utilisation d'un critère H-infini. Ce chapitre a fait l'objet de la publication suivante:
D. Jabri, K. Guelton, N. Manamanni, A. Jaadari, C.C Duong, "Robust stabilization of nonlinear systems based on a switched fuzzy control law," *Journal of Control Engineering and Applied Informatics*, 14(2):40-49, 2012.

Ensuite, afin d'explorer les alternatives possibles aux conditions LMI, l'analyse des systèmes T-S polynômiaux est considérée dans le ***troisième chapitre****. Dans ce contexte, une nouvelle étape sur la relaxation des conditions de stabilité pour l'analyse de systèmes T-S polynomiaux est proposée sur la base de travaux récents [Tanaka et al., 2009] qui considèrent une fonction candidate de Lyapunov polynômiale commune. Inspiré des approches LMI non-quadratique au regard des conditions quadratique, l'emploi d'une nouvelle fonction multiple de Lyapunov polynomiale est proposée. Les conditions de stabilité obtenues sont alors écrites sous la forme de problème d'optimisation polynomiale SOS. Ce chapitre a fait l'objet de la publication suivante :*
K. Guelton, N. Manamanni, C.C Duong and D.L. Koumba-Emianiwe "Sum-of-squares stability analysis of Takagi-Sugeno systems based on multiple polynomial Lyapunov functions.", *International Journal of Fuzzy Systems*, 15(1):34-41, mars 2013.

Les systèmes classiques de Takagi-Sugeno peuvent être considérés comme un cas particulier des systèmes polynômiaux T-S. Par conséquent, le but du ***quatrième chapitre*** *est de montrer comment l'approche SOS peut être employée comme alternative aux conditions LMIs pour la synthèse d'une loi de commande non-PDC dans le cadre non-quadratique. Dans un premier temps, les approches LMI classiques, obtenues au travers d'une fonction candidate de Lyapunov non-quadratique, sont présentées afin de discuter de leurs limites. En effet, elles souffrent de la présence de paramètres inconnus pour implémenter le problème LMI ou conduisent à une analyse locale de la stabilité. Pour surmonter ces inconvénients, on propose une alternative aux LMI en utilisant le formalisme SOS pour synthétiser des lois de commande non-PDC. Ce chapitre fait l'objet de la communication suivante :*
C.C. Duong, K. Guelton, N. Manamanni, "A SOS based alternative to LMI approaches for non-quadratic stabilization of continuous-time Takagi-Sugeno fuzzy systems," *IEEE International Conference on Fuzzy Systems* (FUZZ-IEEE 2013), Hyderabad, India, July 7-10.

Enfin, une conclusion générale et proposée afin de récapituler l'ensemble des contributions de cette thèse ainsi que de présenter quelques per spectives.

General introduction in English

Dynamical system studies are one of the main concerns of the automatic control community. Indeed, mastering the behaviors of dynamical systems needs an essential step of modeling. A model is a mathematical representation of a physical, biological or information system and is required to be as much as possible precise regarding to the system's dynamics [Murray & Åström, 2012]. Therefore, the model of the considered system plays an important role in its automation, for instance for numerical simulations, prediction, stability analysis, controller or observer design... Hence, in automatic control, models of dynamical systems are often described the systems' input/output mathematical relations. There are many kinds of systems from the daily life to the industry. In the reality, most of the systems are dynamic systems whose behaviors change over time.

A physical system can be considered as a set of interrelated phenomena and evolving over time, theoretically modeled via dynamic equations and resulting in a pattern of mathematical knowledge assumed to better represent the reality. These mathematical models can often be represented by Ordinary Differential Equations (ODE), generally expressed into state space representations. These allow connecting the outputs and inputs through a state vector that describes the system's evolution. Many examples of state space modeling can be found for mechanical, electrical or chemical systems and so on [Kumar & Daoutidis, 1995] [Kulakowski et al., 2007][Li et al., 2007][Schulte & Guelton, 2006].

Based, for instance, on time variability or parameters' evolution, dynamical models can be considered as linear, nonlinear, stationary, time invariant, time varying, continuous or discrete... Automatic control is therefore studying such properties. Nowadays, linear control theory is well-established and linear systems satisfy the properties of superposition and homogeneity and so their models may be represented by linear ODEs [Delarminat, 1993] [Kuo, 1995]. However, most of real physical systems are nonlinear and so their studies remain more complex. For instance, a linear system has one equilibrium point (generally set to the origin) but, for a nonlinear one, there may exist more than one equilibrium point. To deal with nonlinear systems, depending on the control objectives, a first attempt may be to consider an approximation around a particular operating point. From this linearized model, linear control theory may be largely utilized. However, these approaches failed when the objective required to work on a wide operating space and so where linear controllers lack of efficiency [Khalil, 1996][Sastry, 1999]. For example, a pendulum is stable around its lower equilibrium point. That means that if we give it an initial condition, it autonomously retrieves its equilibrium. However, if we want to stabilize it around the erect position (inverted pendulum), it is now instable. In this context, the intrinsic stability of a system is an important characteristic to analyze. In the case where the system is stable, a controller may not be necessary unless a performance improvement is required. In the contrary, if the system is unstable, one has to design a convenient controller to stabilize it. Therefore, one of the main challenges is to provide an automatic controller to stabilize the system and to guarantee the achievement of a desired task. Among nonlinear system approaches, there are two categories:

- Non-formal approaches or techniques which don't require the precise knowledge of a formalized model may be used to design controllers. Among them, Mamdani type fuzzy controllers [Mamdani, 1974], Neural network controllers [Ferrari & Stengel, 2002b] [Zilkova et al., 2006], sliding mode controllers [Edwards & Spurgeon, 1998][Perruqetti & Barbot, 2002] has been widely discussed in the literature. These approaches often provide satisfactory results in terms of performances but their main disadvantages are that it is difficult in some cases, to ensure (or prove) the stability.
- The formal approaches require a priori a more precise knowledge of the system's model. This one is necessary to synthesize a convenient controller from analytic methodologies such as Lyapunov theory. In that framework, these approaches are useful to ensure the system's stability in all or in a compact set of the state space. Unfortunately, this warranty is difficult to obtain as the system size increases (number of state variables) or the dynamics of the system is highly nonlinear (switching functions, many nonlinear terms ...) [Khalil, 1996] [Liberzon, 2003]. In other words, these approaches suffer from conservatism and are not so easy to apply.

In the sequel, the work presented in this thesis mainly focuses on formal approaches. Indeed, an elegant way to represent a nonlinear system is to consider it with a multiple-model representation, i.e. a collection of interconnected linear systems by nonlinear functions. Several categories of multiple-models exist in the literature, including linear systems with time-varying parameter (LPV) [Henrion & Garulli , 2004] or quasi-LPV systems [Zerar et al., 2009], also called Takagi-Sugeno systems (T-S) [Takagi & Sugeno, 1985]. Note that these latters hold the property of universal approximation of affine control systems and have the advantage of accurately represent a knowledge model on a compact nonlinear state space [Tanaka & Wang, 2001]. Thus, the important advantage of this approach is that it allows extending many theoretical concepts of linear control theory to the case of nonlinear systems.

During the modeling stage, the challenge of the automation is to propose a model knowledge that can faithfully represent the encountered physical phenomena. Note that modeling assumes that in most cases, approximations may degrade the desired performance for the real system [Dubuisson, 1990][Oustaloup & Mathieu, 1999]. In this context, it may be worthwhile to consider these inaccuracies to be able to ensure the desired performance through the synthesis of robust control laws [Zhou & Doyle, 1998]. Note also that oversimplified model is "false" while a too detailed model may be unusable with the current tools for analyzing systems. In that way, the interest in modeling methods as T-S ones was much demonstrated [Tanaka & Wang, 2001][Lendek et al., 2010]. Indeed, the Takagi-Sugeno (T-S) fuzzy models, which are regarded as a particular type of non-linear system, are used extensively. Generally speaking, these models have been described by a set of fuzzy "If ... then" rules with fuzzy sets in the antecedents and dynamics sub-models in the consequent. And these sub-models are always considered as local Linear Time Invariant (LTI) systems, the aggregation of which represents the nonlinear system behavior. Based on T-S fuzzy models, a great number of results concerning stability analysis and parallel distributed compensation (PDC) controller design have appeared in the literature [Tanaka et al., 1999] [Wang et al., 1996][Arino & Sala, 2007].

Lyapunov stability theory is one of the most efficient methods to analyze the stability to both linear and nonlinear system. Especially, Lyapunov approach is efficient for a wide range of nonlinear systems including Takagi-Sugeno models. The main idea is to find a Lyapunov function, analog to an energy function, which is strictly decreasing along the time evolution of the system’s trajectories [Malisoff & Mazenc, 2009]. In that context, regarding to Takagi-Sugeno based studies, most of the results are based on a common quadratic candidate Lyapunov function [Tanaka & Wang, 1998][Johansson & Rantzer, 1998][Chadli et al., 2005]. This allows us to describe several problems of stability analysis, controller or observer design, as linear matrix inequality (LMI). These ones can be solved, when a solution exists, by optimization algorithms such like interior point ones [Boyd & Vandenberghe, 2004].

However, the stability conditions are still conservative since it is often possible to have some systems stables although solutions to the considered LMI problems are not reached. Conservatism comes from different sources: the type of TS model, the way the membership functions are dropped-off to obtain LMI expressions, the integration of membership-function information, or the choice of Lyapunov function [Sala, 2009]. It is therefore often necessary to “relax” the stability and stabilization conditions, especially for fuzzy systems [Liu & Zhang, 2002][Kim & Lee, 2000][Feng et al., 2006].

This thesis is concerned with relaxation in the sense of the choice of the most convenient Lyapunov function candidate for some class of Takagi-Sugeno systems. With more adequacies to the structure of T-S models, many research efforts have been recently done in the non-quadratic framework. Indeed, several works employing non-quadratic Lyapunov functions (NQLF) candidates have been proposed [Blanco et al., 2001][Tanaka et al., 2003][Guerra & Vermeiren, 2004] see also [Feng, 2006] for a survey on these techniques. These ones are convenient with T-S models since NQLF shares the same activation functions (AFs). However, some drawbacks appear in the continuous time case, e.g. the time derivatives of AFs occurs when applying the direct Lyapunov method to obtain LMI based global stability conditions [Guerra et al., 2012].

Nowadays, despite the success and popularity of LMI based stability conditions for T-S models, the drawbacks described above are understood as a limit of these approaches, especially in the non-quadratic framework. Therefore, a challenging change of perspective has to be considered and this thesis tends to add a step in this way. Very recently, an alternative to LMIs appears by considering polynomial approaches. Indeed, it has been shown that some convex optimization problems, such like LMI problems, can be recasted as more general Sum Of Square (SOS) decomposition problems. Therefore, with the first SOS based stability conditions, T-S model analysis has just passed an important milestone [Tanaka et al, 2003][Sala & Arino, 2009][Tanaka et al., 2009][Naramani & Lam, 2010][Guelton et al., 2011] [Bernal et al., 2011].

According to the concerns described above, this manuscript is organized as follows:

In the **first chapter**, the main concepts regarding to Takagi-Sugeno based modeling and control are introduced. The main LMI quadratic based results will be presented as well as their sources of conservatism. Some of the existing solutions to reduce these sources of conservatism will then be discussed to highlight the contributions provided in the following chapters.

As a first alternative to non-quadratic LMI based conditions, the **second chapter** deals with switched Lyapunov approach. In that context, based on a recent pioneer work on switched PDC controller design [Othake & Tanaka, 2006], a robust switched PDC controller design methodology is proposed for uncertain and disturbed Takagi-Sugeno systems. First, stabilization of uncertain switched fuzzy systems is considered without external disturbances. Then, an extension to systems subject to external disturbances is provided based on a H-infinity criterion. This chapter has been published as:

D. Jabri, K. Guelton, N. Manamanni, A. Jaadari, C.C Duong, "Robust stabilization of nonlinear systems based on a switched fuzzy control law," *Journal of Control Engineering and Applied Informatics*, 14(2):40-49, 2012.

Then, in order to explore what else than LMI can be done in the context of Takagi-Sugeno systems' analysis, polynomial fuzzy systems are considered in the **third chapter**. In this context, a new step on fuzzy relaxation for nonlinear systems' stability analysis is addressed as an extension to the pioneer work on polynomial T-S systems [Tanaka et al., 2009] which considers common polynomial Lyapunov functions (PLF). Inspired from non-quadratic Lyapunov functions (NQLF), regarding to quadratic ones (QLF), a polynomial fuzzy Lyapunov function (PFLF) is proposed as an extension to the PLF. Following the latter post-LMI challenge, the obtained stability conditions are written in terms of a sum-of-squares (SOS) optimization problem. This chapter has been published as:

K. Guelton, N. Manamanni, C.C Duong and D.L. Koumba-Emianiwe "Sum-of-squares stability analysis of Takagi-Sugeno systems based on multiple polynomial Lyapunov functions.", *International Journal of Fuzzy Systems*, 15(1):34-41, mars 2013.

Classical Takagi-Sugeno systems can be regarded as a special case of polynomial ones. Therefore, the purpose of the **fourth chapter** is to show how the SOS based approach can be used as an alternative to LMI for the design of a non-PDC controller in the non-quadratic framework. At first, classical relaxed LMI based approaches, obtained through non-quadratic Lyapunov functions, are presented in order to highlight their limits. Indeed, they suffer from the requirement of unknown parameters or lead to local stability analysis. To overcome these drawbacks, an alternative to LMI is proposed by employing Sum-Of-Squares formalism to design non-PDC controllers, which globally stabilize TS systems. This chapter has been accepted for publication as:

C.C. Duong, K. Guelton, N. Manamanni, “A SOS based alternative to LMI approaches for non-quadratic stabilization of continuous-time Takagi-Sugeno fuzzy systems,” *IEEE International Conference on Fuzzy Systems* (FUZZ-IEEE 2013), Hyderabad, India, July 7-10.

Finally, a concluding section is proposed to summarize the contributions of this thesis as well as highlighting some perspectives.

Chapter 1

Background and basics on Takagi-Sugeno control

Résumé du chapitre 1 en Français:

Notions preliminaries sur l'analyse et la commande des modèles Takagi-Sugeno

Ce chapitre a pour vocation d'introduire les concepts élémentaires relatifs à l'analyse de la stabilité et la synthèse de lois de commande pour les modèles de type Takagi-Sugeno. En effet, ces préliminaires doivent permettre, à un lecteur non-spécialiste de ces systèmes, d'appréhender plus facilement les problèmes traités dans les chapitres suivants. Ainsi, dans un premier temps, une présentation des modèles Takagi-Sugeno ainsi que les méthodes permettant leur obtention à partir d'un système non linéaire sont proposées. Ensuite, les premiers résultats d'analyse de la stabilité et de synthèse de contrôleurs, obtenus au travers de fonctions candidates de Lyapunov quadratique et du formalisme LMI (Inégalité Linéaires Matricielles) sont présentés. Ces résultats s'avèrent néanmoins conservatifs et les techniques couramment employées pour relâcher les conditions LMI sont discutées avant d'introduire les techniques de synthèses non quadratiques. Ces dernières admettent néanmoins des limites dues à leur complexité et leur applicabilité. De ce fait, la question « que peut-il être employé d'autre que des LMIs pour l'analyse des modèles T-S ? » se pose et les premiers travaux utilisant des techniques d'optimisation convexe de polynômes sous forme « Sum-Of-Squares » sont présentés.

1.1. Introduction

In this chapter, we will present the basics on modeling, stability and stabilization of Takagi-Sugeno (T-S) fuzzy models to introduce the preliminary materials, which will be useful in the following chapters. First of all, T-S models are introduced and the way to obtain them from nonlinear systems will be presented. Then, well-known results on stability and stabilization of T-S models will be introduced [Tanaka & Wang, 2001]. These are based on the direct Lyapunov methodology through a quadratic approach. Therefore, the conservatism of such approach is discussed and its sources and some relaxation scheme are presented [Sala, 2009]. Among conservatism relaxation, the use of a non-quadratic Lyapunov function candidate has been mainly used in the literature [Tanaka et al., 2003][Guerra & Vermeiren, 2004]. However, these approaches suffer from the need of unknown parameters, which are difficult to find in practice. Moreover, it is pointed out that many LMI results tend to be more and more complex for a conservatism reduction that may be questionable. Another way to deal with stability analysis is understood as the use of polynomial optimization approach [Prajna et al., 2004]. Thus, the basics on polynomial Sum-Of-Suares (SOS) approaches are proposed with a pioneer work in the field of T-S models [Tanaka et al., 2009].

1.2. State space representation of nonlinear systems

Most of the physical systems can be represented by time continuous dynamical systems. Therefore they can be written as a state space representation, which allows describing input-output relations of a dynamical system by ordinary differential equations. The general form, including implicit systems (algebraic differential equations) of a state space model is given by :

$$\begin{cases} f\left(\dot{x}(t), x(t), u(t)\right) = 0 \\ y(t) = h\left(x(t), u(t)\right) \end{cases} \tag{1.1}$$

where $x(t)$ is the state vector, $u(t)$ is the input vector and $y(t)$ the output vector. The first equation is called “ state space equation” and the second one is the “ output equation” .

Note that (1.1) is given in the general form. A more employed particular class of the explicit state space representation, which will be considered in this thesis, can be expressed by:

$$\begin{cases} \dot{x}(t) = f\left(x(t)\right) + g\left(x(t)\right)u(t) \\ y(t) = s\left(x(t)\right) + m\left(x(t)\right)u(t) \end{cases} \tag{1.2}$$

where $f(x(t))$, $g(x(t))$, $s(x(t))$ and $m(x(t))$ are state dependent functions which can be nonlinear.

1.3. Takagi–Sugeno Models

When used for automatic control studies, Takagi-Sugeno (T-S) models are described in the state space by a set of “If-Then” fuzzy rules [Takagi & Sugeno, 1985]. The premisse part of these rules represents the discourse universe (linguistic framework) where the fuzzy model is valid. The concluding part is represented by local linear time invariant models. The i^{th} fuzzy rule of a T-S model (in continuous time) can be written as :

$$R^i \text{ : } \textbf{IF } z_1(t) \text{ is } F_1^i(z_1(t)) \textbf{ AND } z_2(t) \text{ is } F_2^i(z_2(t)) \ \ldots\ldots\ z_p(t) \text{ is } F_p^i(z_p(t))$$

$$\textbf{THEN} \begin{cases} \dot{x}_i(t) = A_i x(t) + B_i u(t) \\ y_i(t) = C_i x(t) + D_i u(t) \end{cases} \tag{1.3}$$

where R^i represents the i^{th} fuzzy rule for $i = 1,\ldots,r$. Moreover, for $j = 1,\ldots,r$, $F_j^i(z_j(t))$ are the fuzzy subsets, r the number of fuzzy rules, $z_j(t)$ are the premises variables which may depend on the state and/or the input, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^q$ and $u(t) \in \mathbb{R}^m$ are respectively the state vector, the output vector and the input vector. $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{q\times n}$ and $D_i \in \mathbb{R}^{q\times m}$ are the matrices describing the system’s dynamics.

For each rule R^i a weighting function $w_i(z(t))$ is assigned. This weight depends on the degree of membership of the premises variables $z_j(t)$ in the fuzzy subsets $F_j^i(z_j(t))$ connected by the fuzzy operator “ **AND** ” chosen as :

$$w_i(z(t)) = \prod_{j=1}^{p} F_j^i(z_j(t)), \text{ for } i = 1,\ldots,r \tag{1.4}$$

$F_j^i\left(z_j\left(t\right)\right)$ is the membership value in the fuzzy set F_j^i. It yields the following properties:

$$\begin{cases} \sum_{i=1}^{l} w_i\left(z\left(t\right)\right) > 0 \\ w_i\left(z\left(t\right)\right) \geq 0, \forall t \end{cases} \tag{1.5}$$

Now, considering a barycentric defuzzification, on has:

$$h_i\left(z\left(t\right)\right) = \frac{w_i\left(z\left(t\right)\right)}{\sum_{i=1}^{r} w_i\left(z\left(t\right)\right)} \tag{1.6}$$

$h_i\left(z\left(t\right)\right)$ is then the membership function of the i^{th} fuzzy rule. For $i = 1,...,r$, these membership functions hold the convex sum properties, i.e. $\sum_{i=1}^{r} h_i\left(z\left(t\right)\right) = 1$ et $h_i\left(z\left(t\right)\right) \geq 0$.

Finally, the defuzzification of the fuzzy model allow obtaining a state representation of a nonlinear model through the blending of the local linear time invariant models (concluding part) by nonlinear membership functions. It yields:

$$\begin{cases} \dot{x}\left(t\right) = \sum_{i=1}^{r} h_i\left(z\left(t\right)\right)\left(A_i x\left(t\right) + B_i u\left(t\right)\right) \\ y\left(t\right) = \sum_{i=1}^{r} h_i\left(z\left(t\right)\right)\left(C_i x\left(t\right) + D_i u\left(t\right)\right) \end{cases} \tag{1.7}$$

Figure 1.1 illustrates the detailed scheme of a classical T-S model. Note that T-S fuzzy model features an interesting mathematical structure regarding to automatic control theory. Indeed, they allow decreasing the complexity of a studied nonlinear control problem (stability, stabilization, observation, diagnosis…) by splitting it in a convex set of linear problems.

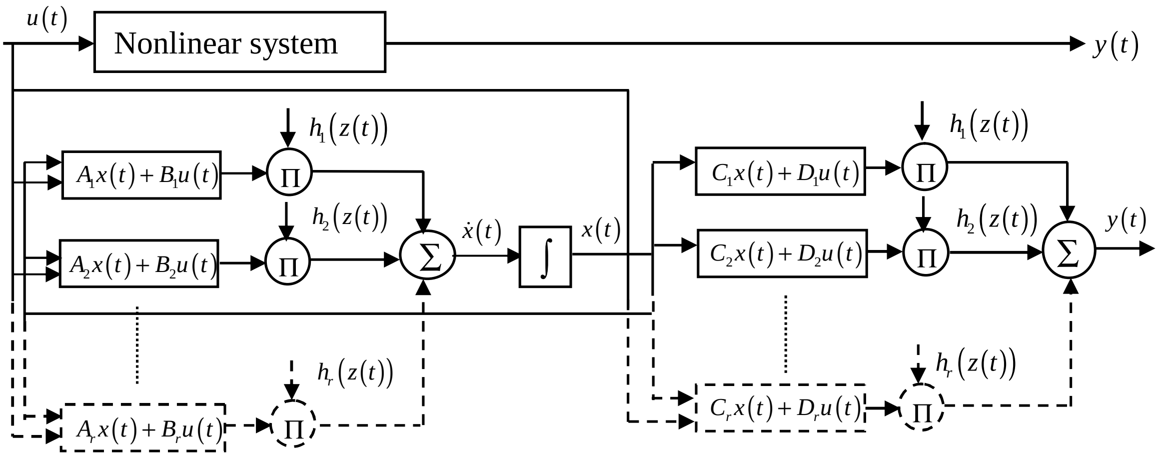


Figure 1.1. Architecture of classical T-S models.

1.4. How to obtain a Takagi-Sugeno model?

At first, the model of system plays an important role in automatic control theory, especially for controller and/or observer design. Therefore, it is important to learn how, from a given nonlinear system, to derive a Takagi-Sugeno? Up to now, there are three ways to obtain a TS fuzzy model which are: 1) identification from input-output data, 2) multiple linearizations from a known nonlinear model and 3) sector nonlinearity decomposition [Takagi & Sugeno, 1985][Tanaka & Wang, 2001].

- **Identification** [Gasso et al., 1999][Gasso et al., 2000]. The input-output data are measured in order to identify the parameters of the local time invariant models around specified operating points. In that case, the obtained T-S model is an approximation of the nonlinear one but is interesting when dealing with complex systems whose dynamics are not analytically expressed.

- **Multiple linearization** [Ma et al., 1998][Tanaka & Wang 2001]. The principle of this technique consists on linearizing the nonlinear systems around a finite set of appropriately chosen operating points. In that case, the whole T-S model is obtained through a fuzzy blending using appropriately chosen membership functions (triangular, trapezoidal, Gaussian…). Note that, in that case, the obtained T-S model remains an approximation of the considered nonlinear one.

- **Sector nonlinearity approach** [Kawamoto et al., 1992][Tanaka & Wang, 2001][Morère, 2001]. The principle of this technique is based on a convex polytopic transformation of the nonlinear terms included in a nonlinear dynamical system. In other words, it consists on finding a sector such that $a_1 x \le f\left(x(t),u(t)\right) \le a_2 x$ where $\dot{x} = f\left(x(t),u(t)\right)$ is a nonlinear dynamical system. The main advantage of this approach is that it guarantees that such obtained T-S model is exactly matching the nonlinear one, on all or part of the state space (see below for more details).

In the sequel, our interest is focused on the third approach (sector nonlinearity), since it has the advantage to be a systematic way to obtain both the local linear time invariant subsystems and the membership functions with a high accuracy regarding to the nonlinear system it represents. Indeed, the sector nonlinearity approach allows avoiding modeling approximation when transforming a nonlinear system into a T-S one. However, it is important to highlight that, in some cases, it could be difficult to find a global sector for all the nonlinearities. In that case, it is necessary to consider local nonlinear sector. This point is illustrated by the figures 1.2 and 1.3.

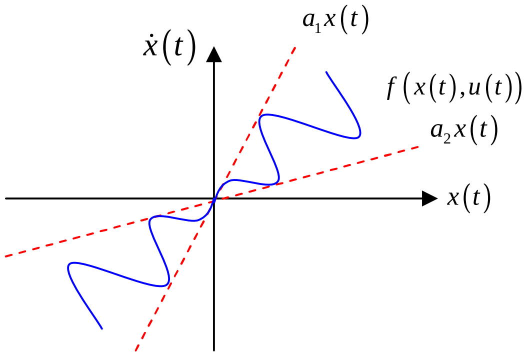
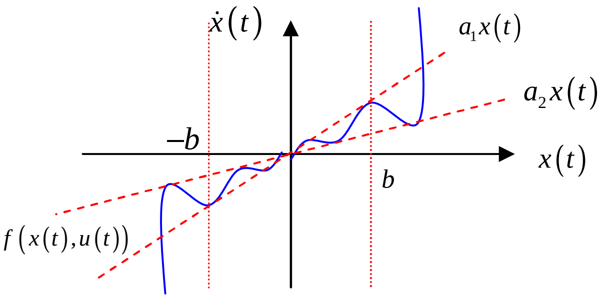


Figure 1.2. Global sector non linearity



.Figure 1.3. Local sector nonlinearity.

Note that the sector nonlinearity approach allows to associate an infinity of T-S models to a given nonlinear model according to the chosen convex polytopic transformation. In other word, polytopic transformations are not unique. However, a systematic way to apply sector nonlinearity approach is based on the following lemma:

Lemma 1.1 [Morère, 2001]**:**

Let $f(x(t)):\mathbb{R}\to\mathbb{R}$ *being a bounded function on* $x\in\Omega\subset\mathbb{R}$*, there always exist two functions* $w_1(x(t))$ *and* $w_2(x(t))$ *as well as two scalars* $\sim$ *and*   *such that :*

$$f(x(t)) = \sim \times w_1(x(t)) + \quad \times w_2(x(t))$$

with $w_1(x(t)) + w_2(x(t)) = 1,\ w_1(x(t)) \geq 0$ *and* $w_2(x(t)) \geq 0$.

***Proof*:**

Suppose that $f(x(t))$ is bounded such that $\sim \leq f(x(t)) \leq \quad$ one can write:

$$f(x(t)) = \sim \times w_1(x(t)) + \quad \times w_2(x(t)) \tag{1.8}$$

with $\sim\ = \max\left(f\left(x(t)\right)\right)$, $= \min\left(f\left(x(t)\right)\right)$, $w_1\left(x(t)\right) = \dfrac{f\left(x(t)\right) -}{\sim\ -}$ and

$w_2\left(x(t)\right) = \dfrac{\sim - f\left(x(t)\right)}{\sim\ -}$. ■

***Remark 1.1*:**

Let us consider $\dot{x} = f\left(x(t)\right)$ with $f(0) = 0$. If, $\forall x(t)$, $f\left(x(t)\right)$ is bounded on $\mathbb{R}$, one can derive a T-S model which is valid on the whole state space. Else if it is not bounded, one may reduce the validity domain to a compact subset Ω^n of the state space.

***Example 1.1*:**

Consider the nonlinear system given by :

$$\dot{x}(t) = x(t)\sin\left(x(t)\right) \tag{1.9}$$

with $f\left(x(t)\right) = \sin\left(x(t)\right) \in \left[-1 \quad 1\right]$, from lemma 1.1, one can always write :

$$\sin\left(x(t)\right) = \underbrace{\frac{\sin\left(x(t)\right)+1}{2}}_{h_1\left(x(t)\right)} \times 1 + \underbrace{\frac{1-\sin\left(x(t)\right)}{2}}_{h_2\left(x(t)\right)} \times (-1) \tag{1.10}$$

So an « exact » 2 rules T-S fuzzy model of (1.9) is obtained as:

$$IF\ \ x(t)\ \ is\ \ F_1^1\left(h_1 x(t)\right)\ \ THEN\ \ \dot{x}(t) = x(t)$$
$$IF\ \ x(t)\ \ is\ \ F_1^2\left(h_1 x(t)\right)\ \ THEN\ \ \dot{x}(t) = -x(t)$$

And so, after barycentric defuzzyfication, on has:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i\left(x(t)\right) a_i x(t) \tag{1.11}$$

with $a_1 = 1$ and $a_2 = -1$.

***Remark 1.2*:**

The T-S models obtained from the sector nonlinearity approach are constitute of 2^{nl} rules, where nl is the number of nonlinear terms to be taking into account in the initial nonlinear model.

1.5. Stability and stabilization of Takagi-Sugeno models

Stability analysis and stabilization of Takagi-Sugeno models are generally based on the Lyapunov theory [Liapounoff, 1907]. Therefore, before presenting classical conditions for T-S models, one proposes some basics and definitions about Lyapunov stability and linear matrix inequalities in the next section.

1.5.1. Basics on Lyapunov stability and linear matrix inequalities

Lyapunov's direct method is often used to analyze the stability of a dynamical system without explicitly solving its differential equation. The basic philosophy of the method is to construct a scalar energy-like function for the system and examine its time variation. Therefore, the ability of stability can be determined by studying the evolution over the time of that energy. Indeed, a system is stable if its whole energy is continuously dissipated, i.e. it is monotonously decreasing.

***Theorem 1.1*:**

Let the origin $x = 0$ be an equilibrium point of $\dot{x} = f(x)$, $f : D \rightarrow \Re^n$ and let $V : D \rightarrow \Re$ be a continuously differentiable function such that:

$$V(0) = 0$$

$$V(x) > 0, \textit{ in } D - \{0\}$$

$$\dot{V}(x) \leq 0, \textit{ in } D - \{0\}$$

Then $x = 0$ is a stable equilibrium point. Moreover, if V is strictly decreasing, i.e. $\dot{V}(x) < 0$ in $D - \{0\}$, then the system is said to be asymptotically stable. Finally, if $V(x)$ is “radially unbounded”, the system is said to be globally asymptotically stable (GAS).

Note that, in the context of Lyapunov stability analysis of state space models, Linear matrix inequalities (LMIs) are often used [Boyd et al., 1994]. Therefore, before going further to T-S models stability, a brief recall of LMIs is proposed.

Definition 1.1 [Boyd et al., 1994]**:**
A Linear Matrix Inequality is an inequality with matrix entries of the following form:

$$\mathrm{T}(x) = \mathrm{T}_0 + \sum_{i=1}^{n} x_i \mathrm{T}_i < 0 \tag{1.12}$$

where $\mathrm{T}(x)$ is an affine function in the decision variables $x = (x_1, x_2, ..., x_n)$ and the real matrices $\mathrm{T}_0, \mathrm{T}_1, ..., \mathrm{T}_n$ are square symetric ones. $T < 0$ stand for negative definite inequalities ($T \leq 0$ for semi-definite negative), that is to say that all its eigenvalues $\lambda(\mathrm{T}(x))$ are negatives.

The feasibility of (1.12) consists on determining the vector $x = (x_1, x_2, ..., x_n)$ such that $\mathrm{T}(x) < 0$ hold. Therefore, this may be obtained by minimizing the eigenvalues, i.e. $\min\limits_{x} \lambda_{\max}(\mathrm{T}(x)) < 0$. This can be achieved with interior point algorithm [Boyd et al., 1994] since $\lambda(\mathrm{T}(x)) < 0$ is a convex constraint.

Since Lyapunov stability analysis and controller synthesis can be expressed in terms of LMIs, the convexity of these constraints plays an important role. Indeed, from convex constraint, a global minimum to the optimization process may be found when a solution exists.

***Definition 1.2*:**

$\P$ is a convex hull if an only if :

$$\forall\left(x_1,x_2\right)\in \P^2 \ \text{then} \ ª x_1+\left(1-ª\right)x_2\in \P \ \text{for} \ 0\leq ª \leq 1 \tag{1.13}$$

Note that the solution set $\P=\left\{x\in \mathbb{R}^n : \mathrm{T}\left(x\right)<0\right\}$ corresponds to the convex optimization problem $\min\limits_{x\in\P} ª_{\max}\left(\mathrm{T}\left(x\right)\right)<0$. Moreover, a function is said to be convex if:

$$\forall\left(x_1,x_2\right)\in \P^2 \ \text{for} \ 0\leq ª \leq 1 \ \text{then} \ \mathrm{T}\left(ª x_1+\left(1-ª\right)x_2\right)=ª\mathrm{T}\left(x_1\right)+\left(1-ª\right)\mathrm{T}\left(x_2\right)<0 \tag{1.14}$$

In most of the control problems, Lyapunov conditions don't lead directly to LMIs, therefore it is often necessary to manipulate the obtained matrix inequalities in order to make them LMIs. To do this, some useful matrix transformations are recalled below [Boyd et al., 1994].

***Definition 1.3*:**

Let $\mathrm{T}\left(x\right)$ be a positive definite symmetric matrix and a non singular matrix Y, then

$$\mathrm{T}\left(x\right)>0 \Leftrightarrow Y^T\mathrm{T}\left(x\right)Y>0 \tag{1.15}$$

The matrix product $Y^T\mathrm{T}\left(x\right)Y$ is called « congruence » of $\mathrm{T}\left(x\right)$. Moreover, in a matrix inequality, the congruence doesn't affect the sign of the eigenvalues.

Lemma 1.2: Schur complement

Let $A_{22}\left(x\right)\in \mathbb{R}^{n\times n}$ a positive definite matrix, $A_{12}\left(x\right)\in \mathbb{R}^{n\times m}$ a full rank matrix in line and $A_{11}\left(x\right)\in \mathbb{R}^{n\times n}$ any kind of matrix. The following inequalities are equivalents:

$$A\left(x\right)=\begin{bmatrix} A_{11}\left(x\right) & A_{12}^T\left(x\right) \\ A_{12}\left(x\right) & A_{22}\left(x\right) \end{bmatrix}>0 \tag{1.16}$$

$$A_{11}(x) - A_{12}^T(x)\left(A_{22}(x)\right)^{-1} A_{12}(x) > 0 \tag{1.17}$$

Lemma 1.3: Square matrix inequality [Zhou & Khargonekar, 1988]

Let us consider two matrices W and N, there a matrix $Q > 0$ or a scalar $\varepsilon > 0$, such that :

$$WN^T + NW^T \leq \varepsilon WW^T + \varepsilon^{-1} NN^T \tag{1.18}$$

$$WN^T + NW^T \leq WQW^T + NQ^{-1}N^T \tag{1.19}$$

1.5.2. Takagi-Sugeno models' stability

The stability study of an autonomous T-S model (1.7) allows determining whether its dynamics is stable when not subject to any external excitation ($u(t) = 0$). In the following, to enable the reader to understand the proposed results in the remainder of this manuscript, some significant result on stability of T-S are first presented. These are given in the form of Linear Matrix Inequalities (LMI) [Boyd et al., 1994]. Note that the following result deals with the stability of T-S described in continuous time:

***Theorem 1.2*:** [Tanaka & Sugeno, 1992]

The continuous autonomous T-S ($u(t) = 0$) (1.7) *is asymptotically stable if, for $i = 1, \ldots, r$, there exists a matrix $P = P^T > 0$, such that the following LMI is verified:*

$$A_i^T P + PA_i < 0 \tag{1.20}$$

Proof:

Let us consider the following Lyapunov candidate function:

$$V\left(x(t)\right) = x^T(t) P x(t) \tag{1.21}$$

the autonomous T-S (1.7) ($u(t) = 0$) is stable if:

$$\dot{V}\left(x(t)\right) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) < 0 \tag{1.22}$$

That is to say:

$$\dot{V}\left(x(t)\right) = x^T(t)\left(\sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i^T P + PA_i\right)\right)x(t) < 0 \tag{1.23}$$

which is verified if the sufficient condition of theorem 1.2 holds.

■

***Remark 1.3*:**

Note that no information on the membership functions $h_i\left(z(t)\right)$ are taken into account in Theorem 1.2. Therefore, LMI stability conditions are only sufficient (not necessary) and consequently conservative. Indeed, for these conditions, we look for a solution to an optimization problem by considering that, for a sum to be negative, each of this term must be negative. However, it is of course possible to seek solutions to the conditions of Theorem 1.2 that would allow certain terms of the sum (1.23) to be positive, while the entire sum remains negative. In this case, the obtained stability conditions are said "relaxed ", that is to say less conservative. In case one can guarantee that the system is stable, it may be worthwhile to improve performance. Otherwise, it is necessary to synthesize a stabilizing fuzzy control law in order to bring the system to equilibrium.

1.5.3. State feedback stabilization of Takagi-Sugeno models

To ensure the stability of a closed loop T-S, the synthesis of a suitable control law is carried out. Several of fuzzy control laws have been proposed in the literature. The most common are based on control laws such parallel distributed compensation (PDC Parallel Distributed Compensation) [Wang et al., 1996][Tanaka et al., 1998]. Further note that, depending on the class of the considered TS models, many alternatives to this kind of control law have also been proposed in the literature, e.g. proportional PDC (PPDC) [Lin & Er, 2001] or control law by compensation and division for fuzzy models [Guerra et al., 1999].

The PDC control law [Wang et al., 1996][Tanaka et al., 1998] has the advantage of considering the same premises as the fuzzy rules in the TS model to stabilize. Therefore, this leads to consider that to each local model corresponds a linear state feedback control that can be interpolated by the same activation functions $h_i\left(z(t)\right)$ as the TS model. Note that in this case, when the fuzzy model obtained by convex polytopic transformation is exact, then this control law is valid throughout the compact space of the state variables. Thus, the control law is given by:

$$u(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right) F_i x(t) \tag{1.24}$$

where $F_i \in \mathbb{R}^{m\times n}$ represent the state feedback gain matrices to be determined for the fuzzy controller.

In order to obtain the closed loop of the fuzzy TS model, one substitutes the control law (1.24) into (1.7), it yields:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right)\left(A_i - B_i F_j\right) \tag{1.25}$$

Sufficient stabilization conditions are given in the following theorem.

Theorem 1.3 [Wang et al., 1996]:
The time continuous T-S (1.7) is asymptotically stabilized through the PDC control law (1.24) if there exist the matrices X and K_j, such that the following LMI are verified.

$$A_i X - B_i K_j + X A_i^T - K_j^T B_i^T < 0, \; for \; i, j = 1,...,r \tag{1.26}$$

with $K_j = F_j X$ *and* $X = P^{-1}$

Proof:

Following the same approach as for the proof of Theorem 1.2, we obtain straightforwardly:

$$\dot{V}(x(t)) = x^T(t)\left(\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\left(A_i^T P + PA_i - F_j^T B_i^T P - PB_i F_j\right)\right)x(t) < 0 \quad (1.27)$$

which is verified $\forall x(t)$ by considering $X = P^{-1}$ $K_j = F_j X$ if conditions of theorem 1.2 hold. ■

1.6. Conservatism reduction of LMI conditions for T-S models

The results presented above, on the stability and stabilization of TS systems, are conservative. Indeed, they are only sufficient but not necessary conditions. In this section, significant solutions to reduce the conservatism of TS approaches are presented. Note that these solutions depend mainly on the source of conservatism to which it is addressed [Ariño & Sala, 2009][Sala, 2009], for example: non-uniqueness of a TS model, the choice of the candidate Lyapunov function the independence of membership function and interconnection structure of the LMI conditions, high computational cost ... In the following, we will explain some of these sources of conservatism and we will present the main results in the literature allowing their reduction.

First of all, a source of conservatism arise from the interaction between crossing terms in the LMI conditions resulting from the double sum structure of the closed-loop dynamics (1.25). Indeed, for the conditions proposed in theorems 1.2 and 1.3, we look for a solution to an optimization problem by considering that, for a sum being negative, each term of this sum must be negative. However, it is of course possible to find solutions that would allow some of these terms being positive while the whole sum remains negative. Several solutions have been proposed to reduce this source of conservatism. These are commonly called "relaxation schemes" [Tanaka & Sano, 1994][Tanaka et al., 1998][Kim & Lee, 2000][Tuan et al., 2001] [Xiaodong & Qingling, 2003]. Among these, we present the most used through the following lemmas.

Lemma 1.4 [Tanaka & Sano, 1994]:

For $i=1,...,r$, $j=1,...,r$, $\forall h_i>0$, $\forall h_j>0$, *the inequality* $\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j \Upsilon_{ij}<0$ *is satisfied if the following conditions hold :*

$$\Upsilon_{ii}<0 \text{ for } i=1,...,r \tag{1.28}$$

$$\Upsilon_{ij}+\Upsilon_{ji}<0 \text{ for } i,j=1,...,r \text{ and } i<j \tag{1.29}$$

Lemma 1.5 [Tuan et al., 2001]:

For $i=1,...,r$, $j=1,...,r$, $\forall h_i>0$, $\forall h_j>0$, *the inequality* $\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j \Upsilon_{ij}<0$ *is satisfied if the following conditions hold :*

$$\Upsilon_{ii}<0 \text{ for } i=1,...,r \tag{1.30}$$

$$\frac{2}{r-1}\Upsilon_{ii}+\Upsilon_{ij}+\Upsilon_{ji}<0 \text{ for } i,j=1,...,r \text{ and } 1\le i\ne j\le r \tag{1.31}$$

Lemma 1.6 [Xiaodong & Qingling, 2003]:

For $i=1,...,r$, $j=1,...,r$, $\forall\, h_i>0$, $\forall\, h_j>0$, *the inequality* $\sum_{i=1}^{r}\sum_{j=1}^{r}h_i h_j \Upsilon_{ij}<0$ *is satisfied if the following conditions hold :*

$$\Upsilon_{ii}>\Xi_{ii} \text{ for } i=1,...,r \tag{1.32}$$

$$\Upsilon_{ij}+\Upsilon_{ji}>\Xi_{ij}+\Xi_{ji} \text{ for } i,j=1,...,r \text{ et } i<j \tag{1.33}$$

$$\begin{bmatrix} \Xi_{11} & \Xi_{12} & .... & \Xi_{1r} \\ \Xi_{21} & \Xi_{22} & .... & \Xi_{2r} \\ \vdots & & \ddots & \vdots \\ \Xi_{r1} & \Xi_{r2} & .... & \Xi_{rr} \end{bmatrix}>0 \tag{1.34}$$

***Remark 1.4*:**

Among the relaxation schemes presented above, Lemma 1.6 [Xiaodong & Qingling, 2003] provides the least conservative results. However, due to the increasing number of decision variables, the computational cost increases significantly. A good compromise between conservatism and computational cost is proposed in Lemma 1.5, leading to satisfactory results in terms of conservatism without introducing new decision variables to be determined [Tuan et al., 2001].

On the other hand, the choice of a candidate quadratic Lyapunov function is also a source of conservatism. Indeed, for TS systems, the choice made to derive the conditions presented above (theorem 1.2 and 1.3) requires finding a common matrix to a set of LMI constraints. Other approaches that consider piecewise Lyapunov functions have been proposed [Johansson, 1999][Johansson et al, 1999][Feng & Wang, 2001][Feng, 2004]. However, they are not very effective when a representative T-S model is obtained by the sector nonlinearity approach. Indeed, this type of Lyapunov functions does not take into account the non-linear interconnection structure between sub-models used in the global system. An alternative, effective when the T-S model is obtained from a nonlinear one, is to check the existence of a non-quadratic Lyapunov candidate [Jadbabaie, 1999][Tanaka et al, 2003][Guerra & Vermeiren 2004][Rhee & Won, 2006][Feng, 2006][Tanaka et al, 2007]. Among these approaches, there are those that consider the use of a candidate Lyapunov function based on the same fuzzy interconnection structure (membership functions) as the TS system studied. Thus, the system, the non-quadratic Lyapunov function and the fuzzy controller are all based on the same structure. Therefore, it is understood that a parallel compensation between these parties facilitates the search for a solution to LMI and leads to a reduction of conservatism.

To clarify this approach, consider the problem of non-quadratic state feedback stabilization of T-S (1.7) model with the following non-quadratic Lyapunov function candidate:

$$V\left(x(t)\right) = x^T(t)\left(\sum_{i=1}^{r} h_i\left(z(t)\right)P_i\right)^{-1} x(t) \tag{1.35}$$

with $P_i = P_i^T > 0$,

and the non-PDC conttrol law given by :

$$u(t) = \sum_{i=1}^{r} h_i\left(z(t)\right) F_i \left(\sum_{j=1}^{r} h_j\left(z(t)\right) P_j \right)^{-1} x(t) \tag{1.36}$$

The closed-loop dynamics of the system can be expressed as :

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right) \left(A_i + B_i F_j \left(\sum_{k=1}^{r} h_k\left(z(t)\right) P_k \right)^{-1} \right) x(t) \tag{1.37}$$

Therefore, (1.37) is stable if [Tanaka & al., 2003]:

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right) \left(P_j A_i^T + A_i P_j + F_j^T B_i^T + B_i F_j - \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) P_k \right) < 0 \tag{1.38}$$

Since $\sum_{k=1}^{r} h_k\left(z(t)\right) = 1,\ \forall R \in \mathbb{R}^{n\times n}$, one can write [Mozelli & al., 2009]:

$$\sum_{k=1}^{r} \dot{h}_k\left(z(t)\right) R = 0 \tag{1.39}$$

so, (1.38) yields :

$$\sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right) \left(P_j A_i^T + A_i P_j + F_j^T B_i^T + B_i F_j - \sum_{k=1}^{r} \dot{h}_k\left(z(t)\right)\left(P_k + R\right) \right) < 0 \tag{1.40}$$

Therefore, assuming that the derivatives of the membership functions are bounded, i.e $\forall t$, $\dot{h}_k\left(z(t)\right) \geq$ ¤$_k$, the closed-loop dynamics (1.37) is stabilized if the conditions proposed in the following theorem are satisfied.

Theorem 1.4 [Mozelli et al., 2009]:

Assuming that $\forall t$, $\dot{h}_k\left(z(t)\right) \geq$ ¤$_k$, *the T-S model (1.7) is asymptotically stabilized via the non-PDC control law* (1.36) *if there exist the matrices* $P_k^T = P_k > 0$, F_j *and* R *such that, for all* $i, j, k = 1, \ldots, r$ *, the following LMI conditions are satisfied:*

$$P_j A_i^T + A_i P_j + F_j^T B_i^T + B_i F_j - \sum_{k=1}^{r} \text{¤}_k \left(P_k + R \right) < 0 \qquad (1.41)$$

$$P_k + R \geq 0 \qquad (1.42)$$

***Remark 1.5*:**

By introducing slack decision matrices (equation (1.39)), the proposed LMI conditions in Theorem 1.4 is an extension (less conservative) of the conditions presented in [Tanaka et al., 2003] since it can be recovered by considering the special case $R = P_r$. Moreover, the quadratic conditions of Theorem 1.3 are included in those of Theorem 1.4. Indeed, by considering $P_k = P$ as a special case, one obtains the conditions of theorem 1.3.

The figure.1.4 illustrates, in terms of areas of feasibility, the conservatism reduction of the LMI conditions provided by the non-quadratic approaches with respect to the quadratic approaches.

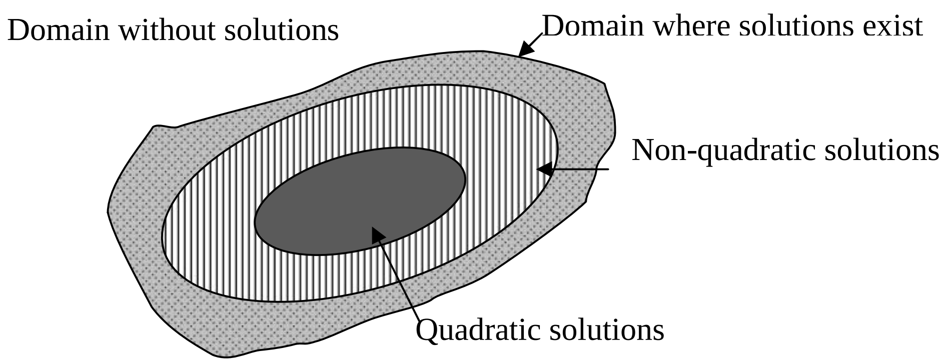


Figure.1.4. Feasibility domains in the non-quadratic and the quadratic frameworks.

To conclude this section, devoted to the presentation of the basic concepts related to the stabilization of T-S models, remember that these models have the ability to accurately represent a nonlinear system on a compact space of the state variables. In addition, the structure of the T-S models, consisting of a set of linear systems blended together by a nonlinear interpolation structure present the advantage of making possible to extend some properties of linear systems to the nonlinear case (control synthesis, robust control ...). However, some drawback appears with non-quadratic Lyapunov functions since they require the knowledge of the bounds of the derivatives of the membership functions. These parameters are often difficult to find in practice, especially in stabilization studies since they are necessary to solve the LMI conditions, but cannot be known before designing the closed-loop dynamics. Some recent work have been done to overcome this drawback but with reconsideration of the global analysis to a local point of view [Guerra et al., 2012]. However, these techniques remain very complex for a conservatism reduction that is sometime questionable. Therefore, the work proposed in the following chapters tends to explore what can be done, else than conventional non-quadratic approaches, or else than LMIs. It is therefore assumed that polynomial approaches may be an alternative. Thus, before going to our contributions, the next section introduces the basics on Sum-Of-Squares (SOS) approaches for the stability analysis and the stabilization of polynomial T-S fuzzy systems.

1.7. Polynomial T-S models analysis via Sum-Of-Squares techniques

Sum-Of-Squares (SOS) techniques are based on the decomposition of multivariate polynomials, which can be optimized using semi-definite programming [Prajna et al., 2004]. Before presenting Polynomial Fuzzy Systems, some generalities on polynomials are recalled.

1.7.1. Generalities on Polynomials and SOS techniques

Let consider a vector $x=\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}^T$, a monomial in x is a function of the form $x_1^{\sim_1} x_2^{\sim_2} \dots x_2^{\sim_2}$ where $\sim_i \in \mathbb{N}$, for $i=1,...,n$. Moreover, a n^{th} order polynomial can be written as $p=\sum_{k=0}^{n} a_k x^k$ with $a_n \neq 0$ and a_k are constant scalars for $k=1,...,n$. Finally, a polynomial matrix is a matrix which entries are polynomials.

The SOS approach is based on the mathematical theory of decomposition of a multivariable polynomial in sum of squares [McCullough, 2001]. One of Hilbert's problems concerned the expression of such polynomials as sums of squares. Indeed, if a positive polynomial in $\mathbb{R}[x]$ can be decomposed into a sum of squares, it was not the same for polynomials with several variables. There are still some results on the positive symmetric polynomials which can be studied in matrix formulation [Helton, 2002], using the Gram decomposition [Choi et al., 1995]. The SOS approach can be seen as a generalization of LMIs. For instance, it may be an alternative since it allows formulating and solving a class of polynomial problems when it is difficult to represent in terms of LMI, or when the results obtained by the LMI are too conservative.

A multivariable polynomial $P(x_1,\ldots,x_n)=P(x)$ is SOS if one can find some polynomials $p_i(x)$, for $i=1,\ldots,m$, such that:

$$P(x)=\sum_{i=1}^{m} p_i^2(x) \tag{1.43}$$

Note that SOS polynomials are obviously a positive (or semi-positive) quantity. Therefore, (1.43) can be rewritten as the following Gram decomposition:

$$P(x)=Z^T(x)QZ(x) \tag{1.44}$$

where $Z(x)$ is a monomial vector of x ($\deg(Z)\le\deg(P)/2$) and Q is a semi-definite positive matrix.

Note that, SOS optimization problems can be computed using semi-definite-programming tools such as SOSTOOLS or YALMIP, available in the MATLAB environment [Prajna et al., 2004].

1.7.2. Stability conditions for polynomial T-S fuzzy systems

Fuzzy polynomial models can also be represented as T-S models, i.e. by fuzzy rules where the concluding part matrices are now polynomial ones. Hence it can be given, in its compact form, as:

$$\dot{x}(t)=\sum_{i=1}^{r}h_i(z(t))\left(A_i(x(t))\hat{x}(x(t))+B_i(x(t))u(t)\right) \tag{1.45}$$

where $A_i(x(t))$, $B_i(x(t))$ are polynomial matrices in $x(t)$, $\hat{x}(x(t))\in\mathbb{R}^N$ is a vector of monomials in $x(t)$ with $\hat{x}(x(t))=0$ if $x(t)=0$.

We now present some works on the stability of Takagi-Sugeno models using the SOS approach. Stability conditions are also given thanks to Lyapunov polynomial functions given by [Tanaka et al., 2009]:

$$V(x)=\hat{x}^T(x(t))P(x(t))\hat{x}(x(t)) \tag{1.46}$$

where $P(x(t))$ is a polynomial matrix in $x(t)$.

In the sequel, in order to simplify mathematical expression, the time t is omitted when there is no ambiguity. Moreover, $A_i^k(x)$ denoted the k^{th} line of a matrix $A_i(x)$.

Theorem 1.5 [Tanaka et al., 2009]:

The polynomial T-S model (1.45) is globally asymptotically stable if there exist a symmetric polynomial matrix $P(x)\in\mathbb{R}^{N\times N}$ and the polynomial $£_1(x)>0$ and $£_{2i}(x)>0$ such that the following polynomials are SOS for $i=1,\ldots,r$:

$$\hat{x}^T(x)\left\{P(x)-£_1(x)I\right\}\hat{x}(x(t)) \tag{1.47}$$

$$-\hat{x}^T(x)\left\{P(x)T(x)A_i(x)+A_i^T(x)T^T(x)P(x)+\sum_{k=1}^{n}\frac{\partial P}{\partial x_k}(x)A_i^k(x)\hat{x}(x)+£_{2i}(x)I\right\}\hat{x}(x) \tag{1.48}$$

where $T(x)\in\mathbb{R}^{N\times n}$ is a polynomial matrix which entries are defined such that:

$$T_{ij}(x)=\frac{\partial\hat{x}_i(x)}{\partial x_j} \tag{1.49}$$

***Proof*:**

Consider the polynomial Lyapunov candidate (1.46) and the unforced ($u(t)=0$) polynomial T-S fuzzy system (1.45). $P(x)>0$ is ensured by (1.47) and the polynomial T-S system (1.45) is stable if:

$$\dot{V}(x)=\dot{\hat{x}}^T(x)P(x)\hat{x}(x)+\hat{x}^T(x)P(x)\dot{\hat{x}}(x)+\hat{x}^T(x)\dot{P}(x)\hat{x}(x)<0 \tag{1.50}$$

Moreover, since $\dot{\hat{x}}(x)=\frac{d\hat{x}(x)}{dt}=\frac{\partial\hat{x}(x)}{\partial x}\frac{dx}{dt}=T(x)\dot{x}$ and $\dot{P}(x)=\sum_{k=1}^{n}\frac{\partial P}{\partial x_k}(x)\dot{x}_k$, (1.50) can be rewritten as:

$$\dot{x}^T(x)T^T(x)P^T(x)\hat{x}(x)+\hat{x}^T(x)P(x)T(x)\dot{x}(x)+\hat{x}^T(x)\left\{\sum_{k=1}^{n}\frac{\partial P}{\partial x_k}(x)\dot{x}_k\right\}\hat{x}(x)<0 \tag{1.51}$$

Now, substituting (1.45) and since $\dot{x}_k=\sum_{i=1}^{r}h_i(z)A_i^k(x)\hat{x}(x)$, (1.51) becomes:

$$\hat{x}^T(x)\sum_{i=1}^{r}h_i(z)\left\{A_i^T(x)T^T(x)P(x)+P(x)T(x)A_i(x)+\sum_{k=1}^{n}\frac{\partial P}{\partial x_k}(x)A_i^k(x)\hat{x}(x)\right\}\hat{x}(x)<0 \tag{1.52}$$

Finally, since $h_i(z)\geq 0$ for $i=1,\ldots,r$, (1.52) is satisfied if (1.48) is SOS.

***Remark 1.6*:**

The quadratic conditions for standard T-S fuzzy systems (1.7) described in theorem 1.2 are included in the SOS based ones given in theorem 1.5. Indeed, in the latter, if we consider $A_i(x)=A_i$ and $P(x)=P$ constant matrices and $\hat{x}(x)=x$, the conditions of theorem 1.2 are recovered.

1.8. Conclusion

In this chapter, some necessary preliminaries about stability and stabilization of Takagi-Sugeno (T-S) fuzzy models have been introduced. It has been shown that LMI results, obtain through a quadratic or a non-quadratic Lyapunov function may suffer from conservatism. Even if the relaxation of LMI is a challenging concern of the T-S community, recent results in this field appear more and more complex with a conservatism reduction that may be questionable. Therefore, the main contribution proposed in the following chapters is to explore what can be done, else than non-quadratic approach or LMI based approaches. In that way, the next chapter deals with the robust stabilization of T-S systems via switched fuzzy approaches. Then, in the chapter 3, Sum-Of-Squares stability conditions for polynomial T-S systems are investigated to extend the one proposed above. Finally, in the 4th chapter, SOS approaches are proposed as an alternative to LMI results in the non-quadratic framework.

Chapter 2

Robust stabilization of nonlinear systems based on a switched fuzzy control law

Résumé du chapitre 2 en Français:

Stabilisation robuste des systèmes non-linéaires par lois de commande floues à commutation

Ce chapitre traite de la stabilisation robuste, par lois de commande PDC à commutations, des systèmes non linéaires représentés par des modèles Takagi-Sugeno incertain et perturbés. L'objectif est ici de proposer une réduction de conservatisme en partitionnant l'espace d'état et en proposant une loi de commande PDC commutant en fonction des régions obtenues. Tout d'abord, des conditions de synthèse LMIs de lois de commande robuste pour ces systèmes sont proposées en ne considérant pas les perturbations extérieures. Ensuite, l'extension de ces conditions de synthèse au cas des systèmes soumis à des perturbations externes est proposée au travers de la minimisation d'un critère H_∞. L'efficacité du critère de stabilisation proposé et de l'approche de synthèse de contrôleur PDC à commutation est illustrée par un exemple académique.

2.1. Introduction

With the growing complexity of some control engineering problems, control techniques drawn from linear theory have shown their limits. Among nonlinear theory, new control approaches have appeared in the last decades such as hybrid or fuzzy techniques.

A hybrid dynamical system (HDS) consists of continuous (or discrete) time dynamics associated with discrete events following some logical or decision-making rules. For instance, power transmission and distribution, constrained robotic systems and intelligent vehicle highway systems may be considered as HDS. Among HDS, switched linear systems have attracted extensive research, see e.g. [Chiou, 2006a][Daafouz et al., 2002][Fang et al., 2004] [Hetel et al., 2006][Mansouri et al., 2008][Liberzon et al., 2009][Ni et al., 2008]. To obtain stability conditions, two techniques are usually employed. Some authors consider the dwell time concept [Liberzon et al., 2009][Chiou, 2006a][Chiou, 2006b]. In these works, authors proved that, when the linear subsystems are Hurwitz, the overall switched system is stable if the time between consecutive switching is sufficiently large. The second technique, generally based on Lyapunov theory, aims at designing a control law able to stabilize the overall linear switched system without considering a particular switching law [Hetel et al., 2006] [Mansouri et al., 2008]. Note that, all these studies consider HDS described as a collection of linear systems switching together. However, stability and stabilisation issue for nonlinear switched systems has been seldom treated in the literature [Hespanha & Morse, 1999] [Palm & Driankov, 1998].

Independently to the works on HDS, other studies have focussed on fuzzy modelling and control approaches. Starting from basic fuzzy control techniques [Mamdani et al., 1974], the last three decades have shown Takagi-Sugeno (T-S) fuzzy modelling and control techniques arising [Takagi & Sugeno, 1985][Tanaka & Wang, 2001]. Indeed, T-S fuzzy models have then attracted interest when dealing with nonlinear systems. These are constituted by a set of linear models interconnected by fuzzy membership functions. Thus, using a convenient convex polytopic transformation, a T-S model can match exactly an affine (bounded) nonlinear system in a compact set of the state space [Tanaka & Wang, 2001]. Based on the polytopic structure of T-S models, the merit of T-S fuzzy control approaches is that they make possible the extension of some linear concept to the case of nonlinear systems [Tanaka & Wang, 2001][Sala et al., 2005][Bouarar et al., 2007, 2010][Zerar et al., 2008] [Mansouri et al., 2009][Lendek et al., 2010]. Nevertheless, an inherent drawback remains since the number of fuzzy rules of a TS model increase exponentially with the number of nonlinearities constituting the matched nonlinear system [Delmotte et al., 2008]. This makes fuzzy controller design and implementation difficult as the complexity of the nonlinear system to be controlled increases.

To outline the problem of rules explosion in T-S modelling, some authors have proposed to combine the merit of switched systems with T-S ones to deal with nonlinear control problems [Othake et al., 2002][Othake et al., 2006][Lam, 2009]. To do so, partitioning the state space of a nonlinear system allows defining a switched nonlinear system. Then, inside each partition, a T-S model can be obtained. So, as stated in [Yang & Zhao, 2007], the resulting switched T-S system inherits some essential features of hybrid systems and maintains all the information and knowledge representation capacity of fuzzy systems. Few papers have studied stabilization issues of switched fuzzy systems based on quadratic approaches [Palm & Driankov, 1998][Lam, 2009][Yand et al., 2008][Ojleska & Stojanovski, 2008][Yang & Zhao, 2007] or switching Lyapunov function [Othake et al., ,2002][Othake et al. 2006]. Note that these studies only consider nominal systems and so, they are irrelevant when dealing with robustness of the designed controller. Therefore, a robust controller design has been proposed in [Yang & Zhao, 2007] for uncertain switched T-S systems. Nevertheless, in the latter study, a classical quadratic Lyapunov approach has been employed leading to conservative results since it needs to check the existence of a common Lyapunov matrix for a set of linear matrix inequalities (LMI) constraints. Following the work on switched linear systems [Dafouz et al., 2002][Fang et al. 2004], less conservative LMI conditions for T-S switched systems have been provided by employing switched Lyapunov function [Othake et al., 2006]. The aim of this chapter is then to extend these works to the case of robust switched fuzzy Parallel Disturbed Compensation (PDC) controller design for the class of uncertain and disturbed switched T-S fuzzy systems.

The chpater is organized as follows. In the first section, the class of uncertain and disturbed switched T-S fuzzy systems is depicted as well as the considered switched PDC control law and switched Lyapunov candidate function are presented. After some useful lemmas and notations, the second section presents the main result: a stabilization criterion is proposed as LMI conditions for uncertain switched T-S systems. Then, this result is extended to the class of uncertain switched fuzzy system subject to external disturbances using a H_∞ criterion. Finally, a simulation example, followed by a conclusion, is provided to illustrate the efficiency of the proposed approaches.

2.2. Problem statement

2.2.1. From a nonlinear system to its switched fuzzy representation

Consider the following nonlinear system:

$$\dot{x}(t) = \left(f\left(x(t)\right) + \Delta f\left(x(t)\right) \right) x(t) + \left(g\left(x(t)\right) + \Delta g\left(x(t)\right) \right) u(t) + d\left(x(t)\right) \text{¨}(t) \qquad (2.1)$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \end{bmatrix} \in \Re^n$, $u(t) = \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_m(t) \end{bmatrix} \in \Re^m$ and $\text{¨}(t) \in \Re^p$ are respectively the state, the input and the external disturbances vectors. $f\left(x(t)\right) \in \Re^{n\times n}$, $g\left(x(t)\right) \in \Re^{n\times m}$ are nonlinear matrices defining the nominal part of (2.1). $\Delta f\left(x(t)\right) \in \Re^{n\times n}$ and $\Delta g\left(x(t)\right) \in \Re^{n\times m}$ represent Lebesgue measurable structural uncertainties due, for instance, to modelling approximations. The quantity $d\left(x(t)\right)\text{¨}(t)$, with $d\left(x(t)\right) \in \Re^{n\times p}$, represent external disturbances to the state dynamics.

Following the way proposed in [Othake et al., 2002] for nominal systems, using the sector nonlinearity (SNL) approach [Tanaka & Wang, 2001] and a convenient state space partitioning, the uncertain and disturbed nonlinear system (2.1) can be rewritten as an uncertain and disturbed switched Takagi-Sugeno model described as follows:

$$\begin{aligned} \dot{x}(t) = \sum_{q=1}^{Q} \sum_{i=1}^{r_q} v_q\left(x(t)\right) h_{qi}\left(x(t)\right) \Big(& \left(A_{qi} + \Delta A_{qi}(t)\right) x(t) \\ & + \left(B_{qi} + \Delta B_{qi}(t)\right) u(t) + G_{qi} \text{¨}(t) \Big) \end{aligned} \qquad (2.2)$$

where Q denotes the number of partitioned regions of the state space and r_q is the number of rules in each region. $A_{qi} \in \Re^{n\times n}$, $B_{qi} \in \Re^{n\times m}$ and $G_{qi} \in \Re^{n\times p}$ are constant matrices with appropriate dimensions for all $i = 1,...,r$ and $q = 1,...,Q$. The matrices $\Delta A_{qi}(t) \in \Re^{n\times n}$,

$\Delta B_{qi}(t) \in \Re^{n\times m}$ represent the uncertain norm-bounded (lebesgue measurable) matrices which can be rewritten such that:

$$\begin{cases} \Delta A_{qi}(t) = H_{aqi} F_a(t) N_{aqi} \\ \Delta B_{qi}(t) = H_{bqi} F_b(t) N_{bqi} \end{cases} \tag{2.3}$$

where H_{aqi}, H_{bqi}, N_{aqi} and N_{bqi} are known real matrices of appropriate dimension, $F_a(t)$ and $F_b(t)$ are unknown normalized functions satisfying respectively $F_a^T(t)F_a(t) \le I$ and $F_b^T(t)F_b(t) \le I$.

Moreover, in (2.2), $h_{qi}(x(t)) \ge 0$ are the fuzzy membership functions verifying the convexes sum propriety $\sum_{i=1}^{r_q} h_{qi}(x(t)) = 1$ and $v_q(x(t))$ are the switched laws defined by:

$$v_q(x(t)) = \begin{cases} 1 & if\ x(t) \in \text{ region } R_q(s_{1q}, s_{2q}, \ldots, s_{nq}) \\ 0 & if\ x(t) \notin \text{ region } R_q(s_{1q}, s_{2q}, \ldots, s_{nq}) \end{cases} \tag{2.4}$$

Consider the state vector $x(t) = [x_1(t) \quad \ldots \quad x_n(t)]$, the q^{th} region $R_q(s_{1q}, s_{2q}, \ldots, s_{nq})$ follows:

$$s_{kq} = \begin{cases} 1 & if\ x_k(t) \ge 0 \\ 0 & if\ x_k(t) < 0 \end{cases}, \qquad k = 1, \ldots, n \tag{2.5}$$

To illustrate this modelling approach, based on the switched laws (2.4), the state space partitions of a second order switched T-S system leads to four regions $R_q(s_{1q}, s_{2q})$ depicted in Fig.2.1.

Note that, using the modelling methodology proposed by [Othake et al., 2002], the switched T-S system (2.2) represents exactly the nonlinear system (2.1) on a compact set of the state space. That is to say that the robust controller design proposed in the sequel is valid on the whole state space if the nonlinearities of the uncertain system (2.1) are bounded (global SNL) or, in the contrary, on a restricted region (local SNL), see [Ohtake et al., 2002] for more details.

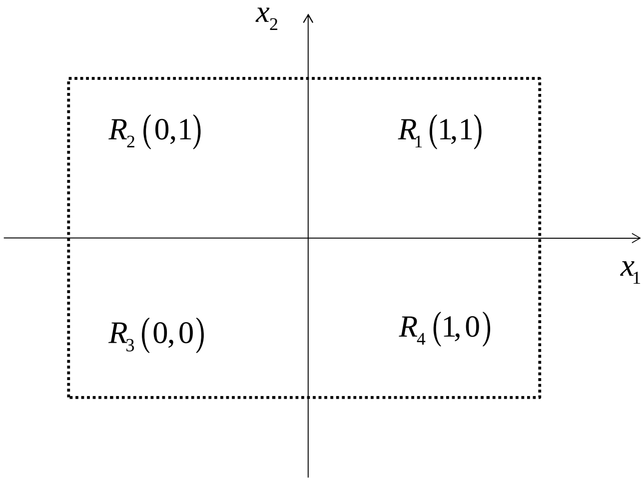


Figure 2.1. Example of a second order state space partition.

To lead to the LMI conditions proposed in the next section, an extended state space system can be employed [Othake et al., 2006]. Following this way, let us consider a stable autonomous linear system such that:

$$\dot{\hat{x}}(t) = C\hat{x}(t) \tag{2.6}$$

where $\hat{x}(t) = \begin{bmatrix} \hat{x}_1(t) & \hat{x}_2(t) & \cdots & \hat{x}_n(t) \end{bmatrix} \in \Re^n$ is a state vector and $C \in \Re^{n\times n}$ is a Hurwitz matrix.

Let $\tilde{x}(t) = \begin{bmatrix} x(t) & \hat{x}(t) \end{bmatrix}$ be an extended state vector, (2.2) can be extended with (2.6) leading to:

$$\dot{\tilde{x}}(t)=\sum_{q=1}^{Q}\sum_{i=1}^{r_q}v_q\left(x(t)\right)h_{qi}\left(x(t)\right)\left(\left(\tilde{A}_{qi}+\Delta\tilde{A}_{qi}(t)\right)\tilde{x}(t)+\left(\tilde{B}_{qi}+\Delta\tilde{B}_{qi}(t)\right)u(t)+\tilde{G}_{qi}\,\ddot{}\,(t)\right) \quad (2.7)$$

with $\tilde{A}_{qi}=\begin{bmatrix}A_{qi} & 0\\ 0 & C\end{bmatrix}$, $\Delta\tilde{A}_{qi}=\begin{bmatrix}\Delta A_{qi} & 0\\ 0 & 0\end{bmatrix}$, $\tilde{B}_{qi}=\begin{bmatrix}B_{qi}\\ 0\end{bmatrix}$, $\Delta\tilde{B}_{qi}=\begin{bmatrix}\Delta B_{qi}\\ 0\end{bmatrix}$ and $\tilde{G}_{qi}=\begin{bmatrix}G_{qi}\\ 0\end{bmatrix}$.

Note that, $\Delta\tilde{A}_{qi}$ and $\Delta\tilde{B}_{qi}$ can be rewritten as follow:

$$\begin{cases}\Delta\tilde{A}_{qi}(t)=\tilde{H}_{aqi}F_a(t)\tilde{N}_{aqi}\\ \Delta\tilde{B}_{qi}(t)=\tilde{H}_{bqi}F_b(t)\tilde{N}_{bqi}\end{cases}$$

with $\tilde{H}_{aqi}=\begin{bmatrix}\tilde{H}_{aqi} & 0\end{bmatrix}$, $\tilde{H}_{bqi}=\begin{bmatrix}\tilde{H}_{bqi} & 0\end{bmatrix}$, $\tilde{N}_{aqi}=\begin{bmatrix}N_{aqi}\\ 0\end{bmatrix}$ and $\tilde{N}_{bqi}=\begin{bmatrix}N_{bqi}\\ 0\end{bmatrix}$.

Note that, as shown in [Ohtake et al., 2006], to lead to LMI conditions, it is convenient to choose $C=-\sim I_{n\times n}$ where $\sim$ is an arbitrary positive scalar and $I_{n\times n}\in\Re^{n\times n}$ is a unit matrix. Moreover, the trajectories of (2.2) with the initial state $x(0)=x_0$ are equal to the first n trajectories of (2.7) with the initial state $\tilde{x}(0)=\begin{bmatrix}x_0^T & 0_{n\times 1}^T\end{bmatrix}^T$.

Remark 2.1: The proposed control approach is dedicated to nonlinear systems (1) (instead of switched nonlinear ones) by rewriting them as switched TS systems (2). Other studies were focused on switched nonlinear systems regarded as sets of TS systems switching together, see e.g. [Guelton et al., 2010][Jabri et al., 2011].

2.2.2. Switched PDC controller and switched Lyapunov candidate

Now, in order to stabilize the switched T-S systems (2.7), consider the following switched Parallel Distributed Compensation (PDC) control law [Othake et al., 2006]:

$$u(t)=\sum_{q=1}^{Q}\sum_{i=1}^{r_q} v_q\left(x(t)\right)h_{qi}\left(x(t)\right)K_{qi}E_q\tilde{x}(t) \tag{2.8}$$

where $K_{qi}\in R^{m\times 2n}$ are the gain matrices and $E_q\in R^{2n\times 2n}$ are non singular matrices defined such that:

$$E_q=\begin{bmatrix} s_{1q} & 0 & \dots & 0 & 0 & 1-s_{1q} & 0 & \dots & 0 & 0 \\ 0 & s_{2q} & \dots & 0 & 0 & 0 & 1-s_{2q} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & s_{(r-1)q} & 0 & 0 & 0 & \dots & 1-s_{(r-1)q} & 0 \\ 0 & 0 & \dots & 0 & s_{rq} & 0 & 0 & \dots & 0 & 1-s_{rq} \\ 1-s_{1q} & 0 & \dots & 0 & 0 & s_{1q} & 0 & \dots & 0 & 0 \\ 0 & 1-s_{2q} & \dots & 0 & 0 & 0 & s_{2q} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1-s_{(r-1)q} & 0 & 0 & 0 & \dots & s_{(r-1)q} & 0 \\ 0 & 0 & \dots & 0 & 1-s_{rq} & 0 & 0 & \dots & 0 & s_{rq} \end{bmatrix} \tag{2.9}$$

with s_{kq}, for $k=1,...,n$, defined above in (2.5).

Substituting (2.8) into (2.7), one obtains the following closed-loop uncertain and disturbed system :

$$\begin{aligned}\dot{\tilde{x}}(t)=\sum_{q=1}^{Q}\sum_{i=1}^{r_q} v_q\left(x(t)\right)h_{qi}\left(x(t)\right)\Big(\Big(\left(\tilde{A}_{qi}+\Delta\tilde{A}_{qi}(t)\right)\\ +\left(\tilde{B}_{qi}+\Delta\tilde{B}_{qi}(t)\right)K_{qj}E_q\Big)\tilde{x}(t)+\tilde{G}_{qi}\ddot{}\,(t)\Big)\end{aligned} \tag{2.10}$$

The goal is now to design the matrices K_{qi}, for $i=1,...,r$ and $q=1,...,Q$, ensuring the stability of the closed loop system (2.10). The following results will be obtained through the use of a candidate switched Lyapunov function given by:

$$V\left(\tilde{x}(t)\right)=\begin{cases}\tilde{x}^T(t)P_1\tilde{x}(t), & \tilde{x}(t)\in R_1\left(s_{11},s_{21},\ldots,s_{n1}\right)\\ \tilde{x}^T(t)P_2\tilde{x}(t), & \tilde{x}(t)\in R_2\left(s_{12},s_{22},\ldots,s_{n2}\right)\\ \quad\vdots & \\ \tilde{x}^T(t)P_Q\tilde{x}(t), & \tilde{x}(t)\in R_Q\left(s_{1Q},s_{2Q},\ldots,s_{nQ}\right)\end{cases}\tag{2.11}$$

where $P_q\in\Re^{2n\times 2n}$, for $q=1,...,Q$, are positive definite real matrices.

Note that, to guarantee the continuity of the candidate Lyapunov function (2.11) on region boundaries [Liberzon & Morse, 2009], P_q can be rewritten such that:

$$P_q=E_q^T P E_q\tag{2.12}$$

where P is a definite positive matrix with E_q defined above.

Let us assume that for each instant t, only one region can be activated, thus (2.11) can be rewritten as:

$$V\left(\tilde{x}(t)\right)=\tilde{x}^T(t)\left(\sum_{q=1}^{Q}E_q^T P E_q\right)\tilde{x}(t)\tag{2.13}$$

Therefore, if there exist $P>0$ such that (2.13) is strictly decreasing, the closed loop system (2.10) is stable. The main result will be provided in the next section in terms of LMIs.

2.3. Notations

The following notations will be used in the sequel to clarify the mathematical expression:

$$A_{vh}=\sum_{q=1}^{Q}\sum_{i=1}^{r_q}v_q\left(x(t)\right)h_{qi}\left(x(t)\right)A_{qi}\ ,\ E_v=\sum_{q=1}^{Q}v_q\left(x(t)\right)E_q,$$

$$X_hY_h=\left(\sum_{i=1}^{r_q}h_i\left(x(t)\right)X_i\right)\left(\sum_{i=1}^{r_q}h_i\left(x(t)\right)Y_i\right).$$

Note that, assuming only one region is allowable at each instant. Thus one has:

$$X_vY_v=\left(\sum_{q=1}^{Q}v_q\left(x(t)\right)X_q\right)\left(\sum_{q=1}^{Q}v_q\left(x(t)\right)Y_q\right)=\sum_{q=1}^{Q}v_q\left(x(t)\right)X_qY_q$$

As usual, a star (*) indicates a transpose quantity in a matrix. The time t will be omitted when there is no ambiguity. I denote identity matrices with appropriate dimensions.

2.4. Robust LMI based switched controller design

In this section, one firstly proposes to study the stabilisation of the system (2.2) without considering external disturbances (i.e. ¨ $(t)=0$). Using the above defined notations, the closed loop uncertain switched T-S system is given as follows:

$$\dot{\tilde{x}}(t)=\left(\left(\tilde{A}_{vh}+\Delta\tilde{A}_{vh}\right)+\left(\tilde{B}_{vh}+\Delta\tilde{B}_{vh}\right)K_{vh}E_v\right)\tilde{x}(t) \tag{2.14}$$

Sufficient LMI conditions for the design of a switched controller (2.8) guaranteeing the stability of (2.14) are proposed in the following theorem.

Theorem 2.1 [Jabri et al., 2012]:

The uncertain switched fuzzy system (2.2) without external disturbances ($¨\left(t\right)=0$ *) is GAS (globally asymptotically stabilized) using the PDC switched fuzzy control law (2.8) if there exist the matrices* $X = X^T > 0$, M_{qi}, *the scalars* $¥_{qi} > 0$, $ª_{qi} > 0$ *satisfying the following LMIs for all* $q = 1,...,Q$, $i, j = 1,...,r_q$ *and* $i < j$ *:*

$$\Gamma_{ii} < 0 \tag{2.15}$$

$$\frac{1}{r_q - 1}\Gamma_{ii} + \frac{1}{2}\left(\Gamma_{ij} + \Gamma_{ji}\right) < 0 \tag{2.16}$$

with $\Gamma_{qij} = \begin{bmatrix} \Gamma_{qij}^{(1,1)} & (*) & (*) \\ N_{aqi}E_q^{-1}X & -¥_{qi}I & 0 \\ N_{bqj}M_{qj} & 0 & -ª_{qj}I \end{bmatrix}$ *and*

$$\Gamma_{qij}^{(1,1)} = XE_q^{-T}\tilde{A}_{qi}^T E_q^T + E_q\tilde{A}_{qi}E_q^{-1}X + M_{qi}^T\tilde{B}_{qj}^T E_q^T + E_q\tilde{B}_{qj}M_{qi} + ¥_{qi}E_qH_{qi}H_{qi}^TE_q^T + ª_{qj}E_qH_{qj}H_{qj}^TE_q^T$$

Then, the switched PDC controller gain matrices are obtained through the bijective change of variables $K_{qi} = M_{qi}X^{-1}$.

***Proof*:**

Consider the Lyapunov candidate (2.13), the closed loop system is asymptotically stable if:

$$\dot{\tilde{x}}^T\left(t\right)E_v^TPE_v\tilde{x}\left(t\right) + \tilde{x}^T\left(t\right)E_v^TPE_v\dot{\tilde{x}}\left(t\right) < 0 \tag{2.17}$$

Considering (2.14), inequality (2.17) is verified for all $\tilde{x}(t)$ if:

$$\begin{aligned}&\left(\left(\tilde{A}_{vh}+\Delta\tilde{A}_{vh}\right)+\left(\tilde{B}_{vh}+\Delta\tilde{B}_{vh}\right)K_{vh}E_v\right)^T E_v^T P E_v\\&+E_v^T P E_v\left(\left(\tilde{A}_{vh}+\Delta\tilde{A}_{vh}\right)+\left(\tilde{B}_{vh}+\Delta\tilde{B}_{vh}\right)K_{vh}E_v\right)<0\end{aligned} \tag{2.18}$$

Multiplying left by E_v^{-T} and right by E_v^{-1}, (2.18) becomes:

$$\begin{aligned}&E_v^{-T}\left(\tilde{A}_{vh}+\Delta\tilde{A}_{vh}\right)^T E_v^T P+K_{vh}^T\left(\tilde{B}_{vh}+\Delta\tilde{B}_{vh}\right)^T E_v^T P\\&+PE_v\left(\tilde{A}_{vh}+\Delta\tilde{A}_{vh}\right)E_v^{-1}+PE_v\left(\tilde{B}_{vh}+\Delta\tilde{B}_{vh}\right)K_{vh}<0\end{aligned} \tag{2.19}$$

which can be rewritten in its extended form considering (2.3) as:

$$\begin{aligned}&E_v^{-T}\tilde{A}_{vh}^T E_v^T P+PE_v\tilde{A}_{vh}E_v^{-1}+K_{vh}^T\tilde{B}_{vh}^T E_v^T P+PE_v\tilde{B}_{vh}K_{vh}\\&+E_v^{-T}\tilde{N}_{avh}^T F_a^T(t)\tilde{H}_{avh}^T E_v^T P+PE_v\tilde{H}_{avh}F_a(t)\tilde{N}_{avh}E_v^{-1}\\&+K_{vh}^T\tilde{N}_{bvh}^T F_b^T(t)\tilde{H}_{bvh}^T E_v^T P+PE_v\tilde{H}_{bvh}F_b(t)\tilde{N}_{bvh}K_{vh}<0\end{aligned} \tag{2.20}$$

Let us consider $X=P^{-1}$, left and right multiplying (2.20) by X, one obtains :

$$\begin{aligned}&XE_v^{-T}\tilde{A}_{vh}^T E_v^T+E_v\tilde{A}_{vh}E_v^{-1}X+M_{vh}^T\tilde{B}_{vh}^T E_v^T+E_v\tilde{B}_{vh}M_{vh}\\&+XE_v^{-T}\tilde{N}_{avh}^T F_a^T(t)\tilde{H}_{bvh}^T E_v^T+E_v\tilde{H}_{avh}F_b(t)\tilde{N}_{avh}E_v^{-1}X\\&+M_{vh}^T\tilde{N}_{bvh}^T F_b^T(t)\tilde{H}_{bvh}^T E_v^T+E_v\tilde{H}_{bvh}F_b(t)\tilde{N}_{bvh}M_{vh}<0\end{aligned} \tag{2.21}$$

Recall that $F_a^T(t)F_a(t)\le I$ and $F_b^T(t)F_b(t)\le I$, applying lemma 1.2 and lemma 1.3, (2.21) is verified if:

$$\Gamma_{vhh} = \begin{bmatrix} \Gamma_{vhh}^{(1,1)} & (*) & (*) \\ \tilde{N}_{avh} E_q^{-1} X & -\text{¥}_{vh} I & 0 \\ \tilde{N}_{bvh} M_{vh} & 0 & -\text{ª}_{vh} I \end{bmatrix} < 0 \tag{2.22}$$

with

$$\Gamma_{vhh}^{(1,1)} = XE_v^{-T} \tilde{A}_{vh}^T E_v^T + E_v \tilde{A}_{vh} E_v^{-1} X + \tilde{M}_{vh}^T B_{vh}^T E_v^T + E_v B_{vh} \tilde{M}_{vh} + \text{¥}_{vh} E_v \tilde{H}_{vh} \tilde{H}_{vh}^T E_v^T + \text{ª}_{vh} E_v \tilde{H}_{vh} \tilde{H}_{vh}^T E_v^T$$

Finally, applying Lemma 1.5 on (2.22), the proof is completed. ■

Now, the purpose is to extend theorem 2.1 to the design of robust controllers (2.8) stabilizing uncertain switched T-S systems (2.2) subject to external disturbances ($\text{¨}(t) \neq 0$). To do so, the following H_∞ criterion is employed to minimize the effect of the external disturbances on the state dynamics:

$$\int_0^{tf} \tilde{x}(t)^T W \tilde{x}(t)\, dt \le \text{¦}^2 \int_0^{tf} \text{¨}(t)^T \text{¨}(t)\, dt \tag{2.23}$$

where $W \in \Re^{2n \times 2n}$ is a weighting positive definite real matrix and ¦ is the disturbances attenuation level and t_f the final time simulation.

The result is summarized in the following theorem.

Theorem 2.2:

The uncertain switched fuzzy system (2.2) subject to external disturbances is GAS using the PDC switched fuzzy control law (2.8) if there exist the matrices $X = X^T > 0$, M_{qi}, the scalars $\text{¥}_{qi} > 0$, $\text{ª}_{qi} > 0$ and satisfying the following LMIs for all $q = 1,...,Q$, $i, j = 1,...,r_q$ and $i < j$:

Minimize ° such that :

$$\Psi_{ii} < 0 \tag{2.24}$$

$$\frac{1}{r_q - 1}\Psi_{ii} + \frac{1}{2}\left(\Psi_{ij} + \Psi_{ji}\right) < 0 \tag{2.25}$$

with $\Psi_{qij} = \left[\begin{array}{ccc|cc} \Gamma_{qij} & & & (*) & (*) \\ \hline E_q^{-1}X & 0 & 0 & -W^{-1} & 0 \\ \tilde{G}_{qi}^{\ T} E_q^T P E_q & 0 & 0 & 0 & -¦^2 I \end{array}\right] < 0$ *and* Γ_{qij} *defined in theorem 2.1.*

Therefore, the robust switched PDC controller gain matrices are obtained using the bijective change of variables $K_{qi} = M_{qi}X^{-1}$ *and the designed control law ensures a* H_∞ *performance* $¦ = \sqrt{°}$.

***Proof*:**

Consider the Lyapunov candidate function (2.13) and the H_∞ criterion given in (2.23). The close loop system (2.10) is stable and the H_∞ performance ¦ is guaranteed if:

$$\dot{\tilde{x}}^T(t) E_v^T P E_v \tilde{x}(t) + \tilde{x}^T(t) E_v^T P E_v \dot{\tilde{x}}(t) + \tilde{x}^T(t) R \tilde{x}(t) - ¦\,¨^T(t)\,¨(t) < 0 \tag{2.26}$$

Considering the same steps as for the proof of theorem 2.1, (2.26) is verified for all $x(t)$ and $¨(t)$ if:

$$\left[\begin{array}{ccc|c} \Gamma_{vhh} + \overline{W} & & & (*) \\ \hline \tilde{G}_{vh}^{\ T} E_v^T P E_v & 0 & 0 & -¦^2 I \end{array}\right] < 0 \tag{2.27}$$

with Γ_{vhh} defined in (2.22) (proof of theorem 2.1) and $\overline{W} = \begin{bmatrix} X E_v^{-T} R E_v^{-1} X & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Then, applying Lemma 1.2, (2.27) is verified if:

$$\Psi_{vhh} = \left[\begin{array}{ccc:cc} \Gamma_{vhh} & & & (*) & (*) \\ \hdashline E_v^{-1} X & 0 & 0 & -W^{-1} & 0 \\ \tilde{G}_{vh}{}^{T} E_v^{T} P E_v & 0 & 0 & 0 & -¦^{2} I \end{array}\right] < 0 \qquad (2.28)$$

Finally, applying lemma 1.3, one obtains the LMI stability conditions proposed in theorem 2.2. This ends the proof. ■

2.5. Numerical example

Let us consider the following uncertain nonlinear system inspired from the nominal one given in [Othake et al., 2006]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) + 0.05 F_a(t) x_1(t) \\ \qquad +0.02 F_a(t) x_2(t) + 0.05 F_b(t) u(t) + ¨\,(t) \\ \dot{x}_2(t) = f(x(t)) + g(x(t)) u(t) + 0.0375 F_a(t) x_1(t) \\ \qquad +0.015 F_a(t) x_2(t) + 0.0375 F_b(t) u(t) + ¨\,(t) \end{cases} \qquad (2.29)$$

with $x_1 \in [-d, d]$ and $x_2 \in [-d, d]$, $d = 0.5$, $f(x) = -x_1^3 - x_2^3 + 5x_1^2 x_2 + 5x_1 x_2^2 - 3x_1 x_2 - x_1 - x_2$ and $g(x) = -0.7 + x_1(t) x_2(t)$.

The state space partition is chosen as in Fig.1 with four regions $R_1(1,1)$, $R_2(0,1)$, $R_3(1,0)$ and $R_4(0,0)$. Thus, the considered switched law is defined by:

$$v_1(x(t)) = \begin{cases} 1 & \text{if } 0 \le x_1 \le \mathrm{d},\ 0 \le x_2 \le \mathrm{d} \\ 0 & \text{otherwise} \end{cases}$$

$$v_2(x(t)) = \begin{cases} 1 & \text{if } -\mathrm{d} \le x_1 < 0,\ 0 \le x_2 \le \mathrm{d} \\ 0 & \text{otherwise} \end{cases}$$

$$v_3\left(x(t)\right)=\begin{cases}1 & \text{if} \quad 0\le x_1\le \mathrm{d},\ -\mathrm{d}\le x_2<0\\ 0 & \text{otherwise}\end{cases}$$

$$v_4\left(x(t)\right)=\begin{cases}1 & \text{if} \quad -\mathrm{d}\le x_1<0,\ -\mathrm{d}\le x_2<0\\ 0 & \text{otherwise}\end{cases}.$$

Using the sector nonlinearity approach [Tanaka & Wang, 2001] and the previous state partition, the nonlinear system (2.29) can be constructed as an uncertain switched TS disturbed system given by:

$$\begin{aligned}\dot{x}(t)=\sum_{q=1}^{Q}\sum_{i=1}^{r}v_q\left(x(t)\right)h_{qi}\left(x(t)\right)\Big(\left(A_{qi}+\Delta A_{qi}(t)\right)x(t)\\ +\left(B_{qi}+\Delta B_{qi}(t)\right)u(t)+G_{qi}\ddot{}\ (t)\Big)\end{aligned} \tag{2.30}$$

with $A_{11}=A_{13}=\begin{bmatrix}0 & 1\\ \text{-0.246} & \text{-0.246}\end{bmatrix}$, $A_{12}=A_{14}=\begin{bmatrix}0 & 1\\ \text{-1.25} & \text{-1.25}\end{bmatrix}$, $A_{21}=A_{23}=\begin{bmatrix}0 & 1\\ \text{-1.952} & \text{-0.246}\end{bmatrix}$,

$A_{22}=A_{24}=\begin{bmatrix}0 & 1\\ 0.75 & \text{-1.25}\end{bmatrix}$, $A_{31}=A_{33}=\begin{bmatrix}0 & 1\\ \text{-0.246} & \text{-1.952}\end{bmatrix}$, $A_{32}=A_{34}=\begin{bmatrix}0 & 1\\ \text{-1.25} & 0.75\end{bmatrix}$

$A_{41}=A_{43}=\begin{bmatrix}0 & 1\\ \text{-1.952} & \text{-1.952}\end{bmatrix}$, $A_{42}=A_{44}=\begin{bmatrix}0 & 1\\ 0.75 & 0.75\end{bmatrix}$, $B_{11}=B_{12}=B_{41}=B_{42}=\begin{bmatrix}0 & \text{-0.45}\end{bmatrix}$,

$B_{13}=B_{14}=B_{43}=B_{44}=\begin{bmatrix}0 & \text{-0.7}\end{bmatrix}$, $B_{21}=B_{23}=B_{31}=B_{32}=\begin{bmatrix}0 & \text{-0.7}\end{bmatrix}$,

$B_{22}=B_{24}=B_{33}=B_{34}=\begin{bmatrix}0 & \text{-0.95}\end{bmatrix}$, $G_{11}=G_{12}=G_{21}=G_{22}=G_{31}=G_{32}=G_{41}=G_{42}=\begin{bmatrix}1\\ 1\end{bmatrix}$ and the external disturbances defined by $\Delta A_{qi}(t)=H_{aqi}F(t)N_{aqi}$ and $\Delta B_{qi}(t)=H_{bqi}F(t)N_{bqi}$ with, for $q=1,\dots,4$ and $i=1,..,4$, $H_{aqi}=H_{bqi}=\begin{bmatrix}0.2 & 0.15\end{bmatrix}^T$ and $N_{aqi}=N_{bqi}=\begin{bmatrix}0.25 & 0.1\end{bmatrix}$.

Recall that E_q, $q=1,\dots,4$, are defined in (2.9) and lead, for the considered partition, to:

$$E_1=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\ E_2=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\ E_3=\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix},\ E_4=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}.$$

The membership functions are given by $h_{qi}\left(x(t)\right)=\Gamma_{qi}w_{qi}$ with:

$$\Gamma_{11}=\Gamma_{13}=\begin{cases}\dfrac{f(x)+1.25x_1+1.25x_2}{1.004x_1+1.004x_2}, & (x_1,x_2)\neq(0,0)\\ 1 & \text{otherwise}\end{cases}$$

$$\Gamma_{12}=\Gamma_{14}=\begin{cases}\dfrac{-0.246x_1-0.246x_2-f(x)}{1.004x_1+1.004x_2}, & (x_1,x_2)\neq(0,0)\\ 0 & \text{otherwise}\end{cases}$$

$$\Gamma_{21}=\Gamma_{23}=\begin{cases}\dfrac{f(x)-0.75x_1+1.25x_2}{-2.702x_1+1.004x_2}, & (x_1,x_2)\neq(0,0)\\ 1 & \text{otherwise}\end{cases}$$

$$\Gamma_{22}=\Gamma_{24}=\begin{cases}\dfrac{-1.952x_1-0.246x_2-f(x)}{-2.702x_1+1.004x_2}, & (x_1,x_2)\neq(0,0)\\ 0 & \text{otherwise}\end{cases}$$

$$\Gamma_{31}=\Gamma_{33}=\begin{cases}\dfrac{f(x)+1.25x_1-0.75x_2}{1.004x_1-2.702x_2}, & (x_1,x_2)\neq(0,0)\\ 1 & \text{otherwise}\end{cases}$$

$$\Gamma_{32}=\Gamma_{34}=\begin{cases}\dfrac{-0.246x_1-1.952x_2-f(x)}{1.004x_1-2.702x_2}, & (x_1,x_2)\neq(0,0)\\ 0 & \text{otherwise}\end{cases}$$

$$\Gamma_{41}=\Gamma_{43}=\begin{cases}\dfrac{f(x)-0.75x_1-0.75x_2}{-2.702x_1-2.702x_2}, & (x_1,x_2)\neq(0,0)\\ 1 & \text{otherwise}\end{cases}$$

$$\Gamma_{42}=\Gamma_{44}=\begin{cases}\dfrac{-1.952x_1-1.952x_2-f(x)}{-2.702x_1-2.702x_2}, & (x_1,x_2)\neq(0,0)\\ 0 & \text{otherwise}\end{cases}$$

and $w_{13} = w_{14} = w_{43} = w_{44} = 1 - 4x_1x_2$, $w_{11} = w_{12} = w_{41} = w_{42} = 4x_1x_2$,

$w_{21} = w_{22} = w_{31} = w_{32} = 4x_1x_2 + 1$, $w_{23} = w_{24} = w_{33} = w_{34} = \dfrac{-1.4 - x_1x_2}{0.25}$.

In order to be able to apply the above proposed theorems, the extended form of (2.30) is built by adding a stable linear system (2.6) with $C = -I$. Hence, the augmented switched fuzzy system yields:

$$\begin{aligned}\dot{\tilde{x}}(t) = \sum_{q=1}^{Q}\sum_{i=1}^{r} v_q\left(x(t)\right)h_{qi}\left(x(t)\right)\Big(\left(\tilde{A}_{qi} + \Delta\tilde{A}_{qi}(t)\right)\tilde{x}(t) \\ + \left(\tilde{B}_{qi} + \Delta\tilde{B}_{qi}(t)\right)u(t) + \tilde{G}_q^{\cdot\cdot}(t)\Big)\end{aligned} \tag{2.31}$$

with $\tilde{A}_{11} = \tilde{A}_{13} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.246 & -0.246 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\tilde{A}_{12} = \tilde{A}_{14} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.25 & -1.25 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$,

$\tilde{A}_{21} = \tilde{A}_{23} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.952 & -0.246 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\tilde{A}_{22} = \tilde{A}_{24} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & -1.25 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$,

$\tilde{A}_{31} = \tilde{A}_{33} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -0.246 & -1.952 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\tilde{A}_{32} = \tilde{A}_{34} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.25 & 0.75 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$,

$\tilde{A}_{41} = \tilde{A}_{43} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.952 & -1.952 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$, $\tilde{A}_{42} = \tilde{A}_{44} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & 0.75 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$,

$$\tilde{B}_{11} = \tilde{B}_{12} = \tilde{B}_{41} = \tilde{B}_{42} = \begin{bmatrix} 0 \\ -0.45 \\ 0 \\ 0 \end{bmatrix}, \ \tilde{B}_{13} = \tilde{B}_{14} = \tilde{B}_{43} = \tilde{B}_{44} = \begin{bmatrix} 0 \\ -0.7 \\ 0 \\ 0 \end{bmatrix},$$

$\tilde{B}_{21} = \tilde{B}_{23} = \tilde{B}_{31} = \tilde{B}_{32} = \begin{bmatrix} 0 \\ -0.7 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{B}_{22} = \tilde{B}_{24} = \tilde{B}_{33} = \tilde{B}_{34} = \begin{bmatrix} 0 \\ -0.95 \\ 0 \\ 0 \end{bmatrix}$ and for $q = 1, 2$ and

$i = 1,..,4$, $\tilde{G}_{iq} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\tilde{H}_{aqi} = \tilde{H}_{bqi} = \begin{bmatrix} 0.2 \\ 0.15 \\ 0 \\ 0 \end{bmatrix}$ and $\tilde{N}_{aqi} = \tilde{N}_{bqi} = \begin{bmatrix} 0.25 & 0.1 & 0 & 0 \end{bmatrix}$.

In order to show the efficiency of theorem 2.1, let us consider the system (2.2) without external disturbances (i.e. ¨ $(t) = 0$). The Matlab LMI toolbox is used to solve the LMI conditions leading to the synthesis of a switched fuzzy controller given by:

$$\text{P} = \begin{bmatrix} 5.358 & 3.198 & 3.268 & 3.360 \\ 3.198 & 5.308 & 3.340 & 3.149 \\ 3.268 & 3.340 & 5.899 & 3.704 \\ 3.360 & 3.149 & 3.704 & 5.884 \end{bmatrix}, \ \tilde{K}_{11} = \begin{bmatrix} 6.334 & 6.204 & 6.206 & 5.992 \end{bmatrix},$$

$\tilde{K}_{12} = \begin{bmatrix} 4.514 & 4.360 & 5.721 & 5.524 \end{bmatrix}$, $\tilde{K}_{13} = \begin{bmatrix} 4.432 & 4.532 & 4.167 & 4.016 \end{bmatrix}$,

$\tilde{K}_{14} = \begin{bmatrix} 2.748 & 2.759 & 3.714 & 3.581 \end{bmatrix}$, $\tilde{K}_{21} = \begin{bmatrix} 4.274 & 4.521 & 3.115 & 4.389 \end{bmatrix}$,

$\tilde{K}_{22} = \begin{bmatrix} 4.008 & 3.152 & 5.692 & 4.116 \end{bmatrix}$, $\tilde{K}_{23} = \begin{bmatrix} 3.225 & 3.653 & 2.026 & 3.301 \end{bmatrix}$,

$\tilde{K}_{24} = \begin{bmatrix} 2.904 & 2.196 & 4.492 & 2.981 \end{bmatrix}$, $\tilde{K}_{31} = \begin{bmatrix} 4.356 & 3.905 & 4.427 & 2.845 \end{bmatrix}$,

$\tilde{K}_{32} = \begin{bmatrix} 3.358 & 3.959 & 4.491 & 5.888 \end{bmatrix}$, $\tilde{K}_{33} = \begin{bmatrix} 3.276 & 2.806 & 3.181 & 1.698 \end{bmatrix}$,

$\tilde{K}_{34}=\begin{bmatrix}2.479 & 3.034 & 3.454 & 4.964\end{bmatrix}$, $\tilde{K}_{41}=\begin{bmatrix}5.802 & 5.563 & 4.421 & 4.328\end{bmatrix}$,

$\tilde{K}_{42}=\begin{bmatrix}5.707 & 5.467 & 8.029 & 8.100\end{bmatrix}$, $\tilde{K}_{43}=\begin{bmatrix}3.688 & 3.533 & 2.270 & 2.281\end{bmatrix}$

and $\tilde{K}_{44}=\begin{bmatrix}3.924 & 3.750 & 5.941 & 6.216\end{bmatrix}$.

A simulation has been performed with the initial states $x(0)=\begin{bmatrix}-0.3 & -0.3\end{bmatrix}^T$. The close-loop dynamics, the control signal and the switched Lyapunov function evolutions are depicted respectively in Figure 2.2, Figure 2.3 and Figure 2.4. These show a stable behaviour.

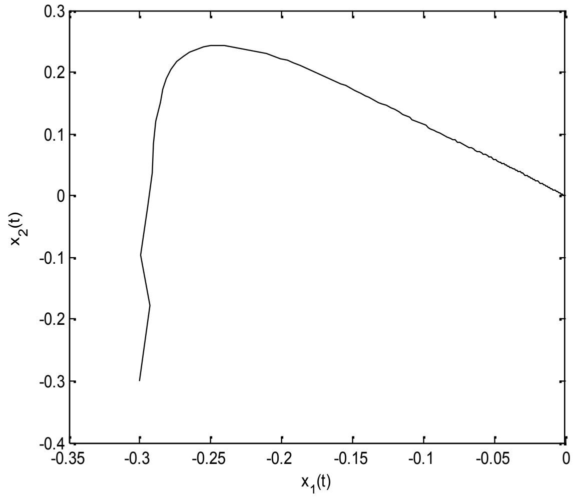


Figure 2.2 Phase plan of the closed-loop uncertain switched fuzzy system.

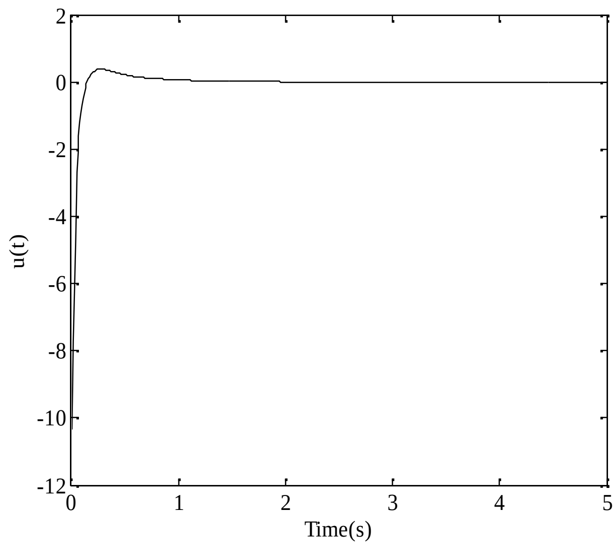


Figure 2.3. Control law evolution of the closed-loop uncertain switched fuzzy system.

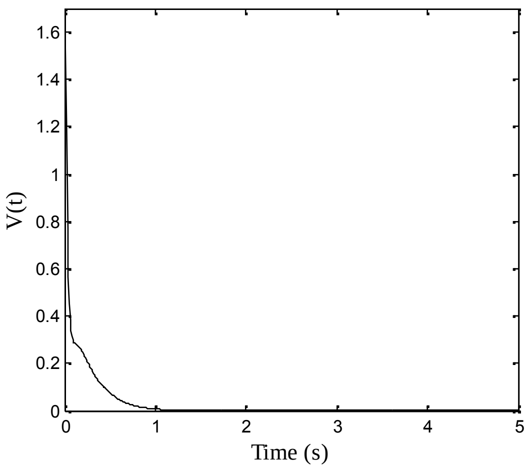


Figure 2.4. Lyapunov function evolution of the closed-loop uncertain switched fuzzy system.

Let us now consider (2.29) subject to external disturbances $\ddot{\ }(t)$. Then, using the Matlab LMI toolbox to solve the LMI conditions proposed in the theorem 2.2, one obtains:

$$P = \begin{bmatrix} 9.071 & 6.780 & 6.996 & 7.123 \\ 6.780 & 8.947 & 6.943 & 7.032 \\ 6.996 & 6.943 & 9.560 & 7.419 \\ 7.123 & 7.032 & 7.419 & 9.844 \end{bmatrix} > 0,\ K_{11} = \begin{bmatrix} 10.924 & 10.707 & 10.990 & 11.147 \end{bmatrix},$$

$K_{12} = [9.749 \quad 9.531 \quad 10.990 \quad 11.147]$, $K_{13} = [7.022 \quad 7.048 \quad 6.692 \quad 6.787]$,

$K_{14} = [5.601 \quad 5.628 \quad 6.692 \quad 6.787]$, $K_{21} = [10.916 \quad 10.890 \quad 9.570 \quad 11.394]$,

$K_{22} = [10.916 \quad 9.715 \quad 12.732 \quad 11.394]$, $K_{23} = [6.639 \quad 7.152 \quad 4.872 \quad 6.927]$,

$K_{24} = [6.639 \quad 7.152 \quad 4.872 \quad 6.927]$, $K_{31} = [10.916 \quad 10.641 \quad 11.163 \quad 9.522]$,

$K_{32} = [9.741 \quad 10.641 \quad 11.163 \quad 12.685]$, $K_{33} = [7.041 \quad 6.501 \quad 6.823 \quad 5.045]$,

$K_{34} = [5.620 \quad 6.501 \quad 6.823 \quad 8.868]$, $K_{41} = [10.674 \quad 10.552 \quad 9.326 \quad 9.474]$,

$K_{42} = [10.674 \quad 10.552 \quad 12.487 \quad 12.637]$, $K_{43} = [6.526 \quad 6.451 \quad 4.766 \quad 5.018]$,

$K_{44} = [6.526 \quad 6.451 \quad 8.589 \quad 8.841]$ and with the disturbance attenuation level $\gamma = 1.932$.

A simulation has been performed with the initial states $x(0) = [-0.4 \ -0.4]^T$ and a disturbance $\omega(t)$ chosen as an uncorrelated Gaussien white noise with a variance equal to 0.1. The close-loop dynamics, the switched law and the control signal are depicted respectively in Figure.2.6, Figure.2.7 and Figure.2.8. One more time, they show, as expected, a stable behaviour.

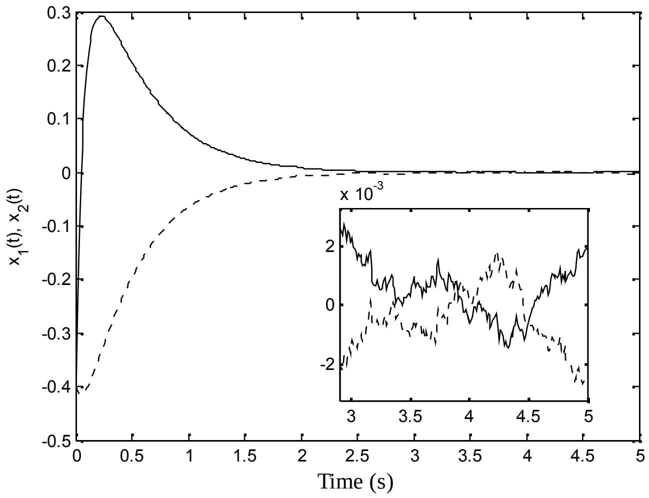


Figure 2.5. State's dynamics of the closed-loop uncertain and disturbed system,(x_1) dotted line, (x_2) solid line.

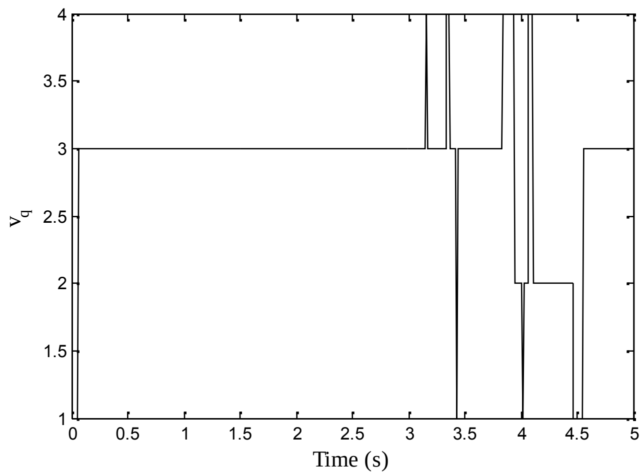


Figure 2. 6. Switched law evolution of the closed-loop uncertain switched fuzzy disturbed system.

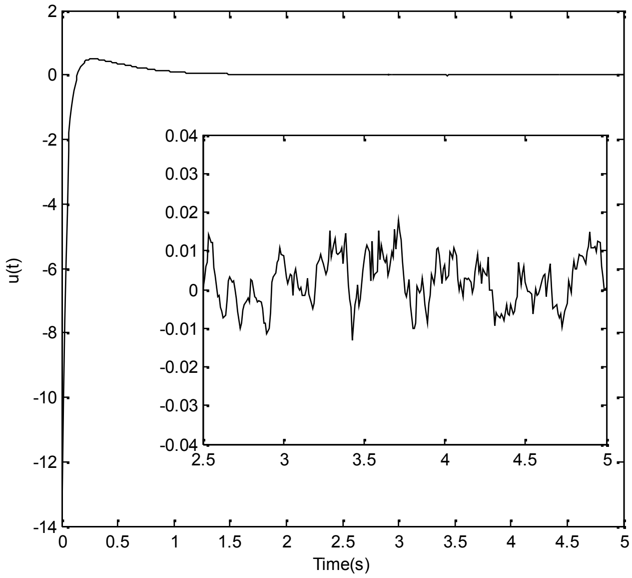


Figure 2.7. Control law evolution of the closed-loop uncertain switched fuzzy disturbed system.

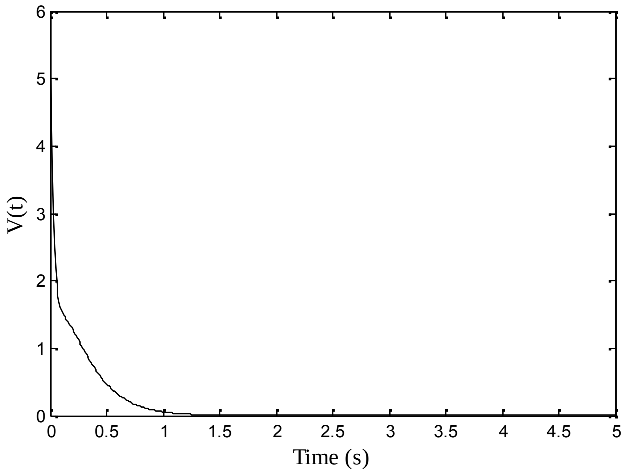


Figure 2.8. Lyapunov function evolution of the closed-loop uncertain switched fuzzy disturbed system.

2.6. Conclusion

In this chapter, the stabilization of a class of nonlinear systems represented by uncertain and disturbed switched Takagi-Sugeno fuzzy systems has been studied. The interest of this approach is to benefit from the well-known information of T-S systems with association to the characteristic of the switched systems. Moreover, to cope with uncertainties and external disturbances, LMI conditions for robust switched fuzzy PDC controller design have been obtained based on switched quadratic Lyapunov function and a H_∞ criterion. Finally, a numerical example has been provided to illustrate the efficiency of the proposed switched fuzzy PDC controller design methodology.

Chapter 3

Sum-Of-Squares based stability of nonlinear systems

Résumé du chapitre 3 en Français:

Analyse de la stabilité des modèles T-S polynomiaux par l'approche «Sum-Of-Squares»

Dans ce chapitre, une nouvelle approche pour l'analyse de la stabilité des systèmes non linéaires décrits par des modèles Takagi-Sugeno polynomiaux est explorée. Ces derniers présentent l'avantage d'inclure les modèles T-S standards en autorisant les matrices des modèles de conclusion des règles floues à maintenir une part de l'information du système non linéaire considéré sous forme polynomiale. Des premières conditions de stabilité pour ce type de systèmes ont été proposées sur la base d'une fonction de Lyapunov polynomiale commune [Tanaka et al., 2009]. Tirant parti de ces travaux, et en s'inspirant des approches non quadratiques proposées pour les systèmes T-S standards, des conditions de moindre conservatisme sont proposée ici, au travers d'une fonction candidate de Lyapunov multiple sous forme polynomiale. Ces conditions sont écrites sous formes d'un problème d'optimisation convexe de « sommes de carrés polynomiaux » (Sum-Of-Squares). Deux exemples sont proposés pour illustrer l'approche proposée.

3.1. Introduction

Since their introduction in 1985 [Tanaka & Sugeno, 1985], Takagi-Sugeno (T-S) models have been the subject of numerous works regarding to their stability analysis and stabilization. T-S models have shown their significance for analyzing nonlinear systems. Indeed, using the well-known sector nonlinearity approach [Tanaka & Wang, 2001], a T-S model remains as quasi-LPV (Linear Parameters Varying) one, since it is able to match exactly a smooth nonlinear system on a compact set of its state space. Using common Quadratic Lyapunov Functions (QLF), Linear Matrix Inequality (LMI) based stability/stabilization criterion have been derived in various cases, see e.g. [Tanaka & Wang, 2001][Sala et al., 2005] and references therein. Although QLF approach is still mainly employed, it is well known that it suffers from conservatism. Indeed, it requires checking the existence of some common decision matrices, which have to be solution of a set of LMI constraints. Thus, many works have been done to relax LMI based stability conditions. For instance, some relaxation schemes have been proposed by decoupling the interconnection structure (sums relaxations) [Tuan et al., 2001][Sala & Arino, 2009] introducing slack decision variables [Liu & Zhang, 2003], recasting T-S models to take advantage of descriptor properties (redundancy) [Tanaka & al, 2007][Guelton et al., 2009][Bouarar et al., 2010] considering switched fuzzy approaches [Ohtake et al., 2006][Jabri et al., 2012][Chen et al., 2012] or, with more adequacy to the T-S model to be analyzed, by employing other piecewise quadratic [Johansson et al., 1999][Xie et al., 1999] or non-quadratic [Blanco et al., 2001] [Tanaka et al, 2003][Guerra & Vermeiren, 2004][Feng, 2006] Lyapunov functions candidates (respectively denoted as PWLF and NQLF); see [Sala, 2009] for a detailed review of sources of conservatism.

This work is concerned with relaxation in the sense of the choice of the most convenient Lyapunov function candidate. Regarding to LMI based PWLF approaches; some interesting results have been provided in terms of conservatism [Johansson et al., 1999][Xie et al., 1999], especially when the considered T-S subsystems are not simultaneously activated all together. However, T-S models obtained from the sector nonlinearity approach lack this property. Complementary to the latter works and with more adequacies to the structure of T-S models, many research efforts have been recently done in the non-quadratic framework. Indeed, several works employing NQLF candidates have been proposed [Blanco et al., 2001] [Tanaka et al, 2003][Guerra & Vermeiren, 2004] see also [Sala, 2009] for a survey on these techniques. These ones are convenient with T-S models since NQLF shares the same activation functions (AFs). However, some drawbacks appear in the continuous time case since the time derivatives of AFs occurs when applying the direct Lyapunov method to obtain LMI based global stability conditions. Therefore, the most commonly used technique is to bound AFs' time derivatives. However, these bound are still difficult to estimate a priori, in practice, before solving LMI conditions. To overcome this problem and to cope with the non-quadratic framework, an alternative way has been considered in [Rhee & Won, 2006] [Guelton et al., 2010][Mozelli et al., 2009a]. The main idea was to employ a Line-Integral

Lyapunov Function (LILF). However, it requires some constraining path-independency conditions which necessitate a particular structure of the Lyapunov decision variables (off-diagonal common matrices) whose may be understood as a source of conservatism. Recently, an interesting step has been addressed with LMI based NQLF stability conditions to avoid any knowledge on AFs' dynamics [Bernal & Guerra, 2010][Guerra et al., 2012]. Nevertheless, these approaches have been obtained by reducing the global stability analysis to a local one. Moreover, in these studies, the obtained local stability conditions remain complex and so make challenging their implementation.

Nowadays, despite the success and popularity of LMI based stability conditions for T-S models, the drawbacks described above are understood as a limit of these approaches, especially in the non-quadratic framework. Therefore, a challenging change of perspective has to be considered and this chapter tends to add a step in this way. Very recently, an alternative to LMIs appears by considering polynomial approaches. Indeed, it has been shown that some convex optimization problems, such like LMI problems, can be recasted as more general Sum Of Square (SOS) decomposition problems. Therefore, with the first SOS based stability conditions, T-S model analysis has just passed an important milestone [Tanaka et al., 2003] [Sala & Arino, 2009][Tanaka et al., 2009][Narimani & Lam, 2010][Guelton et al., 2011] [Bernal et al., 2011]. In this chapter, the goal is to extend the stability conditions for polynomial T-S systems proposed in [Tanaka et al., 2009] by the use of a Multiple Polynomial Lyapunov Function (MPLF) candidate [Guelton et al., 2011][Bernal et al., 2011]. Indeed, the Polynomial Lyapunov Function (PLF) proposed in [Tanaka et al., 2009] don't take into account AFs knowledge and, similarly to the QLF approach, it requires to find a common PLF to a set of SOS conditions. Thus, similarly to the extension of QLF by NQLF, the proposed MPLF shares the same AFs 'structure as the T-S model to be analyzed. Moreover, the proposed MPLF approach extends our preliminary result [Guelton et al., 2011] as well as includes the PLF [Tanaka et al., 2009] and NQLF [Mozelli et al, 2009b] approaches as special cases. Furthermore, the MPLF based result [Bernal et al., 2011] is mainly focused on Taylor-series approach and local stability analysis leading to complex formulation. It would be highlighted that the stability conditions proposed in the sequel do not require any knowledge on AFs' dynamics and straightforwardly guarantee the global stability, when a solution to the SOS optimization problem holds.

The rest of the chapter is organized as follows: first, the statement of the problem is made and the proposed MPLF is presented. Then, the main result is given through a theorem, which allows the MPLF based stability analysis of polynomial T-S systems. Finally, two numerical examples are provided. The first one is a special case studied in [Tanaka et al., 2009] and dedicated to show the efficiency of the proposed approach in terms of conservatism regarding to some previous LMI and SOS based approaches. The second one considers the MPLF based stability analysis of a polynomial T-S system.

3.2. Problem statement

Consider the following nonlinear system:

$$\dot{x}(t) = f\left(x(t)\right) \tag{3.1}$$

where f is a smooth nonlinear function and $x^T(t) = \left[x_1(t) \quad \cdots \quad x_n(t)\right] \in \Re^n$ is the state vector.

Using the well-known sector nonlinearity approach [Tanaka & Wang, 2001] or a Taylor series approach [Guelton et al, 2010], it has been shown that (3.1) can be rewritten (globally or semi-globally) as a polynomial T-S system such that:

$$\dot{x} = \sum_{i=1}^{r} h_i\left(z(t)\right) A_i\left(x(t)\right) \hat{x}\left(x(t)\right) \tag{3.2}$$

where $z(t) = \left[z_1(t) \quad \cdots \quad z_p(t)\right] \in \Re^p$ is the vector of activation variables, $A_i\left(x(t)\right)$ are polynomial matrices in $x(t)$, $\hat{x}^T\left(x(t)\right) = \left[\hat{x}_1\left(x(t)\right) \quad \cdots \quad \hat{x}_N\left(x(t)\right)\right] \in \Re^N$ is a vector of monomials in $x(t)$ and, for $i = 1,...,r$, $h_i\left(z(t)\right)$ are positive AFs holding the convex sum property $\sum_{i=1}^{r} h_i\left(z(t)\right) = 1$.

In [Tanaka et al., 2009], the stability of T-S systems described by (3.2) has been investigated through the use of a common PLF given by:

$$V\left(x(t)\right)=\hat{x}^T\left(x(t)\right)P\left(x(t)\right)\hat{x}\left(x(t)\right) \tag{3.3}$$

This approach remains conservative since it requires checking the existence of a common polynomial Lyapunov matrix $P\left(x(t)\right)\in\Re^{N\times N}$ regarding to the fuzzy interconnection structure of (3.2). Therefore, in order to provide less conservative stability criterion for the class of systems depicted by (3.2), one considers the following MPLF candidate, which shares the same structure as the T-S model to be analyzed [Guelton et al., 2011][Guerra et al, 2012]:

$$v\left(x(t)\right)=\hat{x}^T\left(x(t)\right)\sum_{i=1}^{r}h_i\left(z(t)\right)P_i\left(x(t)\right)\hat{x}\left(x(t)\right) \tag{3.4}$$

where $P_i\left(x(t)\right)\in\Re^{N\times N}$ are polynomial matrices in $x(t)$.

The aim is now to propose new stability conditions for the class of polynomial T-S systems (3.2) based on the MPLF candidate (3.4).

3.3. SOS stability conditions

For more clarity of further mathematical proofs, before presenting the main results, the following useful preliminaries are presented.

3.3.1. Preliminaries

Notations **:**

When there is no ambiguity, the time t will be omitted as entry for simplifying mathematical expressions and I will denote an identity matrix of appropriate dimension. Moreover, let $A_i^k(x)$ be the k^{th} row of $A_i(x)$, from (3.2) one can write, for $k=1,\ldots,n$:

$$\dot{x}_k = \sum_{i=1}^{r} h_i(z) A_i^k(x) \hat{x}(x) \tag{3.5}$$

Assumption 3.1:

For $i = 1,...,r$, each $h_i(z)$ is assumed continuously derivable within each states variables x_k, for $k = 1,...,n$. Therefore, $g_i^k(z) = \dfrac{\partial h_i(z)}{\partial x_k}$ are bounded scalar functions along a compact set Ω.

Proposition 3.1:

Let $g_i^k(z) = \dfrac{\partial h_i(z)}{\partial x_k}$, for $i = 1,...,r$ and $k = 1,...,n$. Under assumption 3.1, $\forall t, g_i^k(z) \in \left[\sim_{i1}^{k}, \sim_{i2}^{k}\right]$. Therefore, using the sector nonlinearity approach, one can write $g_i^k(z) = v_{i1}^k(z) \sim_{i1}^{k} + v_{i2}^k(z) \sim_{i2}^{k}$ with $v_{i1}^k(z) \geq 0$, $v_{i2}^k(z) \geq 0$ and $v_{i1}^k(z) + v_{i2}^k(z) = 1$ (convex function properties). Then, with the previously defined notations, it yields, for $i = 1,...,r$:

$$\dot{h}_i(z) = \sum_{k=1}^{n} g_i^k(z) \dot{x}_k = \sum_{j=1}^{r} \sum_{k=1}^{n} \sum_{s=1}^{2} h_j(z) v_{is}^k(z) \sim_{is}^{k} A_j^k(x) \hat{x}(x) \tag{3.6}$$

***Remark 3.1*:**

Note that, the bounds $\sim_{i1}^{k}$ and $\sim_{i2}^{k}$ of $g_i^k(z)$ are always known on a compact set Ω according to assumption 3.1. For example, consider a T-S system composed of two subsystems with the convex AFs $h_1(z) = \cos^2 x_1$ and $h_2(z) = \sin^2 x_1$. Following proposition 3.1, one has $\dot{h}_1(x_1) = \dot{x}_1 g_1^1(x_1)$ with $g_1^1(x_1) = -2\sin x_1 \cos x_1 = -\sin(2x_1)$ and $\dot{h}_2(x_1) = \dot{x}_1 g_2^1(x_1)$ with $g_2^1(x_1) = 2\cos x_1 \sin x_1 = \sin(2x_1)$. In that case, one has $\sim_{11}^{1} = \sim_{21}^{1} = -1$, $\sim_{12}^{1} = \sim_{22}^{1} = 1$, $v_{11}^1(x_1) = v_{21}^1(x_1) = \dfrac{1+\sin(2x_1)}{2}$ and $v_{12}^1(x_1) = v_{22}^1(x_1) = \dfrac{1-\sin(2x_1)}{2}$ satisfying the convex sum properties.

Definition 3.1 [Prajna et al., 2004]:

A multivariate polynomial $p(x_1,...,x_n) \triangleq p(x)$ is a SOS, if there exist polynomials $f_1(x),..., f_m(x)$ such that :

$$p(x)=\sum_{i=1}^{m} f_i^2(x) \tag{3.7}$$

It follows that the set of SOS polynomials in n variables is a convex cone. The existence of a SOS decomposition (3.7) can be shown to be equivalent to the existence of a positive semi-definite matrix Q such that:

$$p(x)=\hat{x}^T(x)Q\hat{x}(x) \tag{3.8}$$

3.3.2. Main result

Theorem 3.1 [Guelton et al., 2013]:

The polynomial T-S system (3.2) is globally asymptotically stable if there exists, for all combinations of $i=1,...r$, $j=1,...,r$, $j\neq i$, $p=1,...,r$, $q=1,...,n$ *and* $s=1,2$, $P_i(x)\in \Re^{N\times N}$ *and* $R_i(x)\in \Re^{N\times N}$ *symmetric polynomial matrices, such that the polynomials (3.9), (3.10) and (3.11) are SOS, with the polynomials* $£_{1i}(x)>0$ *for* $x\neq 0$, $£_{2i}(x)\geq 0$ *and* $£_{3ij}(x)\geq 0$, $\forall x$.

$$\hat{x}^T(x)\left(P_i(x)-£_{1i}(x)I\right)\hat{x}(x) \tag{3.9}$$

$$-\hat{x}^T(x)\left(\Upsilon_{iips}^q(x)+£_{2i}(x)I\right)\hat{x}(x) \tag{3.10}$$

$$-\hat{x}^T(x)\left(\frac{1}{r-1}\Upsilon_{iips}^q(x)+\frac{1}{2}\left(\Upsilon_{ijps}^q(x)+\Upsilon_{jips}^q(x)\right)+£_{3ij}(x)I\right)\hat{x}(x) \tag{3.11}$$

with

$$\Upsilon_{ijps}^q(x)=\frac{1}{nr}\Bigg(A_i^T(x)T^T(x)P_j(x)+P_j(x)T(x)A_i(x)$$
$$+\sum_{k=1}^{n}\frac{\partial P_j(x)}{\partial x_k}A_i^k(x)\hat{x}(x)\Bigg)+\sim_{ps}^{q}A_i^q(x)\hat{x}(x)\left(P_p(x)+R_j(x)\right)$$

and $T(x) \in \Re^{N \times n}$*, a polynomial matrix in* x *such that* $\dot{\hat{x}}(x) = T(x)\dot{x}$*, i.e. whose* $(i, j)^{th}$ *entry is given by* $T_{(i,j)}(x) = \dfrac{\partial \hat{x}_i(x)}{\partial x_j}$.

Proof.

Let us consider the MPLF candidate (3.4), the system (3.2) is stable if:

$$\begin{aligned} \dot{v}(x) &= \sum_{i=1}^{r} h_i(z)\left(\dot{\hat{x}}^T(x) P_i(x)\hat{x}(x) + \hat{x}^T(x) P_i(x)\dot{\hat{x}}(x)\right) \\ &+ \sum_{i=1}^{r} h_i(z)\hat{x}^T(x)\dot{P}_i(x)\hat{x}(x) + \sum_{i=1}^{r}\dot{h}_i(z)\hat{x}^T(x)P_i(x)\hat{x}(x) < 0 \end{aligned} \tag{3.12}$$

Let $T(x) \in \Re^{N \times n}$ be a polynomial matrix such that $\dot{\hat{x}}(x) = T(x)\dot{x}$, i.e. whose $(i, j)^{th}$ entry is given by $T_{(i,j)}(x) = \dfrac{\partial \hat{x}_i(x)}{\partial x_j}$, considering equation (3.2), the inequality (3.12) can be rewritten as:

$$\begin{aligned} &\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\hat{x}^T(x)\left(A_i^T(x)T^T(x)P_j(x)\right. \\ &\quad \left. + P_j(x)T(x)A_i(x)\right)\hat{x}(x) + \sum_{j=1}^{r} h_j(z)\hat{x}^T(x)\dot{P}_j(x)\hat{x}(x) \\ &\quad + \sum_{j=1}^{r}\dot{h}_j(z)\hat{x}^T(x)P_j(x)\hat{x}(x) < 0 \end{aligned} \tag{3.13}$$

Moreover, using the above notations, it yields :

$$\sum_{j=1}^{r} h_j(z)\dot{P}_j(x) = \sum_{j=1}^{r} h_j(z)\sum_{k=1}^{n}\frac{\partial P_j(x)}{\partial x_k}\dot{x}_k(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z)h_j(z)\sum_{k=1}^{n}\frac{\partial P_j(x)}{\partial x_k}A_i^k(x)\hat{x}(x) \tag{3.14}$$

Then, since the AFs hold the convex sum property $\sum_{i=1}^{r} h_i(z)=1$, one has $\sum_{i=1}^{r} \dot{h}_i(z)=0$ and so, for any polynomial T-S matrices $\sum_{j=1}^{r} h_j(z) R_j(x)$ in x, one has $\sum_{i=1}^{r}\sum_{j=1}^{r} \dot{h}_i(z) h_j(z) R_j(x)=0$. Therefore, from proposition 3.1, one can write:

$$\begin{aligned}
&\sum_{j=1}^{r} \dot{h}_j(z)\hat{x}^T(x)P_j(x)\hat{x}(x)\\
&=\sum_{j=1}^{r}\sum_{p=1}^{r} h_j(z)\dot{h}_p(z)\hat{x}^T(x)\left(P_p(x)+R_j(x)\right)\hat{x}(x)\\
&=\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{n}\sum_{s=1}^{2} h_i(z)h_j(z)v_{ps}^q(z)\hat{x}^T(x)\Lambda_{ijps}^q\hat{x}(x)
\end{aligned} \tag{3.15}$$

with $\Lambda_{ijps}^q = \sim_{ps}^q A_i^q(x)\hat{x}(x)\left(P_p(x)+R_j(x)\right)$.

Hence, from (3.14) and (3.15), inequality (3.13) can be written as:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{p=1}^{r}\sum_{q=1}^{n}\sum_{s=1}^{2} h_i(z)h_j(z)v_{ps}^q(z)\hat{x}^T(x)\Upsilon_{ijps}^q(x)\hat{x}(x)<0 \tag{3.16}$$

with

$$\begin{aligned}
\Upsilon_{ijps}^q(x)=\frac{1}{nr}\Bigg(&A_i^T(x)T^T(x)P_j(x)+P_j(x)T(x)A_i(x)\\
&+\sum_{k=1}^{n}\frac{\partial P_j(x)}{\partial x_k}A_i^k(x)\hat{x}(x)\Bigg)\\
&+\sim_{ps}^q A_i^q(x)\hat{x}(x)\left(P_p(x)+R_j(x)\right)
\end{aligned}$$

Finally, since for all p,q and s one has $v_{ps}^q(z)\geq 0$, after applying lemma 1.5, one obtains the conditions of theorem 3.1. ■

Remark 3.2:

The PLF based stability conditions ([Tanaka et al., 2001], Theorem 3.1) are included in the conditions of theorem 3.1. Indeed, from (3.9), (3.10) and (3.11), choosing $P_j(x)=P(x)$ a common polynomial Lyapunov matrix as well as $R_j(x) = -P(x)$, one obtains the conditions proposed in [Tanaka et al., 2001]. Thus, theorem 3.1's conditions are less conservative. Moreover, in the proposed MPLF based SOS stability conditions, $R_j(x) \in \Re^{N \times N}$ are polynomial slack decision variables introduced as an extension of the relaxation proposed in [Mozelli et al., 2009b]. Note that the MPLF result in [Bernal et al., 2011] doesn't include such relaxation.

Remark 3.3:

As quote in [Tanaka et al., 2009], the second order PLF remains to the QLF $v(x) = x^T Px$ with $P \in \Re^{n \times n}$. Similarly, the second-order MPLF (3.4) is equivalent to the well-known NQLF $v(x) = \sum_{i=1}^{r} h_i(z) x^T P_i x$ [Blanco et al., 2001][Tanaka et al, 2003][Guerra & Vermeiren, 2004][Feng, 2006]. Let us recall that the main drawback of previous LMI based non-quadratic Lyapunov results was that, to study the global stability of a T-S model, some unknown parameters (e.g. lower bounds of AFs derivatives) have to be known before solving LMI algorithms. A LMI based alternative to this problem has been proposed in [Bernal & Guerra, 2010][Guerra et al., 2012] but it leads to complex LMI formulation and requires to reduce the global stability analysis into a local point of view. What is important to highlight with the proposed SOS based approach is that, the conditions of theorem 3.1 overcome the above quoted problems of non-quadratic Lyapunov approaches since it stands for global stability analysis of T-S systems without requiring unknown parameters from AFs' dynamics (see Proposition 3.1 and Remark 3.1).

3.4.Numerical example

In this section, two examples are proposed. The first one, which is not a polynomial system but a standard T-S one, is taken from [Tanaka et al., 2009] in order to compare and illustrate the benefit of the proposed approach in terms of conservatism regarding to previous results. The second one is dedicated to show the effectiveness of the proposed SOS based approach on a polynomial T-S system.

3.4.1. First example

Let us consider the following nonlinear system corresponding to example 2 in [Tanaka et al., 2009]:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i\left(z(t)\right) A_i x(t) \tag{3.17}$$

Where $A_1 = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ -2-k & -1 \end{bmatrix}$, $h_1\left(z(t)\right) = \dfrac{k - g(t)}{k}$ and $h_2(t) = \dfrac{g(t)}{k}$ and $z(t) = g(t) \in [0, k]$ for all t.

As quoted in [Tanaka et al., 2009], the quadratic stability conditions [Tanaka & Wang, 2001] guarantees the stability of (3.17) for $k \le 3.82$ whereas the piecewise quadratic approach [Xie et al., 1997] guarantees the stability for $k \le 4.7$.

Let us now apply the proposed approach (theorem 3.1) on the system (3.17) by considering $\hat{x} = x$. Analytical results from second-order to tenth-order MPLF (3.4) are considered in order to compare the results to the one obtained with the SOS based PLF approaches proposed in [Tanaka et al., 2009]. These results are obtained using SOSTOOLS for Matlab [Prajna et al., 2004] from theorem 3.1 with the following tuning:

- The order of polynomials in $R(x)$ is the same as the ones of $P(x)$ (resp. 0, 2, 4, 6 and 8).
- For all $i = 1,\dots,r$ and $j = 1,\dots,r$, $£_{1i}(x) = ©_{1i}\left(x_1^2 + x_2^2\right)$, $£_{2i}(x) = ©_{2i}\left(x_1^2 + x_2^2\right)$ and $£_{3ij}(x) = ©_{3ij}\left(x_1^2 + x_2^2\right)$ with $©_{1i}$, $©_{2i}$ and $©_{3ij}$ positives scalars to be minimized.

Therefore, for different polynomial settings (second to tenth-order polynomial Lyapunov functions), the maximal value of k, for which a feasible solution is found from stability conditions of theorem 3.1 and the ones of [Tanaka et al., 2009], have been computed. These results are summarized in Table 3.1 and Figure 3.1. As expected, the proposed SOS based stability conditions are outperforming the other approaches in terms of conservatism.

Moreover, for each of these different values, the MPLF $v(x)=\sum_{i=1}^{2} h_i(z)\, p_i(x)$ are obtained with $p_1(x)$ and $p_2(x)$ given for the second to the tenth-order as:

- Polynomials of the second-order MPLF:

$$p_1(x)=33.452x_1^2+8.688x_1x_2+8.826x_2^2,\ p_2(x)=35.71x_1^2+8.142x_1x_2+7.771x_2^2.$$

- Polynomials of the fourth-order MPLF:

$$\begin{aligned}p_1(x)&=24.73x_1^4+8.2x_1^3x_2\\&\quad+13.9324x_1^2x_2^2+4.2x_1x_2^3+1.2383x_2^4\end{aligned},\quad \begin{aligned}p_2(x)&=24.831x_1^4+8.3x_1^3x_2\\&\quad+14.067x_1^2x_2^2+3.3x_1x_2^3+1.09x_2^4\end{aligned}.$$

- Polynomials of the sixth-order MPLF:

$$\begin{aligned}p_1(x)&=323.119x_1^6+110x_1^5x_2^1+269.29x_1^4x_2^2\\&\quad+121.9x_1^3x_2^3+76.63x_1^2x_2^4+100x_1x_2^5+3.1533x_2^6\end{aligned},$$

$$\begin{aligned}p_2(x)&=323.79x_1^6+103x_1^5x_2^1+269.806x_1^4x_2^2\\&\quad+126.406x_1^3x_2^3+77.27x_1^2x_2^4+18x_1x_2^5+3.1533x_2^6\end{aligned}.$$

- Polynomials of the eighth-order MPLF:

$$\begin{aligned}p_1(x)&=2480.5x_1^8+972x_1^7x_2^1+2891.56x_1^6x_2^2\\&\quad+1612.3x_1^5x_2^3+1158.16x_1^4x_2^4+425.7x_1^3x_2^5\\&\quad+177.98x_1^2x_2^6+34x_1x_2^7+4.6342x_2^8\end{aligned},$$

$$\begin{aligned}p_2(x)&=2480.5x_1^8+973x_1^7x_2^1+2883.92x_1^6x_2^2\\&\quad+1607.06x_1^5x_2^3+1174.06x_1^4x_2^4+433.2x_1^3x_2^5\\&\quad+177.32x_1^2x_2^6+30x_1x_2^7+4.6342x_2^8\end{aligned}.$$

- Polynomials of the tenth-order MPLF:

$$\begin{aligned} p_1(x) = {} & 23262x_1^{10} + 8486x_1^9x_2^1 + 33281.8x_1^8x_2^2 \\ & + 24197.3x_1^7x_2^3 + 20630.5x_1^6x_2^4 + 8677x_1^5x_2^5 \\ & + 4686.3x_1^4x_2^6 + 1650.82x_1^3x_2^7 + 505.9x_1^2x_2^8 \\ & + 70x_1x_2^9 + 8.5894x_2^{10} \end{aligned},$$

$$\begin{aligned} p_2(x) = {} & 23262x_1^{10} + 8484x_1^9x_2^1 + 33284x_1^8x_2^2 \\ & + 24225.6x_1^7x_2^3 + 20629.4x_1^6x_2^4 + 8642.4x_1^5x_2^5 \\ & + 4683.1x_1^4x_2^6 + 1659.86x_1^3x_2^7 + 506.48x_1^2x_2^8 \\ & + 73x_1x_2^9 + 8.5894x_2^{10} \end{aligned}$$

Moreover, the behaviors of the obtained MPLF are depicted in Figure 3.2. These numerical simulations have been done for an initial condition $x^T(0) = [8 \quad 0]$ and, for scale compatibility, the results have been normalized before being plotted. As expected, all the obtained MPLFs are monotonously decreasing.

TABLE 3.1. Maximal value of k for 2nd to 10th-order polynomial Lyapunov functions.

Orders	2nd	4th	6th	8th	10th
PLF [29]	$k \le 3.82$	$k \le 5.62$	$k \le 6.17$	$k \le 6.39$	$k \le 6.64$
MPLF	$k \le 4.48$	$k \le 5.87$	$k \le 6.30$	$k \le 6.45$	$k \le 6.66$

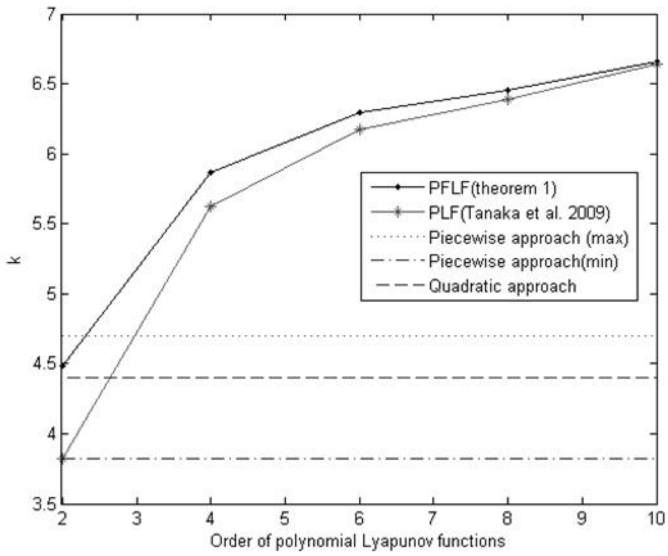


Figure 3.1. Comparison of the proposed SOS based results (example 3.4.1) with previous results [Xie et al., 1997][Tanaka et al., 2009].

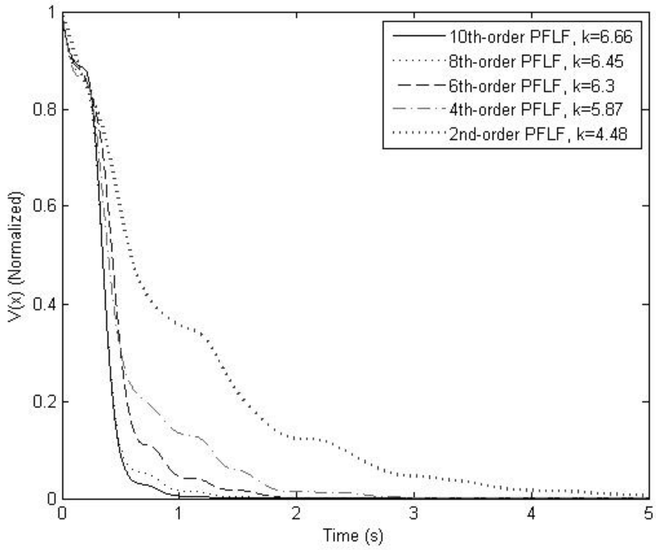


Figure 3.2. Time transient of the obtained MPLF (example 3.4.1).

3.4.2. Second example

Let us consider the following polynomial T-S system:

$$\dot{x} = \sum_{i=1}^{2} h_i(z) A_i(x)\hat{x} \tag{3.18}$$

where $\hat{x} = x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $z = x_1$, $h_1(z) = \dfrac{1+\sin x_1}{2}$, $h_2(z) = \dfrac{1-\sin x_1}{2}$, $A_1(x) = \begin{bmatrix} -0.8 - x_1^4 & -4 \\ -1 & -2 \end{bmatrix}$ and $A_2(x) = \begin{bmatrix} -2 & -4 \\ 20 & -2-5x_1^2 - x_2^2 \end{bmatrix}$.

Note that the sixth-order PLF based stability conditions [Tanaka et al., 2009] failed. However, the proposed SOS based conditions (3.9), (3.10) and (3.11) are feasible. The following sixth-order MPLF, obtained from theorem 3.1, guarantees the global asymptotical stability of (3.18):

$$V(x) = \sum_{i=1}^{2} h_i(z)\, p_i(x) \tag{3.19}$$

with

$$p_1(x) = 15.826x_1^6 + 41.007x_1^5x_2^1 + 11.546x_1^4x_2^2 + 14.596x_1^3x_2^3 + 35.996x_1^2x_2^4 + 18.318x_1x_2^5 + 4.6672x_2^6$$

and

$$p_2(x) = 15.827x_1^6 + 24.8804x_1^4x_2^2 + 39.61x_1^3x_2^3 + 43.45x_1^2x_2^4 + 18.64x_1^1x_2^5 + 47.615x_2^6 .$$

Figure 3.3 and Figure 3.4 show respectively the time responses of the state variables and the behaviour of the obtained MPLF (3.19) from the initial condition $x^T(0) = [2 \quad -1]^T$.

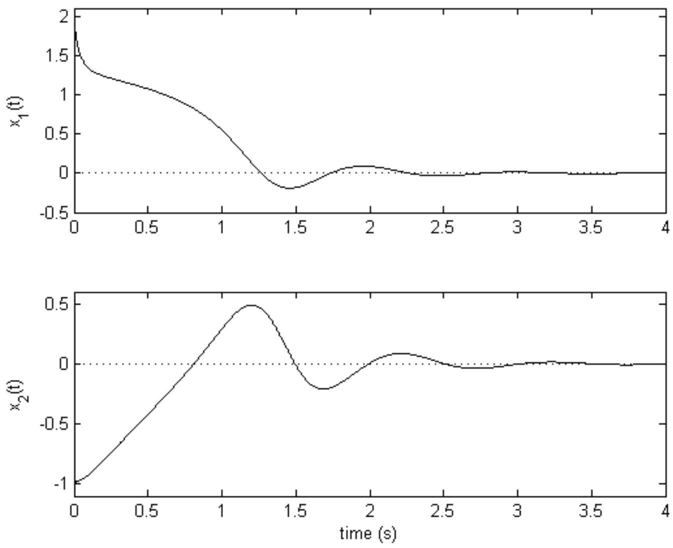


Figure 3.3. Time responses (example 3.4.2).

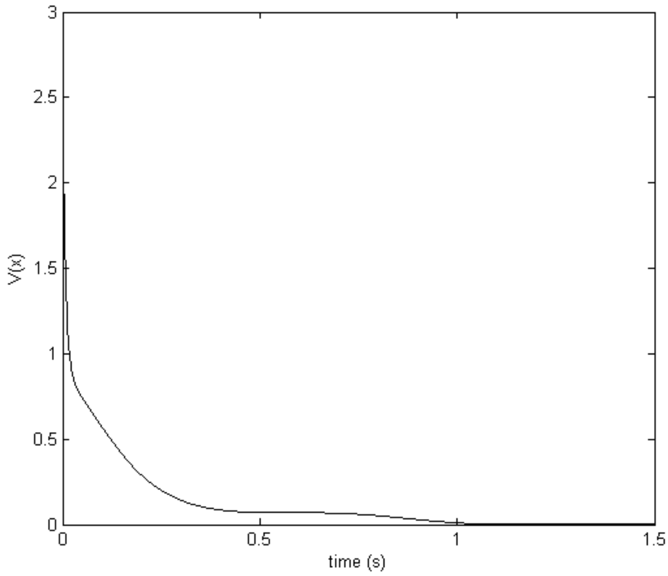


Figure 3.4. Time transient of the obtained MPLF (example 3.4.2).

3.4.1. Discussion

It can be noticed that the gap between MPLF and PLF is decreasing with example (3.17) when the chosen order of polynomial Lyapunov functions increases, see Figure 3.1. According to the authors' analysis, this is due to the influence of the knowledge left in the Lyapunov matrices $P_i(x)$ regarding to the knowledge included in the AFs $h_i(z)$. In other words, more the order of the Lyapunov matrices increase less is the superiority of MPLF regarding to PLF. Nevertheless, it may be hazardous to conclude that the proposed approach is useless regarding to the PLF approach. Indeed, due to computational limitations, SOS based algorithms are still unfortunately crashing with high-order polynomials. A similar statement may be found in [Bernal et al., 2011] where MPLF results converge to PLF for the third order computation. Even so, theorem 3.1 still provides a slight improvement at the height order, suggesting a better conservatism reduction.

It is to be noticed that a particular attention has to be made with the results provided by SOSTOOLS. Indeed, due to numerical approximation during the convex optimization process, especially when the feasibility is checked close to its limit, a solution may be obtained from the algorithm which is not strictly verifying the proposed theorem (for instance the Lyapunov function is not strictly monotonously decreasing). Therefore, it is often necessary to post verify the obtained SOS based results [Löfberg, 2009]. In our case, two verifications are made:

- The obtained polynomials are post verified to be SOS.
- The obtained MPLF are computed in numerical simulation and their monotonous decreasing is verified.

If these two points are not strictly verified, the obtained solution is reputed false. Note finally that, for a rigorous comparison of the proposed results with the one of [Tanaka et al., 2009], the latter have been recomputed with the same optimization parameters. This explains the slight differences between the numerical values presented in this chapter and those in [Tanaka et al., 2009].

3.5.Conclusions

In this chapter, a step on relaxation for T-S systems' stability analysis has been addressed. Inspired from NQLF regarding to quadratic ones, a MPLF has been proposed extending the results in [Tanaka et al., 2009]. Following the latter post-LMI challenge, the obtained stability conditions have been written in terms of a SOS optimization problem. The proposed MPLF includes the well-studied NQLF ones as a special case. Moreover, the proposed SOS based stability analysis doesn't require parameters (bounds of AFs derivatives) to be known in advance. In addition, it guarantees the global asymptotical stability when a solution to the SOS based optimization problem holds. Therefore, the drawbacks of previous LMI based non-quadratic approaches overcame. To conclude, following these SOS based approach, the door is now open for more complex post-LMIs control problems such as controller design, observer design, and so on for T-S systems.

Chapter 4

A sum-of-squares approach for global non-quadratic stabilization of T-S models

Résumé du chapitre 4 en Français:

Approche « Sum-Of-Squares » pour la stabilisation non quadratique globale des modèles T-S

De nos jours, lorsqu'il s'agit d'étudier la stabilisation de modèles T-S continus à l'aide d'approches non quadratiques, les conditions obtenue sous forme LMI deviennent de plus en plus complexe pour une réduction du conservatisme qui est parfois paraître discutable. E effet, l'apparition des dérivées des fonctions d'appartenances au sein des conditions de Lyapunov ruinent l'obtention de conditions de synthèse de loi de commande globalement stabilisantes via des approches LMI. Ainsi, plusieurs approches ont été proposées en considérant les bornes des dérivées des fonctions d'appartenances [Tanaka et al., 2003], paramètres difficiles à obtenir en pratique, ou encore en reconsidérant le problème en maximisant l'étendue d'un domaine d'attraction local [Guerra et al., 2012]. Par conséquent, ce chapitre a pour objectif de proposer une première alternative aux approches LMI par l'obtention de conditions réécrites sous la forme de contraintes « Sum-Of-Squares » dans le cadre de la stabilisation non quadratique des modèles T-S standards. Ces conditions obtenues à titre exploratoire, permettent la stabilisation asymptotique globale des modèles T-S au détriment d'une restriction de modélisation qui est ici discutée. Elles constituent donc une première étape vers des approches alternatives aux LMIs qu'il conviendra d'améliorer dans de futurs travaux.

4.1Introduction

Takgi-Sugeno (TS) models have been the subject of many researches in the past few decades. Indeed, firstly introduced in 1985 [Tanaka & Sugeno, 1985], they were originally based on an IF-THEN rules fuzzy formalism to approximate nonlinear systems. Then, by the use of the Sector Nonlinearity modeling approach [Tanaka & Wang 2001], the interest on TS models has grown up due to their ability to match a nonlinear system exactly on a compact set of the state space. Indeed, similarly to Quasi-LPV (Linear Parameter Varying) models, a TS model remains on a convex polytopic representation of a nonlinear system, i.e. a collection of linear systems blended together by convex nonlinear functions. Based on the convex property, numerous works have been done to study TS models stability or stabilization through the optimization of Linear Matrix Inequality (LMI) conditions obtained from the direct Lyapunov methodology, see e.g. [Tanaka & Sugeno, 1992][Feng,2006].

Most of LMI based studies on TS models stability analysis, controller or observer design, were obtained through the use of a common quadratic Lyapunov function. However, these works suffer from conservatism since a common Lyapunov matrix has to be solution of a set of several LMI constraints.

For a review of conservatism sources in TS studies, one may refer to [Liu & Zhang, 2003]. In order to reduce the conservatism, some LMI relaxation schemes have been proposed [Tuan et al, 2001][Tanaka et al, 2003] and recent studies have focused on the use of Non-Quadratic Lyapunov Functions (NQLF), see e.g. [Tuan et al, 2001][Tanaka et al 2003]. These NQLF, also called multiple Lyapunov or fuzzy Lyapunov functions, appear convenient since they are based on similar convex polytopic structures as the TS models to be analyzed or stabilized. However, when dealing with continuous-time TS fuzzy models, such non-quadratic LMI approaches remain difficult to apply since they often require some parameters (bounds of the derivatives of the membership functions) which are often unknown in practice. To overcome this drawback, line integral Lyapunov functions has been employed leading to LMIs with a particular structure of Lyapunov decision variables (off-diagonal common matrices) which may be understood as a source of conservatism [Rhee & Won, 2006] [Guelton et al., 2010][Mozelli et al., 2009a) . Some other recent works reduced to local non-quadratic controller design for continuous-time TS models have been proposed, see e.g. [Guerra et al., 2013]. However they lead to a somewhat complex LMI formulation and the global non-quadratic stabilization remains an open problem.

Complementary to LMI based conditions for TS models, some recent works were focused on the stability analysis and the stabilization of polynomial fuzzy models using Sum-Of-Squares (SOS) techniques [Tanaka et al, 2009][Narimani & Lam, 2010]. Following the way of LMI studies regarding to conservatism reduction using non-quadratic Lyapunov functions, these results have been extended to the stability analysis via multiple polynomial Lyapunov functions [Guelton et al., 2010][Bernal et al, 2011).

Nowadays, it is sometime argued that LMI based successive improvements lead to more and more complex problem formulations for conservatism reductions that are sometime minor and so, it may be interesting to investigate other ways to formulate the non-quadratic stabilization problem of TS models. In this context, one doesn't pretend to overcome existing non-quadratic LMI approaches in terms of conservatism but just exploring another ways to partially answer the question "what else than LMIs in non-quadratic stabilization of TS fuzzy models?". Therefore, as a first attempt, the aim of this chapter is to show how SOS based approach can be used to derive global non-quadratic controller design conditions for conventional TS models (rather than polynomial ones) as an alternative to LMIs without requiring unknown parameters, even if another restrictive assumption is still required in stabilization (see assumption 4.1 bellow) [Tanaka et al., 2009][Narimani & Lam, 2010].

The rest of the chapter is organized as follows: First, usual non-quadratic approaches are recalled and discussed. Then, SOS based conditions for continuous-time TS models using non-quadratic Lyapunov functions are provided. Finally, a numerical example illustrates the validity of the proposed approach.

4.2 LMI Based Non-quadratic Stabilization and Problem Statement

Consider the following nonlinear system:

$$\dot{x}(t) = A\big(x(t)\big)x(t) + B\big(x(t)\big)u(t) \tag{4.1}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input vector, $A(x(t)) \in \Re^{n \times n}$ and $B(x(t)) \in \Re^{n \times m}$ are matrices which entries may contains smooth and bounded nonlinear functions on a compact set $\Omega \subseteq \Re^n$ of the state space.

Using the well-known sector nonlinearity approach [Tanaka & Wang, 2001], (4.1)can be exactly rewritten on Ω as a TS fuzzy system such that:

.

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))(A_i x(t) + B_i u(t)) \tag{4.2}$$

where $z(t) \in \Re^p$ is the premise vector, $A_i \in \Re^{n \times n}$ and $B_i \in \Re^{n \times m}$ are constant matrices and $h_i(z(t)) > 0$ are positive fuzzy membership functions holding the convex sum property

$$\sum_{i=1}^{r} h_i(z(t)) = 1.$$

Now, let us consider the following non-PDC control law [Tanaka et al., 2003]:

$$u(t) = \sum_{i=1}^{r} h_i(z(t)) F_i \left(\sum_{i=1}^{r} h_i(z(t)) X_i \right)^{-1} x(t) \tag{4.3}$$

where $F_i \in \Re^{m \times n}$ and $X_i \in \Re^{n \times n}$ are gain matrices to be synthesized.

***Remark 4.1*:**

From the sector nonlinearity approach [Tanaka & Wang, 2001], the TS fuzzy model (4.2)is reputed to be valid on a local subset Ω of the state space if some nonlinearities of (4.1) are unbounded on the whole state space. Nevertheless, if all the nonlinearities are bounded on the whole state space, (4.2)is said to be globally matching (4.1). Therefore, in the sequel as well as for all TS fuzzy model based controller design studies, a non-PDC controller (4.3) is assumed to be globally stabilizing the TS fuzzy system (4.2) on its whole definition set Ω instead of the whole state space of the initial nonlinear model (4.1).

***Notations*:**

In the sequel, to lighten mathematical expressions, when there is no ambiguity, the time t will be omitted. Moreover, for a matrix M, one denotes $He(M) = M + M^T$, $M_h = \sum_{i=1}^{r} h_i(z(t)) M_i$, $M_{hh} = \sum_{i=1}^{r} \sum_{i=1}^{r} h_i(z(t)) h_j(z(t)) M_{ij}$ and so on.

From (4.2) and (4.3), the closed loop dynamics may be expressed as:

$$\dot{x} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) \left(A_i + B_i F_j \left(\sum_{i=1}^{r} h_i(z) X_i \right)^{-1} \right) x \tag{4.4}$$

In order to design a non-PDC controller(4.3) such that the closed-loop dynamics (4.4) is stable, the direct Lyapunov method has been used in several studies to lead to LMI conditions, see e.g.[Tanaka & al, 2003]. Among Lyapunov function candidates, the NQLF is the subject of many interests since its structure shares the same fuzzy membership functions as the TS model to be stabilized and so leads less conservative LMI conditions than common quadratic Lyapunov functions [Tanaka & Sugeno, 1992]. Therefore, for the stabilization of (4.4), one considers the NQLF given by .[Tanaka & al, 2003]:

$$v(x) = x^T \left(\sum_{i=1}^{r} h_i(z) X_i \right)^{-1} x \tag{4.5}$$

which is strictly positive for $x(t) \neq 0$ with $X_i = X_i^T > 0$.

The following theorem, inspired from [Mozelli et al., 2009a], expresses non-quadratic LMI conditions for the design of non-PDC controllers (4.3).

***Theorem 4.1*:**

Assuming $\forall i \in \{1,...,r\}, \left|\dot{h}_i(z)\right| < \bar{\ }_i$, *the TS fuzzy system* (4.2) *is asymptotically stabilized by the non-PDC control law* (4.3) *if there exist the matrices* F_j, $X_i = X_i^T > 0$ *and* R_{ij} *such that,* $\forall (i, j, k) \in \{1,...,r\}^3 \ / \ j \neq i$ *, the following LMIs are satisfied*:

$$\Upsilon_{ii} < 0. \tag{4.6}$$

$$\frac{1}{r-1}\Upsilon_{ii}(x) + \frac{1}{2}\left(\Upsilon_{ij} + \Upsilon_{ji}\right) < 0 \tag{4.7}$$

$$X_k + R_{ij} > 0 \tag{4.8}$$

with $\Upsilon_{ij} = He\left(A_i X_j + B_i F_j\right) - \sum_{k=1}^{r} \bar{\ }_k \left(X_k + R_{ij}\right)$.

***Proof*:**

From (4.5), the closed-loop dynamics (4.4) is stable if $\dot{v}(x) < 0$, i.e.:

$$x^T \left(He\left(X_h^{-1} A_h + X_h^{-1} B_h F_h X_h^{-1}\right) + \dot{X}_h^{-1}\right) x < 0 \tag{4.9}$$

That is to say, $\forall x$:

$$He\left(X_h^{-1} A_h + X_h^{-1} B_h F_h X_h^{-1}\right) + \dot{X}_h^{-1} < 0 \tag{4.10}$$

Multiplying (4.11) left and right by X_h, one obtains:

$$He\left(A_h X_h + B_h F_h\right) + X_h \dot{X}_h^{-1} X_h < 0 \tag{4.12}$$

Now, since $\dot{X}_h = -X_h \dot{X}_h^{-1} X_h$, see e.g. [Bouarar et al., 2010], the inequality (4.12) can be rewritten as:

$$He\left(A_h X_h + B_h F_h\right) - \dot{X}_h < 0 \tag{4.13}$$

Note that the term $\dot{X}_h$ occurs in (4.13) and so a bounded real lemma cannot be obtained directly in terms of LMIs. To cope with this problem, a way consists on considering the bounds of the membership functions' derivatives $\left|\dot{h}_i(z)\right| < {}^{-}{}_{i}$. Indeed:

$$\dot{X}_h = \sum_{i=1}^{r} \dot{h}_i(z) X_i \tag{4.14}$$

Moreover, following recent non-quadratic improvement regarding to conservatism [Mozelli & al, 2009a], slack decision matrices may be introduced since, whatever a matrix R_{hh} is:

$$\sum_{i=1}^{r} \dot{h}_i(z) = \sum_{k=1}^{r} \dot{h}_k(z) \left(\sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) R_{ij} \right) = 0 \tag{4.15}$$

and so:

$$\dot{X}_h = \sum_{i=1}^{r} \sum_{j=}^{r} \sum_{k=1}^{r} h_i(z) h_j(z) \dot{h}_k(z) \left(X_k + R_{ij}\right) \tag{4.16}$$

Therefore, assuming the bounds of membership functions derivatives $\left|\dot{h}_i(z)\right| < {}^{-}{}_{i}$ and from (4.16), the inequality (4.13) is satisfied if:

$$\Upsilon_{hh} = \sum_{i=1}^{r} \sum_{j=1}^{r} h_i(z) h_j(z) \Upsilon_{ij} < 0 \tag{4.17}$$

with $\Upsilon_{ij} = He\left(A_i X_j + B_i F_j\right) - \sum_{k=1}^{r} {}^{-}{}_{k} \left(X_k + R_{ij}\right)$ and, for all $(i, j, k) \in \{1, \ldots, r\}^3, X_k + R_{ij} > 0$.

Thus, from (4.17) and the relaxation proposed in lemma 1.5, one can express a bounded real lemma expressed in theorem ‘;1.

***Remark 4.2*:**

The bounded real lemma given in theorem 4.1 constitutes a slight improvement of the LMI conditions proposed in [Mozelli et al., 2009a]. Indeed, the latter study's LMI conditions may be recovered from theorem 4.1 by considering the slack variables $R_{ij} = R$ common for each LMIs. Moreover, if $R_{ij} = 0$, one obtains the non-quadratic conditions proposed in [Tanaka et al., 2003]. Then, considering $X_k = X$ and $R_{ij} = -X$ common matrices, it yields the well-known quadratic conditions [Tanaka & Sugeno, 1992][Tuan et al., 2001]. Consequently, theorem 4.1 includes as special cases some previous results and obviously leads to less conservatism.

***Remark 4.3*:**

Theorem 4.1 requires the knowledge of the bounds $\bar{\ }_i$ of the time derivatives of the membership functions $\dot{h}_i\left(z(t)\right)$ as parameters to the LMI computation. It is often criticized since the membership functions depend on the time evolution premises variables, which are often state variables. Therefore, in practice, it is sometime difficult, even more for stabilization, to estimate these bounds before solving the LMIs and so before having synthesized the closed-loop dynamics. Moreover, without prior knowledge on the closed-loop state dynamics, it is not correct to tell about global asymptotical stability on Ω since it is somewhat hypothetic to say that $\left|\dot{h}_i(z)\right| < \bar{\ }_i$ arises for every initial condition $x(0) \in \Omega$.

***Remark 4.4*:**

Other ways to cope with the problem of the membership functions' derivatives depicted in remark 4.2 have been proposed. First, a Line-integral fuzzy Lyapunov functional (LIFLF) has been proposed in [Rhee & Won, 2006][Guelton et al., 2010][Mozelli et al., 2009a]. However, even if some recent improvements lead to LMIs in stabilization [Mozelli et al., 2009a], these approaches need to recast the decision variables to a particular case where the off-diagonal elements of all X_i are common and thus leading to conservatism. More recently, a local result has been proposed to overcome the problem of the membership functions' derivatives [Guerra et al., 2012]. In the latter study, the idea is to maximize a domain of attraction

$D=\left\{x : x\in\Omega, \left\|\sum_{j=1}^{r}\dot{h}_j P_j\right\| < a\right\}$. Although this result is LMI (with a somewhat complex formulation), it is still a local asymptotical stabilization approach since here, the domain of attraction D doesn't recover the whole set Ω.

At this step, the concerns pointed out in remarks 4.3 and 4.4 are understood as drawbacks of LMI based non-quadratic controller design for TS models. Moreover, since LMI based successive improvements leads to more and more complex problem formulations for conservatism reductions that are sometime minor, it may be interesting to investigate other ways to formulate the non-quadratic stabilization problem of TS models. This is the attempt of the following section, which constitutes the main contribution of the chapter, where one proposes Sum-Of-Squares (SOS) based global conditions (on Ω) as a possible alternative to LMIs.

4.3 SOS Based Non-quadratic Controller Design for TS Models

In this section, the goal is to propose new non-quadratic conditions for the design of non-PDC controllers (4.3) for TS Fuzzy systems (4.2) leading to global stabilization (on Ω) of the closed-loop dynamics (4.4). These conditions will be based on the SOS formalism [Prajna et al., 2004] with the NQLF candidate (4.5). For more clarity of further mathematical proofs, before presenting the main results, some useful preliminaries (Assumptions and Definitions) are presented.

***Assumption 4.1*:**

Following previous work on polynomial fuzzy models stabilization [Tanaka et al, 2009][Narimani & Lam, 2010][Guelton et al., 2011], to avoid non-convex conditions, we assume that the premise vector z only depends on the states that are not directly affected by the control input, namely, states whose corresponding rows in B_i are zero. Let A_i^k be the k^{th} row of A_i. $\mathrm{K}=\{k_1,k_2,...,k_p\}$ denotes the set of row indices of B_i whose corresponding rows are equal to zero, one can write:

$$\dot{x}_k = \sum_{i=1}^{r} h_i(z) A_i^k x, \forall k \in \mathrm{K} \tag{4.18}$$

and $\forall s \in \{1,...,r\}$,

$$\frac{\partial h_i(z)}{\partial x_s} = 0, \forall s \in \{1,...,n\} / \mathrm{K} \tag{4.19}$$

***Assumption 4.2*:**

$\forall i \in \{1,...,r\}$, $h_i(z)$ are continuously derivable within each states variables x_s (with $s \in \mathrm{K}$) on the compact set Ω. Thus, their derivatives $g_i^s(z) = \frac{\partial h_i(z)}{\partial x_s}$ are bounded on Ω such that $\forall t, g_k^s(z) \in [\sim_{k1}^s, \sim_{k2}^s]$ and these bounds can always be known on Ω whereas the bounds $|\dot{h}_i(z)| < \bar{\ }_i$ are unknown. Therefore, one may applies the sector nonlinearity approach and one can define convex functions $\mu_{k1}^s(z) \geq 0$ and $\mu_{k2}^s(z) \geq 0$, $\mu_{k1}^s + \mu_{k2}^s = 1$, such that:

$$g_k^s(z) = \frac{\partial h_k(z)}{\partial x_s} = \sum_{^2=1}^{2} \mu_{k^2}^s(z) \sim_{k^2}^s \tag{4.20}$$

***Example*:**

To illustrate assumption 4.2, let us consider the membership function $h_1(x_1) = \sin^2 x_1$. Its time derivative $\dot{h}_1(x_1) = 2\dot{x}_1 \cos x_1 \sin x_1$ depends on $\dot{x}_1$ and so its bound $|\dot{h}_1(z)| < \bar{\ }_1$ is unknown for stabilization studies since the dynamics $\dot{x}_1$ is unknown before designing the non-PDC controller and so the closed-loop dynamics. However, $g_1^1(x_1) = \frac{\partial h_1(x_1)}{\partial x_1} = 2\cos x_1 \sin x_1 \in [\sim_{11}^1, \sim_{12}^1]$ is bounded whatever x_1 is, and $\sim_{11}^1 = -1$ and $\sim_{12}^1 = 1$ are known parameters! Hence, considering $\mu_{11}^1(x_1) = \frac{1}{2} - \cos x_1 \sin x_1 \geq 0$ and $\mu_{12}^1(x_1) = \frac{1}{2} + \cos x_1 \sin x_1 \geq 0$, such that $\mu_{11}^1 + \mu_{12}^1 = 1$, one has

$$g_1^1\left(x_1\right)=\frac{\partial h_1\left(x_1\right)}{\partial x_1}=\sum_{²=1}^{2}\mu_{1²}^{1}\left(z\right)\sim_{1²}^{1}\ .$$

The following theorem expresses the non-quadratic SOS based conditions for the design of non-PDC controllers (4.3) which globally stabilize (4.2) on Ω under assumption 1.

***Theorem 4.2*:**

The TS fuzzy system (4.2) is globally asymptotically stabilized (on Ω) by the non-PDC control law (4.3) if there exist the matrices F_j, $X_i = X_i^T > 0$ and R_j as well as the polynomials $£_{1i}\left(x\right)>0$, $£_{2i}\left(x\right)\geq 0$, $£_{3ij}\left(x\right)\geq 0$ and $£_{4ij}\left(x\right)\geq 0$ such that, $\forall\left(i,j,k\right)\in\left\{1,...,r\right\}^3$, $j\neq i$, $\forall s\in \mathrm{K}$ and $\forall ² \in\left\{1,2\right\}$, the SOS conditions (4.21), (4.22), (4.23) and (4.24) are satisfied.

$$x^T\left(X_i - £_{1i}\right)x \text{ is SOS} \tag{4.21}$$

$$x^T\left(X_k + R_j - £_{2kj}\right)x \text{ is SOS} \tag{4.22}$$

$$-x^T\left[\mathrm{P}_{iips}^{q} + £_{2i}I\right]x \text{ is SOS} \tag{4.23}$$

$$-x^T\left(\frac{1}{r-1}\mathrm{P}_{iips}^{q}+\frac{1}{2}\left(\mathrm{P}_{ijps}^{q}+\mathrm{P}_{jips}^{q}\right)+£_{3ij}I\right)x\ .\text{ is SOS} \tag{4.24}$$

with $\mathrm{P}_{ijk²}^{s} = He\left(A_i X_j + B_i F_j\right) - \sim_{k²}^{s} A_i^s x\left(X_k + R_j\right)$.

***Proof*:**

Let us consider the NQLF candidate(4.5). The TS system (4.2) is stable if (4.13) holds, i.e. $\forall x$:

$$x^T\left(He\left(A_h X_h + B_h F_h\right) - \dot{X}_h\right)x < 0 \tag{4.25}$$

Let us now focus on the term $\dot{X}_h$ occurring in (4.25). Similarly to equation (4.16), one can write:

$$\begin{aligned}\dot{X}_h &= \sum_{j=}^{r}\sum_{k=1}^{r} h_j(z)\dot{h}_k(z)\left(X_k + R_j\right) \\ &= \sum_{j=}^{r}\sum_{k=1}^{r}\sum_{s\in \mathrm{K}} h_j(z)\frac{\partial h_k(z)}{\partial x_s}\dot{x}_s\left(X_k + R_j\right)\end{aligned} \tag{4.26}$$

Under assumptions 4.2, $\forall t, g_k^s(z)\in\left[\sim_{k1}^{s}, \sim_{k2}^{s}\right]$ with known bounds on the compact set Ω. Therefore, applying the sector nonlinearity approach on $g_k^s(z)$, there always exist convex functions $\mu_{k1}^s(z)>0$ and $\mu_{k2}^s(z)>0$, $\mu_{k1}^s + \mu_{k2}^s = 1$, such that (4.20) holds. Thus, equation (4.26) can be rewritten as:

$$\dot{X}_h = \sum_{j=}^{r}\sum_{k=1}^{r}\sum_{s\in \mathrm{K}}\sum_{\text{²}=1}^{2} h_j(z)\mu_{k\text{²}}^{s}(z)\sim_{k\text{²}}^{s}\dot{x}_s\left(X_k + R_j\right) \tag{4.27}$$

Now, under assumption 4.1 and substituting (4.18) in (4.27), it yields:

$$\dot{X}_h = \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{s\in \mathrm{K}}\sum_{\text{²}=1}^{2} h_i(z)h_j(z)\mu_{k\text{²}}^{s}(z)\sim_{k\text{²}}^{s}A_i^s x\left(X_k + R_j\right) \tag{4.28}$$

Thus, under assumption 4.1 and 4.2, considering (4.28), the inequality (4.25) is, strictly equivalent on Ω to:

$$x^T\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{s\in \mathrm{K}}\sum_{\text{²}=1}^{2} h_i(z)h_j(z)\mu_{k\text{²}}^{s}(z)\mathrm{P}_{ijk\text{²}}^{s}x < 0 \tag{4.29}$$

with $\mathrm{P}_{ijk\text{²}}^{s} = He\left(A_i X_j + B_i F_j\right) - \sim_{k\text{²}}^{s}A_i^s x\left(X_k + R_j\right)$

Note that $x^T\mathrm{P}_{ijk\text{²}}^{s}x$ are scalar polynomials in x, so the negativity of (4.29), and so (4.25), may be checked using SOS optimization tools. Therefore, applying lemma 1.5 on (4.29), one obtains the SOS conditions summarized in theorem 4.2.

***Remark 4.4*:**

The most improvements of the above-defined SOS conditions (theorem 4.2) regarding to LMI ones (e.g. theorem 4.1) are: The conditions of theorem 4.2 are free of unknown parameters such as the bounds of derivatives of the membership functions. Indeed, the bounds of $g_k^s(z)$, $\sim_{k1}^s$ and $\sim_{k2}^s$ do not depend directly on $\dot{x}$ and are always known and well-defined under assumption 4.2 (see e.g. the example provided above).

The whole parameters of theorem 4.2 being known, the SOS based approach ensures, when a solution exists, a global stabilization on the whole set Ω, where the TS system is valid, which cannot be guarantee with LMI based approaches such that [Tuan et al., 2001][Tanaka et al., 2003] or [Guerra, 2012] (since they are local results), excepted LMI conditions obtained through a line-integral Lyapunov function [Mozelli et al., 2009a]. However, in the latter approach, the particular case with common off-diagonal decision matrices is required.

However, it appears that the main drawback of SOS based approaches relies on assumption 1, namely the premise vector z only depends on the states that are not directly affected by the control input. Obviously, it is restrictive and, as state in [Tanaka et al., 2009] regarding to the stabilization of polynomial fuzzy models, it may be advantageous to employ an initial state transformation to introduce as many zero rows as possible in the input matrices B_1. This point, understood as a lock, should be the focus of further SOS based development dealing with controller design for fuzzy models.

4.4 Numerical example

Let us consider the following TS system:

$$\dot{x}(t) = \sum_{i=1}^{2} h_i(z)\left(A_i x(t) + B_i u(t)\right) \tag{4.30}$$

where $A_1 = \begin{bmatrix} 2 & -10 \\ 2 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & -5 \\ 1 & 2 \end{bmatrix}$, $B_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$,

$h_1(x_2) = \frac{1}{2}(1 - \sin x_2)$ and $h_2(x_2) = 1 - h_1(x_2)$.

Recall that, using theorem 4.1 and previous LMI-based non-quadratic conditions, a non-PDC controller guaranteeing a global stabilization cannot be designed, excepted LMI conditions obtained through a line-integral Lyapunov function [Mozelli et al., 2009a]. Moreover, as illustrated by the phase portrait plot in figure 4.1, the open-loop TS system (4.30) is unstable.

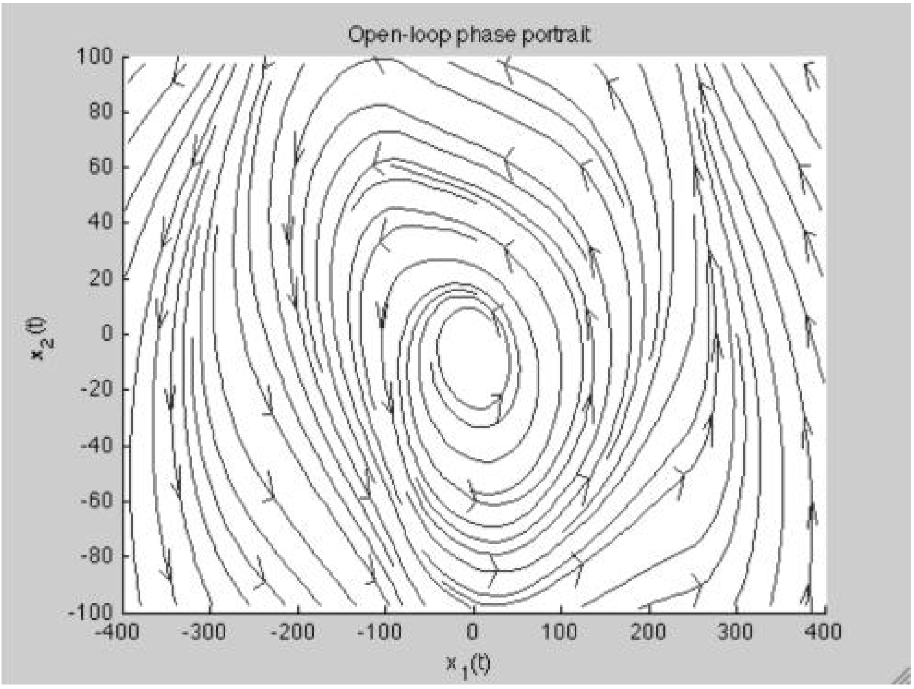


Figure 4.1. Open-loop phase portrait**.**

From the definition of h_1 and h_2, one has :

$$\frac{\partial h_1(x_1)}{\partial x_1} = -\frac{1}{2}\cos x_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{4.31}$$

$$\frac{\partial h_2(x_1)}{\partial x_1} = \frac{1}{2}\cos x_1 \in \left[-\frac{1}{2}, \frac{1}{2}\right] \tag{4.32}$$

Therefore, one may apply theorem 4.2 with the known parameters $\sim_{11}^{1} = \sim_{21}^{1} = -\frac{1}{2}$ and $\sim_{12}^{1} = \sim_{22}^{1} = \frac{1}{2}$. Note that, the decision variables $£_{1i}$, $£_{2i}$, $£_{3ij}$ and $£_{4ij}$ have been set as second-order polynomials. The result, obtained using the toolbox SOSTOOLS for Matlab [Prajna et al., 2004], is given by the matrices:

$$F_1 = \begin{bmatrix} -7.8438 & -0.99286 \end{bmatrix},\ F_2 = \begin{bmatrix} -8.9582 & -0.4258 \end{bmatrix},$$

$$X_1 = \begin{bmatrix} 3.5266 & -7.1173 \\ -7.1173 & 0.36954 \end{bmatrix},\ X_2 = \begin{bmatrix} 3.7194 & -0.7743 \\ -0.7743 & 0.20693 \end{bmatrix},$$

defining the designed non-PDC control law:

$$u(t) = \sum_{i=1}^{2} h_i(z(t)) F_i \left(\sum_{i=1}^{2} h_i(z(t)) X_i \right)^{-1} x(t) \tag{4.33}$$

which globally stabilizes (4.30) as illustrated by the phase portrait given in figure 4.2.

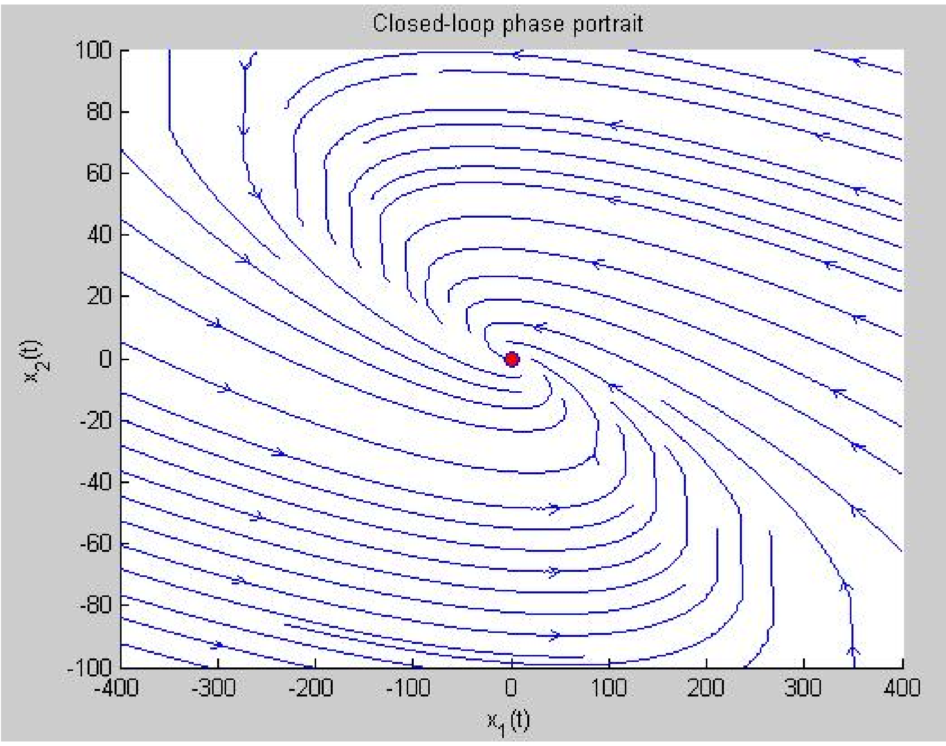


Figure 4.2 : Closed-loop phase portrait**.**

4.5 Conclusion

In this chapter, an alternative to LMI approaches for global non-quadratic stabilization of Takagi-Sugeno models has been proposed. Indeed, LMIs have proved their efficiency in fuzzy controller design for Takagi-Sugeno models. However, they suffer from conservatism in the quadratic framework and successive recent advances, such that non-quadratic approaches, lead to complex LMI formulation for a conservatism reduction that may sometimes be questionable. In order to explore what can be done, else than LMIs, SOS based conditions have been proposed for non-quadratic Lyapunov stabilization of TS models. The proposed approach is an attempt as an alternative to LMIs without claiming to make better in terms of conservatism. Having said that, despite most of LMIs approaches, the proposed non-quadratic conditions allow the global stabilization of a TS system on its domain of definition have been obtained. However, the proposed conditions still require some restrictive modeling assumptions and this point, understood as a drawback of SOS formalism, will be the concern of further studies.

Conclusion générale en francais

Dans le cadre de ce travail de thèse, l'analyse et la stabilisation des systèmes non linéaires a été abordé. Sur la base de la modélisation de type Takagi-Sugeno (T-S), de nouveaux critères de d'analyse de la stabilité et de synthèse de contrôleurs ont été proposées au travers de la méthode directe de Lyapunov. De nombreux résultats existent dans la littérature concernant la synthèse de lois de commande pour les systèmes T-S via l'optimisation de contraintes d'inégalités linéaires matricielles (LMI) [Tanaka & Wang, 2001]. Ces résultats sont souvent basés sur des fonctions quadratiques de Lyapunov. Cependant, ces approches souffrent du conservatisme puisqu'elles exigent de trouver des variables de décision communes à un ensemble de contraintes LMI. Pour réduire cette source de conservatisme, beaucoup de travaux ont été effectués en considérant des fonctions non-quadratiques de Lyapunov, voir par exemple [Tanaka et al., 2003]. Ces fonctions non-quadratiques, également appelées fonctions de Lyapunov multiples ou encore fonction de Lyapunov floues, partagent la même structure d'interconnexion floue que les modèles T-S qu'elles permettent d'analyser (ou de stabiliser). Néanmoins, ces techniques non-quadratiques exigent souvent la connaissance à priori de paramètres, comme les bornes des dérivées temporelles des fonctions d'appartenance, qui sont difficiles à obtenir que dans la pratique en amont de la synthèse d'un contrôleur. Des travaux récents ont permis de surmonter cet inconvénient en reconsidérant l'objectif de synthèse en un problème local [Guerra et al., 2012]. Dans ce cas, le but est de maximiser un domaine d'attraction où le système considéré sera stable. Cependant, ces résultats mènent à une formulation LMI quelque peu complexe, avec une amélioration du conservatisme qui peut parfois ne pas être très significative. Par conséquent, une des tentatives de ce travail de thèse a consisté à explorer « ce qui est possible pour faire, de manière alternative aux approches LMIs dans le cadre non-quadratique, pour l'analyse des modèles T-S afin de réduire le conservatisme ». Dans ce cadre, ont été abordées une approche mettant en oeuvre des lois de commande PDC à commutations synthétisées via des fonctions de Lyapunov à commutation, puis des *approches basées sur l'optimisation polynomiale par les techniques « Sum-Of-Squares » (SOS) pour les modèles T-S polynomiaux.*

Dans le ***premier chapitre****, quelques préliminaires nécessaires au sujet de stabilité et stabilisation des modèles flous de type Takagi-Sugeno (T-S) ont été présentés. L'accent a également été mis sur les résultats LMI existant, obtenus via les approches quadratiques ou non-quadratique de Lyapunov, et leur conservatisme a été discuté. Aussi, même si la relaxation des conditions LMI est un souci permanent de la communauté travaillant sur l'analyse et la commande des systèmes T-S, les résultats récents dans ce domaine apparaissent de plus en plus complexes avec une réduction de conservatisme qui s'avère parfois discutable.*

Dans le ***deuxième chapitre****, la stabilisation d'une classe des systèmes non linéaires représentés par des systèmes T-S à commutations, incertains et perturbés, a été étudiée. L'intérêt de cette approche est de tirer parti de l'information contenue au seing des systèmes T-S en association avec les caractéristiques des systèmes à commutations. Afin de faire face aux incertitudes et aux perturbations externes, des conditions LMI pour la synthèse de contrôleurs PDC à commutations robustes ont été obtenues via une fonction de Lyapunov à commutations et d'un critère* H_∞*.*

Une autre approche a été abordée dans le ***troisième chapitre*** *en proposant des conditions d'analyse de la stabilité des modèles T-S polynomiaux. Ainsi, inspiré des approches non-quadratiques classiques, l'emploi d'une fonction de Lyapunov polynomiale multiple a été proposé de manière à réduire le conservatisme et étendre les résultats proposés initialement via des techniques d'optimisation SOS [Tanaka et al., 2009]..*

Enfin, dans le ***quatrième chapitre****, sur la base des travaux présentés au chapitre précédent, une alternative aux approches LMI pour la stabilisation non-quadratique globale des modèles Takagi-Sugeno, présentés comme un cas particulier des modèles polynomiaux, a été proposée. En effet, dans le contexte non-quadratique, l'obtention de conditions de synthèse de lois de commande à base de contraintes SOS peut-être vue comme une alternative aux conditions LMIs. Ces travaux constituent une première étape exploratoire et ne prétendent pas, à ce jour, améliorer le conservatisme vis-à-vis des approches LMI existantes.*

Néanmoins, au prix d'hypothèses de modélisation quelque peu restrictives, les conditions SOS obtenues permettent de garantir la stabilisation non-quadratique globale des systèmes T-S lorsque les résultats LMI rentent locaux.

À ce stade, il convient de souligner que les conditions SOS non-quadratiques proposés incluent à titre de cas particuliers certains travaux LMI, dans le cadre non-quadratique et par conséquent quadratique. De plus ces approches SOS permettent de s'affranchir de la connaissance à priori des bornes des dérivées temporelles des fonctions d'appartenance qui sont difficiles à obtenir dans la pratique et souvent nécessaire pour les approches LMI. Par conséquent, certains des inconvénients des approches non-quadratiques LMI sont ainsi évité. Cependant, les conditions proposées exigent des hypothèses de modélisation restrictives et ce point, compris comme inconvénient du formalisme SOS, devrait faire l'objet de futurs travaux. Néanmoins, sur la base de ces travaux préliminaires sur les techniques SOS d'évoluent à l'analyse des modèles T-S; la porte est maintenant ouverte pour des problèmes d'automatique plus complexes comme les points présentés ci-après comme possibles perspectives de ces travaux :

- *Comment choisir une fonction de Lyapunov candidate adéquat pour un système non linéaire particulier? En effet, la méthodologie classiquement utilisée pour l'analyse modèle de T-S est générique et n'est pas consacrée à un problème particulier sur une application donnée. Il peut donc être établi que, pour un problème particulier, une structure spécifique de la fonction candidate de Lyapunov peut être assignée pour réduire le conservatisme.*

- *L'inconvénient majeur des approches SOS se situe dans le fait que, pour la synthèse de contrôleur, une restriction forte de modélisation est exigée. En effet, pour éviter des conditions non convexes, il est supposé que le vecteur de permises ne dépend uniquement des variables d'états qui ne sont pas directement affectées par l'entrée, c'est-à-dire des états dont les lignes correspondantes dans la matrice d'entrée sont nulles (pour plus de détails voir l'hypothèse 4,1 du chapitre 4). Par conséquent, avant d'aller plus loin vers des problèmes plus complexes de commande basés sur des conditions SOS, il apparaît important de concentrer nos efforts afin de surmonter cet inconvénient.*

- *D'autre part, concernant l'analyse des systèmes non linéaires, il pourrait également être intéressant de proposer des conditions SOS locales lorsque les conditions globales n'admettent pas de solutions et, par conséquent, de proposer des algorithmes de maximiser l'estimation de l'étendue de leurs domaines d'attraction.*

- *Enfin, sur la base de ces travaux préliminaires sur l'analyse de la stabilité et la synthèse de contrôleurs par approche SOS, il serait intéressant d'étudier d'autres problèmes d'automatique tels que la synthèse d'observateur, la commande par retour de sortie, la commande tolérante aux fautes, la spécification de performances, etc.*

General conclusion in English

In this thesis, one has dealt with nonlinear systems' stability analysis and stabilization. Based on the Takagi-Sugeno modeling approaches, some new stability and controller design criteria has been proposed through the direct Lyapunov methodology. Numerous results can be found in the literature in terms of Linear Matrix Inequalities (LMI) optimization problem [Tanaka & Wang, 2001]. These results are often based on common quadratic Lyapunov functions. However, these approaches suffer from conservatism since they require finding common decision variables within a set of LMIs. To reduce that source of conservatism, many works have been done by considering non-quadratic Lyapunov functions, see e.g. [Tanaka et al., 2003]. The latters, also called fuzzy Lyapunov functions share the same membership functions as the T-S models they are dedicated to analyze. Therefore, the computation of such obtained stability conditions require the knowledge of some parameters, such like the bounds of the derivatives of the membership functions, which are difficult to obtain in practice. Some other recent works has been proposed to overcome this drawback by reconsidering the objective to a local problem [Guerra et al., 2012]. In that case, the goal is to maximize a domain of attraction where the considered system will be stable. However, these results lead to a somewhat complex LMI formulation, with a conservatism improvement which is sometime questionable. Therefore, one of the attempts of the present work was to seek “what is possible to do, else than LMIs in the nonquadratic framework, for the analysis of T-S models with a conservatism reduction?”. To add steps in this field, one has explored some alternatives to non-quadratic LMIs by considering first, switched PDC controllers and switched Lypunov functions and then, SOS based approaches for polynomial T-S models.

In the **first chapter**, some necessary preliminaries about stability and stabilization of Takagi-Sugeno (T-S) fuzzy models have been introduced. It has been shown that LMI results, obtain through a quadratic or a non-quadratic Lyapunov function may suffer from conservatism. Even if the relaxation of LMI is a challenging concern of the T-S community, recent results in this field appear more and more complex with a conservatism reduction that may be questionable.

In the **second chapter**, the stabilization of a class of nonlinear systems represented by uncertain and disturbed switched Takagi-Sugeno fuzzy systems has been studied. The interest of this approach is to benefit from the well-known information of T-S systems with association to the characteristic of the switched systems. Moreover, to cope with uncertainties and external disturbances, LMI conditions for robust switched fuzzy PDC controller design have been obtained based on switched quadratic Lyapunov function and a H_∞ criterion.

Another step has been addressed in the **third chapter** by considering the stability analysis of polynomial T-S systems. Inspired from classical non-quadratic approaches regarding to quadratic ones, a multiple polynomial Lyapunov function has been proposed to extend the results in [Tanaka et al., 2009]. Following the latter post-LMI challenge, the obtained stability conditions have been written in terms of a SOS optimization problem.

Finally, in the **fourth chapter**, an alternative to LMI approaches for global non-quadratic stabilization of Takagi-Sugeno models has been proposed as a particular case of the result proposed in chapter 3. Indeed, SOS based conditions as an alternative to LMIs without claiming to make better in terms of conservatism. Having said that, despite most of LMIs approaches, the proposed non-quadratic conditions allow the global stabilization of a T-S system on its domain of definition. However, the proposed conditions still require some restrictive modeling assumptions and this point, understood as a drawback of SOS formalism, will be the concern of further studies.

At this stage, it may be highlighted that the proposed SOS based conditions includes some well-studied non-quadratic and quadratic results as a special cases. Moreover, the proposed SOS based stability analysis doesn't require parameters (bounds of membership function derivatives) to be known in advance. In addition, it guarantees the global asymptotical stability when a solution to the SOS based optimization problem holds. Therefore, some of the drawbacks of previous LMI based non-quadratic approaches overcame. However, the proposed conditions still require some restrictive modeling assumptions and this point, understood as a drawback of SOS formalism, should be the concern of further studies. Nevertheless, following these SOS based approaches; the door is now open for more complex post-LMIs control problems such as the points given hereafter as **perspectives of this thesis**:

- How to choose a convenient Lyapunov function candidate for particular nonlinear systems? Indeed, the methodology classically employed for T-S model analysis is generic and not especially dedicated to a particular control problem. It may then be argue that, from a particular control problem, a specific structure of the Lyapunov function candidate may be assigned to reduce the conservatism.

- The main drawback of SOS approaches lies in the fact that, for controller design purposes, a strong modeling restriction is required. Indeed, to avoid non-convex conditions, it has been assumed that the premise vector only depends on the states that are not directly affected by the control input, namely, states whose corresponding rows in the input matrix are zero (for more details see assumption 4.1 in chapter 4). Therefore, before going further to more complex SOS base control problems, it may be important to concentrate our efforts to overcome this drawback.

- Moreover, dealing with nonlinear systems' analysis, it must be interesting to check SOS based local conditions and to propose algorithms to expend or maximize the estimation of their domains of attraction.

- Finally, preliminary SOS based condition for stability analysis and state feedback controllers being addressed, it should be interesting to investigate some other control engineering problems such like: Observer design, output feedback control, fault tolerant control, performance specifications, and so on…

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Résumé: Les travaux de cette thèse portent sur la stabilité et la stabilisation des systèmes non-linaires représentés par des modèles Takagi–Sugeno (T-S). L'objectif de ces travaux est d'explorer des techniques alternatives aux LMI pour l'analyse et la synthèse de lois de commande dans le cadre non quadratique afin de réduire le conservatisme. Tout d'abord, la stabilisation robuste de systèmes T-S à commutations incertains et perturbés a été considérée. Ainsi, des conditions de stabilisation ont été obtenues sous forme LMI sur la base d'une fonction candidate de Lyapunov à commutations**.** Puis, une nouvelle approche, pour l'analyse de la stabilité des systèmes non linéaires décrits par des modèles T-S polynomiaux a été proposée. L'objectif est ici d'explorer des techniques alternatives aux LMI dans le cadre non-quadratique. Ainsi, sur la base de travaux préliminaires dévolus à l'analyse de la stabilité via les techniques d'optimisation polynomiale « Sum-Of-Squares » (SOS), l'emploi d'une fonction candidate de Lyapunov polynomiale multiple a été proposée. Celle-ci permet de réduire le conservatisme des approches polynomiales existantes dans la littérature. Enfin, les modèles T-S classiques pouvant-être vus comme un cas particulier des modèles polynomiaux, une méthodologie de synthèse de lois de commande dans le cadre non quadratique est proposée. Celle-ci permet de s'affranchir de paramètres difficiles à obtenir en pratique via les approches LMI ainsi que de fournir une solution globale lorsque celle-ci existe. Néanmoins, à ce jour, des hypothèses fortes de modélisation restent toutefois nécessaires et constituent l'inconvénient majeur des approches SOS. Inconvénient qu'il conviendra de traiter dans des travaux futurs et qui suggèrent donc quelques perspectives à ces travaux.

Mots Clés: modèles Takagi-Sugeno, systèmes non linéaires, Commande robuste à commutations, inégalité matricielle linéaire (LMI), sommes de carrés polynomiaux (SOS), fonctions multiples de Lyapunov, Analyse non quadratique.

Abstract : This thesis deals with the stability and stabilization of nonlinear systems represented by Takagi-Sugeno (T-S) models. The objective of this work is to explore and find out some alternatives to classical LMI conditions in order to reduce the conservatism. First, we focus on robust stabilisation of uncertain switched T-S models. Based on a switched Lyapunov function, the stabilisation conditions are obtained in terms of LMI. Then, a new approach for the stability analysis of polynomial T-S models is proposed. The goal is here to explore alternatives to LMI in the non-quadratic framework. Therefore, an extension of some preliminary result on the stability analysis of polynomial T-S models is proposed by the use of a multiple polynomial Lyapunov function which lead to less conservatism. The stability conditions are given in terms of Sum-of-Squares (SOS) polynomial optimization problem. Finally, classical T-S models being a particular case of polynomial ones, an attempt is done as an alternative to LMI in the non quadratic framework for the design of non-PDC controllers via SOS techniques. Within this framework, global stability may be guaranteed if there exists a solution to the SOS constraints. Moreover, it didn't require unknown parameters in advance like in LMI based non quadratic approaches. However, these SOS based controller design conditions are obtained through a restrictive modelling assumption, suggesting future prospects to this work.

Keywords: Takagi–Sugeno models, nonlinear systems, Robust switched controllers, Linear Matrix Inequalities (LMI), Sum of Squares (SOS), multiple polynomial Lyapunov function, Nonquadratic analysis.