AD-A097 412 CALIFORNIA UNIV BERKELEY CENTER FOR PURE AND APPLIE--ETC F/6 12/1
THE HAMILTONIAN STRUCTURE OF THE MAXWELL-VLASOV EQUATIONS.(U)
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UNCLASSIFIED PAM-28 ARO-16567.8-M NL

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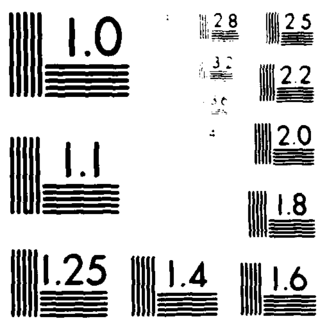

MICROCOPY RESOLUTION TEST CHART

Technical Report # 12

J.E. Marsden and A. Weinstein

February, 1981

U.S. Army Research Office

Contract DAA G29-79-C-0086

University of California, Berkeley,

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER
Technical Report # 12

2. GOVT ACCESSION NO.
AD-A097 412

3. RECIPIENT'S CATALOG NUMBER
(14) PAM-8

4. TITLE (and Subtitle)
The Hamiltonian Structure of the Maxwell-Vlasov equations.

5. TYPE OF REPORT & PERIOD COVERED
(9) Interim

6. PERFORMING ORG. REPORT NUMBER

7. AUTHOR(s)
J.E. Marsden and A. Weinstein

8. CONTRACT OR GRANT NUMBER(s)
(15) DAA G29-79-C-0086

9. PERFORMING ORGANIZATION NAME AND ADDRESS
(18) ARO
(19) 16501.8-M

10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS

11. CONTROLLING OFFICE NAME AND ADDRESS
U.S. Army Research Office
Post Office Box 12211
Research Triangle Park, NC 27709

12. REPORT DATE
(11) Feb[redacted]81

13. NUMBER OF PAGES
27

14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
(10) Jerrold E. Marsden
Alan Weinstein

15. SECURITY CLASS. (of this report)
Unclassified

15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
NA

16. DISTRIBUTION STATEMENT (of this Report)
Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
NA

18. SUPPLEMENTARY NOTES
The findings in this report are not to be construed as an official Department of the Army position, unless so designated by other authorized documents.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
Plasma physics, Hamiltonian system, gauge group, reduction, momentum map.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)
Morrison has shown that the Maxwell-Vlasov and the Vlasov-Poisson equations for a collisionless plasma can be written in Hamiltonian form relative to a certain Poisson bracket. We derive another very similar Poisson structure for these equations using co-adjoint orbits for the group of canonical transformations and the electromagnetic field regarded as a gauge theory. Our construction shows canonical variables can be found and can be extended to Yang-Mills gauge theories.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

The Hamiltonian Structure of the Maxwell-Vlasov Equations

Jerrold E. Marsden* and Alan Weinstein**

Department of Mathematics
University of California
Berkeley, CA 94720

Abstract

Morrison [1980] has shown that the Maxwell-Vlasov and Vlasov-Poisson equations for a collisionless plasma can be written in hamiltonian form relative to a certain Poisson bracket. We derive another (very similar) Poisson structure for these equations by using general methods of symplectic geometry. The main ingredients in our construction are the symplectic structure on the co-adjoint orbits for the group of canonical transformations and the symplectic structure for the phase space of the electromagnetic field regarded as a gauge theory. Our construction shows where canonical variables can be found and can be extended to Yang-Mills gauge theories.

1. Introduction

In this paper we show how to construct a Poisson structure for the Maxwell-Vlasov and Vlasov-Poisson equations for collisionless plasmas by using general methods of symplectic geometry. We shall compare our structure to that obtained by Morrison [1980].

*Research partially supported by NSF grant MCS-78-06718 and ARO contract DAAG-29-79-C-0086.
**Research partially supported by NSF grant MCS 77-23579.

We consider a plasma consisting of particles with charge e and mass m moving in Euclidean space $\mathbb{R}^3$ with positions x and velocities v. For simplicity we consider only one species of particle; the general case is similar. Let $f(x,v,t)$ be the plasma density at time t, and $E(x,t)$ and $B(x,t)$ be the electric and magnetic field. The Maxwell-Vlasov equations are:

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial x} + \frac{e}{m}\left(E + \frac{v\times B}{c}\right)\cdot\frac{\partial f}{\partial v} = 0 \tag{1.1}$$

$$\left.\begin{aligned} \frac{1}{c}\frac{\partial B}{\partial t} &= -\operatorname{curl} E \\ \frac{1}{c}\frac{\partial E}{\partial t} &= \operatorname{curl} B - \frac{e}{c}\int_v vf(x,v,t)dv \end{aligned}\right\} \tag{1.2}$$

$$\left.\begin{aligned} &\operatorname{div} E = \rho_f, \quad \text{where } \rho_f = e\int_v f(x,v,t)dv \\ &\operatorname{div} B = 0 \end{aligned}\right\} \tag{1.3}$$

Letting $c \to \infty$ leads to the Vlasov-Poisson equation:

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \phi_f}{\partial x}\cdot\frac{\partial f}{\partial v} = 0 \tag{1.4}$$

where

$$\Delta\phi_f = -\rho_f. \tag{1.5}$$

In what follows we shall set $c = 1$ in the Maxwell-Vlasov system.

The Hamiltonian for the Maxwell-Vlasov system is

$$H(f,E,B) = \int_{x,v} \frac{1}{2}mv^2 f(x,v,t)dxdv + \int_x \frac{1}{2}[E(x,t)^2 + B(x,t)^2]dx \tag{1.6}$$

while that for the Vlasov-Poisson equation is

$$H(f) = \int_{x,v} \frac{1}{2}mv^2 f(x,v,t)\,dx\,dv + \frac{1}{2}\int_x \phi_f(x)\rho_f(x)\,dx \tag{1.7}$$

The Poisson bracket used by Morrison is defined on functions F(f,E,B) of the fields f,E,B by

$$\{F,G\} = \int_{x,v} f\left\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\right\}dx\,dv + \int_x \frac{\delta F}{\delta E}\cdot \mathrm{curl}\,\frac{\delta G}{\delta B} - \frac{\delta G}{\delta E}\cdot \mathrm{curl}\,\frac{\delta F}{\delta B}\,dx$$

$$+ \int_{x,v} \frac{\delta F}{\delta E}\cdot\frac{\partial f}{\partial v}\,\frac{\delta G}{\delta f} - \frac{\delta G}{\delta E}\cdot\frac{\partial f}{\partial v}\,\frac{\delta F}{\delta f}\,dx\,dv \tag{1.8}$$

$$+ \int_{x,v} \frac{\delta F}{\delta B}\cdot\left(\frac{\partial f}{\partial v}\times v\right)\frac{\delta G}{\delta f} - \frac{\delta G}{\delta B}\cdot\left(\frac{\partial f}{\partial v}\times v\right)\frac{\delta F}{\delta f}\,dx\,dv$$

where in the first term { , } denotes the usual Poisson bracket for functions of (x,v), and where the functional derivatives are defined in terms of the usual (Frèchet) derivative by

$$(D_f F)\cdot f' = \int_{x,v} \frac{\delta F}{\delta f} f'\,dx\,dv, \text{ etc.}$$

For the Vlasov-Poisson equation, one keeps only the first term of (1.8). The equations (1.1) and (1.2) (or (1.4)) are then equivalent to

$$\dot{F} = \{F,H\} \tag{1.9}$$

with H given by (1.6) (or (1.7) for the Vlasov-Poisson equation .

Our purpose is to show how another Poisson structure can be constructed by a general procedure involving reduction (Marsden and Weinstein [1974]) and coupling of hamiltonian systems to gauge fields (Weinstein [1978b]). Our Poisson bracket differs from (1.8) in that the last integral is replaced by

$$\int_{x,v} \frac{\delta F}{\delta f} \frac{\partial f}{\partial v} \cdot \left[\frac{\partial}{\partial v}\left(\frac{\delta G}{\delta f}\right) \times \underset{\sim}{B}\right] dxdv - \int_{x,v} \frac{\delta G}{\delta f} \frac{\partial f}{\partial v} \cdot \left[\frac{\partial}{\partial v}\left(\frac{\delta G}{\delta f}\right) \times \underset{\sim}{B}\right] dxdv \tag{1.10}$$

Both structures satisfy the Jacobi identity and yield the correct equations of motion for the hamiltonians we have specified; however, ours is constructed by general principles rather than being ad hoc, and it is clear where the canonical variables are to be found. In addition, the equations div E = ρ_f and div B = 0 arise naturally from the gauge symmetry of the problem and need not be postulated separately. We believe that the presence of the two symplectic structures might imply the existence of some new conserved quantities. Our Poisson structure fits into a pattern, special cases of which have been found by others. For example, Arnold [1966] showed that the Euler equations for a perfect incompressible fluid are a Hamiltonian system in the canonical Poisson structure associated with the group of volume preserving diffeomorphisms of a region in $\mathbb{R}^3$. Using Arnold's methods, one can also see that the compressible equations are associated to the semidirect product of the group of diffeomorphisms and the (additive group of) densities on $\mathbb{R}^3$. It is easy to check that this approach yields the same Poisson structure found for perfect fluids by Morrison and Greene [1980] and ought to be extendible to the MHD equations by the methods of this paper. The KdV equation is associated with the Lie algebra of the group of canonical transformations in the work of, for example, Adler [1979]. (We recall that there is a standard link between the Maxwell-Vlasov equations and the KdV equation -- see Davidson [1972]). In Ebin and Marsden [1970], the functional analytic machinery required to fully justify Arnold's approach was given. It was proved, for example, that the volume preserving diffeomorphisms form a true infinite dimensional manifold which is, in an appropriate sense, a Lie group. It was also shown that the group of canonical transformations has similar features, but no physical interpretation was given. The Vlasov equation provides one.

All of these clues suggest that it is fruitful to find a more geometric and group-theoretic framework for the basic equations of plasma physics. Such a framework is provided here. We shall not deal with the delicate functional analytic issues needed to make precise all the infinite dimensional geometry, nor shall we deal with questions of existence and uniqueness (cf. Batt [1977] and Horst [1980]).

Acknowledgment

We are grateful to Allan Kaufman for showing us Morrison's paper and for his continuing encouragement. We also thank Philip Morrison, Bertram Kostant, Robert Littlejohn, Paul Rabinowitz, Tudor Ratiu, Richard Spencer and Rudolf Schmid for their interest and comments.

2. The Poisson Structure for the Density Variables

We begin by explaining the geometric meaning of the first term in (1.8), $\int_{x,v} f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dx dv$. In the following sections, we shall explain the term for Maxwell's equations (the second integral in (1.8)), and then finally the coupling terms (the remaining two integrals).

In the absence of a magnetic field, by normalizing mass, we can identify velocity with momentum; hence we let $\mathbb{R}^6$ denote the usual position-momentum phase space with coordinates $(x_1,x_2,x_3,p_1,p_2,p_3)$ and symplectic structure $\Sigma dx_i \wedge dp_i$. (See Abraham and Marsden [1978] or Arnold [1978].) Let S denote the group of canonical transformations of $\mathbb{R}^6$ which have polynomial growth at infinity in the momentum directions. The Lie algebra $\mathfrak{s}$ of S consists of the hamiltonian vector fields on $\mathbb{R}^6$ with polynomial growth in the momentum directions. We can identify elements of $\mathfrak{s}$ with their generating functions,* so that $\mathfrak{s} = C^\infty(\mathbb{R}^6)$; the (right) Lie algebra structure is given by the <u>negative</u> of the usual Poisson bracket on phase space. (This follows from Exercise 4.1G and Corollary 3.3.18 of Abraham and Marsden [1978]).

The dual space $\mathfrak{s}^*$ can be identified with the distribution densities on $\mathbb{R}^6$ which are rapidly decreasing in the momentum directions; the pairing between $h \in \mathfrak{s}$ and $f \in \mathfrak{s}^*$ is given by

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*The generating function of a hamiltonian vector field is determined only up to an additive constant. The "correct" group S is really the group of transformations of $\mathbb{R}^6 \times \mathbb{R}$ preserving the 1-form $\Sigma p_i dq_i + d\tau$ (Van Hove [1951]), but we can ignore this technical point here without encountering any essential difficulties.

$$\langle h,f\rangle = \int_{\mathbb{R}^6} hf\, dxdp \ .$$

(The "density" is really f dxdp, but we denote it simply by f.) Now as for any Lie algebra, the dual space $\mathfrak{s}^*$ carries a natural Poisson structure which is non-degenerate on the co-adjoint orbits (see, e.g. Guillemin and Sternberg [1980]). In our case the orbit through $f \in \mathfrak{s}^*$ is

$$\mathcal{O}_f = \{\eta^* f \mid \eta \in \mathfrak{s}\} \tag{2.1}$$

and so the Kirillov-Kostant symplectic form on $\mathcal{O}_f$ is given by

$$\omega_f(\{f,h\},\{f,k\}) = \langle f,\{h,k\}\rangle \tag{2.2}$$

where $\{f,h\} = L_{X_h} f$ (Lie differentiation) represents a typical tangent vector to $\mathcal{O}_f$ at f. (See p. 303 of Abraham and Marsden [1978] -- two minus signs have cancelled here.) The hamiltonian vector field X_F on $\mathcal{O}_f$ determined by a smooth function $F: \mathfrak{s}^* \to \mathbb{R}$ satisfies

$$\omega_f(X_F(f),\{f,k\}) = dF(f)\cdot\{f,k\} \tag{2.3}$$

for all $k \in \mathfrak{s}^*$. We claim that

$$X_F(f) = -\{f,\frac{\delta F}{\delta f}\} \tag{2.4}$$

Indeed, by (2.3) and (2.2) we need only check that

$$-\langle f,\{\frac{\delta F}{\delta f},k\}\rangle = dF(f)\cdot\{f,k\} = \langle\frac{\delta F}{\delta f},\{f,k\}\rangle \tag{2.5}$$

But (2.5) holds by integration by parts. In fact, the following identity is of general utility:

$$-\langle f,\{h,k\}\rangle = \langle\{f,k\},h\rangle \tag{2.6}$$

Thus, the Poisson bracket on $\mathcal{s}^*$ is given by

$$\{F,G\}(f) = \omega_f(X_F(f),X_G(f)) \qquad \text{(by definition)}$$

$$= \omega_f\ \{f,\frac{\delta F}{\delta f}\},\{f,\frac{\delta G}{\delta f}\} \qquad \text{(by (2.4))}$$

$$= \langle f,\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\}\rangle \qquad \text{(by 2.2)}$$

We have proved:

2.1 Proposition. The natural Poisson structure on the dual of the Lie algebra of the group of canonical transformations is given by

$$\{F,G\} = \int f\{\frac{\delta F}{\delta f},\frac{\delta G}{\delta f}\}dxdp \tag{2.7}$$

<u>Remarks</u>. 1. Notice that (2.7) coincides with the first term for the Poisson structure (1.8) if p is replaced by v.

2. The bracket (2.7) automatically satisfies the Jacobi identity since it coincides with the Poisson bracket on each of the symplectic manifolds $\mathcal{O}_f$.

3. (a) If f is a delta density, $\mathcal{O}_f$ "coincides" with $\mathbb{R}^6$. (In a similar way, every symplectic manifold is a co-adjoint orbit.) For f a sum of n delta functions, $\mathcal{O}_f$ is the phase space for n particles. For continuous plasmas, f is taken to be a continuous density, in which case $\mathcal{O}_f$ can be shown to be a smooth infinite dimensional manfiold.

3. (b) If f is a density concentrated along a curve, then $\mathcal{O}_f$ is identifiable with all curves having a fixed action integral. This is a reduced form of the loop space, a symplectic manifold used in the variational principle of Weinstein [1978a]. If f is concentrated on a Lagrangian torus, then $\mathcal{O}_f$ consists of lagrangian tori with fixed action integrals. This is related to a variational principle of Percival [1979].

4. By using an appropriate Darboux theorem, (see Marsden [1981], lecture 1), one can show that $\mathcal{O}_f$ admits canonically conjugate coordinates.

5. The Vlasov-Poisson equation is a hamiltonian system on $\mathfrak{s}^*$ with energy function given by (1.7). If f evolves according to (1.4) then (1.9) is true. This is a direct calculation, already noted by Morrison [1980]. More can be learned from our derivation of the Poisson structure: the equation (1.4) is tangent to each

orbit $\mathcal{O}_f$, so it defines a hamiltonian system on each orbit. This can be seen directly by noting that (1.4) can be written in terms of ordinary Poisson brackets as

$$\frac{\partial f}{\partial t} = \{f, H(f)\} \tag{1.4'}$$

where

$$H(f) = \frac{1}{2}mv^2 + \frac{e}{m}\phi_f(x).$$

Thus, the evolution of f can be described by

$$f_t = \eta_t^* f_0$$

where f_0 is the initial value of f, f_t is its value at time t and $\eta_t \in S$. In particular, if F is a function of a single real variable we get the well-known conservation laws

$$\int_{\mathbb{R}^6} F(f_t) = \text{constant in time}$$

by the change of variables formula and the fact that each $\eta \in S$ is volume preserving. (These conservation laws are useful in proving existence and uniqueness theorems since, as in the case of two-dimensional ideal incompressible flow, they lead to a priori L^p-estimates.)

6. In Ebin-Marsden [1970] the convective term $v \cdot \nabla v$ in fluid mechanics led to a crucial difference between working spatially (in the Lie algebra--the "Euler" picture) or materially (on the group...the "Lagrange" picture). Here there is no such term, since it would be given by $\{f,f\}$, which vanishes.

6. Analogies with fluid mechanics raise several interesting analytical problems: (A) if δ times a dissipation term associated with collisons is added, do the solutions converge to those of the Vlasov equation as $\delta \to 0$? (Analogous to the limit of zero viscosity). Standard techniques (Ebin-Marsden [1970], Kato [1975]) can probably be used to answer this affimatively for short time.

(B) Can the hamiltonian structure be used to study chaotic or turbulent dynamics, as was done in, for example, Holmes and Marsden [1981]?

(C) Is the time-t map for the Vlasov-Poisson or Maxwell-Vlasov equations smooth? See Ratiu [1979] for a discussion of why this question is of interest for the KdV equation.

§3. Maxwell's Equations and Reduction

Before coupling the Vlasov equation to the electromagnetic field equations, we shall review the hamiltonian description of Maxwell's equations. The appropriate Poisson bracket for the E and B fields (the second term in (1.8)) will be constructed by reduction (Marsden and Weinstein [1974]).

Let $P \to M$ be a given principal S^1 bundle over a manifold M. In our case $M = \mathbb{R}^3$ and $P = S^1 \times \mathbb{R}^3$. Let $\mathcal{A}$ denote the (affine) space of all connections for this bundle. Let $\mathcal{G}$ denote the group of gauge transformations; i.e. bundle automorphisms of $P \to M$. Elements of $\mathcal{G}$ may be denoted $e^{i\phi}$ where $\phi: M \to \mathbb{R}$. There is a natural action of $\mathcal{G}$ on $\mathcal{A}$ given by

$$(e^{i\phi}, A) \mapsto A + d\phi \tag{3.1}$$

Consider $T^*\mathcal{A}$, the cotangent bundle of $\mathcal{A}$ with the canonical symplectic structure. Elements of $T^*\mathcal{A}$ may be identified with pairs (A,E) where A is a connection and E is a vector field density on M. The canonical symplectic structure ω on $T^*\mathcal{A}$ is given by

$$\omega((A_1,E_1),(A_2,E_2)) = \int_M (E_2 \cdot A_1 - E_1 \cdot A_2)\,dx \tag{3.2}$$

with associated Poisson bracket given by

$$\{F,G\} = \int_M \left(\frac{\delta F}{\delta A}\frac{\delta G}{\delta E} - \frac{\delta E}{\delta E}\frac{\delta G}{\delta A} \right) dx$$

Maxwell's equations in terms of E and A are Hamilton's equations on $T^*\mathcal{A}$ relative to (3.2) for the hamiltonian

$$H(A,E) = \frac{1}{2}\int |E|^2 dx + \frac{1}{2}\int |dA|^2 dx \tag{3.3}$$

Now G acts on $\mathcal{A}$ by (3.1) and hence on $T^*\mathcal{A}$. It has a momentum map $J: T^*\mathcal{A} \to \mathfrak{g}^*$ where $\mathfrak{g}$, the Lie algebra of G, is identified with the real valued functions on M. The momentum map may be determined by a standard formula (Abraham and Marsden [1978, Corollary 4.2.11]) to be:

$$J(A,E)\cdot\phi = \int (E\cdot d\phi)\, dx = -\int (\operatorname{div} E\cdot\phi)\, dx \tag{3.4}$$

Thus $J^{-1}(0) = \{(A,E) \in T^* \mid \operatorname{div} E = 0\}$. By a general theorem on reduction (Marsden and Weinstein [1974]), the manifold $J^{-1}(0)/G$ has a naturally induced symplectic structure.

3.1. <u>Proposition</u>. <u>We may identify</u> $J^{-1}(0)/G$ <u>with</u> $M = \{(E,B) \mid \operatorname{div} E = 0, \operatorname{div} B = 0\}$. <u>The Poisson bracket on</u> M <u>is given by</u>

$$\{F,G\} = \int_M \left(\frac{\delta F}{\delta E}\operatorname{curl}\frac{\delta G}{\delta B} - \frac{\delta G}{\delta E}\operatorname{curl}\frac{\delta F}{\delta B}\right) dx \tag{3.5}$$

<u>Maxwell's equations in the vacuum are Hamilton's equations for</u>.

$$H(E,B) = \frac{1}{2}\int |E|^2 dx + \frac{1}{2}\int |B|^2 dx \tag{3.6}$$

Proof. The identification between $J^{-1}(0)/G$ and M is by the equation $B = \text{curl } A$. (We assume M has trivial first cohomology, as is the case here; for Yang Mills fields, B is the "curvature" of A.) Let $F, G: M \to \mathbb{R}$. We may regard F and G as G-invariant functions $\hat{F}$, $\hat{G}$ on $T^*\mathcal{X}$ by $B = \text{curl } A$. Since Poisson brackets push down under reduction we have

$$\{\hat{F},\hat{G}\}_0 = \{F,G\} \tag{3.7}$$

where 0 means the induced function on M. Now in the canonical structure on $T^*\mathcal{X}$ we have

$$\{\hat{F},\hat{G}\} = \int\left(\frac{\delta\hat{F}}{\delta A}\frac{\delta\hat{G}}{\delta E} - \frac{\delta\hat{G}}{\delta A}\frac{\delta\hat{F}}{\delta E}\right) dx \tag{3.8}$$

The chain rule and the definition of functional derivatives give the identity

$$\int\frac{\delta\hat{F}}{\delta A}\cdot A'dx = \int\frac{\delta F}{\delta B}\cdot \text{curl } A'dx = -\int A'\,\text{curl }\frac{\delta F}{\delta B}\, dx \tag{3.9}$$

Substitution of (3.9) in (3.8) gives (3.5). The rest is readily checked. ■

This formalism generalizes readily to Yang-Mills fields and to these fields coupled to gravity; see Arms [1979].

§4. A General Construction for Interacting Systems

The work of Weinstein [1978b] on the equations of motion for a particle in a Yang-Mills field uses the following general set-up. Let $\pi: \mathcal{B} \to M$ be a principal G-bundle and Q a hamiltonian G-space (or a Poisson manifold which is a union of hamiltonian G-spaces). Then G acts on $T^*\mathcal{B}$ and on Q, so it acts on $Q \times T^*\mathcal{B}$. This action has a momentum map J and so may be reduced at 0:

$$(Q \times T^*\mathcal{B})_0 = J^{-1}(0)/G$$

The reduced manifold carries a symplectic (or Poisson, if Q was a Poisson manifold) structure naturally induced from those of Q and $T^*\mathcal{B}$.

For a particle in a Yang-Mills field one chooses $\mathcal{B}$ to be a G-bundle over 3-space M and Q a co-adjoint orbit for G (the internal variables). The hamiltonian is constructed using a connection (i.e. a Yang-Mills field) for $\mathcal{B}$. For electromagnetism $G = S^1$ and $Q = \{e\}$ is a point.

For the Vlasov-Maxwell system we choose our gauge bundle to be

$$\mathcal{B} = \mathcal{Q} \to \mathcal{M}$$

where $\mathcal{M} = \{(E,B) \mid \operatorname{div} B = 0\}$, with $G = \mathcal{G}$ the gauge group described in the previous section.

We now choose $\mathcal{S}$ to be the group of canonical transformations of T^*M ($= \mathbb{R}^6$). We can let Q be either the symplectic manifold

T^*S or the Poisson manifold $\mathcal{S}^*$. It remains to specify an action of G on T^*S. We set $e = 1$ and let G act on S by

$$(\phi,\eta) \mapsto \eta + d\phi\circ\pi\circ\eta \tag{4.1}$$

where $\pi: \mathbb{R}^6 \to \mathbb{R}^3$ is the projection $(x,p) \mapsto x$; i.e. we translate the momentum space at x by $d\phi(x)$. A simple computation gives:

4.1. Lemma. The action of G induces an action on $\mathcal{S}^*$ given by

$$(e^{i\phi},f) \mapsto f\circ\tau_{d\phi}$$

where $\tau_{d\phi} \in S$ is translation by $d\phi$. The momentum map $J: \mathcal{S}^* \to G^*$ for this action is given by

$$J(f)\cdot\phi = -\int f(x,p)\phi(x)\,dx\,dp$$

To construct the interaction space we need to compute the momentum map for the corresponding action of G on $\mathcal{S}^* \times T^*\mathcal{O}\!\mathcal{C}$ given by

$$(e^{i\phi}(f,A,E)) \mapsto (f\circ\tau_{d\phi}, A + d\phi, E) \tag{4.2}$$

4.2. Lemma. This momentum map is given by

$$J(f,A,E)\cdot\phi = \int_{x,p} f(x,p)\phi(x)dxdp + \int E\cdot d\phi dx$$

$$= \int_x \rho_f(x)\phi(x)dx - \int_x \operatorname{div} E\ \phi(x)dx$$

Again this is a straightforward calculation. Our reduced (interaction) space $J^{-1}(0)/G$ may now be identified with

$$VM = \{(f,B,E) \mid \operatorname{div} E = \rho_f,\ \operatorname{div} B = 0\} \tag{4.3}$$

where

$$J^{-1}(0) = \{(f,A,E) \mid \operatorname{div} E = \rho_f\}.$$

§5. Computation of the Poisson Structure

By reduction, the space $\mathcal{VM}$ defined by (4.3) has a natural Poisson structure; we now compute it.

5.1. Theorem. The Poisson structure on $\mathcal{VM}$ is given by

$$\{F,G\} = \int_{x,p} f\left\{\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right\} dx\,dp + \int_x \left(\frac{\delta F}{\delta E} \operatorname{curl} \frac{\delta G}{\delta B} - \frac{\delta G}{\delta E} \operatorname{curl} \frac{\delta F}{\delta B}\right) dx + \left[\int_{x,p} f\left(\frac{\partial}{\partial p} \frac{\delta F}{\delta f}\right)\left(\frac{\partial}{\partial x} \Delta^{-1} \operatorname{div} \frac{\delta G}{\delta E}\right) dx\,dp - \int_{x,p} f\left(\frac{\partial}{\partial p} \frac{\delta G}{\delta f}\right)\frac{\partial}{\partial x}\left(\Delta^{-1} \operatorname{div} \frac{\delta F}{\delta E}\right) dx\,dp\right] \tag{5.1}$$

Proof. Having identified $J^{-1}(0)/\mathcal{G}$ with $\mathcal{VM}$, one checks that the natural projection $\pi\colon J^{-1}(0) \to \mathcal{VM}$ is given by

$$\pi(f,A,E) = (f \circ \tau_{d\phi_A}, \operatorname{curl} A, E) \tag{5.2}$$

where

$$\phi_A = -\Delta^{-1}(\operatorname{div} A) \tag{5.3}$$

and, as before, $\tau_{d\phi_A}$ is translation by $d\phi_A$.

Next, define a projection map

$$\chi: \mathcal{S}^* \times T^*\mathcal{O} \to J^{-1}(0);\ \chi(f,A,E) = (f,A,E + \nabla\phi_{f,E}) \tag{5.4}$$

where $\phi_{f,E} = \Delta^{-1}(\rho_f - \operatorname{div} E)$.

Now if $F,G : VM \to \mathbb{R}$, we can extend F,G to maps on $\mathcal{S}^* \times T^*\mathcal{O}$ by

$$\hat{F} = F\circ\Pi\circ\chi,\quad \hat{G} = G\circ\Pi\circ\chi \tag{5.5}$$

If we denote the map $\Pi\circ\chi$ by

$$(f,A,E) \mapsto (\bar{f},\bar{B},\bar{E})$$

we have, from (5.2) and (5.4),

$$\left.\begin{aligned} \bar{f} &= f\circ\tau_{d\phi_A} \\ \bar{B} &= \operatorname{curl} A \\ \bar{E} &= E + \nabla\phi_{f,E} \end{aligned}\right\} \tag{5.6}$$

Since $J^{-1}(0)$ is co-isotropic, Poisson brackets can be computed in $\mathcal{S}^* \times T^*\mathcal{O}$, and the answer is independent of how F and G are extended to $\mathcal{S}^* \times T^*\mathcal{O}$. We chose the specific map χ to effect the extension explicitly.

Using the definition of functional derivatives and the chain rule, one finds the following relationships

$$\left.\begin{aligned}\frac{\delta\hat{F}}{\delta f} &= \frac{\delta F}{\delta \bar{f}} - \Delta^{-1}\,\mathrm{div}\,\frac{\delta F}{\delta \bar{E}} \\ \frac{\delta\hat{F}}{\delta A} &= -\int_p \frac{\partial}{\partial x}\,\Delta_x^{-1}\mathrm{div}_x\left(\frac{\delta F}{\delta \bar{f}}\,\frac{\partial f}{\partial p}\right)dp - \mathrm{curl}\,\frac{\delta F}{\delta \bar{B}} \\ \frac{\delta\hat{F}}{\delta E} &= \frac{\delta F}{\delta \bar{E}} - \frac{\partial}{\partial x}\,\Delta_x^{-1}\mathrm{div}_x\,\frac{\delta F}{\delta \bar{E}}\end{aligned}\right\} \tag{5.7}$$

Substituting (5.7) into the Poisson structure on $\mathfrak{s}^* \times T^*\mathcal{X}$ obtained from that on $\mathfrak{s}^*$ (see 2.7) plus that on $T^*\mathcal{X}$ (see 3.2), i.e. into

$$\{\hat{F},\hat{G}\} = \int_{x,p} f\left\{\frac{\delta\hat{F}}{\delta f}, \frac{\delta\hat{G}}{\delta f}\right\}dx\,dp + \int_x \left(\frac{\delta\hat{F}}{\delta A}\,\frac{\delta\hat{G}}{\delta E} - \frac{\delta\hat{F}}{\delta E}\,\frac{\delta\hat{G}}{\delta A}\right)dx \tag{5.8}$$

we obtain formula (5.1). ■

<u>Note</u>: In carrying out the substitution of (5.7) into (5.8), some cancellation occurs since $\frac{\partial}{\partial x}\Delta_x^{-1}\mathrm{div}_x V$ is the gradient part of V, which is L^2-orthogonal on $\mathbb{R}^3$ to divergence-free vector fields.

The Poisson structure (5.1) automatically satisfies the Jacobi identity, since it is the structure associated with reduced manifolds, which are symplectic.

The hamiltonian for the Vlasov-Maxwell equations in the momentum space representation is the function H on $\mathcal{S}^* \times T^*\mathcal{U}$ given by

$$H(f,A,E) = \int_{x,p} \frac{1}{2m}(p-A)^2 f(x,p)\,dx\,dp + \frac{1}{2}\int_x (E^2 + (\text{curl } A)^2)\,dx \qquad (5.9)$$

This hamiltonian is gauge-invariant; i.e. it is invariant under the action of G given by (4.2). Thus H produces a well-defined hamiltonian H_0 on VM. We find that

$$H_0(f,B,E) = \int_{x,p} \frac{p^2}{2m} f(x,p-\nabla \times \Delta^{-1}B)\,dx\,dp + \frac{1}{2}\int (E^2 + B^2)\,dx \qquad (5.10)$$

5.2. Theorem. The hamiltonian (5.10) in the Poisson structure (5.1) yields the Vlasov-Maxwell equations in momentum representation.

This can be checked directly; however, to facilitate comparison with Morrison's results we shall transform our Poisson structure to produce a velocity space verson of 5.2.

§6. Transformation to Velocity Representation

To compare our bracket (5.1) with Morrison's (1.8), we transform (5.1) to velocity space. Let us denote the space defined by (4.3) by $\mathcal{VM}_p$ to emphasize that f is a density on position-momentum space. We write $\mathcal{VM}_v$ for the corresponding space (4.3) for f a density on position-velocity space. We shall take $e = 1, m = 1$ for simplicity. Define the transformation

$$\mathcal{P} : \mathcal{VM}_v \to \mathcal{VM}_p$$

by

$$(f,B,E) \mapsto (\tilde{f},\tilde{B},\tilde{E})$$

where

$$\left.\begin{aligned} \tilde{f}(x,p) &= f(x, p + \nabla \times \Delta^{-1}B(x)) \\ \tilde{B} &= B \\ \text{and} \quad \tilde{E} &= E \end{aligned}\right\} \qquad (6.1)$$

6.1. Theorem. The transformation (6.1) transforms the bracket (5.1) to (1.8) with the last two terms replaced by (1.10). Both brackets yield the correct equations of motion; i.e. (1.1)-(1.3) imply that (1.9) holds.

Sketch of Proof. We make these remarks: (a) Transforming the first term of (5.1) produces the first term of (1.8) plus the term (1.10). This is seen using the chain rule and the equation $p + \nabla \times \Delta^{-1} B = v$.

(b) The second and third terms of (5.1) both produce extra terms via the chain rule. These recombine to give the second and third terms of 1.8.

(c) In the special case in which G is the Hamiltonian (1.6), both (1.10) and the last term of (1.8) give the same expression. This is why both structures give the same equations of motion. ■

§7. Additional Remarks

(A) Poisson structures may be viewed as bundle maps taking covectors to vectors. (In this guise, they are called "cosymplectic structures.") Viewed this way, Morrison's bracket is the map

$$(f^*, B^*, E^*) \mapsto (\delta f, \delta B, \delta E)$$

given by

$$\left.\begin{aligned}
\delta f &= -\{f, f^*\} - \frac{\partial f}{\partial v}\cdot E^* - \frac{\partial f}{\partial v}\cdot (v \times B^*) \\
\delta B &= \int_P \left(\frac{\partial f^*}{\partial v} \times v\right) f^* dv - \text{curl } E^* \\
\delta E &= \int_P \frac{\partial f}{\partial v} f^* dv + \text{curl } B^*
\end{aligned}\right\} \qquad (7.1)$$

while our cosymplectic structure is given by

$$\left.\begin{aligned}
\delta f &= -\{f, f^* + \Delta^{-1} \text{ div } E^*\} \\
\delta B &= -\text{ curl } E \\
\delta E &= \mathbb{P}\int_P f^* \frac{\partial f}{\partial p} dp + \text{curl } B^*
\end{aligned}\right\} \qquad (7.2)$$

in momentum representation, where $\mathbb{P}$ is the projection to the divergence free part. In velocity representation, (7.2) becomes

$$\left.\begin{aligned}
\delta f &= -\{f, f^* + \Delta^{-1}\,\mathrm{div}\,E^*\} - \frac{\partial f}{\partial v}\cdot\left(\frac{\partial f^*}{\partial v} \times B\right) \\
\delta B &= -\,\mathrm{curl}\,E \\
\delta E &= \int_p \frac{\partial f}{\partial v} f^* dv + \mathrm{curl}\,B^*
\end{aligned}\right\} \qquad (7.3)$$

Thus (7.1) and (7.3) differ only in the terms

$$v \times B^* \text{ verses } \frac{\partial f^*}{\partial v} \times B$$

From (7.2) we see explicitly that δf is tangent to co-adjoint orbits in S. (This fact is obscured in velocity representation.) On the other hand, when Morrison's structure (7.1) is transformed to momentum space, δf is not tangent to co-adjoint orbits of S.

(B) A "cold plasma" may be defined as one for which f is a δ measure supported on the graph of a vector field $p = \theta(x)$. This property persists as f evolves by composition with a canonical transformation. In fact, the property that θ is curl-free is also maintained, since this corresponds to the graph's being a lagrangian submanifold. After a long time, the submanifold may no longer be a graph. This is the "shock" phenomenon, leading to <u>multiple streaming</u> (Davidson [1972] .) We remark that Maslov ([1976], p. 44) has already noticed this evolution of lagrangian submanifolds for the Vlasov-Poisson equation.

(C) We would like to understand in general terms the contraction of one hamiltonian system to another. Examples are the passage to the restricted three body problem from the full three body problem and the limit $c \to \infty$ to get the Vlasov-Poisson equation. It would also be of interest to realize both the Vlasov-Maxwell and MHD equations as limiting cases of a grand hamiltonian system constructed from the Boltzmann equation.

(D) We have remarked that our formalism readily generalizes to Yang-Mills interactions. Is such a hamiltonian structure useful in nuclear physics for Yang-Mills plasmas?

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