

ON 64%-MAJORITY RULE

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Many electoral rules (such as those governing the U.S. Constitution) require a super-majority vote to change the status quo. It is well known that without some restriction on preferences, super-majority rules have paradoxical properties. For example, electoral cycles are possible with anything other than 100%-majority rule. Can these problems still arise if there is sufficient similarity of attitudes among the voting population?

We introduce a definition of social consensus which involves two restrictions on domain: one on individual preferences, the other on the distribution of preferences. Individuals vote for the proposal closest (in Euclidean distance) to their most preferred point. The density of voters' ideal points is concave over its support in R^n . Under these conditions, there exists an unbeatable proposal according to 64%-majority rule. In addition, no electoral cycles are possible. For n -dimensional decision problems, the precise majority size necessary to avoid cycles is $1 - (n/(n+1))^n$ which rises monotonically to $1 - (1/e)$, just below 64%.

Our approach is based on the Simpson-Kramer min-max rule. We compare this rule with Condorcet's original proposal for an electoral system immune to his paradox of voting. We conclude by considering the properties of a voting constitution based on 64%-majority rule.

KEYWORDS: Condorcet's paradox, min-max majority, social choice, voting.

1. INTRODUCTION

THE MATHEMATICAL APPROACH to the theory of social choice dates back 200 years to the great work of Condorcet (1785). He was the first to illustrate a paradoxical property of voting under majority rule. Three individuals with preferences $\{A > B > C\}$, $\{B > C > A\}$, and $\{C > A > B\}$ will choose A over B , B over C , and C over A , producing a cycle of electoral victories. A broader perspective on the paradox of voting is provided by Arrow's general possibility theorem (1951). Without some restriction on preferences, no satisfactory procedure exists for arriving at social decisions.

Both Condorcet and Arrow provide positive suggestions on how to resolve the problems of social decision making. Condorcet proposes that elections won by a large majority should take precedence over elections decided by a narrow majority. To implement this proposal, a vote above 50% (a super-majority) may be required to overturn the status quo.

Arrow conjectures that in societies with sufficient similarity of attitudes, the social choice problem will be soluble.

"The solution of the social welfare problem may lie in some generalization of the unanimity condition...But, the correct mathematical generalization of the unanimity condition is not easy to see" (1951, p. 89).

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If everyone has the same preferences, social decisions can be made unanimously. Arrow's suggestion requires a restriction of the distribution of preferences less extreme than unanimity.

We present a new approach to the theory of social choice combining suggestions of Condorcet and Arrow. The idea that preferences may be similar across society is formalized in a mathematical definition of social consensus. When this consensus exists, voting cycles are impossible under a 64%-majority rule.

Our results provide new evidence on the positive properties of super-majority rules. Such voting mechanisms are widely used in practice. The Constitutions of the U.S., twenty of the fifty states, and a wide variety of countries all require a two-thirds vote to approve an amendment.

The paper is organized as follows. Section 2 reviews the literature. Our definition of social consensus is presented in Section 3. The theorems are stated and proved in Section 4 with extensions provided in Section 5. In Section 6, we show how our results are related to Condorcet's original proposal. Section 7 discusses the design of a voting constitution based on a super-majority voting rule. Concluding remarks are in Section 8.

2. MAJORITY RULE AND ITS GENERALIZATIONS

Two different directions have been taken to resolve Condorcet's paradox of voting. One approach is to find conditions which rule out voting cycles even under simple (50%) majority rule. This literature begins with Black's (1948a) work on the median voter. A second approach, also due to Black (1948b), is to consider properties of super-majority rules. The applicability of each approach is considered in turn.

Black (1948a) demonstrates that voting cycles cannot arise when preferences satisfy a "single-peakedness" condition. There are two requirements for single-peakedness: the social decision must be one-dimensional and voters' utility functions must be unimodal in this dimension. Individuals can then be identified by their most preferred position. The median voter's optimum secures a majority over all alternatives.

The one-dimensional nature of the median voter result is problematic. It imposes strong restrictions on individuals preferences. The restrictions can be illustrated using the political parties in the United Kingdom. If we place the Labour party (L) on the left, the SDP/Liberal Alliance (A) in the center, and the Conservatives (C) on the right, single-peakedness rules out ($L > C > A$) and ($C > L > A$). In reality, other dimensions such as experience in office are important. As a result, these preferences can no longer be ruled out and single-peakedness fails.

Multi-dimensional analogues of the median voter result are offered by Davis et al. (1972), Grandmont (1978), Plott (1967), and Tullock (1967). A multi-dimensional median is created by imposing symmetry conditions. For example, when the distribution of most preferred points is radially symmetric around a median voter, this voter's optimum will secure a majority over all alternatives.

These results show that extremely strong assumptions are needed to support simple majority rule. Weaker assumptions are possible if we consider the generalization to super-majority rules, also known as δ -majority rules (see Black (1948b)). Under a δ -majority rule, priority is given to the status quo. Any alternative proposal must gain the support of more than δ of the population to replace the status quo. A proposal which is unbeatable should it become the status quo is called a δ -majority winner. A δ -majority winner is then a proposal preferred by more than a fraction $(1 - \delta)$ of the population to any other alternative in pairwise comparisons.

Condorcet's paradox demonstrates the possibility that there may be no 50%-majority winner. In fact, the works of Kramer (1973), Plott (1967), and Rubinstein (1979) show that this possibility is almost a certainty; the set of societies for which there exists a 50%-majority winner is of measure zero. It is then natural to investigate the minimal majority size which ensures the existence of a δ -majority winner. This is known as the Simpson-Kramer min-max majority (Simpson (1969), Kramer (1977)). This concept is central to our work and is now presented more formally.

There is a social decision to be made. The set of proposals among which society can choose is denoted by X . Elements of X are represented as vectors in n -dimensional Euclidean space, $X \subset R^n$. For a given element $x \in X$, the k th coordinate, x_k , pinpoints the proposal's position on the k th issue, $1 \leq k \leq n$.

Individual preferences are defined over R^n . These preferences vary across society. The range of preferences is summarized by an index of types, I : an individual of type i has preferences $>_i$. A given society can then be characterized by the distribution of types, described by the density function $f(i)$ on $i \in I$. Definition 1 summarizes this description of the social choice problem.

DEFINITION 1: A social decision problem, C , is defined by the triple $\{X, >_i, f(i)\}$ with $X \subset R^n$, representing the choice set; $>_i$ representing type i 's preference ordering over R^n , $i \in I$; and $f(i)$ representing the density function over types in society.

The definition of the min-max majority closely follows Kramer (1977) and Rosenthal (1975). The difference is that our definition allows for a continuum of voters.

- DEFINITION 2:** For a decision problem C and $x, y \in X$,
- (a) $m(x, y)$ is the fraction of the population for whom $y >_i x$;
 - (b) $m(x)$ is the maximal fraction against x , $m(x) \equiv \sup_{y \in X} m(x, y)$;
 - (c) $m^*(C)$ is the min-max majority, $m^*(C) \equiv \inf_{x \in X} m(x)$;
 - (d) the min-max set consists of all points x^* for which $m(x^*) = m^*(C)$.

Note that a δ -majority winner exists if and only if $\delta \geq m^*(C)$.²

² With $\delta = m^*(C)$, existence of a δ -majority winner requires that the infimum in Definition 2(c) be attained.

The social choice correspondence which associates with a decision problem C the corresponding min-max set is known as the min-max rule. An axiomatic characterization of this rule is provided by Blair (1979).

A major result on the min-max majority is due to Greenberg (1979): for decision problems C in R^n and individuals with convex preferences, $m^*(C)$ is bounded above by $n/(n+1)$. Lemma 1 in the Appendix provides a simple proof of this result for the case of Euclidean preferences.

The problem of dividing a fixed pie among $n+1$ selfish individuals illustrates Greenberg's bounds.³ Here, n of the $n+1$ people prefer to expropriate the excluded individual's piece of the pie. Thus any proposed division can be outvoted by a majority of $n/(n+1)$. The need for such a large majority size reflects society's complete polarization.

The importance of these bounds is seen in the context of a voting rule.⁴ The voting rule specifies the majority size, δ , required to overturn the status quo. Once δ is chosen, it will be applied to a wide variety of currently unknown decision problems. To avoid voting cycles, δ must exceed the min-max majority. But Greenberg's result demonstrates that the only universal upper bound on the min-max majority is 1. Use of $\delta = 1$ means that unanimity is required to change the status quo: anybody can veto change.

The unanimity rule is needed when allowing for issues (such as pie division) which polarize voters. This brings us back to the suggestion of Arrow. We consider a restricted class of decisions for which there is a social consensus. Under these conditions, a 64%-majority rule winner always exists.

3. THE DEFINITION OF SOCIAL CONSENSUS

We introduce a definition of social consensus which involves two restrictions on domain: one on individual preferences $>_i$, the other on the distribution of preferences $f(i)$.

ASSUMPTION A1 (Euclidean Preferences): *For an individual of type i , $a >_i b \leftrightarrow \|a - x_i\| < \|b - x_i\|$, with $\| \cdot \|$ representing the Euclidean norm, and x_i representing type i 's most preferred alternative in R^n .*

With A1, individuals rank propositions according to their Euclidean distance from a most preferred point. The function describing the distribution of types can then be represented by a density function $f(x)$ on most preferred points $x \in R^n$. This paper focuses on decisions for which $f(x)$ is concave.

ASSUMPTION A2 (Concavity): *The density of voters' most preferred points, $f(x)$, is a concave function over its support, S , which is a convex subset of R^n with*

³ An allocation can be represented as a point in the solid n -dimensional unit simplex. For each i , $1 \leq i \leq n$, individual i receives share x_i , and individual $n+1$ receives the remainder.

⁴ The topic of designing a voting rule is taken up in Section 7.

positive but finite volume;

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \lambda f(x_1) + (1 - \lambda)f(x_2),$$

$$0 \leq \lambda \leq 1 \quad \text{and} \quad (x_1, x_2) \in S.$$

These conditions are taken up in reverse order.

Assumption A2 is new to the social choice literature. Concavity requires a degree of consensus and thus ensures that society is not polarized. Its meaning is illustrated by an example. Consider the choice of a flat tax rate t , $0 < t < 1$. Viewing this as a simple one-dimensional problem, A1 implies that individuals rank tax rates according to their absolute difference from some most preferred rate. The assumption that $f(t)$ is concave rules out a situation in which many people favor very low rates, many favor very high rates, but few favor intermediate rates. More precisely, concavity allows statements of the following kind: if at least 40% of the population favor $0 \leq t < 0.2$ and at least 20% favor $0.4 \leq t < 0.6$, then at least 30% favor $0.2 \leq t < 0.4$.

In a more subtle manner, A2 (concavity) also implies that no type i has positive measure.⁵ Theorem 3 demonstrates that the results extend to the limiting case of large finite populations drawn from the original concave density. In this sense, the concavity assumption can be applied to a large finite economy.

The restrictive nature of A2 is apparent. Restrictive though it may be, it cannot be relaxed in an obvious way. In Section 5, we consider the weaker requirement of quasi-concavity as an alternative to concavity as a definition of social consensus,

$$f(\lambda x_1 + (1 - \lambda)x_2) \geq \text{Min} [f(x_1), f(x_2)],$$

$$0 \leq \lambda \leq 1 \quad \text{and} \quad (x_1, x_2) \in S.$$

Proposition 6 exhibits quasi-concave densities which fail to satisfy any reasonable measure of consensus: in these examples, quasi-concavity has no power in bounding the min-max majority. However, our results do generalize to densities which are "close to" concave (e.g., the truncated normal). Proposition 5 provides this extension.

The applicability of A2 to a given social choice problem is an empirical question. Since it requires a degree of social consensus it may be implausible for decisions in which polarization is to be expected, such as the pie division problem. However, the assumption may be more realistic in cases when the underlying issues are not so clearly divisive. In a limited set of tests, data from political scientists suggest that A2 is applicable to Presidential elections. For example, Aldrich and McKelvey (1977) and Poole and Rosenthal (1984) provide spatial mappings of most preferred points for voters in the 1968 to 1980 elections which broadly support concavity.

⁵ In addition, it allows us to downplay the possibility of strategic behavior [Gibbard (1973), Satterthwaite (1975), Zeckhauser (1973)].

Assumption A1 (Euclidean preferences) was introduced in Davis and Hinich (1967) and has since become common in the social choice literature. In one dimension, A1 implies Black's condition of single-peakedness. This leads to a restriction on preferences, a value restriction, whenever there are three or more alternatives (Sen (1966), Kramer (1973)). Recalling the example of the U.K. political parties, the restriction on preferences over the three alternatives is removed in a two-dimensional setting. Proposition 1, proven in the Appendix, shows that as dimensionality increases, A1 has less restrictive implications for preference orderings.

PROPOSITION 1: *Consider a decision problem C in R^n satisfying A1 together with m distinct proposals.⁶*

(a) *For $m \leq n + 1$, no value restriction is implied; all $m!$ ordinal rankings can coexist.*

(b) *For $m > n + 1$, value restrictions are implied; some of the $m!$ orderings of the alternatives are ruled out.*

The proposition highlights the role of dimensionality. When the choice set is of a low dimension, A1 implies particularly strong a priori restrictions on preferences.⁷ On the other side, A2 (Concavity) becomes more restrictive as dimensionality increases. Thus, the choice of an appropriate dimensional setting for a given problem is itself a subtle issue. Fortunately, our central results can be stated without reference to dimensionality.

For our purposes, the essential property of A1 is that supporters of distinct proposals are divided by a hyperplane in the space of most preferred points. Section 5 shows that our results generalize to any family of preferences with this property. Such families of preferences are called intermediate preferences (Grandmont (1978)); this class includes the important case of constant elasticity of substitution utility functions.

We make one final assumption which simplifies our proofs. The choice set X is assumed to be compact and to contain the Pareto optimal set, S . Proposition 7 extends all results to the case where X is finite.

ASSUMPTION A3 (Inclusivity): *The set X of proposals is compact and contains the set S , the support of $f(x)$.*

4. RESULTS

We establish that the min-max majority is always less than 64%. There exists a proposal which at least 36% of the population favor against any alternative. For n -dimensional decision problems, Theorem 2 provides an upper bound on $m^*(C)$ of $1 - (n/(n+1))^n$. These bounds are the best available. The dimension-free

⁶ Provided that the m proposals span a space of dimension $m - 1$.

⁷ Another restrictive aspect of A1 is the implied symmetry of individual preferences around the most preferred point. The generalization to intermediate preferences in Section 5 shows that this form of symmetry is not essential to our results.

result arises from the fact that $(n/(n+1))^n$ falls monotonically to $1/e$, which is just above 36%.

Before presenting the theorems, we provide a series of examples and preliminary results. The examples provide the intuition needed to motivate the theorems. The first significant step is then taken in Theorem 1 which establishes the bound for a restricted class of concave densities, uniform densities over convex sets in R^n . Theorem 2 next establishes the general bound. Finally, examples are provided showing the bounds to be tight.

Our first preliminary result is closely related to Hotelling's principle of minimal differentiation (1929). The best way to gather votes against a given proposal is to locate "next door," on the side with the largest population.

PROPOSITION 2: *Under Assumptions A1–A3, $m(x)$ equals the largest fraction of the population on either side of any hyperplane through x .*

PROOF: Pick points x interior to S , $y \in X$, $y \neq x$, and consider proposals along the line segment joining y to x . Proposals along this segment closer to x get an ever larger share of the vote against x . This follows from the concavity of the Euclidean preferences: an individual who prefers y to x also prefers $[\lambda y + (1 - \lambda)x]$ to x , $0 < \lambda < 1$. Hence in looking for $\sup_y \{m(x, y)\}$, we can restrict attention to points $y \in S$ arbitrarily close to x . In the limit, voters are divided by a hyperplane through x with normal $y - x$. Since x is an interior point of S , it can be approached from all directions so that the hyperplane through x which most unevenly divides the population defines $m(x)$.

If $x \in X$ not interior to S , then since S is a convex set, there exists a hyperplane through x containing S in one of the half-spaces. This concurs with $m(x) = 1$; for x exterior to S the closest point in S is unanimously preferred, while for x on the boundary, points interior to S approaching x can capture the entire vote. *Q.E.D.*

Proposition 2 greatly simplifies the interpretation of the min-max majority, $m^*(C)$. This majority can now be related to a cake-cutting problem. Two people must divide an asymmetric cake defined by the density function $f(x)$. The second person both cuts the cake and chooses the side. But this cut is constrained to pass through a point of the first person's choosing. If the first person chooses the point x , then the second person playing optimally receives a fraction $m(x)$ of the cake. The first person's objective is to find the point minimizing $m(x)$. In equilibrium, the second person receives fraction $m^*(C)$, the min-max majority.

To understand the general results on $m^*(C)$, consider some simple 1- and 2-dimensional voting problems. Insights from these examples are central to understanding higher dimensional problems. Here, Proposition 2 is used to simplify the exposition; instead of proposals and counterproposals, we have points and hyperplanes.

With voters' most preferred points uniformly distributed along a line segment, $[0, 1]$, the min-max majority is 50%, as location at the mid-point leaves the set evenly divided. The fact that $m^*(C) = 1/2$ is well known; it is a special case of Black's median voter result.

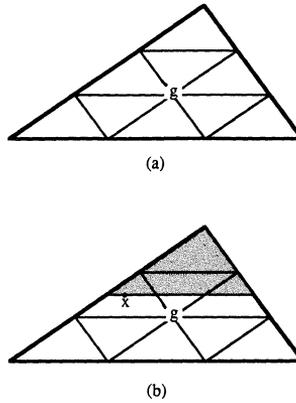


FIGURE 1

In two dimensions, the simplest cases involve voters' most preferred points uniformly distributed over centrally symmetric figures. For a uniform density over a rectangle (Tullock (1967)), the min-max majority is again 50% and the min-max point is at the center of the rectangle. However, the 50% min-max majority does not generalize to distributions that are not radially symmetric.

An example of an asymmetric density satisfying A2 is a uniform distribution of most preferred points over a triangle in the plane. In this case, no proposal can sustain a 50%-majority against all alternatives. In fact, Proposition 3 shows that the min-max majority is $5/9$ ths and that the unique min-max point is at the triangle's center of gravity. Note that existence and uniqueness of the min-max point extends to the entire class of concave population densities under consideration (see Demange (1982)).

PROPOSITION 3: *For a decision problem C in R^2 satisfying Assumptions A1–A3 with the additional assumption that $f(x)$ is uniform on a triangle, $m^*(C) = 5/9$. The unique min-max point is at the triangle's center of gravity.*

PROOF: The result is illustrated in Figure 1(a). The lines inside the triangle divide it into nine identical triangles. Lines through the centroid, g , can cut off five of the nine triangles by approaching g at an angle perpendicular to any side of the outer triangle. All other lines through g secure less than $5/9$ ths of the triangle.

For any point other than g , there exists a division capturing more than $5/9$ ths of the triangle. In Figure 1(b), point $x \neq g$ is chosen. A line through x can now cut off more than $5/9$ ths of the triangle by approaching x at an angle perpendicular to the farthest side. Therefore, g is the unique min-max point.⁸ *Q.E.D.*

The examples show that the min-max majority is connected to the degree of symmetry of the density, $f(x)$. Indeed, this precise measure of symmetry was

⁸ Note that for other distributions, the centroid is not the min-max point. For example, the centroid is not the min-max point when the population is uniformly distributed over a trapezoid.

introduced by Winternitz in his study of convex sets (see Bonnesen and Fenchel (1934)). This unexpected connection is important in the demonstration of our theorems.

THEOREM 1: *For a decision problem C in R^n satisfying Assumptions A1–A3 with the additional assumption that the density $f(x)$ is uniform over its convex support S , the min-max majority satisfies*

$$m^*(C) \leq 1 - \left(\frac{n}{n+1} \right)^n.$$

PROOF: Consider the point $g \equiv \int_S x f(x) dx$, the center of gravity (centroid) of the set S . By Proposition 2, $m(g)$ equals the largest fraction of the population on either side of any hyperplane through g . Lemma 2, a major result on the Winternitz measure of symmetry, shows that $m(g) \leq 1 - (n/(n+1))^n$. The proof of Theorem 1 is completed by noting that $m^*(C) \leq m(g)$. *Q.E.D.*

LEMMA 2 (Grunbaum (1960), Hammer (1960)): *Let S be a convex body of Volume 1 in R^n . Let v_1 be the larger of the two volumes in a division of S by a hyperplane through its centroid. Then*

$$v_1 \leq 1 - \left(\frac{n}{n+1} \right)^n.$$

PROOF: The proof is sketched to introduce a technique to economists which may prove useful in other applications. Figure 2 provides an illustration for the two-dimensional case. Without loss of generality, let S have its centroid at the origin. If the lemma is false then there exists a line through the origin with less than $4/9$ ths of S on one side. Let this line be $x_2 = 0$ and assume that the area of S with $x_2 > 0$ is less than $4/9$ ths. We show that this leads to a contradiction.

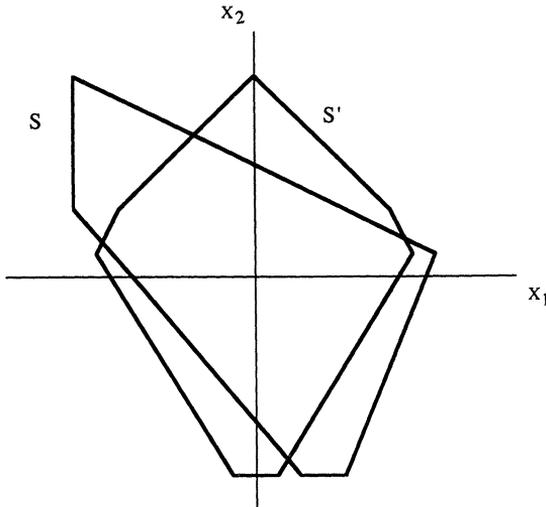


FIGURE 2(a)

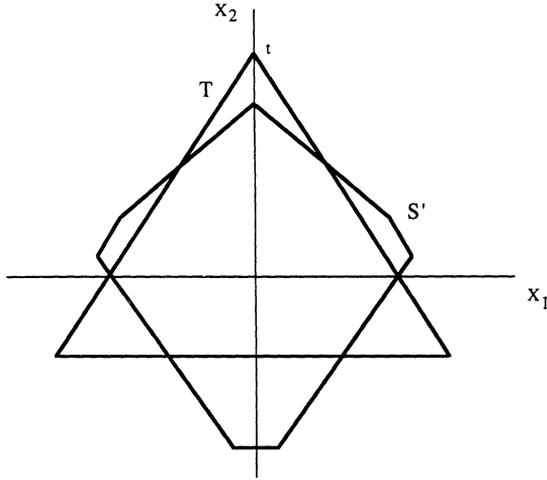


FIGURE 2(b)

First, derive a new set S' from S by the process of Schwartz symmetrization (Bonnesen and Fenchel (1934)). For any value $x_2 = k$, the length of the line segments in sets S and S' are the same. S' differs from S in that in S' all of these line segments are centered around $x_1 = 0$. Note that S' is convex and that the origin remains the centroid of S' . In addition, S' has the same area as S in both the half-spaces $x_2 > 0$ and $x_2 < 0$. This symmetrized figure is illustrated in Figure 2(a).

Using S' , construct a triangle T . The triangle agrees with S' on $x_2 = 0$ and is also symmetric around $x_1 = 0$. In addition, T has the same area as S' in both the half-spaces $x_2 > 0$ and $x_2 < 0$. Finally, as illustrated in Figure 2(b), T is pointed upwards: its upper vertex, t , on $x_1 = 0$ has $x_2 > 0$.

Note that in both half-spaces, $x_2 > 0$ and $x_2 < 0$, wherever T and S' do not overlap, T lies above S' . Hence, the centroid of T lies on $x_1 = 0$ (by symmetry) above the origin. Thus the area of S' in the half-space $x_2 > 0$ exceeds the area of T above its centroid. From Proposition 3, the area lying above the centroid of T is $4/9$ ths. This contradicts the assumption that the area of S with $x_2 > 0$ is less than $4/9$ ths.

To extend the two-dimensional result to n -dimensions is straightforward (see Grunbaum (1960), Hammer (1960)). The area on the smaller side of any hyperplane through the centroid of S is at least equal to the area above the centroid of an upward pointed n -dimensional cone. This area is $(n/(n+1))^n$.
Q.E.D.

Theorem 1 taken alone is of limited applicability. The assumption of a uniform distribution mandates perfect evenness within a set, falling to zero at the boundaries. It is important, therefore, to extend the results to less extreme cases.

Theorem 2 extends the bounds to general concave densities. However, unlike Theorem 1, the centroid no longer suffices to establish the necessary bound, as a simple example demonstrates.

EXAMPLE 1: Consider a one-dimensional problem with a triangular density of voters' most preferred points, $\{f(x) = 2(1 - x), x \in [0, 1]\}$. The centroid is at $x = 1/3$. Note that a point just to the left of $x = 1/3$ commands a $5/9$ ths majority against x . However, the median of the distribution, $x = 1 - (\sqrt{2/2})$, commands at least a 50% vote against any alternative. With $n = 1, 1/2$ and not $5/9$ ths is the desired bound on the min-max majority.

The problem arises because the centroid is calculated by weighting mass by distance, but distance appears to be an inessential feature of the problem. (Yet there is an interesting relationship between the min-max point and the centroid which is discussed in Comment 2 following Lemma 1 in the Appendix.)

Note that in Example 1, we get the two-dimensional bound for a one-dimensional problem. More generally, a concave density in n -dimensions gives rise to an $(n + 1)$ -dimensional convex set with uniform density, where the additional dimension represents the height of the density. This observation shows that the $(n + 1)$ -dimensional bound on $m^*(C)$ from Theorem 1 can still be applied to concave densities in n dimensions. But this is not the best bound available. Theorem 2 demonstrates that the bounds of Theorem 1 apply to general concave densities.

THEOREM 2: For a decision problem C in R^n satisfying A1–A3,

$$m^*(C) \leq 1 - \left(\frac{n}{n+1}\right)^n.$$

PROOF: Let x^* be the min-max point. We radially transform the set S around x^* into a new set with a uniform density. Construction 1 provides this corresponding set U . This transformation has three properties: (i) All hyperplanes through x^* divide the volume of U in the same proportion as they divide the population in S . (ii) U is a convex set. (iii) x^* is the min-max point of U . Properties (i) and (ii) are verified in Lemmas 3, 4, and 5. Property (iii) is proven in Lemma 6. Construction 1 and the lemmas appear in the Appendix.

With this construction, we have shown that x^* is also the min-max point of a convex set with uniform density. Theorem 1 applies directly to show that the fraction of the area of U on any side of a hyperplane through x^* is less than $1 - (n/(n + 1))^n$. Finally, according to (i) above, planes through x^* divide the volume of U in the same proportions that they divide the population density in S , proving the theorem. Q.E.D.

Theorem 2 provides a 50% bound for the one-dimensional problem, a $5/9$ ths bound or roughly 56% for two dimensions. The bound rises with dimension, converging to approximately 63.2%. Thus, the min-max point is a proposal supported by more than 36% of the population against any alternative.

COROLLARY: *For any decision problem, C, satisfying Assumptions A1–A3,*
 $m^*(C) < 1 - 1/e \approx 0.632.$

PROOF: $(n/(n + 1))^n$ falls monotonically to a limit of $1/e.$ *Q.E.D.*

Dimension by dimension the bounds of Theorem 2 are the best available. In two dimensions, the 5/9ths bound is attained by the triangle (Proposition 3). As noted by Grunbaum (1960) and Hammer (1960), the solid n -dimensional unit simplex provides the general worst case. The min-max point of the solid simplex is always found at its center of gravity, $x^* = [1/(n + 1), 1/(n + 1), \dots, 1/(n + 1)].$ This point captures $(n/(n + 1))$ of the area in each of n dimensions. Correspondingly, the min-max majority equals $(n/(n + 1))^n$ for the n -dimensional solid simplex.

5. EXTENSIONS

The results of the last section are robust to various changes in the underlying assumptions. A1–A3 are taken up and relaxed in turn.

A1. *Euclidean Preferences*

Central to our results is a simple property possessed by Euclidean preferences: those who prefer proposal x to proposal y can be separated by a hyperplane from those who prefer y to $x.$ With Euclidean preferences, this hyperplane is drawn in the space of most preferred points; it is the plane which perpendicularly bisects the line joining x and $y.$ Individuals with most preferred points on one side prefer $x,$ those on the other side prefer $y,$ and those on the plane itself are equidistant between and thus indifferent between x and $y.$

Grandmont (1978) observed that this “division-by-hyperplane” property of Euclidean preferences is central to results supporting simple majority rule. Applying Grandmont’s insight to the current study allows generalization of the results to the class of “intermediate preferences.”

The characteristic feature of intermediate preferences is that they can be parameterized so that supporters of opposing proposals can always be separated by a hyperplane. The class of intermediate preferences includes Constant Elasticity of Substitution (C.E.S.) utility functions, $U(x_1, x_2, \dots, x_n) = [a_1x_1^r + a_2x_2^r + \dots + a_nx_n^r]^{1/r}.$ These preferences may be parameterized by the vector a contained in the n -dimensional unit simplex, $S^n.$ ⁹ Here, those who prefer bundle x to bundle y can be separated by a hyperplane in the parameter space, $S^n.$

The C.E.S. utility function is used as an example to show how our results can be extended to the entire class of intermediate preferences. The translations of A2 and A3 are straightforward. First, the distribution of parameters, $f(a),$ must be concave over $S \subset S^n.$ Corresponding to A3, the choice set X is compact; this ensures that individuals have at least one most preferred point in $X.$ Proposition 4 extends the bounds to these cases.

⁹ The elasticity of substitution, $r,$ is held constant across society.

PROPOSITION 4: Consider a decision problem C defined as follows: individuals have C.E.S. utility functions; the distribution of types, $f(a)$, is concave over its support S , a convex subset of S^n , $n \geq 2$; X is compact. Then

$$m^*(C) \leq 1 - \left(\frac{n-1}{n} \right)^{n-1}.$$

PROOF: By Theorems 2 and 3, there exists a point a^* in S^n such that every hyperplane through a^* has at least an $((n-1)/n)^{n-1}$ fraction of the population on either side.¹⁰ Since X is compact, an individual of type a^* has a most preferred point z^* in X . Consider the vote for any proposal against z^* . Since type a^* weakly prefers z^* , the set preferring z^* contains all points on one side of a hyperplane through a^* . Hence, $m(z^*) < 1 - ((n-1)/n)^{n-1}$, proving the result since $m^*(C) \leq m(z^*)$. Q.E.D.

This argument which depends only on division of sets by hyperplanes extends directly to the general class of intermediate preferences and beyond.¹¹ This extension is especially important in locational models of industrial organization. In the framework of Gorman (1980) and Lancaster (1966), consumers rank products according to their characteristics. Preferences over the space of characteristics are more sensibly treated as C.E.S. (notably, Cobb-Douglas) than as Euclidean. Extension to C.E.S. utility functions leads to a multi-dimensional model of sequential product entry (Caplin and Nalebuff (1986)) in the tradition of Hay (1976) and Prescott and Visscher (1977).

A2. Concavity

Three relaxations of concavity are considered in turn. The results are first extended to functions close to concave. In this case, the limits on the min-max majority remain close to our previous bounds. More surprisingly, our results change dramatically when general quasi-concave densities are allowed. Finally, we consider finite populations drawn from a concave density. We show that as the sample population increases, the min-max majority converges almost surely to its nonatomic limit value.

To extend the bounds to functions close to concave, we use the L_1 norm. This norm provides one possible measure of distance between two integrable functions, $f(x)$ and $g(x)$, defined on R^n ,

$$\rho(f, g) \equiv \int_{R^n} |f(x) - g(x)| dx.$$

¹⁰ One dimension is gained because of the restriction to the simplex.

¹¹ Proposition 2 applies to all families of convex preferences for which those supporting "neighboring" propositions can be separated by a hyperplane: these are called local intermediate preferences (see Demange (1982)). In two dimensions, this allows our results to be generalized from preferences based on the Euclidean norm to preferences based on any norm.

PROPOSITION 5: Let $f(x) \geq 0$ on $x \in R^n$ satisfy A2, and let $g(x)$ be an arbitrary probability density on R^n . Define $F \equiv \int f(x) dx$. Then,

$$m^*(C) \leq \left[1 - \left(\frac{n}{n+1} \right)^n \right] F + \rho(f, g).$$

PROOF: It follows directly from Theorem 2 that for the function $f(x)$ there is a point z^* with no more than a fraction $1 - (n/(n+1))^n$ of F on either side of any hyperplane through z^* . Using point z^* with function g , the most that can be added on any side is $\rho(f, g)$. Thus, for function g ,

$$m^*(C) \leq \left[1 - \left(\frac{n}{n+1} \right)^n \right] F + \rho(f, g). \tag{Q.E.D.}$$

Proposition 5 verifies that for functions nearly concave, the previous bounds can be applied with little alteration. For example, in two dimensions, a truncated normal density (centered in the positive orthant) is close to concave; adjusting the bound according to Proposition 5 shows that $m^*(C) < 5/9[0.925] + 0.075 \approx 0.6$.¹²

Success with the normal suggests consideration of general quasi-concave densities. Quasi-concavity is a weaker alternative to concavity as a characterization of social consensus. By definition, for any two proposals x and y in the support, $f[\lambda x + (1 - \lambda)y] \geq \min[f(x), f(y)]$, $0 \leq \lambda \leq 1$. Proposition 6 illustrates the lack of power in the assumption of quasi-concavity.

PROPOSITION 6: There are decision problems C in R^n satisfying A1 and A3 with $f(x)$ quasi-concave on X for which

$$m^*(C) = \left(\frac{n}{n+1} \right).$$

PROOF: A series of examples generated inductively prove the proposition. Note that while the examples below use atomic densities, the extension to nonatomic densities is direct.¹³

¹² To establish the bound, we linearize the truncated normal at points of inflection.

¹³ Consider the nonatomic density function $f(x_1, x_2)$ defined on the unit square:

$$f(x_1, x_2) = \begin{cases} \frac{2}{3}, & x_1 + x_2 \geq a, \\ \frac{2}{3} + \frac{2}{3a^2}, & x_1 + x_2 < a. \end{cases}$$

The min-max majority can be brought close to $\frac{1}{3}$ by choosing a to be sufficiently small. Specifically, $m^*(C) \leq \frac{1}{3} + \mu$, where $\mu = \frac{1}{3}a^{2/3}$. To see this, note that all points are in one of the three regions: (I) $x_1 + x_2 < \sqrt{3}\mu$, (II) $x_2 - [1 - (a/\sqrt{3}\mu)]x_1 \geq a$, or (III) $x_1 - [1 - (a/\sqrt{3}\mu)]x_2 \geq a$. The population density contained in region I is $\frac{1}{3} + \frac{2}{3}(3\mu/2) = \frac{1}{3} + \mu$. The total population in each of regions II and III is bounded above by $(2/3)[(1 - a) + (a/\sqrt{3}\mu) - a]/2 < \frac{1}{3} + (a/\sqrt{27}\mu) = \frac{1}{3} + \mu$.

In two dimensions consider the quasi-concave density function $f(x_1, x_2)$ defined on the unit square:

$$f(x_1, x_2) = \begin{cases} \frac{2}{3}\delta(1), & x_1 = x_2 = 1; \\ \frac{2}{3}, & \text{elsewhere,} \end{cases}$$

where $\delta(1)$ is the Dirac-delta function. For any point in the unit square, there exists a line through that point with no more than $1/3$ of the mass on the smaller side. This bound is tight for the point $(1, 1)$. For any other point, x , there is a line through x where the larger side contains both $(1, 1)$ and arbitrarily close to half of the unit square. Hence, $m^*(C) = 2/3$.

This example directly extends to higher dimensions. In R^3 , let $(3/4)f(x_1, x_2)$ be applied to the bottom face of the unit cube with the remaining $1/4$ mass spread uniformly over the entire cube. It is readily verified that $m^*(C) = 3/4$. Generally, in n dimensions, $(n/(n+1))$ of the previous density is applied to one $(n-1)$ -dimensional face of the unit hypercube, and the remaining $1/(n+1)$ mass is distributed uniformly over the hypercube. This generates an n -dimensional quasi-concave density function such that $m^*(C) = (n/(n+1))$. *Q.E.D.*

Proposition 6 implies that the only dimension-free bound on the min-max majority for quasi-concave densities is 1, unanimity. Together with our earlier results, this shows that the restriction to concavity is increasingly important in higher dimensions. In one dimension, the bound is $1/2$ for both the concave and the quasi-concave cases. In the plane, the bounds are $5/9$ and $2/3$ respectively, in R^3 , $37/64$ and $3/4$. The difference between these bounds grows monotonically with n from 0 to 0.11 to 0.17 to a limiting value of $1/e$.

The quasi-concave bound of $(n/(n+1))$ in fact provides a universal upper-bound. For any n -dimensional decision problem, $m^*(C) \leq (n/(n+1))$. With intermediate preferences, the universal nature of this bound follows from Lemma 1 in the Appendix. Even the restriction to intermediate preferences is unnecessary; the bounds of $(n/(n+1))$ hold for arbitrary convex preferences (Greenberg (1979), Coughlin (1981)).

We now consider decision problems with a finite population. The most preferred points are independently drawn from a probability density $f(x)$, where $f(x)$ satisfies A2. In the limit as the population increases, the min-max majority converges to $m^*(C)$, its value for the limit decision problem.

The sample space is the set Ω of all infinite sequences of most preferred points $x \in S$. A realization $\omega \in \Omega$ involves the sequence of values $(x_1(\omega), \dots, x_k(\omega), \dots)$ where $x_k(\omega)$ represents the most preferred point for individual k in the given realization. Given ω , consider the population consisting of the first k realized values, $(x_1(\omega), \dots, x_k(\omega))$. Note that the min-max majority for this population is uniquely defined (this is the finite case originally studied by Simpson and Kramer) and is denoted by $m_k^*(\omega)$. Theorem 3, proven in the Appendix, confirms that with independent draws, the sample min-max majority converges almost surely to its limiting value. Hence, the bounds of the paper extend to large finite populations drawn from a concave density.

THEOREM 3: *For a decision problem C in R^n satisfying Assumption A1–A3,*

$$m_k^*(\omega) \rightarrow m^*(C) \quad a.e.$$

A3. Inclusivity

Until now, we have assumed A3 (Inclusivity), that there is a continuum of alternative proposals. Inclusivity was used in the proof of Proposition 2 to approach interior points of S from all directions. In addition, inclusivity guarantees that the min-max point is a possible choice.

Yet, political decisions rarely involve more than a few alternatives. The restricted set of choices may not include the unrestricted min-max point, in which case our existing proofs break down. It is then important to show that all the results extend to cases in which X is finite, thus violating A3.

Proposition 7 demonstrates that the previous bounds on the min-max majority still apply provided only that the choice set, X , is compact. The proof uses the proposal in X closest to the unrestricted min-max point.

PROPOSITION 7: *Consider a decision problem C' satisfying A1 and A2 with X a compact set. Let C be the same problem with X extended to include S , $C = \{X \cup S, >, f\}$. Then, $m^*(C') \leq m^*(C)$.*

PROOF: Consider the point $y \in X$ closest to x^* , the min-max point for problem C . (If two or more points in X are equidistant from x^* , then any one of them can be selected.) At least a proportion $[1 - m^*(C)]$ favors y against any alternative in X . If not, those favoring the alternative must strictly include x^* (since x^* is interior to all half-spaces containing more than fraction $m^*(C)$ of the population). But, this yields a contradiction: no point in X is closer to x^* than is the point y . *Q.E.D.*

This result is related to Kramer's (1977) demonstration that with intermediate preferences, the partial ordering defined by the min-max majority is acyclic. If proposal A is supported by more than $m^*(C)$ against proposal B , then A must also be closer to the global min-max point. A cycle would contradict the fact that each vector is successively closer to the min-max point.

6. CONDORCET ON VOTING

In this section, we suggest an interpretation of Condorcet's proposal for an electoral system immune to his paradox of voting. We show that this interpretation is closely related to the Simpson-Kramer min-max rule.

Condorcet argues that an electoral procedure should be based on pairwise comparison of proposals to avoid other paradoxes arising with plurality voting. He then demonstrates that majority votes in pairwise competition may lead to a "contradictory system". When this contradiction arises, Condorcet suggests that

a vote decided by a large majority should take precedence over a vote decided by a small majority: a vote of 90 : 10 counts for more than a vote of 51 : 49.

“The preceding reflections suggest this general rule: that whenever it is essential to make the election, it is necessary to take successively all the propositions that have a majority, beginning with those possessing the largest. As soon as these first propositions produce a result, it should be taken as the decision, without regard for the less probable decisions that follow” (Condorcet (1785, p. 56)).

With this procedure, only majorities above some critical size are counted in the social ordering. A result is achieved when this (partial) ordering produces a proposal which is both undominated, and dominates directly or indirectly all other proposals. The following definition formalizes this interpretation of Condorcet’s idea; other interpretations are offered in Black (1953).

DEFINITION 3—CONDORCET’S RULE: (i) For a given δ , consider all pairwise comparisons in which the winner receives more than $\delta\%$ of the vote. (ii) When these comparisons give rise to an acyclic partial ordering $[\succ_{\delta}]$, define $\{\succ_{\delta}\}$ to be its transitive closure. (iii) If possible, pick δ^* so that $\{\succ_{\delta^*}\}$ has a dominant element. Implement this proposal.

An example based on Condorcet (p. 55, para. 3) illustrates this procedure.

EXAMPLE 2: There are three proposals A , B , and C . In pairwise comparisons, 75% prefer A to B ; 65% prefer B to C ; and 60% prefer C to A .¹⁴ Counting only majorities above 60% yields the partial ordering $A[\succ_{60}]B, B[\succ_{60}]C$. In the transitive closure, A dominates both B and C . A is then the proposal selected by Condorcet’s rule.

Condorcet’s rule is closely related to the Simpson-Kramer min-max rule. Both are based on giving priority to elections determined by the largest majorities. However, rather than using the largest majority elections directly to generate an ordering, the min-max rule uses these elections to eliminate losing propositions. The min-max point is then determined when all but one of the proposals are beaten. Returning to Example 2, B is beaten 75 to 25 and thus is eliminated first. Next, C is eliminated (with a vote of 65 to 35 against). Using a 60%-majority, only A remains as an undominated element.

In Example 2, both the Condorcet rule and the min-max rule lead to proposal A . Theorem 4 establishes that this equivalence holds more generally.

THEOREM 4: *For a decision problem C satisfying $A1$ and $A3$, Condorcet’s rule selects the min-max point x^* .*

¹⁴ Preferences generating this voting pattern are $\{A \succ B \succ C\}$ for 40% of the voters $\{B \succ C \succ A\}$ for 25% of the voters, and $\{C \succ A \succ B\}$ for 35% of the voters. Note that these preferences can arise in a two-dimensional decision problem with Euclidean preferences although the density of most preferred points is not concave.

PROOF: Under A1 and A3, the min-max point is the unique element undominated according to the partial ordering $[>_{m^*(C)}]$ and thus also for its transitive closure $\{>_{m^*(C)}\}$. No point other than x^* could be chosen by Condorcet's rule: for majority sizes $\delta > m^*(C)$, x^* is undominated; for majority sizes $\delta < m^*(C)$, all points are dominated. Thus it remains to show that according to $\{>_{m^*(C)}\}$, x^* is a dominant proposal, i.e., for any given proposal z other than x^* there exists a finite list of J proposals z_1 through z_J such that

$$x^* [>_{m^*(C)}] z_1 [>_{m^*(C)}] z_2 [>_{m^*(C)}] \cdots [>_{m^*(C)}] z_J [>_{m^*(C)}] z.$$

This is demonstrated by Lemmas 7 and 8 in the Appendix. *Q.E.D.*

Theorem 4 has striking implications. For example, with a min-max majority of 60%, the min-max point is guaranteed at least 40% support against any alternative. Theorem 4 turns this statement around. The min-max point actually beats every other proposal through a sequence of votes in which the winning proposition receives more than 60% support. For smaller min-max majorities the margin of victory available to the min-max point actually increases. If the min-max point can only be guaranteed a 1% vote, then it defeats all other proposals, directly or indirectly, by a 99 to 1 vote.

The difficulty with Condorcet's rule is that without A1 it may fail to produce a result. There may be no majority size v^* such that $\{>_{v^*}\}$ has a unique dominant element. A cycle can arise before the appearance of a dominant element, as in Example 3.

EXAMPLE 3: There are 4 proposals, A , B , C , and D . In pairwise comparisons, 75% prefer A to B , 65% prefer B to C , and 60% prefer C to A . D receives 45% of the vote against each of A , B , and C .¹⁵ The partial ordering $[>_v]$ is acyclic only for $v > 60\%$. However, these orderings fail to produce a dominant element. Thus, Condorcet's rule cannot be applied to the example.

In contrast to Condorcet's rule, the min-max rule is always well-defined: in Example 3, D is chosen. Theorem 4 together with this example suggest that the Simpson-Kramer min-max rule is a fully consistent analogue to Condorcet's rule.

7. 64%-MAJORITY RULE AS AN ELECTORAL SYSTEM

This section considers the properties of a voting constitution based on a δ -majority rule. The constitution fixes a majority size, δ , which will then be applied to a large number of decision problems.¹⁶ Different decisions will typically have different min-max majorities so that the fixed value of δ will not

¹⁵ Preferences generating this voting pattern are $\{A > B > C > D\}$ for 22% of the voters, $\{B > C > A > D\}$ for 13.75% of the voters, $\{C > A > B > D\}$ for 19.25% of the voters, $\{D > A > B > C\}$ for 18% of the voters, $\{D > B > C > A\}$ for 11.25% of the voters, and $\{D > C > A > B\}$ for 15.75% of the voters. These preferences can arise in a 3-dimensional decision problem with Euclidean preferences.

¹⁶ Ferejohn et al. (1980) discuss properties of related voting schemes.

coincide with the min-max majority for each specific decision problem. As the pre-specified majority size is increased, there will be more decision problems with $\delta > m^*(C)$ and fewer with $\delta < m^*(C)$. To evaluate the tradeoff between high and low values of δ , we examine in turn the different issues that arise when $\delta < m^*(C)$ and when $\delta > m^*(C)$.

Serious problems may arise when a δ -majority below the min-max majority is used. McKelvey (1979) shows that use of a 50%-majority rule when no simple majority winner exists (i.e. $m^*(C) > 1/2$) leads to global cycles. A sequence of majority votes can lead from any given proposal to any other proposal: whoever controls the agenda controls the outcome. The same problem occurs even with voting under a super-majority rule. For example, global cycles arise with the use of a 60%-majority rule when no 60%-majority winner exists (i.e. $m^*(C) > 0.6$). Generally, McKelvey's (1979) arguments apply to prove existence of global cycles whenever the majority size δ is less than the min-max majority.

No voting cycles are possible when δ is larger than the min-max majority.¹⁷ Instead, there are many δ -majority winners and once any of them becomes the status quo it can never be replaced. This results in an indeterminacy; many different positions are unbeatable as the status quo. The significance of this indeterminacy depends on the size of the set of δ -majority winners. Theorem 5 demonstrates that the set of δ -majority winners shrinks to zero in a uniform manner as δ falls toward $m^*(C)$. Hence, indeterminacy may not be disturbing when δ is close to $m^*(C)$ as the set of δ -majority winners is small and its elements are all very similar.

Definition 2 provides a measure of indeterminacy for a given δ -majority rule when $\delta > m^*(C)$. The set of δ -majority winners is measured according to its population.

DEFINITION 2: Consider a decision problem, C , satisfying Assumptions A1–A3, and $\varepsilon \geq 0$. Define $\delta \equiv m^*(C) + \varepsilon$. Let $W(\delta)$ denote the nonempty set of δ -majority winners in X . The indeterminacy of the δ -majority rule is defined by

$$I(C, \varepsilon) \equiv \int_{W(\delta)} f(x) dx.$$

$I(C, \varepsilon)$ measures the proportion of the population whose most preferred points are δ -majority winners when $\delta = m^*(C) + \varepsilon$. At $\varepsilon = 0$, $I(C, 0) = 0$ since the min-max point is a singleton and $f(x)$ is nonatomic. Theorem 5, proven in the Appendix, demonstrates uniform convergence of $I(C, \varepsilon)$ to zero as ε shrinks to 0.

THEOREM 5: For a decision problem C in R^n satisfying Assumptions A1–A3,

$$I(C, \varepsilon) \leq \varepsilon \left(\frac{n+1}{n} \right)^n.$$

¹⁷ To avoid cycles, the min-max majority is calculated treating the entire Pareto optimal set S as part of the choice set (see Proposition 7). Acyclicity also requires intermediate preferences.

In the proof, we show that the set of δ -majority winners is contained in a simplex with no more than $\varepsilon((n+1)/n)^n$ of the total population. This has the additional implication that for small ε , all of the δ -majority winners are close together. Since all winning propositions are similar, the issue of which one is chosen becomes less significant.

The bounds of Theorem 5 may not be the best available, especially for decision problems in higher dimensions. In the present context, we are especially interested in the potential indeterminacy of a 64%-majority rule. Example 4 considers decision problems where a simple majority winner always exists, so that $m^*(C) = 0.5$ and $\varepsilon = 0.14$. For these examples we show that the indeterminacy of a 64%-majority rule shrinks rapidly with the dimension of the decision problem. The measure of indeterminacy falls from 28% with $n = 1$ to below 1% for $n > 4$.

EXAMPLE 4: Consider a decision problem C in R^n with $f(x)$ uniform over the solid n -dimensional cube. Because this density is radially symmetric, $m^*(C) = 1/2$. Here the indeterminacy of 64%-majority rule is $I(C, 0.14) \leq (0.28)^n$. This bound is established by considering the planes parallel to the outer faces of the cube. Shift each of the $2n$ faces inward until 36% of the density lies between it and the outer face. Any point in this outer region can be beaten using a 64%-majority rule. The area remaining between these planes is $(0.28)^n$.

Return to the problem of fixing a voting rule. The tradeoff in choosing δ before knowing C is that $\delta < m^*(C)$ results in global cycles while $\delta > m^*(C)$ leads to indeterminacy. The results of this section highlight an asymmetry in this trade-off. Global cycles exist for any $\delta < m^*(C)$. In contrast, the indeterminacy disappears continuously as δ falls to $m^*(C)$.¹⁸

It may be judged that avoiding global cycles (and the resulting possibilities for agenda control) is more important than avoiding a small amount of indeterminacy. This suggests fixing the majority size above rather than below the average value of $m^*(C)$. Without Assumptions A1–A3, a unanimity rule may be needed to avoid electoral cycles (Greenberg (1979)) leading to complete indeterminacy. The value of Theorem 2 is to provide a 64% bound on $m^*(C)$ applicable to a wide variety of problems. For any social decision problem satisfying A1–A3, a 64%-majority rule avoids electoral cycles and may lead to only minor indeterminacy.

8. CONCLUSION

Many electoral rules require a super-majority vote to change the status-quo. Without some restriction on preferences, super-majority rules have paradoxical properties. For example, electoral cycles are possible with anything other than 100%-majority rule. It has long been suspected that these problems will not arise if there is sufficient similarity of attitudes among the voting population.

¹⁸ However, it may be that as δ approaches $m^*(C)$ from below, the number of steps needed to complete a voting cycle grows without bound. This would make the discontinuity less dramatic.

We introduce a definition of social consensus which involves two restrictions on domain: one on individual preferences, the other on the distribution of preferences. When this consensus exists, 64%-majority rule has many desirable properties including the elimination of all electoral cycles. Given the widespread acceptance of voting in democratic societies, it is particularly pleasing to present positive results.

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APPENDIX

Lemma 1 serves two roles. It provides a simple proof of Greenberg's (1979) bound on the min-max majority for the special case of Euclidean preferences. More importantly, the lemma plays an integral role in the proofs of Theorems 2, 3, and 5.

Some shorthand notation will be useful. Any hyperplane H divides the space X into larger and smaller closed half-spaces, H^L and H^S . By larger, we mean that the half-space H^L contains more of the population than H^S . In the event of an even split, the naming is arbitrary.

LEMMA 1: *For a decision problem C in R^n satisfying A1–A3, there exists a family of $(n + 1)$ hyperplanes $H = \{H_1, H_2, \dots, H_{n+1}\}$ with properties (i)–(iii): (i) All planes H_i pass through x^* ; (ii) all half-spaces H_i^L contain precisely fraction $m^*(C)$ of the total population; (iii) every point in R^n is contained in one of the smaller half-spaces, $\bigcup_{i=1}^{n+1} H_i^S = R^n$.¹⁹*

Properties (i)–(iii) imply that $m^(C) \leq n/(n + 1)$.*

PROOF: Consider the family F of all hyperplanes satisfying conditions (i) and (ii) with typical element $H_f \in F$. We first show that

$$(1) \quad \text{Interior} \left[\bigcap_{H_f \in F} H_f^L \right] = \emptyset.$$

To prove (1), assume to the contrary that there exists a point $y \in \text{Int}[\bigcap H_f^L]$. Consider all half-spaces defined by hyperplanes through x^* which exclude y from their interior. These can be identified by the set of unit normals, Π , passing through x^* and pointing away from y :

$$\Pi \equiv \{ \pi \in R^n \mid \pi \cdot \pi = 1 \text{ and } \pi \cdot (y - x^*) \leq 0 \}.$$

Note that Π is a compact set. With A2, the population in any half-space is a continuous function of π . Thus there exists a maximal population m' among the half-spaces identified by Π . But any half-space in Π has a population strictly less than $m^*(C)$ since y lies interior to all half-spaces H_f^L with population $m^*(C)$. Hence $m' = m^*(C) - h$ for some $h > 0$.

Now consider a point z on the line segment (y, x^*) which is close enough to x^* for the population between all parallel hyperplanes through x^* and z to be below h .²⁰ We show that no hyperplane through z has population more than $m^*(C)$ on either side: $m(z) \leq m^*(C)$. Note that no half-space defined by a hyperplane through z contains both x^* and y in its interior (since x^* , z , and y are colinear). If x^* is not in the interior, then the half-space is a subset of a half-space defined by a plane through x^* ; hence its population is no more than $m^*(C)$. If y is not in the interior, then by construction the most added to the constrained optimum of m' is h . Again, the population is no more than $m' + h = m^*(C)$. This proves $m(z) \leq m^*(C)$.

¹⁹ Our proof assumes that the set X is of full dimension. More generally, the proof holds for $n = \dim(X)$, which strengthens the result.

²⁰ The existence of such a point z follows from the continuity and boundedness of full-dimensional nonatomic concave densities.

But $m(z) < m^*(C)$ is inconsistent with x^* being the min-max point. And $m(z) = m^*(C)$ implies that z is also a min-max point which is inconsistent with the uniqueness of x^* (see Demange (1982)). This contradiction rules out the existence of any point $y \in \text{Int}[\cap H_i^L]$ establishing equation (2).

Application of Helly's theorem on the intersection of convex sets (see Rado (1946)) to equation (2) implies that there exists an $(n + 1)$ member subfamily of F such that the intersection of their larger sides also has empty interior. For this $(n + 1)$ member subfamily, the union of the smaller half-spaces is R^n , completing the proof of (i)-(iii).

Finally, since $\cup_{i=1}^{n+1} H_i^S \supset X$, the population in H_i^S must be at least $1/(n + 1)$ of the total. *Q.E.D.*

COMMENT 1: With slight amendments Lemma 1 could also provide a direct argument for the existence and uniqueness of the min-max point.

COMMENT 2: The H_i are the hyperplanes through x^* which maximize the population in their larger half-space. The first-order condition for this maximization is simply that x^* is the centroid of H_i . The min-max point is the centroid of each hyperplane in this $(n + 1)$ -member family. A good approximation for this point may be the centroid of the entire density. This may make the role of the centroid in Theorem 1 less surprising.

PROPOSITION 1: Consider a decision problem C in R^n satisfying A1 together with m distinct proposals:

(a) For $m \leq n + 1$, no value restriction is implied (provided that the m proposals span a space of dimension $m - 1$); all $m!$ ordinal rankings can coexist.

(b) For $m > n + 1$, value restrictions are implied; some of the $m!$ orderings of the alternatives are ruled out.

PROOF: (a) To generate the ordering $\{x_1 > x_2 > \dots > x_m\}$, for example, consider the set of points contained in the intersection of the following half-spaces: the x_1 side of the hyperplane perpendicularly bisecting x_1 and x_2 ; the x_2 side of the hyperplane perpendicularly bisecting x_2 and x_3 ; ... and so on. It must be shown that under the hypothesized conditions, this set is nonempty. A standard sufficient condition for nonemptiness of a set defined by m linear inequalities is that the gradient vectors be linearly independent. In this case, the gradient vectors are $(x_2 - x_1), (x_3 - x_2), \dots, (x_m - x_{m-1})$. But, these vectors must be linearly independent; otherwise the vectors $(x_2 - x_1), (x_3 - x_1), \dots, (x_m - x_1)$ would also be linearly dependent, contradicting the dimensionality assumption.

(b) The proof is based on a theorem of Greenberg (1979, Theorem 2). A special case of Greenberg's result holds that under A1 when the choice set X is n -dimensional, there exists a proposal in X supported by at least $1/n + 1$ of the population against any alternative. This result implies that some of the following m orderings must be excluded:

$$(x_1 > x_2 > \dots > x_m); (x_2 > x_3 > \dots > x_m > x_1); \dots; (x_m > x_1 > x_2 > \dots > x_{m-1}).$$

Otherwise, consider a population of m individuals with these preference orderings. Note the $m - 1$ of the individuals prefer x_2 to x_1 , $m - 1$ prefer x_3 to x_2 , and so on. Thus no proposal gains the support of more $1/m$ of the population against all alternatives, contradicting Greenberg's theorem. *Q.E.D.*

CONSTRUCTION 1: U is a set with uniform (unit) density derived from S by a radial transformation around x^* . The transformation is constructed in such a way that it preserves mass in all regions which are unions of half-lines emanating from x^* . In particular, the transformation preserves the mass on either side of hyperplanes through x^* .

To construct the set U , draw the line from x^* extending in direction π to the point $b_\pi \in Bd(S)$, the boundary of S . The mass along this line segment is calculated using n -dimensional spherical integration around x^* . Here, the density at distance d is weighted by d^{n-1} ; for example in spherical integration (R^3) the density is weighted by d^2 .

Without loss of generality take x^* to be the origin. Let μ_π equal the distance to b_π , $\mu_\pi = \|b_\pi\|$. Define A_π to be the mass of f along $[O, b_\pi]$, calculated using spherical integration around the origin;

$$(2) \quad A_\pi \equiv \int_0^1 (r\mu_\pi)^{n-1} f(rb_\pi) \mu_\pi dr.$$

To construct a set with unit density while preserving the mass in direction π requires extending (or contracting) $Bd(S)$ by a factor α_π , where

$$(3) \quad \alpha_\pi \equiv \frac{(nA_\pi)^{1/n}}{\|b_\pi\|},$$

so that

$$(4) \quad A_\pi = \int_0^1 (r\alpha_\pi \mu_\pi)^{n-1} \alpha_\pi \mu_\pi dr.$$

Define U to be the set which in each direction π away from 0 has length $\alpha_\pi \|b_\pi\|$. By construction, spherical integration shows that the mass in the half-space defined by any hyperplane through 0 is the same for the set S with density $f(x)$ as for the set U with unit density. Q.E.D.

LEMMA 3: *The set U defined in Construction 1 is convex.*

PROOF: Without loss of generality, take x^* to be the origin. Consider two points b_1 and b_2 on the boundary of set S . Since S is convex, the point $x_\lambda = \lambda b_1 + (1 - \lambda)b_2$ is contained in S , for $0 \leq \lambda \leq 1$. In the construction of the new set U , the ray joining 0 to b_1 is extended by factor α_1 and the ray joining 0 to b_2 is extended by factor α_2 . No generality is lost setting $\alpha_1 = 1$ and removing the subscript from α_2 , $\alpha = \alpha_2/\alpha_1$. To prove that U is convex it suffices that the expansion factor α_λ applicable to the line in S joining 0 and x_λ is large enough to extend the point x_λ beyond the line connecting b_1 to αb_2 . By Lemma 4 below, this will be the case provided that α_λ satisfies inequality (5):

$$(5) \quad \alpha_\lambda \geq \alpha / [\alpha\lambda + 1 - \lambda].$$

But, according to Lemma 5 below, $\alpha_\lambda \geq \lambda + (1 - \lambda)\alpha$. Thus, to complete the proof, we need only show that,

$$(6) \quad [\lambda + (1 - \lambda)\alpha] \geq \alpha / [\alpha\lambda + 1 - \lambda], \quad 0 \leq \lambda \leq 1 \quad \text{and} \quad 0 \leq \alpha.$$

Multiplying both sides of (7) by $[\alpha\lambda + 1 - \lambda]$ and collecting terms shows that this inequality is equivalent to $\lambda(1 - \lambda)(\alpha - 1)^2 \geq 0$, completing the proof of Lemma 3. Q.E.D.

LEMMA 4: *If $\alpha_\lambda \geq \alpha / [\alpha\lambda + 1 - \lambda]$, then the set U is convex.*

PROOF: Consider, α_λ^* , the minimal value of α_λ which still ensures that the transformed line in U between 0 and x_λ extends beyond the line joining b_1 and αb_2 . The derivation of α_λ^* is illustrated in Figure 3. The line joining b_1 to αb_2 intersects the ray extending from 0 to x_λ at point $\alpha_\lambda^* x_\lambda$ where

$$(7) \quad \alpha_\lambda^* x_\lambda = \tau b_1 + (1 - \tau)\alpha b_2,$$

for some $\tau, 0 \leq \tau \leq 1$. Substituting in the value of $x_\lambda = \lambda b_1 + (1 - \lambda)b_2$ demonstrates that $\tau = \lambda \alpha_\lambda^*$ and as required, $\alpha_\lambda^* = \alpha / [\alpha\lambda + 1 - \lambda]$. Q.E.D.

LEMMA 5: $\alpha_\lambda \geq [\lambda + (1 - \lambda)\alpha^n]^{1/n} \geq \lambda + (1 - \lambda)\alpha, 0 \leq \lambda \leq 1$ and $0 \leq \alpha$.

PROOF: To prove the first half of the inequality, note that along any ray from the origin, concavity of the density $f(x)$ implies

$$(8) \quad f(rx_\lambda) \geq \lambda f(rb_1) + (1 - \lambda)f(rb_2), \quad r \geq 0.$$

Substituting this inequality into the definitions of A_λ and α_λ in Construction 1 shows

$$(9) \quad \alpha_\lambda \geq [\lambda + (1 - \lambda)\alpha^n]^{1/n}.$$

To demonstrate the second half of the inequality, consider the function

$$(10) \quad H(\lambda, \alpha) \equiv \{\lambda + (1 - \lambda)\alpha^n\} - \{\lambda + (1 - \lambda)\alpha\}^n.$$

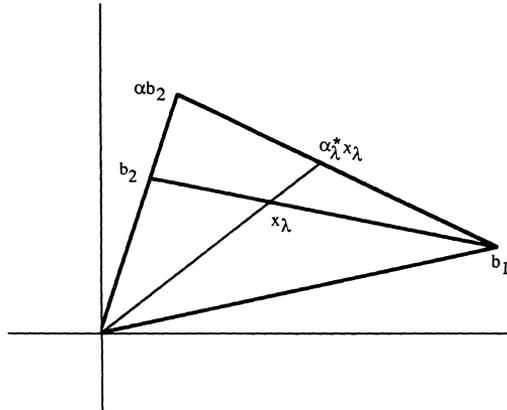


FIGURE 3

Taking the second partial derivative of $H(\lambda, \alpha)$ with respect to λ yields

$$(11) \quad \frac{\partial^2 H(\lambda, \alpha)}{\partial \lambda^2} = -(n)(n-1)(1-\alpha)^2(\lambda + (1-\lambda)\alpha)^{n-2} \leq 0, \text{ for } n \geq 1.$$

Thus H is weakly concave with respect to λ . Inspection verifies that $H(0, \alpha) = H(\lambda, \alpha) = 0$ for $\alpha \geq 0$. Combining this with concavity shows that $H(\lambda, \alpha) \geq 0$, for $0 \leq \lambda \leq 1$ and $\alpha \geq 0$. *Q.E.D.*

LEMMA 6: *The min-max point of the set U is x^* .*

PROOF: Since all hyperplanes through x^* divide the areas of U and S in the same proportion, the maximal vote against x^* in U is also preserved and equals $m^*(C)$. In addition, we know from Lemma 1 that for the original decision problem there are $n + 1$ hyperplanes through x^* containing exactly $m^*(C)$ on their larger side. All points of R^n are contained in the smaller side of one of these $n + 1$ hyperplanes. Construction 1 preserves these properties. For any point $u \in U$, therefore, one of these $n + 1$ hyperplanes will contain even less than $1 - m^*(C)$ on its smaller side, confirming the Lemma. *Q.E.D.*

THEOREM 3: *For a decision problem C in R^n satisfying A1–A3,*

$$m_k^*(\omega) \rightarrow m^*(C) \text{ a.e.}$$

Before presenting the proof, we must provide additional definitions and preliminary remarks. Pick a proposition $z \in S$ and a gradient vector π , normalized to lie on the n -dimensional unit ball, B^n . We define four functions as follows:

$$(12a) \quad I_k(z, \pi, \omega) = \begin{cases} 1, & \text{if } \pi \cdot x_k(\omega) > \pi \cdot z; \\ 0, & \text{if } \pi \cdot x_k(\omega) \leq \pi \cdot z; \end{cases}$$

$$(12b) \quad m_k(z, \pi, \omega) = \frac{1}{k} \sum_{j=1}^k I_j(z, \pi, \omega);$$

$$(12c) \quad m_k(z, \omega) = \max_{\pi \in B^n} m_k(z, \pi, \omega);$$

$$(12d) \quad m_k^*(\omega) = \min_{z \in S} m_k(z, \omega).$$

The function $m_k(z, \pi, \omega)$ represents the proportion of the first k observations lying strictly above the hyperplane through z with normal π , while $m_k(z, \omega)$ represents the maximal proportion across all hyperplanes through z and $m_k^*(\omega)$ represents the minimum such value across $z \in S$.²¹

As in Proposition 2, we know that $m_k(z, \omega)$ also defines the maximal vote that any proposition receives against z , and that $m_k^*(\omega)$ defines the min-max majority for a population of size k with preferences $\{x_1(\omega), \dots, x_k(\omega)\}$.

For a given pair (z, π) , note that $m_k(z, \pi, \omega)$ is a random variable, since it is a sum of measurable indicator functions. In addition, the indicator functions, $I_k(z, \pi, \omega)$, are independent with common distribution, taking the values 1 and 0 with probability $m(z, \pi)$ and $1 - m(z, \pi)$, where

$$(13) \quad m(z, \pi) \equiv \lim_{\lambda \downarrow 0} m(z, z + \lambda\pi),$$

with $m(z, z + \lambda\pi)$ as in Definition 2(a).²² Hence, the strong law of large numbers (see e.g., Chung (1974, Theorem 5.12)) applies to show that

$$(14) \quad m_k(z, \pi, \omega) \rightarrow m(z, \pi) \quad \text{a.e.}$$

We can assert, therefore, that for any finite list of pairs $(z_1, \pi_1), \dots, (z_K, \pi_K)$

$$(15) \quad \sup_{1 \leq k \leq K} |m_k(z_k, \pi_k, \omega) - m(z_k, \pi_k)| \rightarrow 0 \quad \text{a.e.}$$

PROOF OF THEOREM 3: We first prove that $\liminf m_k^*(\omega) \geq m^*(C)$ a.e. By Lemma 1, we know that there exists a family of $n + 1$ hyperplanes through x^* with normal vectors $(\pi_1, \dots, \pi_{n+1})$ such that

$$(16) \quad m(x^*, \pi_i) = m^*(C), \quad \text{for all } 1 \leq i \leq n + 1,$$

and all points $z \in S$ lie in the smaller side of one of these hyperplanes,

$$(17) \quad \pi_i \cdot z \leq \pi_i \cdot x^* \quad \text{for some } 1 \leq i \leq n + 1.$$

Given $z \in S$, choose π_i so that equation (17) is satisfied. By definition (12a) this implies $I_k(z, \pi_i, \omega) \geq I_k(x^*, \pi_i, \omega)$ for all k and ω . Application of (12b) and (12c) yields

$$(18) \quad m_k(z, \omega) \geq m_k(z, \pi_i, \omega) \geq m_k(x^*, \pi_i, \omega).$$

Note that an equation of this form is valid for all points $z \in S$, where only the index i varies on the right-hand side. Definition (12d) then gives

$$(19) \quad m_k^*(\omega) \geq \min_{1 \leq i \leq n+1} m_k(x^*, \pi_i, \omega).$$

Application of the strong law of large numbers to the finite list (x^*, π_i) , $1 \leq i \leq n + 1$, yields

$$(20) \quad \liminf m_k^*(\omega) \geq \min_{1 \leq i \leq n+1} m(x^*, \pi_i) = m^*(C) \quad \text{a.e.}$$

where the substitution of $m^*(C)$ in (20) follows from (16). To prove that $\limsup m_k^*(\omega) \leq m^*(C)$ a.e., it suffices to prove the result for a fixed $\epsilon > 0$, i.e., $\limsup m_k^*(\omega) \leq m^*(C) + \epsilon$, since the intersection of countable sequences of such sets with ϵ shrinking to zero will itself be a set of full measure.

Given $\epsilon > 0$, we first select $\delta > 0$ so that $\forall z \in S, \pi \in B^n$,

$$(21) \quad \|x^* - z\| \leq \delta \implies \|m(x^*, \pi) - m(z, \pi)\| \leq \epsilon.$$

Existence of such a $\delta > 0$ follows from the fact that $m(x, \pi)$, defined in equation (13), is a continuous function on a compact set and is therefore uniformly continuous.

Next we select a finite number of hyperplanes (π_1, \dots, π_j) so that $\forall \pi \in B^n$,

$$(22) \quad \{z \in S | \pi \cdot z > \pi \cdot x^*\} \subset \{z \in S | \pi_j \cdot z > \pi_j \cdot (x^* - \delta\pi_j)\}$$

²¹ This is also the minimum for $z \in X$, since for points $z \notin S$, $m_k(z, \omega) = 1$.

²² It is convenient to use $x \in R^n$. Extension to arbitrary compact X follows as in Proposition 7.

for some $j, 1 \leq j \leq J$. Existence of such a family of hyperplanes follows from the open cover theorem applied to gradient vectors $\pi \in B^n$. To see this, note that (22) holds strictly when $\pi_j = \pi$, while compactness of S ensures that for a fixed vector π_j on the right hand side, (22) holds for a neighborhood of π_j in B^n .

Property (22) ensures that $\forall \pi \in B^n$,

$$(23) \quad m_k(x^*, \pi, \omega) \leq \max_{1 \leq j \leq J} m_k(x^* - \delta\pi_j, \pi_j, \omega).$$

Since this inequality holds for all π ,

$$(24) \quad m_k(x^*, \omega) \leq \max_{1 \leq j \leq J} m_k(x^* - \delta\pi_j, \pi_j, \omega).$$

But then application of the strong law of large numbers to the list $(x^* - \delta\pi_j, \pi_j), 1 \leq j \leq J$, ensures

$$(25) \quad \begin{aligned} \limsup m_k(x^*, \omega) &\leq \max_{1 \leq j \leq J} m(x^* - \delta\pi_j, \pi_j) \quad \text{a.e.} \\ &\leq m(x^*, \pi_j) + \varepsilon \quad \text{a.e.} \\ &\leq m(x^*) + \varepsilon = m^*(C) + \varepsilon \quad \text{a.e.} \end{aligned}$$

as required. The first inequality follows by (15), the second by (21), and the third by the definition of x^* . Q.E.D.

Note that although we have restricted our attention to concave densities, this result applies more generally to continuous densities on full-dimensional compact supports.

LEMMA 7: *The min-max point x^* dominates directly or indirectly all other proposals in X . For any given proposal p other than x^* there exists a sequence of j proposals z_1 through z_j such that:*

$$x^* [>_{m^*(C)}] z_1 [>_{m^*(C)}] z_2 [>_{m^*(C)}] \cdots [>_{m^*(C)}] z_j [>_{m^*(C)}] p.$$

PROOF: The following notational conventions are used in the proof: m^* is shorthand for $m^*(C)$, $N_\varepsilon(x)$ is the open ε -ball around x , and $N_\varepsilon(x) [>_{m^*}] y$ means that $x_1 [>_{m^*}] y$ holds for $x_1 \in N_\varepsilon(x)$. First note that the set of points y for which $y [>_{m^*}] p$ is open. Hence, there exists some point y^* and some $\varepsilon > 0$ such that $N_\varepsilon(y^*) [>_{m^*}] p$. Now define the family of sets $S(y), y \in X$:

$$S(y) = \{ x \in X | y [>_{m^*}] x \}.$$

Consider the collection of sets Θ consisting of all sets $S(y), y \in X$, together with $N_\varepsilon(x^*)$. We show that Θ forms an open cover of X . Directly, x^* is in $N_\varepsilon(x^*)$. For any other point x there is some point y such that $y [>_{m^*}] x$; therefore, $x \in S(y)$. Finally, $S(y)$ is open since the population density is nonatomic.

Since Q is an open cover of a compact set, there exists some finite subcover. Any subcover must include $N_\varepsilon(x^*)$ since x^* is unbeaten according to $[>_{m^*}]$. Let the subcover be $\{ N_\varepsilon(x^*), S(y_1), S(y_2), \dots, S(y_Q) \}$.

Recall the point y^* for which $N_\varepsilon(y^*) [>_{m^*}] p$. We now show that there is a sequence of no more than Q elections leading from y^* to within ε of x^* in which the winner always receives more than m^* percentage of the vote. Either $y^* \in N_\varepsilon(x^*)$ or it is contained in one of the $S(y_q)$ in which case without loss of generality assume $y^* \in S(y_1)$ so that $y_1 [>_{m^*}] y^*$. Since y_1 is not contained in $S(y_1)$ either $y_1 \in N_\varepsilon(x^*)$ or without loss of generality $y_1 \in S(y_2)$ so that $y_2 [>_{m^*}] y_1$. Continue this process. Since successive winning propositions are closer to x^* , there can be no cycles among the series of points y_q (see Proposition 7). Thus, after no more than Q steps, the series of successive propositions must terminate with a point $y_q \in N_\varepsilon(x^*)$. Hence, we have a series, $y_q [>_{m^*}] y_{q-1} \cdots y_2 [>_{m^*}] y_1 [>_{m^*}] y^*$, and $y_q \in N_\varepsilon(x^*)$.

We replace this series of elections by a series starting from x^* ,

$$x^* [>_{m^*}] z_q \cdots z_3 [>_{m^*}] z_2 [>_{m^*}] z_1$$

where x^* and z_q have the same perpendicular bisector as y_q and y_{q-1} . This construction continues

so that finally y_1 and y^* have the same perpendicular bisector as z_2 and z_1 . Lemma 8, demonstrated below, shows that $\|x^* - y_q\| = \|z_q - y_{q-1}\| = \dots = \|z_1 - y^*\| < \epsilon$. Thus $z_1 \in N_\epsilon(y^*)$ and hence $z_1 \in]_{> m^*}]_p$, completing the proof. Q.E.D.

LEMMA 8: If x_1 and y_1 have the same bisecting hyperplane as x_2 and y_2 , then $\|x_2 - x_1\| = \|y_2 - y_1\|$.

PROOF: The following two conditions must hold if x_2 and y_2 are to have the same bisecting hyperplane as x_1 and y_1 .

$$(26) \quad (x_2 - y_2) = \mu(x_1 - y_1), \quad \mu \in R;$$

$$(27) \quad [(x_2 + y_2)/2 - (x_1 + y_1)/2] \cdot (x_1 - y_1) = 0.$$

Equation (26) states that the line joining x_1 to y_1 is parallel to the line joining x_2 to y_2 ; (27) states that the two midpoints lie on the same bisecting plane. Combining these equations reveals,

$$(28) \quad [(x_2 - x_1) + (y_2 - y_1)] \cdot [(x_2 - x_1) - (y_2 - y_1)] = 0,$$

so that $\|x_2 - x_1\| = \|y_2 - y_1\|$. Q.E.D.

THEOREM 5: For a decision problem C in R^n satisfying A1-A3,

$$I(C, \epsilon) \leq \epsilon \left(\frac{n+1}{n} \right)^n.$$

PROOF: The proof relies on the family of $n+1$ hyperplanes $H = \{H_1, H_2, \dots, H_{n+1}\}$ from Lemma 1. Construct a family $G = \{G_1, G_2, \dots, G_{n+1}\}$ of hyperplanes where G_i is parallel to H_i but shifted into H_i^S until the population in $H_i^S \cap G_i^L$ equals ϵ . From condition (ii) of Lemma 1, any point interior to a smaller side, G_i^S , is beaten by more than $m^*(C) + \epsilon$ by its projection onto the plane G_i . Therefore no point interior to any half-space G_i^S can be a δ -majority winner, with $\delta = m^*(C) + \epsilon$. Hence any δ -majority winner must be contained in all the larger half-spaces: $I(C, \epsilon) \subseteq \Delta^L(\epsilon) \equiv \bigcap G_i^L$.

It remains only to show that the simplex $\Delta^L(\epsilon)$ contains no more than $((n+1)/n)^n \epsilon$ of the total population.

Assume to the contrary that $\Delta^L(\epsilon)$ contains more than $((n+1)/n)^n \epsilon$ of the total population. Applying Theorem 2, there is at least $(n/(n+1))^n$ of the simplex's population and therefore strictly more than ϵ of the total population contained in $\Delta^L(\epsilon)$ on the smaller side of any hyperplane through d^* , the min-max point of $\Delta^L(\epsilon)$. By condition (iii) of Lemma 1, $d^* \in H_i^S$ for some i . In addition, $d^* \in G_i^L$ since $d^* \in \Delta^L(\epsilon)$. However, the region $\{H_i^S \cap G_i^L\}$ is constructed to have population ϵ . This implies that the plane parallel to H_i through d^* can have a population of no more than ϵ in $\Delta^L(\epsilon)$ on its smaller side (since this region is a subset of $\{H_i^S \cap G_i^L\}$). This contradicts the conclusion that all hyperplanes through d^* have more than ϵ from $\Delta^L(\epsilon)$ on their smaller side and completes the proof. Q.E.D.

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