DUALLY STEADY RINGS

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Abstract. Two duals of steady rings are introduced and studied.

1. Preliminaries

Throughout the paper, R stands for a (non-zero and associative) ring with unit and modules are unitary left R-modules. The terminology and notation used in what follows is quite standard (see e.g. [11]).

The only exception is the (left R-) module $R^\omega / R^{(\omega)}$ which is denoted by W in the sequel. Finally, a module will be called complete if it is complete in a non-discrete linear Hausdorff topology.

2. $\cup$-compact modules and steady rings

It is well known and easy to check that the following six conditions are equivalent for a module M:

(A1) If M_i, $i < \omega$, is a countable family of submodules of M such that $\Sigma M_i = M$, then $\Sigma_{i \leq n} M_i = M$ for some $n < \omega$.

(A2) If $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ is a countable chain of submodules of M such that $\bigcup M_i = M$, then $M_n = M$ for some $n < \omega$.

(A3) If $\epsilon : \coprod_\omega A_i \to M$ is an epimorphism, then $\epsilon(\coprod_{i \leq n} A_i) = M$ for some $n < \omega$.

(A4) If $\varphi : M \to \coprod_I A_i$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq \coprod_J A_i$ for a finite subset J of I.

(A5) If $\varphi : M \to \coprod_\omega A_i$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq \coprod_{i \leq n} A_i$ for some $n < \omega$.

(A6) If Q is a cogenerator for R-Mod and if $\varphi : M \to Q^{(\omega)}$ is a homomorphism, then $\mathrm{Im}(\varphi) \subseteq Q^{(n)}$ for some $n < \omega$.

Such a module M will be called $\cup$-compact in this paper (other known names: Σ-compact, $\coprod$-slender, dually slender, small, ...) and one sees immediately that every finitely generated module is $\cup$-compact. If the converse is true, then the ring R is said to be left steady. Many such rings were studied in the literature (see e.g. [2], [10], [12]) but we will mention only the following few examples:

The authors were supported by the Grant Agency of the Czech Republic grant No. 201/00/0766. The authors also acknowledge the support of the institutional grant MSM 113 200 007.

Proposition 2.1. *R is left steady in each of the following cases:*
(1) *R is left noetherian.*
(2) *R is left perfect.*
(3) *R is left semiartinian of countable $\widehat{\mathrm{Soc}}$-length.*
(4) *R is a countable commutative domain.*

Proof. See [2]. □

3. ∩-compact modules and 1-dually steady rings

The following conditions are equivalent for a module M:

(B1) If M_i, $i < \omega$, is a countable family of submodules of M such that $\bigcap M_i = 0$, then $\bigcap_{i \leq n} M_i = 0$ for some $n < \omega$.

(B2) If $M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ is a countable chain of submodules of M such that $\bigcap M_i = 0$, then $M_n = 0$ for some $n < \omega$.

(B3) If $\iota : M \to \Pi_\omega A_i$ is a monomorphism, then $\iota^{-1}(\Pi_{i \geq n} A_i) = 0$ for some $n < \omega$.

Such a module M will be called ∩-compact in the paper and one sees again immediately that every finitely cogenerated module is ∩-compact. If the converse is true, then we will say that the ring R is left 1-dually steady. Some examples of such rings:

Proposition 3.1. *R is 1-dually steady in each of the following cases:*
(1) *R is right noetherian and every left ideal is a two-sided ideal.*
(2) *R is left semiartinian.*
(3) *R is countable.*

Proof. See [7], Prop. 3. □

Lemma 3.2. *Let R be a commutative domain with a quotient field $Q \neq R$. Then the following conditions are equivalent:*
(i) *R is ∩-compact.*
(ii) *Every countably generated submodule of ${}_RQ$ is a fractional ideal (i.e., ${}_RQ$ is strongly ∪-compact).*
Moreover, if R is uniserial, then the above conditions are equivalent to:
(iii) *${}_RQ$ is ∪-compact.*
(iv) *${}_RQ$ is not countably generated.*

Proof. Easy. □

4. Slender and slim modules and 2-dually steady rings

Consider the following three conditions for a module M:

(B4) If $\psi : \Pi_I B_i \to M$ is a homomorphism, then $\Pi_J B_i \subseteq \mathrm{Ker}(\psi)$ for a cofinite subset J of I.

(B5) If $\psi : \Pi_\omega B_i \to M$ is a homomorphism, then $\Pi_{i \geq n} B_i \subseteq \mathrm{Ker}(\psi)$ for some $n < \omega$.

(B6) If $\psi : R^\omega \to M$ is a homomorphism, then $R^{\omega \setminus n} \subseteq \mathrm{Ker}(\psi)$ for some $n < \omega$.

Clearly, the conditions (B5) and (B6) are equivalent and the corresponding modules are just the well known slender modules (see [3], Chapter III). Furthermore, (B4) implies (B5), (B6) and the modules satisfying (B4) are called slim in [4].

Proposition 4.1. ([1]) *A module M is slender if and only if* $\mathrm{Hom}_R(W, M) = 0$, $W = R^\omega/R^{(\omega)}$, *and M is not complete (i.e., not complete in any non-discrete linear Hausdorff topology).*

Proposition 4.2. (i) *If a module M is $\cap$-compact or if $|M| < 2^\omega$, then M is slender if and only if* $\mathrm{Hom}_R(W, M) = 0$.

(ii) *If S is a submodule of R^ω maximal with respect to $R^{(\omega)} \subseteq S$ and $(1, 1, 1, ...) \notin S$, then the factor-module $T = R^\omega/S$ is cocyclic and not slender.*

(iii) *If A is a non-zero slender module, then $A^{(\omega)}$ is a slender module that is not finitely cogenerated.*

Proof. (i) The assertion follows immediately from 4.1.

(ii) The socle of T is an essential simple submodule, and hence T is cocyclic. By 4.1, T is not slender.

(iii) Slender modules are closed under direct sums. $\square$

Proposition 4.3. *Assume that every maximal left ideal of R is a two-sided ideal. Then no non-zero finitely cogenerated module is slender.*

Proof. Due to 4.1, no simple module is slender and it is enough to take into account that slender modules are closed under submodules. $\square$

Proposition 4.4. *The following conditions are equivalent:*

(i) *All slender modules are finitely cogenerated.*

(ii) *There exist no non-zero slender modules.*

(iii) *There exist no non-zero slender cyclic modules.*

(iv) $\mathrm{Hom}_R(W, R/I) \neq 0$ *for every proper left ideal I such that the cyclic factor-module R/I is not complete.*

Moreover, if every left ideal of R is a two-sided ideal, then the above conditions are equivalent to:

(v) *Slender modules are closed under factor-modules.*

Proof. First, (i) implies (ii) by 4.2 (iii) and (ii) implies (i), (iii) and (v) trivially. Further, (iii) is equivalent to (iv) by 4.1 and (iii) implies (i) due to the fact that slender modules are closed under submodules. Now, assume that all left ideals are two-sided and that the condition (v) is satisfied. We are going to show that then (iii) is true. For, let I be an ideal of R such that the cyclic module $M = {}_RR/I$ is slender and let κ be a cardinal number such that $\kappa \geq |R^\omega|$. Since slender modules are closed under direct sums, the module $N = M^{(\kappa)}$ is slender. On the other hand, N may be viewed as a free R/I-module and consequently there is an epimorphism $\varphi : N \to M^\omega$. According to (v), M^ω is a slender module and it follows immediately that $M = 0$. $\square$

If the ring R satisfies the equivalent conditions 4.4 (i), ..., (iv), then we will say that R is left 2-dually steady.

Lemma 4.5. *Let I be an ideal, finitely generated as a right ideal, of R, and let $S = {}_RR/I$. If M is a module such that $IM = 0$, then M is a slender R-module if and only if M is a slender S-module.*

Proof. There is a natural isomorphism $\theta : W/IW \to S^\omega/S^{(\omega)}$. □

Corollary 4.6. *If R is left 2-dually steady and I is an ideal, finitely generated as a right ideal, of R, then the factor-ring R/I is also left 2-dually steady.*

Lemma 4.7. *No simple module is slender in each of the following cases:*
(1) *Every maximal left ideal is a two-sided ideal.*
(2) *$I \neq I^2$ for every maximal left ideal I.*
(3) *R is a left V-ring.*
(4) *Every simple module is finite.*
(5) *$R/\mathrm{J}(R)$ is a left 2-dually steady ring.*

Proof. Easy. □

Proposition 4.8. *Assume that R is left semiartinian. Then R is left 2-dually steady in each of the following cases:*
(1) *Every maximal left ideal is a two-sided ideal.*
(2) *R is a left V-ring.*
(3) *Every simple module is finite.*
(4) *$R/\mathrm{J}(R)$ is left 2-dually steady ring.*

Proof. Combine 4.7 and the fact that slender modules are closed under submodules. □

Corollary 4.9. *Every right perfect ring is both left 1- and 2-dually steady.*

Proposition 4.10. *Every complete commutative principal ideal domain is both 1- and 2-dually steady.*

Proof. If R is a principal ideal domain, then no simple module is slender and, up to isomorphism, the only Soc-torsionfree cyclic module is R itself. Now, it is clear that R satisfies the condition 4.4 (iv). □

Remark 4.11. Every slim module is slender and, according to [8] the converse is true if and only if there are no measurable cardinal numbers (and then the conditions (B4), (B5) and (B6) are equivalent). On the other hand, if κ is a measurable cardinal and $|R| < \kappa$, then there exist no non-zero slim (R-) modules. Consequently, in case there are too many measurable cardinals, non-zero slim modules do not exist over any ring (the converse is also true - see [4], Theorem 8.2).

5. Small 2-dually steady rings

Throughout this section, a small ring is any ring R with $|R| < 2^\omega$.

Proposition 5.1. *If R is small, then the following conditions are equivalent:*
(i) *R is left 2-dually steady.*
(ii) *$\mathrm{Hom}_R(W, R/I) \neq 0$ for every proper left ideal I.*
(iii) *$\mathrm{Hom}_R(W, M) \neq 0$ for every non-zero module M.*

Proof. We have $|A| \geq 2^\omega$ for every complete module A and the rest is clear from 4.1 and 4.4. □

Proposition 5.2. *A small prime ring R is (left, right) 2-dually steady if and only if R is (isomorphic to) a full matrix ring over a division ring.*

Proof. The direct implication follows from [5], statement 4.1 of the paper, and the converse one is clear. □

Theorem 5.3. *The following conditions are equivalent for a small ring R:*

(i) *R is right noetherian and left 2-dually steady.*

(ii) *R is right artinian.*

If these conditions are satisfied, then R is (left and right) steady, 1-dually steady and 2-dually steady.

Proof. Assume (i) be true. Since R satisfies maximal condition on ideals, the prime radical P of R is the intersection of a finite family of prime ideals, say $P = P_1 \cap \dots \cap P_n$. Further, by 4.6 and 5.2, all the factor-rings R/P_i are completely reducible and consequently $P = \mathrm{J}(R)$ and R is semilocal. Since R is right noetherian, P is nilpotent and it follows easily that R is right artinian.

Conversely, if R is right artinian, then R is (left, right) steady by 2.1 (2) (and its right hand form), R is 1-dually steady by 3.1 (2) and R is 2-dually steady by 4.9. □

6. Commutative noetherian 2-dually steady rings

Theorem 6.1. *Let R be a commutative noetherian ring.*

(i) *R is 2-dually steady if and only if every non-zero* Soc*-torsionfree cyclic module is complete.*

(ii) *If R is 2-dually steady, then R is semilocal and, moreover, if R is not artinian, then $\widehat{R/\mathrm{Soc}(R)}$ is complete.*

Proof. (i) Combine 4.4 (iii) and [6], statement 4.4 of the paper.

(ii) Use (i) and [6], statements 3.1 and 3.2 of the paper. □

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