Credit Risk Modelling: Intensity Based Approach

Tomasz Bielecki
Department of Mathematics
The Northeastern Illinois University, Chicago, USA

Marek Rutkowski
Faculty of Mathematics and Information Science
Warsaw University of Technology, 00-661 Warszawa, Poland

Contents

1 Introduction

In this article we present a survey of recent research efforts aimed at pricing and hedging of default-prone debt instruments. We concentrate on intensity and ratings based approaches. In particular we review results derived by Duffie, Schröder and Skiadas (1996), Duffie and Singleton (1998a, 1999), Jarrow and Turnbull (1995, 2000), Jarrow, Lando and Turnbull (1997), Lando (1998), Madan and Unal (1998, 1999), Jeanblanc and Rutkowski (2000a), and Bielecki and Rutkowski (1999, 2000), among results obtained by other researchers. In addition we present a brief survey of some important types of credit derivatives, that is derivative products linked to either corporate or sovereign debt. It should be emphasised that the need to rationally price and hedge credit derivatives, whose presence in financial markets has been continuously growing in the recent years, was one of the motivations, besides the need to manage credit risk, behind the explosion of research on quantitative aspects of the credit risk, that has been observed in the 1990s.

Let us mention here that the firm-specific approach – that is, an approach based on observations of the value of debt's issuer – is not addressed in the present article. This alternative approach was initiated in the 70s by Merton (1974), Black and Cox (1976), and Geske (1977). It was subsequently developed in various directions by several authors; to mention a few: Brennan and Schwartz (1997, 1980), Pitts and Selby (1983), Rendleman (1992), Kim et al. (1993), Nielsen et al. (1993), Leland (1994), Longstaff and Schwartz (1995), Leland and Toft (1996), Mella-Barral and Tychon (1996), Briys and de Varenne (1997), Crouhy et al. (1998), Duffie and Lando (1998), and Anderson and Sundaresan (2000). Reviewing this approach would require a separate article (see Chapter IV in Ammann (1999)). The list of references is not representative of all important papers and books published in this area in recent years, but it includes works that are most related to this presentation.

2 Credit Derivatives

Credit derivatives are privately negotiated derivatives securities that are linked to a credit-sensitive asset (index) as the underlying asset (index). More specifically, the reference security of a credit derivative can be an actively-traded corporate or sovereign bond or a portfolio of these bonds. A credit derivative can also have a loan (or a portfolio of loans) as the underlying reference credit. Credit derivatives can be structured in a large variety of ways; they are typically complex agreements, customized to the precise needs of an investor. The common feature of all credit derivatives is the fact that they allow for the transference of the credit risk from one counterparty to another, so that they can be used to control the credit risk exposure. *Credit risk* refers to the possibility that a borrower will fail to service or repay a debt on time. The overall risk we are concerned with involves two components: market risk and asset-specific credit risk. In contrast to 'standard' interest-rate sensitive derivatives, credit derivatives allow to isolate and handle not only the market risk, but also the firm-specific credit risk. They provide also a way to synthetize assets that are otherwise not available to a particular investor (in this application, an investor 'buys' – rather then 'sells' – a specific credit risk).

Similarly as in the case of derivative securities associated with the risk-free term structure, we may formally distinguish three main types of agreements: forward contracts, swaps, and options. A *forward contract* commits the buyer to purchasing a specified bond at a specified future date at a price predetermined at contract inception. In a forward contract, the default risk is normally borne by the buyer. If a credit event occurs, the transaction is marked to market and unwound. Forward contracts can also be transacted in spread form; that is, the agreement can be based on the specifed bond's spread over a benchmark asset. It should be stressed that the classification above does not corresponds to market terminological conventions, as described below.

In market practice, the most popular credit-sensitive swap contract is a *total rate of return swap,* explained in some detail in Section 2.1 below. Credit options are typically embedded in complex credit-sensitive agreements, though the over-the-counter traded credit options – such as *default puts* also described in Section 2.1 – are also available. Let us finally mention the so-called *vulnerable* *options,* or more generally, *vulnerable claims.* These are contingent agreements that are issued by credit-sensitive institutions, so that they are subject to default in much the same way as defaultable bonds.

2.1 Overview of Instruments

We first review the most actively traded types of credit-sensitive agreements.[1] It should be stressed that we do not intend to examine here all aspects of credit derivatives as a tool in the risk management. The nonexhaustive list of examples given below makes it clear that a wide range of objectives can be achieved by trading in credit derivatives. For an extensive analysis of economical reasons which support the use of these products, we refer to Das (1998a, 1998b) or Tavakoli (1998).

Total rate of return swaps. *Total rate of return swaps* (*total return swaps,* for short) are agreements in which the total return of an underlying credit-sensitive asset (basket of assets, index, etc.) is exchanged for some other cash flow. More specifically, one party agrees to pay the total return (income plus or minus any change in the capital value) on a notional principal amount to another party in return for periodic fixed or floating-rate payments on the same notional amount. Let us enumerate the most important features of a total return swap: (a) no principal amounts are exchanged and no physical change of ownership occurs, (b) the maturity of the total return swap agreement need not match that of the underlying, (c) at the contract termination – i.e., at the contract maturity or upon default – according to Das (1998a), "a price settlement based on the change in the value of the bond or loan is made." Total return swaps can incorporate put and call options (to establish caps and floors on the returns of the reference assets), as well as caps and floors on a floating interest rates.

Credit-spread swaps and options. With *credit-spread swaps* (that is, relative performance total return swaps), also known as *credit-spread forwards*, investors pay the total return of one asset while receiving the total return of another credit-sensitive asset. *Credit-spread options* are option agreements whose payoff is associated with the yield differential of two credit-sensitive assets. For instance, the reference rate of the option can be a spread of a corporate bond over a benchmark asset of comparable maturity. The option can be settled either in cash or through physical delivery of the underlying bond, at a price whose yield spread over the benchmark asset equals the strike spread. Options on credit spreads allow to isolate the firm-specific credit risk from the market risk.

Credit (default) swaps. These are agreements in which a periodic fixed payments (or upfront fee) from the protection buyer is exchanged for the promise of some specified payment from the protection seller to be made only if a particular, predetermined *credit event* occurs. If, during the term of the default swap, a credit event occurs, the seller pays the buyer an amount to cover the loss, and the swap then terminates. If no credit event has occurred by maturity of the swap, both sides end their obligations to each other. The most important covenants of a credit swap contract are: (a) the specification of the *credit event,* which is formally defined as a 'default' (in practice, it may include: bankruptcy, insolvency, or payment default, a stipulated price decline for the reference asset, a rating downgrade for the reference asset), (b) the contingent default payment, which may be structured in a number of ways; for instance, it may be linked to the price movement of the reference asset, or it can be set at a predetermined level (e.g., a fixed percentage of the notional amount of the transaction), (c) the specification of periodic payments which depend, in large part, on the credit quality of the reference asset. Credit swaps are usually settled in cash, but the agreement may also provide for physical delivery; for example, it may involve payment at par by the seller in exchange for the delivery of the defaulted reference asset. If the payment is triggered by the default and equals to the difference between the face value of a bond and its market price, the contract is named the *default swap.* Let us finally mention the so-called *first-to-default swaps,* which are examples of *basket default swaps* (i.e., default swaps linked to a portfolio of credit-sensitive securities).

[1] Let us mention that the terminologal conventions relative to credit derivatives are not yet fully standardized; we shall try to follow the most widely accepted terminology.

Credit (default) options. A credit call (put, resp.) option gives the right to buy (to sell, resp.) an underlying credit-sensitive asset (index, credit spread, etc.) at a predetermined price. The most widely used type of a credit option is a *default put.* The buyer of the default put pays a premium (either an upfront fee or a periodic payment) to the seller who then assumes the default risk for the reference asset. If there is a credit (default) event during the term of the option, the seller pays the buyer a (fixed or variable) *default payment.*

Credit linked notes. *Credit linked notes* are debt instruments in which the coupon or price of the note is linked to the performance of a reference credit-sensitive asset (rate or index). For instance, a credit-linked note may stipulate that the principal repayment is reduced to a certain level below par if the external corporate or sovereign debt defaults before the maturity of the note. This means that the buyer of the note sells credit protection to the issuer of the note; in exchange the note pays a higher-than-normal yield.

2.2 Market Pricing Methods

Since a reliable benchmark model for credit derivatives is not yet available, it is common in market practice to value a credit derivative on a stand-alone basis, using a judiciously chosen ad hoc approach, rather than a sophisticated mathematical model.

We shall now review the most widely used of these approaches. For explanatory purposes, we focus on the valuation of a default swap, and we base our description of the four pricing methods on BeSaw (1997).

Same-cost as reference method. To estimate the price of a default swap, one assumes that there exists an insured bond which is otherwise identical to the reference bond of the swap. The spread between the yield of the insured bond and that of the reference bond can then be taken as the proxy of the default swap price. Notice that this method identifies a default swap with bond insurance, and disregards the credit difference between the bond insurer and the default swap counterparty.

Credit-spread-based method. This way of default swap valuation is based on a comparison of the yield of the reference bond and the yield of a risk-free bond with similar maturity. It is thus implicitly assumed that the spread over the risk-free asset is entirely due to the credit risk so that the impact of tax and/or liquidity effects are neglected. Another difficulty arises when one wishes to price a swap with maturity which does not correspond to the maturity of the reference corporate bond.

Replication of cost method. In this method, the price of a default swap is calculated through evaluation of the cost of a portfolio necessary to replicate the swap. The replication of cost method thus mimics the standard approach to contingent claims valuation in an arbitrage-free setup. Unfortunately, it is typically not possible or to costly to establish a (static or dynamic) portfolio which fully hedges (i.e. replicates) a credit derivative.

Ratings-based default method. This approach, which will be analysed in more details in what follows, determines the price of a credit derivative (for instance, a default swap) as the expected loss resulting from default. To derive default probabilities, it is common to model the Markov chain representing ratings migration process using the estimated credit ratings transition matrix. If the valuation is made on a stand-alone basis, it would be more adequate to use the firm-specific transition matrix corresponding to the reference asset. It is clear that such a matrix is not easily available, however. Similarly, constant (or random) recovery rates, which are needed to evaluate the expected loss, are either inferred using the historical data, or assessed on a stand-alone basis. The *credit-spread-based default method* can be seen as a variant of a ratings-based default method. It uses an issuer-specific credit spread over default-free instruments of similar maturity to estimate the probability of default and the expected recovery rate in default.

For an exhaustive analysis of practical aspects of credit swaps and a review of non-technical methods of their valuation (including the estimation of hazard rates), we refer to Duffie (1999).

3 Valuation of Defaultable Claims

The exposition in this section is mainly based on Duffie et al. (1996). In this section, our goal is to present the most fundamental results which can be obtained using the intensity-based approach. In Section 4, special attention will be paid to the various kinds of recovery rates, such as, for instance, zero recovery, fractional recovery of par, and fractional recovery of market value. On the other hand, in order to obtain as explicit valuation formulae as possible, we shall still assume that only two states are possible, namely, non-default and default. An analysis of the case of several credit rating classes is postponed to Sections 5–7. We make the following standing assumptions.

(A.1) We are given a probability space $(\Omega, \mathcal{G}, \mathbb{P}^*)$, endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ (of course, $\mathcal{F}_t \subset \mathcal{G}$ for any t). The probability measure $\mathbb{P}^*$ is interpreted as a martingale measure for our underlying securities market model (complete or not). Let τ be a non-negative random variable on the probability space $(\Omega, \mathcal{G}, \mathbb{P}^*)$. In what follows, we shall refer to τ as the *default time.*

For convenience, we assume that for every $t \in \mathbb{R}_+$ $\mathbb{P}^*\{\tau = 0\} = 0$ and $\mathbb{P}^*\{\tau > t\} > 0$. Given a default time τ, we introduce the associated jump process H by setting $H_t = \mathbb{1}_{\{\tau \le t\}}$ for $t \in \mathbb{R}_+$. It is obvious that H is a right-continuous process. Let $\mathbb{H}$ be the filtration generated by the process H, i.e., $\mathcal{H}_t = \sigma(H_u : u \le t)$. We introduce the enlarged filtration $\mathbb{G}$ which satisfies $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ – that is, $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t = \sigma(\mathcal{H}_t, \mathcal{F}_t)$ for every t. The default time τ is not necessarily a $\mathbb{F}$-stopping time, but it is obviously a $\mathbb{G}$-stopping time.

(A.2) For a given default-risky security, its *default process* is modelled through a jump process H with strictly positive *intensity* (or *hazard rate*) process[2] λ under $\mathbb{P}^*$. The intensity λ is a $\mathbb{F}$-progressively measurable process such that the *compensated process*

$$M_t := H_t - \int_0^{t\wedge\tau} \lambda_u \, du = H_t - \int_0^t h_u \, du, \quad \forall\, t \in [0, T^*], \tag{3.1}$$

follows a $\mathbb{G}$-martingale under $\mathbb{P}^*$. Notice that $h_t := \mathbb{1}_{\{t \le \tau\}}\lambda_t$.

Remarks. Let us stress that the stochastic intensity λ is assumed to be $\mathbb{F}$-adapted, and, in general, $\mathbb{F}$ is strictly smaller than $\mathbb{G}$. On the other hand, the case of a $\mathbb{F}$-stopping time is also covered (in this case, $\mathbb{F} = \mathbb{G}$).

(A.3) Given a maturity date $T > 0$, a $\mathcal{F}_T$-measurable random variable X represents the *promised claim,* that is, the amount of cash which the owner of a defaultable claim is entitled to receive at time T, provided that the default has not occurred before the maturity date T.

(A.4) A $\mathbb{F}$-predictable process Z models the payoff which is actually received by the owner of a defaultable claim, if default occurs before maturity T. We refer to Z as the *recovery process* of X.

(A.5) A $\mathbb{F}$-adapted process r stands for the short-term interest rate, and $B_t = \exp(\int_0^t r_u \, du)$, $t \in \mathbb{R}_+$, is the associated savings account process.

The main result in the intensity-based approach states that a defaultable security can be priced as if it were a default-risk free security, provided that the credit spread is already incorporated in the risk premium. In other words, the risk premium process of a defaultable security differs from that associated with a risk-free bond, both in the real-world and in the risk-neutral world. In particular, in a risk-neutral world the risk premium associated with a risk-free bond vanishes, but the risk premium associated with a defaultable security is still present.

Example 3.1 If the intensity process $\lambda_t = \lambda > 0$ is constant, the process H can be seen as a continuous-time Markov chain with the state space $\{0, 1\}$, and with constant intensity matrix $\Lambda = [\lambda_{ij}]_{0\le i,j\le 1}$, where $\lambda_{00} = -\lambda$, $\lambda_{01} = \lambda$, and $\lambda_{1i} = 0$ for $i = 0, 1$ (so that the state 1 is absorbing). In this case, τ can be seen as the first jump time of a standard Poisson process N with constant

[2] We refer to Artzner and Delbaen (1995), Elliott et al. (2000), Jeanblanc and Rutkowski (2000b), Kusuoka (1999) or Rutkowski (1999) for more details on stochastic intensities. See Lemma 16 in Wong (1998) for a more general setup.

intensity λ. This simple example can be generalized in two directions. First, in some circumstances it might be natural to assume that $\lambda_t = \lambda(Y_t)$, where Y is a given k-dimensional $\mathbb{F}$-adapted stochastic process, and $\lambda : \mathbb{R}^k \to \mathbb{R}_+$ is a strictly positive deterministic function. Second, the basic model can be extended to accomodate for different credit rating classes, $\Lambda_t = [\lambda_{ij}(Y_t)]_{0\le i,j\le K}$, with K being an absorbing state (see, for instance, Jarrow et al. (1997)).

We need first to formally define the value process S of a (European) *defaultable claim,* represented by a triplet (X, Z, τ) and maturity date T. Since we assume throughout that $\mathbb{P}^*$ is a spot martingale measure, it is natural to postulate that the value S_0 at time 0 of a defaultable claim (X, Z, τ) equals

$$S_0 := B_0 \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]0,T]} B_u^{-1} \, dD_u \Big), \tag{3.2}$$

where $B_t = \exp(\int_0^t r_u \, du)$ stands for the savings account process, and D is the 'dividend process' (cf. (A.3)-(A.4))

$$D_t = \int_{]0,t]} Z_u \, dH_u + X(1 - H_T) \mathbb{1}_{\{t=T\}}. \tag{3.3}$$

Formula (3.2) can be easily generalized to give the price of a defaultable claim for any date t, namely

$$S_t := B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} \, dD_u \, \Big| \, \mathcal{G}_t \Big), \tag{3.4}$$

or equivalently,

$$S_t := B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u \, dH_u + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \, \Big| \, \mathcal{G}_t \Big). \tag{3.5}$$

In particular, at maturity of the contract we have $S_T = X \mathbb{1}_{\{T<\tau\}}$, as expected. Notice that (3.5) can be also rewritten as follows

$$S_t = B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \, \Big| \, \mathcal{G}_t \Big), \tag{3.6}$$

or finally,

$$S_t = \mathbb{E}_{\mathbb{P}^*}\Big(e^{-\int_t^{\tau\wedge T} r_u \, du} \big(Z_\tau \mathbb{1}_{\{t<\tau\le T\}} + X \mathbb{1}_{\{T<\tau\}} \big) \, \Big| \, \mathcal{G}_t \Big). \tag{3.7}$$

Definition 3.1 By a *defaultable claim* we mean a triplet (X, Z, τ), where X is the *promised payoff,* Z represents the *recovery process* of X, and τ is the *default time.* The *price* (or *value*) *process* S of a defaultable claim (X, Z, τ) is given by either of the formulae (3.4)-(3.7).

Remarks. Notice that Definition 3.1 specifies the price of a defaultable security on the ex-dividend basis. In particular, for any t we have $S_t = 0$ on the event $\{\tau \le t\}$. Intuitively, this means that the payoff at the event of default is received in cash (and invested, e.g., in the risk-free savings account), and the defaultable security becomes worthless forever. This convention agrees, of course, with our current set of Assumptions (A.1)-(A.5), but does not necessarily reflect the actual bankruptcy procedures. Once again, it should be generalized to fit more adequately the real-world behaviour of defaultable securities.

The following lemma provides still another representation for the price process S of a defaultable claim. It appears that, due to Assumption (A.2), the integration with respect to the process H_t can be substituted with the integration with respect to the associated *intensity measure* $h_t dt$.

Lemma 3.1 *The price process S admits the following representations*

$$S_t = B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T B_u^{-1} Z_u h_u \, du + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \, \Big| \, \mathcal{G}_t \Big) \tag{3.8}$$

and

$$S_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \big(Z_u h_u - r_u S_u \big) \, du + X \mathbb{1}_{\{T<\tau\}} \, \Big| \, \mathcal{G}_t \Big). \tag{3.9}$$

Proof. The first formula follows from (3.5), combined with the equality

$$\mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u\, dH_u \,\Big|\, \mathcal{G}_t\Big) = \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1} Z_u\, (dM_u + h_u\, du)\Big|\, \mathcal{G}_t\Big)$$

which in turn is an immediate consequence of (3.1). For the second, it is enough to rewrite (3.8) as follows

$$S_t = B_t\Big(\tilde{M}_t - \int_0^t B_u^{-1} Z_u h_u\, du\Big), \tag{3.10}$$

where we have put

$$\tilde{M}_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_0^T B_u^{-1} Z_u h_u\, du + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big).$$

Applying Itô's formula to (3.10), we obtain

$$dS_t = (r_t S_t - Z_t h_t)\, dt + B_t\, d\tilde{M}_t,$$

and thus

$$\mathbb{E}_{\mathbb{P}^*}(S_T \,|\, \mathcal{G}_t) = S_t + \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T (r_u S_u - Z_u h_u)\, du \,\Big|\, \mathcal{G}_t\Big).$$

Since obviously $S_T = X\mathbb{1}_{\{T<\tau\}}$, the last equality yields (3.9). □

Notice that for Lemma 3.1 to hold, it is enough to assume that processes B and Z are $\mathbb{G}$-predictable, and X is $\mathcal{G}_T$-measurable. The following result – due to Duffie et al. (1996) – plays a crucial role in what follows.

Theorem 3.1 *For a given $\mathbb{F}$-predictable process Z and $\mathcal{F}_T$-measurable random variable X, we define the process V by setting*

$$V_t = \tilde{B}_t\, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t\Big), \tag{3.11}$$

where $\tilde{B}$ is the 'savings account' corresponding to the default-adjusted short-term rate $R_t = r_t + \lambda_t$, that is,

$$\tilde{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u)\, du\Big). \tag{3.12}$$

Then

$$\mathbb{1}_{\{t<\tau\}} V_t = B_t\, \mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1}(Z_\tau + \Delta V_\tau)\mathbb{1}_{\{t<\tau\le T\}} + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big). \tag{3.13}$$

Proof. In view of (3.11), we have

$$V_t = \tilde{B}_t\Big(N_t - \int_0^t \tilde{B}_u^{-1} Z_u \lambda_u\, du\Big), \tag{3.14}$$

where N is a $\mathbb{G}$-martingale given by the formula

$$N_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_0^T \tilde{B}_u^{-1} Z_u \lambda_u\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t\Big). \tag{3.15}$$

Using Itô's product rule, we obtain

$$dV_t = r_t V_t\, dt - (Z_t - V_{t-})\lambda_t\, dt + \tilde{B}_t\, dN_t. \tag{3.16}$$

Define $U_t = \tilde{H}_t V_t$, where $\tilde{H}_t = 1 - H_t = \mathbb{1}_{\{t<\tau\}}$, so that $U_t = \mathbb{1}_{\{t<\tau\}} V_t$. It is useful to observe that (3.13) may be rewritten as follows

$$U_t = B_t\, \mathbb{E}_{\mathbb{P}^*}\Big(\int_{]t,T]} B_u^{-1}(Z_u + \Delta V_u)\, dH_u + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big). \tag{3.17}$$

On the other hand, an application of Itô's product rule yields (obviously the process $\tilde{H}$ is of finite variation)

$$dU_t = d(V_t \tilde{H}_t) = \tilde{H}_{t-}\, dV_t + V_{t-}\, d\tilde{H}_t + \Delta V_t \Delta \tilde{H}_t.$$

In view of (3.16) and equality $h_t = \lambda_t 1\!\!1_{\{t \le \tau\}}$, this yields

$$dU_t = d(V_t \tilde{H}_t) = \tilde{H}_{t-}\big(r_t V_t\, dt - (Z_t - V_{t-}) h_t\, dt + \tilde{B}_t\, dN_t\big) + V_{t-}\, d\tilde{H}_t + \Delta V_t \Delta \tilde{H}_t.$$

After rearranging and noticing that $\Delta \tilde{H}_t = -\Delta H_t$, we obtain

$$dU_t = r_t U_t\, dt - (Z_t + \Delta V_t)\, dH_t + d\tilde{N}_t, \tag{3.18}$$

where $\tilde{N}$ stands for the local $\mathbb{G}$-martingale, more precisely,

$$d\tilde{N}_t = \tilde{H}_{t-} \tilde{B}_t\, dN_t + (Z_t - V_{t-})\, dM_t.$$

Since $U_T = X 1\!\!1_{\{T<\tau\}}$, formula (3.18) gives expression (3.17) (if the local martingale $\tilde{N}$ is in fact a 'true' martingale). □

Once again, for the validity of Theorem 3.1 and Corollary 3.1, it is enough to assume that processes B and Z are $\mathbb{G}$-predictable, and X is $\mathcal{G}_T$-measurable.

Corollary 3.1 *Let the processes S and V be defined by* (3.5) *and* (3.11), *respectively. Then* (i)

$$S_t = 1\!\!1_{\{t<\tau\}}\Big(V_t - B_t\, \mathbb{E}_{\mathbb{P}^*}\big(B_\tau^{-1} 1\!\!1_{\{\tau \le T\}} \Delta V_\tau \,\big|\, \mathcal{G}_t\big)\Big), \tag{3.19}$$

(ii) *if $\Delta V_\tau = 0$, then $S_t = 1\!\!1_{\{t<\tau\}} V_t$ for every $t \in [0, T]$.*

Proof. A comparison of expressions (3.6) and (3.13) yields

$$S_t = U_t - B_t\, \mathbb{E}_{\mathbb{P}^*}\big(B_\tau^{-1} 1\!\!1_{\{t<\tau \le T\}} \Delta V_\tau \,\big|\, \mathcal{G}_t\big).$$

Formula (3.19) now easily follows. □

For easy further reference, we shall write down the particular case of (3.19) when $\Delta V_\tau = 0$. In this case, we have simply $S_t = U_t$, that is,

$$S_t = 1\!\!1_{\{t<\tau\}} \tilde{B}_t\, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t\Big). \tag{3.20}$$

A more general version of (3.20) is proved in Proposition 5 in Wong (1998). The formula there is called the *price representation theorem.*

Remarks. In view of the relationship established in part (ii) of Corollary 3.1, the process V given by formula (3.11) is commonly referred to as the *pre-default value* of a defaultable claim X. To examine the continuity condition $\Delta V_\tau = 0$, we find it convenient to introduce additional restrictions on the underlying filtrations.[3] It will soon become clear, that we need to restrict our attention to the case of $\mathbb{F}$-predictable processes B and Z, and to $\mathcal{F}_T$-measurable random variable X.

3.1 Hypotheses (H)

We shall now examine some specific assumptions related to the underlying filtrations. Let us first formulate the following hypothesis (recall that $\mathcal{F}_t \subseteq \mathcal{G}_t$ so that $\mathcal{G}_t \vee \mathcal{F}_t = \mathcal{G}_t$).

Assumption (H.1) For any t, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{G}_t$ are conditionally independent given $\mathcal{F}_t$. Equivalently, for any t, and any bounded $\mathcal{F}_\infty$-measurable r.v. ξ we have $\mathbb{E}_{\mathbb{P}^*}(\xi \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}^*}(\xi \,|\, \mathcal{F}_t)$.

[3]Let us mention that these hypotheses are satisfied in the classic case of Cox processes.

Definition 3.2 We say that a filtration $\mathbb{F}$ has the *martingale invariance property* with respect to a filtration $\mathbb{G}$ if every $\mathbb{F}$-martingale is also a $\mathbb{G}$-martingale.

Lemma 3.2 *A filtration $\mathbb{F}$ has the martingale invariance property with respect to a filtration $\mathbb{G}$ if and only if condition* (H.1) *is satisfied.*

Proof. Assume first that (H.1) holds. Let M be an arbitrary $\mathbb{F}$-martingale. Then for any $t \le s$ we have

$$\mathbb{E}_{\mathbb{P}^*}(M_s \,|\, \mathcal{G}_t) = \mathbb{E}_{\mathbb{P}^*}(M_s \,|\, \mathcal{F}_t) = M_t,$$

so that M is a $\mathbb{G}$-martingale. Conversely, let us assume that every $\mathbb{F}$-martingale is a $\mathbb{G}$-martingale. We shall check that this implies (H.1). To this end, for any fixed $t \le s$ we consider an arbitrary set $A \in \mathcal{F}_\infty$. We introduce the $\mathbb{F}$-martingale $M_u := \mathbb{E}_{\mathbb{P}^*}(\mathbb{1}_A \,|\, \mathcal{F}_u)$, $u \in \mathbb{R}_+$. Since M is also a $\mathbb{G}$-martingale, we obtain

$$\mathbb{E}_{\mathbb{P}^*}(\mathbb{1}_A \,|\, \mathcal{G}_t) = M_t = \mathbb{E}_{\mathbb{P}^*}(\mathbb{1}_A \,|\, \mathcal{F}_t).$$

By standard arguments this shows that (H.1) is satisfied. □

Recall that in the present setup we have $\mathbb{G} = \mathbb{H} \vee \mathbb{F}$ for a certain filtration $\mathbb{H}$. Let us introduce the following condition.

Assumption (H.2) For any t, the σ-fields $\mathcal{F}_\infty$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$.

Since $\mathcal{H}_t \subset \mathcal{G}_t$, it is easily seen that (H.1) is stronger than (H.2). It appears that assumptions (H.1) and (H.2) are in fact equivalent.

Lemma 3.3 *Conditions* (H.1) *and* (H.2) *are equivalent.*

Proof. It is enough to check that (H.2) implies (H.1). Condition (H.2) is equivalent to the following one: for any bounded $\mathcal{F}_\infty$-measurable random variable ξ, we have $\mathbb{E}_{\mathbb{P}^*}(\xi \,|\, \mathcal{H}_t \vee \mathcal{F}_t) = \mathbb{E}_{\mathbb{P}^*}(\xi \,|\, \mathcal{F}_t)$. Since $\mathcal{G}_t = \mathcal{H}_t \vee \mathcal{F}_t$, this immediately gives (H.1). □

Under assumption (H.1) the conditioning with respect to $\mathcal{G}_t$ in (3.11) may be replaced by conditioning with respect to $\mathcal{F}_t$, that is, we may set

$$V_t = \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big).$$

This follows from the fact that the process N given by (see formula (3.15) in the proof of Theorem 3.1)

$$N_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_0^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big) \tag{3.21}$$

is not only a $\mathbb{F}$-martingale but also a $\mathbb{G}$-martingale. Therefore, (3.16) gives the semimartingale decomposition of V with respect to both filtrations, $\mathbb{F}$ and $\mathbb{G}$. The remaining part of the proof of Theorem 3.1 is thus still valid. If, in addition, $\Delta V_\tau = 0$ then we have

$$S_t = \mathbb{1}_{\{t<\tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big). \tag{3.22}$$

In some particular case, for instance when the filtration $\mathbb{F}$ is generated by a Brownian motion (under $\mathbb{P}^*$), the continuity of the process N given by (3.21), and thus also the continuity of V is obvious. In many other important practical cases, the validity of (3.22) can be verified directly (see, for instance, Proposition 6.1 below).

In general case, it seems more convenient to derive formula (3.22) using the standard results on intensities of random times (see, e.g., Elliott et al. (2000), Jeanblanc and Rutkowski (2000b), or Rutkowski (1999)), rather than Theorem 3.1. To this end, notice that since obviously $\mathcal{F}_t \subset \mathcal{F}_\infty$, we may restate condition (H.2) as follows:

Condition (H.3) For any $t \in \mathbb{R}_+$ and every $u \leq t$, we have $\mathbb{P}(\tau \leq u \,|\, \mathcal{F}_t) = \mathbb{P}(\tau \leq u \,|\, \mathcal{F}_\infty)$.

It is thus clear that in the present setup, the process $F_t := \mathbb{P}^*(\tau \leq t \,|\, \mathcal{F}_t)$ admits a modification with increasing sample paths. Assume that $F_t < 1$ for every $t \in \mathbb{R}_+$. The $\mathbb{F}$-*hazard process* of τ, denoted by Γ, is defined through the formula $1 - F_t = e^{-\Gamma_t}$, or equivalently, $\Gamma_t = -\ln(1 - F_t)$ for every $t \in \mathbb{R}_+$. If F follows an absolutely continuous process, then it can be shown (see the abovementioned papers for details) that $\Gamma_t = \int_0^t \lambda_u \, du$, and

$$\begin{aligned} S_t &= B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1} Z_\tau \mathbb{1}_{\{t<\tau\leq T\}} + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t\Big) \\ &= \mathbb{1}_{\{t<\tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u Z_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t\Big). \end{aligned}$$

This means that under the above set of assumptions, we have $\mathbb{E}_{\mathbb{P}^*}\big(B_\tau^{-1} \mathbb{1}_{\{\tau \leq T\}} \Delta V_\tau \,\big|\, \mathcal{G}_t\big) = 0$.

4 Alternative Recovery Schemes

In this section, we shall further specify the model presented in the previous section, by introducing various kinds of recovery processes. The recent work by Wong (1998) provides an interesting study of various recovery schemes in the framework of a fairly general model. We do not present Wong's results here, however, and we refer an interested reader to the original paper. We assume throughout that (H.1) (or equivalently (H.2)) holds.

4.1 Exogenous Recovery Rates

Assume, as before, that Z is an exogenously given $\mathbb{F}$-predictable process. The price process S of a defaultable claim is uniquely specified through expressions (3.5)-(3.6). It is thus clear that only the values of the process Z at default time τ are essential. Therefore, instead of specifying the $\mathbb{F}$-predictable process Z, it is enough to consider a random variable Z_τ. For instance, we could postulate that we are given a bounded random variable, denoted by W, which models the recovery value at default time. By assumption, W is $\mathcal{F}_\tau$-measurable random variable, meaning that[4] $W = Z_\tau$ for some $\mathbb{F}$-adapted process Z (a slightly stronger assumption would be to postulate that W is $\mathcal{F}_{\tau-}$-measurable random variable, which would mean that $W = Z_\tau$ for some $\mathbb{F}$-predictable process Z).

Following Duffie (1998b), we shall now consider both the case of discrete-time and continual recovery of a defaultable claim with an arbitrary recovery value W. In the case of *continual recovery,* the price process S of a defaultable claim X is set to satisfy (as before, we assume that the claim is of European style and it settles at time T)

$$S_t := B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_\tau^{-1} W \mathbb{1}_{\{t<\tau\leq T\}} + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big). \tag{4.1}$$

It appears (see Duffie (1998a) in this regard) that the results of Section 3 remain valid in the case of continual recovery with the recovery value W, provided that the recovery process Z is substituted with a $\mathbb{F}$-predictable process $\widetilde{W}$ which satisfies $\widetilde{W}_\tau = \mathbb{E}_{\mathbb{P}^*}(W \,|\, \mathcal{G}_{\tau-})$ (notice that $\mathbb{E}_{\mathbb{P}^*}(W \,|\, \mathcal{G}_{\tau-})$ is $\mathcal{G}_{\tau-}$-measurable random variable in the usual sense, and $\mathcal{F}_{\tau-}$-measurable random variable in the generalized sense).

On the other hand, a *discrete-time recovery* assumes that the payoff at the event of default is received by the owner of a claim on the first date after default among a predetermined set of *admissible* dates $0 = T_0 < T_1 < \ldots < T_n = T$. Under this convention, the value process $\tilde{S}$ of a defaultable claim equals

$$\tilde{S}_t := \sum_{T_i \geq t} B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} W \mathbb{1}_{\{T_{i-1}<\tau\leq T_i\}} \,\big|\, \mathcal{G}_t\big) + B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big). \tag{4.2}$$

[4]Notice that τ is not necessarily a $\mathbb{F}$-stopping time, so that $\mathcal{F}_\tau$ is not defined as the 'usual' σ-field generated by a $\mathbb{F}$-stopping time. For the more general definition of $\mathcal{F}_\tau$-measurability we use here, see Page 202 in Dellacherie and Meyer (1975).

In practical terms, when default occurs, the associated payoff (if any) is postponed to the nearest date T_i after default. It should be stressed that it is now enough to assume that a random variable W is such that for every $i = 1, \ldots, n$, the random variable $W_i = W \mathbb{1}_{\{T_{i-1} < \tau \le T_i\}}$ is $\mathcal{G}_{T_i}$-measurable. Put another way, the amount which is paid to the owner of the claim at the date T_i is based on the total information which is available at this time, including the default event $\{T_{i-1} < \tau \le T_i\}$. For technical reasons, we shall postulate that for every i we have

$$W_i = \hat{W}_i \mathbb{1}_{\{T_{i-1} < \tau \le T_i\}} \tag{4.3}$$

where for each i the random variable $\hat{W}_i$ is $\mathcal{F}_{T_i}$-measurable.

It is worthwhile to observe that the valuation formula (4.2) has slightly different practical features than the basic valuation formula (3.5). Indeed, formula (3.5) implicitly assumes that a defaultable claim becomes worthless as soon as a default occurs. On the other hand, when formula (4.2) is used to value a defaultable claim, a claim becomes worthless not at the time of default, but after the nearest date from the set of admissible dates.

Our next goal is to get a more explicit expression for (4.2). For a fixed $t \le T$, we shall write $i_0 = i_0(t) = \inf\{\, i : T_i \ge t\,\}$. It is thus clear that

$$\tilde{S}_t = \sum_{i=i_0}^{n} (\hat{U}_t^i - \tilde{U}_t^i) + U_t^n,$$

where

$$\hat{U}_t^i = B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} \hat{W}_i \mathbb{1}_{\{T_{i-1} < \tau\}} \,\big|\, \mathcal{G}_t\big), \quad \tilde{U}_t^i = B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} \hat{W}_i \mathbb{1}_{\{T_i < \tau\}} \,\big|\, \mathcal{G}_t\big),$$

and

$$U_t^n = B_t \, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_n}^{-1} X \mathbb{1}_{\{T_n < \tau\}} \,\big|\, \mathcal{G}_t\big).$$

Since for every $i = i_0, \ldots, n$ we have: (a) $\mathcal{G}_t \subset \mathcal{G}_{T_i}$, and (b) the random variable W_i is $\mathcal{G}_{T_i}$-measurable, the evaluation of $\tilde{U}_t^i$, $i = 1, \ldots, n$ and U_t^n is standard. Indeed, we may apply previously established results, with $Z = 0$ and $T = T_i$. To get a more transparent expression for the valuation formula, we shall assume that $\Delta V_\tau = 0$, where V stands for the pre-default value process introduced in Theorem 3.1 (since in the present context V depends on i, so that the assumption that V doesn't jump at default time is made for every i). Using (3.22), we obtain

$$\tilde{U}_t^i = \mathbb{1}_{\{t < \tau\}} \, \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\big(\tilde{B}_{T_i}^{-1} \hat{W}_i \,\big|\, \mathcal{F}_t\big)$$

for $i = 1, \ldots, n$, and

$$U_t^n = \mathbb{1}_{\{t < \tau\}} \, \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\big(\tilde{B}_{T_n}^{-1} X \,\big|\, \mathcal{F}_t\big).$$

We may proceed in a similar way when dealing with $\hat{U}_t^i$, provided that $i \ge i_0 + 1$ (this ensures that $\mathcal{G}_t \subset \mathcal{G}_{T_{i-1}}$). To this end, we find it convenient to represent $\hat{U}_t^i$ as follows

$$\hat{U}_t^i = B_t \, \mathbb{E}_{\mathbb{P}^*}\Big(B_{T_{i-1}}^{-1} \mathbb{E}_{\mathbb{P}^*}\big(B_{T_{i-1}} B_{T_i}^{-1} \hat{W}_i \,\big|\, \mathcal{G}_{T_{i-1}}\big) \mathbb{1}_{\{T_{i-1} < \tau\}} \,\Big|\, \mathcal{G}_t\Big).$$

This means that

$$\hat{U}_t^i = B_t \, \mathbb{E}_{\mathbb{P}^*}(B_{T_{i-1}}^{-1} Y_i \mathbb{1}_{\{T_{i-1} < \tau\}} \,|\, \mathcal{G}_t),$$

where Y_i is a $\mathcal{F}_{T_{i-1}}$-measurable random variable (in the second equality below, we make use of assumption (H.2))

$$Y_i = B_{T_{i-1}} \mathbb{E}_{\mathbb{P}^*}(B_{T_i}^{-1} \hat{W}_i \,|\, \mathcal{F}_{T_{i-1}} \vee \mathcal{H}_{T_{i-1}}) = B_{T_{i-1}} \mathbb{E}_{\mathbb{P}^*}(B_{T_i}^{-1} \hat{W}_i \,|\, \mathcal{F}_{T_{i-1}}). \tag{4.4}$$

Notice that Y_i represents the price at time T_{i-1} of a non-defaultable claim that pays $\hat{W}_i$ at time T_i. Arguing along the same lines as before, we get

$$\hat{U}_t^i = \mathbb{1}_{\{t < \tau\}} \, \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*}\big(\tilde{B}_{T_{i-1}}^{-1} Y_i \,\big|\, \mathcal{F}_t\big).$$

It thus remains to analyse the following term

$$\hat{U}_t^{i_0} = B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\mathbb{E}_{\mathbb{P}^*} \big(B_{T_{i_0}}^{-1} \hat{W}_{i_0} \,\big|\, \mathcal{G}_{T_{i_0-1}} \big) \mathbb{1}_{\{T_{i_0-1} < \tau\}} \,\Big|\, \mathcal{G}_t \Big).$$

Since $\mathcal{G}_{T_{i_0}} \subset \mathcal{G}_t$ and the event $\{T_{i_0-1} < \tau\}$ belongs to $\mathcal{G}_{T_{i_0}}$, we obtain

$$\hat{U}_t^{i_0} = \mathbb{1}_{\{T_{i_0-1} < \tau\}} B_t \, \mathbb{E}_{\mathbb{P}^*} \big(B_{T_{i_0}}^{-1} \hat{W}_{i_0} \,\big|\, \mathcal{G}_t \big) = \mathbb{1}_{\{T_{i_0-1} < \tau\}} Y_{i_0},$$

where Y_{i_0} represents the price at time t of a non-defaultable claim that pays $\hat{W}_{i_0}$ at time T_{i_0}. We are in a position to state the following result. Let us stress that we assume that formula (3.22) may be applied to each term $\hat{U}_t^i$ and $\tilde{U}_t^i$.

Proposition 4.1 *Let the price $\tilde{S}_t$ at time $t \le T$ of a defaultable claim X with discrete-time recovery be given by formula* (4.2)*. Then*

$$\begin{aligned} \tilde{S}_t \;=\;& \mathbb{1}_{\{T_{i_0-1} < \tau\}} B_t \, \mathbb{E}_{\mathbb{P}^*} \big(B_{T_{i_0}}^{-1} \hat{W}_{i_0} \,\big|\, \mathcal{F}_t \big) + \mathbb{1}_{\{t < \tau\}} \sum_{i=i_0+1}^{n} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \big(\tilde{B}_{T_{i-1}}^{-1} Y_i \,\big|\, \mathcal{F}_t \big) \\ & - \mathbb{1}_{\{t < \tau\}} \sum_{i=i_0}^{n} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \big(\tilde{B}_{T_i}^{-1} \hat{W}_i \,\big|\, \mathcal{F}_t \big) + \mathbb{1}_{\{t < \tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \big(\tilde{B}_{T_n}^{-1} X \,\big|\, \mathcal{F}_t \big), \end{aligned}$$

where $i_0 = i_0(t) = \inf\{\, i \,:\, T_i > t\}$, the random variables $\hat{W}_i$ are given by (4.3)*, Y_i is given by* (4.4)*, and $\tilde{B}$ by* (3.12).

We shall now focus on the case of a defaultable term structure, that is, we set $X = 1$. The most tractable cases are: (i) the case of zero recovery: $W = 0$, (ii) the case of fractional recovery of par: $W = \delta$ with $0 < \delta < 1$ (in principle, δ can be any real number). For any adapted process γ, we find it convenient to denote

$$B^{\gamma}(t,T) = \mathbb{E}_{\mathbb{P}^*} \Big\{ \exp \Big(- \int_t^T (r_u + \gamma_u) \, du \Big) \,\Big|\, \mathcal{F}_t \Big\}. \tag{4.5}$$

Notice that $B^0(t,T) = B(t,T)$, and $B^{\gamma}(t,T) < B(t,T)$ if γ is strictly positive.

Zero recovery. In the case of zero recovery, formulae (4.1) and (4.2) yield, as expected, the same result for the price process $D^0(t,T)$ of the T-maturity defaultable bond. Namely, we have

$$D^0(t,T) = B_t \, \mathbb{E}_{\mathbb{P}^*} (B_T^{-1} \mathbb{1}_{\{T < \tau\}} \,|\, \mathcal{G}_t). \tag{4.6}$$

As usual, we assume that we are in a position to use formula (3.22) (i.e. $\Delta V_\tau = 0$). Then

$$D^0(t,T) = \mathbb{1}_{\{t < \tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} (\tilde{B}_T^{-1} \,|\, \mathcal{F}_t) = \mathbb{1}_{\{t < \tau\}} B^{\lambda}(t,T).$$

This means that the price of a bond before default can be calculated in a 'standard' way, provided that the risk-free rate r is substituted with the default-adjusted rate $R = r + \lambda$. In particular, if λ is strictly positive then $D^0(t,T) < B(t,T)$ for $t < T$, and $D^0(T,T) \le B(T,T) = 1$.

Fractional recovery of par. In the case of a non-zero recovery coefficient δ, for the price $D^{\delta}(t,T)$ of a defaultable bond with continual recovery we get

$$D^{\delta}(t,T) := B_t \, \mathbb{E}_{\mathbb{P}^*} \big(\delta B_\tau^{-1} \mathbb{1}_{\{t < \tau \le T\}} + B_T^{-1} \mathbb{1}_{\{T < \tau\}} \,\big|\, \mathcal{G}_t \big) = \mathbb{1}_{\{t < \tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\delta \int_t^T \tilde{B}_u^{-1} \lambda_u \, du + \tilde{B}_T^{-1} \,\Big|\, \mathcal{F}_t \Big),$$

where the second equality holds provided that $\Delta V_\tau = 0$. The price of a defaultable bond with discrete-time recovery equals (cf. (4.2))

$$\tilde{D}^{\delta}(t,T) := \sum_{T_i \ge t} B_t \, \mathbb{E}_{\mathbb{P}^*} \big(\delta B_{T_i}^{-1} \mathbb{1}_{\{T_{i-1} < \tau \le T_i\}} \,\big|\, \mathcal{G}_t \big) + B_t \, \mathbb{E}_{\mathbb{P}^*} \big(B_T^{-1} \mathbb{1}_{\{T < \tau\}} \,\big|\, \mathcal{G}_t \big).$$

Let us analyse the latter case in more details. Suppose that $T_{i_0-1} \le t < T_{i_0}$. First, we have

$$\begin{aligned}\tilde{D}^{\delta}(t,T) &= \delta B_t \sum_{i=i_0}^{n} \Big(\mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} 1\!\!1_{\{T_{i-1}<\tau\}} \,\big|\, \mathcal{G}_t\big) - \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} 1\!\!1_{\{T_i<\tau\}} \,\big|\, \mathcal{G}_t\big)\Big) \\ &\quad + B_t\, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_n}^{-1} 1\!\!1_{\{T_n<\tau\}} \,\big|\, \mathcal{G}_t\big),\end{aligned}$$

or in an abbreviated form

$$\tilde{D}^{\delta}(t,T) = \sum_{i=i_0}^{n} \delta\, \hat{U}(t,T_i) - \sum_{i=i_0}^{n} \delta\, \tilde{U}(t,T_i) + U(t,T_n). \tag{4.7}$$

Since $T_{i_0-1} \le t$ and thus $\mathcal{G}_{T_{i_0-1}} \subset \mathcal{G}_t$, it is clear that

$$\hat{U}(t,T_{i_0}) = B_t\, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_{i_0}}^{-1} 1\!\!1_{\{T_{i_0-1}<\tau\}} \,\big|\, \mathcal{G}_t\big) = 1\!\!1_{\{T_{i_0-1}<\tau\}} B(t,T_{i_0}). \tag{4.8}$$

Furthermore, for any $i = i_0+1,\dots,n$ we have $\mathcal{G}_t \subset \mathcal{G}_{T_{i-1}}$, and thus

$$\hat{U}(t,T_i) = B_t \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} 1\!\!1_{\{T_{i-1}<\tau\}} \,\big|\, \mathcal{G}_t\big) = B_t \mathbb{E}_{\mathbb{P}^*}\big(B_{T_{i-1}}^{-1} 1\!\!1_{\{T_{i-1}<\tau\}} B(T_{i-1},T_i) \,\big|\, \mathcal{G}_t\big).$$

By applying (3.22), we obtain (as usual, we assume that V does not jump at τ)

$$\hat{U}(t,T_i) = 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_{i-1}} (r_u+\lambda_u)\,du\Big) B(T_{i-1},T_i) \,\Big|\, \mathcal{F}_t\Big\},$$

or equivalently (cf. (4.5))

$$\begin{aligned}\hat{U}(t,T_i) &= 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u+\lambda_u 1\!\!1_{[0,T_{i-1}]}(u))\,du\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &= 1\!\!1_{\{t<\tau\}} B^{\lambda^{i-1}}(t,T_i),\end{aligned} \tag{4.9}$$

where we set $\lambda_t^{i-1} = \lambda_t 1\!\!1_{[0,T_{i-1}]}(t)$ for $t \in [0,T]$. Finally, once again using (3.22), we get for any $i = i_0,\dots,n$

$$\begin{aligned}\tilde{U}(t,T_i) &= B_t\, \mathbb{E}_{\mathbb{P}^*}\big(B_{T_i}^{-1} 1\!\!1_{\{T_i<\tau\}} \,\big|\, \mathcal{G}_t\big) \\ &= 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u+\lambda_u)\,du\Big) \,\Big|\, \mathcal{F}_t\Big\},\end{aligned} \tag{4.10}$$

so that $\tilde{U}(t,T_i) = 1\!\!1_{\{t<\tau\}} B^{\lambda}(t,T_i) = D^0(t,T_i)$. By plugging (4.8)-(4.10) into (4.7), we arrive at the following representation of the price $\tilde{D}^{\delta}(t,T)$.

Proposition 4.2 *Let us set $I_0 = 1\!\!1_{\{T_{i_0-1}<\tau\}} \delta B(t,T_{i_0})$. For every $t \le T$, the price $\tilde{D}^{\delta}(t,T)$ of a defaultable bond with discrete-time fractional recovery of par equals*

$$\begin{aligned}\tilde{D}^{\delta}(t,T) &= I_0 + 1\!\!1_{\{t<\tau\}} \sum_{i=i_0+1}^{n} \delta\, \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u+\lambda_u^{i-1})\,du\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &\quad - 1\!\!1_{\{t<\tau\}} \sum_{i=i_0}^{n} \delta\, \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_i} (r_u+\lambda_u)\,du\Big) \,\Big|\, \mathcal{F}_t\Big\} \\ &\quad + 1\!\!1_{\{t<\tau\}} \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(-\int_t^{T_n} (r_u+\lambda_u)\,du\Big) \,\Big|\, \mathcal{F}_t\Big\},\end{aligned}$$

where $i_0 = i_0(t) = \inf\{\, i : T_i > t \,\}$ and $\lambda_t^{i-1} = \lambda_t 1\!\!1_{[0,T_{i-1}]}(t)$. Put another way,

$$\tilde{D}^{\delta}(t,T) = I_0 + 1\!\!1_{\{t<\tau\}} \Big(\sum_{i=i_0+1}^{n} \delta\, B^{\lambda^{i-1}}(t,T_i) - \sum_{i=i_0}^{n} \delta\, B^{\lambda}(t,T_i) + B^{\lambda}(t,T_n)\Big).$$

Example 4.2 Let us consider a very special case of a T-maturity defaultable bond with a discrete-time recovery, with only two admissible dates $T_0 = 0$ and $T_1 = T$. Since default at time 0 is excluded with probability 1, it is clear that the payment always occurs at time T, no matter whether a bond has defaulted before maturity or not. For any $t \le T$ we have

$$\tilde{D}^\delta(t,T) = B_t \, \mathbb{E}_{\mathbb{P}^*} \big(\delta B_T^{-1} \mathbb{1}_{\{0<\tau\le T\}} + B_T^{-1} \mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t \big).$$

On the other hand, since $i_0(t) = 1$ for any $t \le T$, formula established in Proposition 4.2 gives

$$\tilde{D}^\delta(t,T) = \delta B(t,T) + \mathbb{1}_{\{t<\tau\}}(1-\delta) B^\lambda(t,T). \tag{4.11}$$

Under the present assumptions, since a defaulted bond pays the amount δ at time T, we get $\tilde{D}^\delta(t,T) = \delta B(t,T)$ on the random set $[\tau, T]$, that is, after default. Before default, its value is strictly greater than $\delta B(t,T)$, but we have always $\tilde{D}^\delta(t,T) < B(t,T)$. The last inequality is trivial, since the process λ is strictly positive, and thus $B^\lambda(t,T) < B(t,T)$ for every $t \le T$. We conclude that under the present assumptions, the price of defaultable bond never exceeds the price of the risk-free bond,[5] which is a natural property to require from a model valuing risky debt. On the other hand, for the general model of the continual recovery we have only following equivalence, which holds on the set $\{\tau > t\}$ (of course, $D^\delta(t,T) = 0 < B(t,T)$ on $\{\tau \le t\}$)

$$D^\delta(t,T) \le B(t,T) \quad \text{iff} \quad \delta \, \mathbb{E}_{\mathbb{P}^*}(B_\tau^{-1} \mathbb{1}_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t) \le \mathbb{E}_{\mathbb{P}^*}(B_T^{-1} \mathbb{1}_{\{t<\tau\le T\}} \,|\, \mathcal{G}_t).$$

This shows that the valuation in the case of the continual fractional recovery appears to be rather delicate.

4.2 Endogenous Recovery Rules

If Z is not an exogenously given process (but, for instance, a deterministic function of the value process S), the problem of existence and uniqueness of a process S defined by (3.5) arises. We take the uniqueness of solution to (3.5) for granted, and we address the problem of pricing of defaultable claims of the form (X, Z, τ), where Z is a specific 'recovery rule,' rather than a given process.

Fractional recovery of market value. Following Duffie and Singleton (1999), we assume that $Z_t = (1 - L_t) S_{t-}$, where S is a process we are looking for, and L is a given $\mathbb{F}$-predictable process. We start with the following lemma, which deals with the process V only. Notice that formula (4.12) represents a stochastic equation which needs to be solved for the unknown $\mathbb{F}$-adapted process V.

Lemma 4.1 *Under* (H.1), *let V satisfy* (3.11) *with $Z_t = (1 - L_t) V_{t-}$ for some predictable process L, that is,*

$$V_t = \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T \tilde{B}_u^{-1} (1 - L_u) V_u \lambda_u \, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big). \tag{4.12}$$

Then V is unique, and it is given by the formula

$$V_t = \hat{B}_t \, \mathbb{E}_{\mathbb{P}^*} \big(\hat{B}_T^{-1} X \,\big|\, \mathcal{F}_t \big) \tag{4.13}$$

where the $\mathbb{F}$-adapted process $\hat{B}$ equals

$$\hat{B}_t = \exp \Big(\int_0^t (r_u + \lambda_u L_u) \, du \Big). \tag{4.14}$$

Proof. In view of (3.16) with N is given by (3.21), we obtain

$$dV_t = V_t(r_t + \lambda_t) \, dt - (1 - L_t) V_t \lambda_t \, dt + \tilde{B}_t \, dN_t,$$

[5] This holds true also in the case of zero recovery.

or equivalently,

$$dV_t = V_t(r_t + \lambda_t L_t)\,dt + \tilde{B}_t\,dN_t.$$

This immediately yields (4.13) (as usual, we assume that the last term follows a martingale). Of course, this proves also that equation (4.12) admits a unique solution. □

The next step is to examine the relationship between the process V (or rather $U_t = 1\!\!1_{\{t<\tau\}}V_t$) and the price process of a defaultable claim. In view of Theorem 3.1 (which we may apply since $Z_t = (1-L_t)V_{t-}$ follows a $\mathbb{F}$-predictable process), we find that U satisfies

$$U_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1}\big((1-L_\tau)V_{\tau-} + \Delta V_\tau\big)1\!\!1_{\{t<\tau\le T\}} + B_T^{-1}X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big). \tag{4.15}$$

Corollary 4.1 *Let the process V be given by formula* (4.12) *for some predictable process L. Assume that $\Delta V_\tau = 0$. Then the process $U_t = 1\!\!1_{\{t<\tau\}}V_t$ satisfies*

$$U_t = 1\!\!1_{\{t<\tau\}}\hat{B}_t\,\mathbb{E}_{\mathbb{P}^*}\big(\hat{B}_T^{-1}X\,\big|\,\mathcal{F}_t\big) \tag{4.16}$$

and

$$U_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1}(1-L_\tau)U_{\tau-}1\!\!1_{\{t<\tau\le T\}} + B_T^{-1}X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big). \tag{4.17}$$

Proof. Equality (4.16) is an immediate consequence of (4.13). The second formula follows from (4.15) (we use the trivial equality $U_{\tau-} = V_{\tau-}$). □

In view of Corollary 4.1, the process U satisfies equation (4.17), that is, the implicit definition of the price process S. Note that we have not proved that the uniqueness of solutions holds for the equation

$$S_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1}(1-L_\tau)S_{\tau-}1\!\!1_{\{t<\tau\le T\}} + B_T^{-1}X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big). \tag{4.18}$$

We have merely shown that (4.18) admits a solution. The uniqueness of solutions to (4.18) can be deduced from standard results on backward SDEs, however.[6] To this end, it might be convenient to use the equivalent representation of equation (4.18), namely (cf. (3.9))

$$S_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T S_u\big((1-L_u)h_u - r_u\big)\,du + X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big). \tag{4.19}$$

General recovery rule. In principle, we may also deal with a 'general recovery rule', more precisely, we may assume that the payoff process Z satisfies $Z_t = p(t, S_{t-})$, where the function $p(t,s)$ is Lipschitz continuous with respect to s, and satisfies $p(t,0)=0$. In this case, however, we have merely the following result, which again is a consequence of Theorem 3.1 (once again, the problem of existence and uniqueness of solutions to (4.21) and (4.23) is not addressed here; this follows from standard results on backward SDEs).

Corollary 4.2 *Let S be the unique solution to the backward SDE*

$$S_t = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(B_\tau^{-1}p(\tau, S_{\tau-})1\!\!1_{\{t<\tau\le T\}} + B_T^{-1}X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big), \tag{4.20}$$

or equivalently, to the equation (*cf.* (3.9))

$$S_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \big(p(u,S_u)h_u - r_uS_u\big)\,du + X\,1\!\!1_{\{T<\tau\}}\,\Big|\,\mathcal{G}_t\Big). \tag{4.21}$$

Let V be the unique solution to the backward SDE

$$V_t = \tilde{B}_t\,\mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \tilde{B}_u^{-1}p(u,V_u)\lambda_u\,du + \tilde{B}_T^{-1}X\,\Big|\,\mathcal{F}_t\Big), \tag{4.22}$$

[6] For other applications of backward SDEs in mathematical finance, and further references, see the papers by El Karoui and Quenez (1997a, 1997b) and El Karoui et al. (1997).

or equivalently, to the equation

$$V_t = \mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T \big(p(u,V_u)\lambda_u - (r_u+\lambda_u)V_u\big)\,du + X \,\Big|\, \mathcal{F}_t\Big). \tag{4.23}$$

If $\Delta V_\tau = 0$, *then* $S_t = 1\!\!1_{\{t<\tau\}}V_t$. *Otherwise,* S *is given by formula* (3.19).

5 Credit-ratings-based Markov Model

To produce a tractable model which accounts for the migration between rating grades, Jarrow et al. (1997) make the following, rather stringent, assumptions: (i) there exists a unique equivalent martingale measure $\mathbb{P}^*$ making all default-free and default-risky zero coupon bond prices martingales, after normalization by the savings account, (ii) the default time τ is independent of the risk-free rate r under the martingale measure $\mathbb{P}^*$, (iii) the recovery coefficient is a constant δ. They first develop a discrete-time model which takes into account the migration of a defaultable bond in the finite set of *credit rating classes.* Subsequently, a continuous-time counterpart is also examined. Methodology developed in Jarrow et al. (1997) is a direct extension of the approach in Jarrow and Turnbull (1995). They assume that a defaulted bond pays at maturity a fraction of its par value.[7] Therefore, the price at time $t \le T$ of a T-maturity defaultable bond equals

$$\tilde{D}^\delta(t,T) = B_t\,\mathbb{E}_{\mathbb{P}^*}\Big(B_T^{-1}\big(\delta 1\!\!1_{\{T\ge\tau\}} + 1\!\!1_{\{T<\tau\}}\big)\,\Big|\,\mathcal{G}_t\Big), \tag{5.24}$$

where τ is the default time, and δ is the constant recovery rate. Suppose that we have chosen a model for the short-term rate r. It is clear from expression (5.24) that we need only to model a random time τ. In addition, under assumption (i), formula (5.24) can be substantially simplified, namely,

$$\tilde{D}^\delta(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{P}^*}\big(\delta 1\!\!1_{\{T\ge\tau\}} + 1\!\!1_{\{T<\tau\}}\,\big|\,\mathcal{G}_t\big). \tag{5.25}$$

Consequently (it might be instructive to compare (5.26) with (4.11)),

$$\tilde{D}^\delta(t,T) = B(t,T)\,\big(\delta + (1-\delta)\mathbb{P}^*\{T<\tau\,|\,\mathcal{G}_t\}\big). \tag{5.26}$$

As will soon become clear, the stopping time τ is explicitly dependent on the initial rating of a particular bond. Therefore, expressions (5.24)-(5.26) should be seen as generic valuation formulae for defaultable bonds. Given an initial rating of a defaultable bond, the future changes in its assessments by a rating agency are described by a stochastic process, referred to as the *migration process.* Formally, for a given bond, the value at time t of the associated migration process coincides with its current rating. There is no loss of generality, if we assume that the set of rating classes is $\{1,\dots,K\}$, where the state K is assumed to correspond to the default event. It is assumed that the migration process, C say, follows a Markov chain (under both real-world probability $\mathbb{P}$ and the spot martingale measure $\mathbb{P}^*$), that is, the future evolution of ratings classes a particular bond does not depend on the bond's history, but only on its current rating.

5.1 Discrete-time Model

In a discrete-time setup, the migration process and the default time are assumed to satisfy: (iv) the migration process C follows, under the real-world probability $\mathbb{P}$, a time-homogeneous Markov chain with the transition matrix (by definition, $p_{ij} = \mathbb{P}\{C_{t+1}=j\,|\,C_t=i\}$)

$$P = [p_{ij}]_{1\le i,j\le K}, \quad p_{i,j}\ge 0, \quad \sum_{j=1}^K p_{ij} = 1,$$

[7] This convention coincides with the concept of discrete-time fractional recovery of par introduced in Section 4, provided that we take $T_0 = 0$ and $T_1 = T$ (cf. Example 4.2).

with $p_{Kj} = 0$ for every $j < K$ (so that $p_{KK} = 1$; that is, the state K is absorbing), and (v) C follows a (time-inhomogeneous) Markov chain under $\mathbb{P}^*$, with time-dependent transition matrix

$$Q(t) = [q_{ij}(t, t+1)]_{1 \le i,j \le K}$$

where

$$q_{ij}(t, t+1) \ge 0, \quad \sum_{j=1}^{K} q_{ij}(t, t+1) = 1,$$

and finally $q_{Kj}(t, t+1) = 0$ for every $j < K$ and t (so that once again the state K is absorbing).

The default time τ is the first moment the rating process hits the state K (the horizon date T^* is assumed to be a natural number). Formally,

$$\tau := \inf \{ t \in \{0, 1, \ldots, T^*\} : C_t = K \} \tag{5.27}$$

where, by convention, the infimum over an empty set equals $+\infty$.

To ensure analytical tractability of the model, an additional 'technical' assumption is made. Namely, it is postulated that the following relationship holds

$$q_{ij}(t, t+1) = \pi_i(t) p_{ij}, \quad \forall\, i \ne j, \tag{5.28}$$

where time-dependent coefficients $\pi_i(t)$ are interpreted as discrete-time *risk premia.* The last assumption implies, in particular, that

$$q_{ii}(t, t+1) = 1 + \pi_i(t)(p_{ii} - 1), \quad \forall\, i.$$

In other words, for any state i, the probability under the martingale measure $\mathbb{P}^*$ of jumping to the state $j \ne i$ is assumed to be proportional to the corresponding probability under the real-world probability $\mathbb{P}$, with the proportionality factor which may depend on i and t, but not on j.

Assume that we are given the initial term structures of default-free and defaultable bonds, and the real-world transition matrix P (in principle, all these quantities can be 'observed'). Then, under the above set of assumptions, Jarrow et al. (1997) offer a recursive procedure which leads to the unique determination of the 'risk premium' process $\pi(t)$, $t = 0, \ldots, T^* - 1$. Consequently, the time-dependent transition matrix $Q(t)$ under $\mathbb{P}^*$ is also uniquely specified.

5.2 Continuous-time Model

A similar approach is developed in the continuous-time setup. It is postulated that: (iv') under the real-world probability $\mathbb{P}$, the migration process C follows a time-homogeneous Markov chain, with intensity matrix $\tilde{\Lambda}$ satisfying mild 'technical' conditions (which guarantee that the state K is absorbing, and a suitable monotonicity of default probabilities holds), (v') under the martingale measure $\mathbb{P}^*$, the migration process also follows a Markov chain, but with a possibly time-dependent intensity matrix Λ_t. As before, the default time τ is the first time the rating process hits the absorbing state K. Tractability condition (5.28) now takes the following form: there exists a diagonal matrix U, whose first $K-1$ entries, $U_{ii}(t)$, $i = 1, \ldots, K-1$, are strictly positive deterministic functions, and the last entry, $U_{KK}(t) = 1$ for every t, such that the risk-neutral and real-world intensity matrices satisfy

$$\Lambda_t = U(t)\tilde{\Lambda}, \quad \forall\, t \in [0, T^*]. \tag{5.29}$$

Suppose that the initial term structures of default-free and default-risky zero coupon bonds are known. Then for any choice of the 'historical' intensity matrix $\tilde{\Lambda}$, one can produce a model for defaultable term structure in two steps. In the first step, we construct the migration process C under the real-world probability $\mathbb{P}$, using the intensity matrix $\tilde{\Lambda}$ (by assumption, the migration process is independent of the underlying risk-free short-term rate r). Subsequently, we search for an equivalent probability measure $\mathbb{P}^*$, which would reproduce the observed prices of all defaultable bonds through the risk-neutral valuation formula (5.26). If we denote by $\tilde{D}_i^\delta(0,T)$ the initial price of the defaultable bond which belongs to the $i^{\rm th}$ rating class at time 0, then we have

$$\tilde{D}_i^\delta(0,T) = B(0,T)\left(\delta + (1-\delta)\mathbb{P}^*\{T<\tau \,|\, C_0 = i\}\right). \tag{5.30}$$

Since τ is the hitting time of K, and the state K absorbing, it is also clear that

$$\mathbb{P}^*\{T<\tau \,|\, C_0=i\} = 1 - \mathbb{P}^*\{C_T = K \,|\, C_0 = i\} = 1 - q_{iK}(0,T),$$

where $Q(0,T) = [q_{ij}(0,T)]_{1\le i,j\le K}$ is the transition matrix corresponding to the time interval $[0,T]$.

6 Modelling with State Variables

In this section – in which we follow Duffie and Singleton (1999) and Lando (1998) – we place ourselves again within the general framework, as presented in Section 3. In order to make the model of Section 3 analytically more tractable, we impose additional conditions on the default time τ – more specifically, on the intensity process λ of the default process H. It should be stressed that additional conditions of this kind are complementary to those considered in Section 5. For instance, it seems natural to examine a model of defaultable debt which combines the presence of the migration process C with the presence of the state variables process Y (as, for instance, in Lando (1998)).

We assume that we are given a k-dimensional stochastic process Y defined on the underlying filtered probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$. The $\mathbb{F}$-adapted process Y, which typically is assumed to be Markovian under the spot martingale measure $\mathbb{P}^*$, is assumed to model the dynamics of 'state variables' which underpin the evolution of all other variables in our model of the economy. As far as the default time is concerned, we postulate that τ is the first jump time of a Cox process with the intensity of the form $\lambda_t = \lambda(Y_t)$, for some function $\lambda : \mathbb{R}^k \to \mathbb{R}_+$. It is thus clear that the intensity of a default time is a $\mathbb{F}$-adapted stochastic process.

Let us mention that at this stage no explicit distinction between defaultable bonds with different rating assessments is made. In other words, we focus on a bond which currently belongs to a particular class, and we exclude the possibility of migration to any other class then to the 'default class.'

The construction of the default time τ with these properties can be achieved as follows. Let $\mathbb{F}$ be the filtration with respect to which the process Y is adapted, and let η be a random variable independent of $\mathbb{F}$. Of course, η and Y are assumed to be defined on a common probability space $(\Omega, \mathcal{G}, \mathbb{P}^*)$, so that a suitable enlargement of the underlying probability space might be required. More specifically, we assume that η has a unit exponential probability law under $\mathbb{P}^*$. To define default time τ (that is, the first jump of the Cox process), we set

$$\tau = \inf\Big\{t\in\mathbb{R}_+ : \int_0^t \lambda(Y_u)\,du \ge \eta\Big\}. \tag{6.1}$$

It should be stressed that for construction above the hypothesis (H.1) is valid.

To make such a specification of the default time τ useful, we need to assume, in addition, that X is a $\mathcal{F}_T$-measurable random variable, the recovery process Z is $\mathbb{F}$-predictable, and, for instance, $r_t = r(Y_t)$ (this agrees with our interpretation of Y as state-variables process). Under this set of assumptions, in all previously established formulae in which the default time τ does not appear explicitly, that is, the presence of the default process N is manifested only through its intensity process $\lambda_t = \lambda(Y_t)$, we may substitute the conditional expectation with respect to $\mathcal{G}_t$ with the conditional expectation with respect to $\mathcal{F}_t$. For instance, using (3.22), we obtain

$$S_t = \mathbb{1}_{\{t<\tau\}}\mathbb{E}_{\mathbb{P}^*}\Big(\int_t^T e^{-\int_t^u R(Y_v)\,dv} Z_u \lambda(Y_u)\,du + e^{-\int_t^T R(Y_v)\,dv} X \,\Big|\, \mathcal{F}_t\Big), \tag{6.2}$$

where $R(Y_u) = r(Y_u) + h(Y_u)$. Let us notice that formula (6.2) is a direct consequence of equality (3.20), combined with the observation that $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{F}_t \vee \sigma(\eta)$, where, by assumption, the σ-fields

$\mathcal{F}_T$ and $\sigma(\eta)$ are mutually independent. As shown by Lando (1998), formula (6.2) can be derived in a more straightforward way, without making explicit reference to the pre-default value process V (that is, using directly Lemma 3.1 rather than a suitable version of Corollary 3.1).

Proposition 6.1 *Let the default time* τ *be given by* (6.1). *Then we have*

$$S_t = \mathbb{1}_{\{t<\tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda(Y_u)\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{F}_t \Big), \tag{6.3}$$

where the process $\tilde{B}$ *is given by* (3.12) *with* $r_u = r(Y_u)$ *and* $\lambda_u = h(Y_u)$.

Proof. Notice that for any $0 \le t \le u$ we have

$$\mathbb{P}^*\{\tau > u \,|\, \mathcal{F}_T \vee \mathcal{H}_t\} = \begin{cases} \exp\big(- \int_t^u \lambda(Y_v)\, dv \big), & \text{on the set } \{\tau > t\}, \\ 0, & \text{otherwise,} \end{cases}$$

where, as before, $\mathcal{H}_t = \sigma(H_u \,:\, u \le t)$. Therefore (cf. (3.8)),

$$\begin{aligned} S_t &= B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u) \mathbb{1}_{\{u \le \tau\}}\, du + B_T^{-1} X \mathbb{1}_{\{T<\tau\}} \,\Big|\, \mathcal{G}_t \Big) \\ &= B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u) \, \mathbb{P}^*\{\tau \ge u \,|\, \mathcal{F}_T \vee \mathcal{H}_t\}\, du \,\Big|\, \mathcal{G}_t \Big) \\ &\quad + B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(B_T^{-1} X \, \mathbb{P}^*\{\tau > T \,|\, \mathcal{F}_T \vee \mathcal{H}_t\} \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{t<\tau\}} B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T B_u^{-1} Z_u \lambda(Y_u) \exp\Big(- \int_t^u \lambda(Y_v)\, dv \Big)\, du \,\Big|\, \mathcal{G}_t \Big) \\ &\quad + \mathbb{1}_{\{t<\tau\}} B_t \, \mathbb{E}_{\mathbb{P}^*} \Big(B_T^{-1} X \exp\Big(- \int_t^T \lambda(Y_v)\, dv \Big) \,\Big|\, \mathcal{G}_t \Big) \\ &= \mathbb{1}_{\{t<\tau\}} \tilde{B}_t \, \mathbb{E}_{\mathbb{P}^*} \Big(\int_t^T \tilde{B}_u^{-1} Z_u \lambda(Y_u)\, du + \tilde{B}_T^{-1} X \,\Big|\, \mathcal{G}_t \Big). \end{aligned}$$

We wish now to substitute $\mathcal{G}_t$ with $\mathcal{F}_t$ in the last expression. It is enough to observe that conditioning with respect to $\mathcal{G}_t$ coincides in our case with conditioning with respect to $\mathcal{F}_t \vee \mathcal{H}_t \subset \mathcal{F}_t \vee \sigma(\eta)$. Equality (6.3) now follows immediately from the fact that the random variable η is independent of $\mathcal{F}_T$, and thus σ-fields $\mathcal{F}_T$ and $\mathcal{H}_t$ are conditionally independent given $\mathcal{F}_t$ (cf. the hypothesis (H.2)). Since the random variable under the sign of the conditional expectation is measurable with respect to the σ-field $\mathcal{F}_T$, the result follows. □

Proposition 6.1 combined with Corollary 3.1 suggest that the jump ΔV_τ, even if it does not vanish, plays no longer an important role in the present setup. Indeed, it shows that we always have $S_t = \mathbb{1}_{\{t<\tau\}} V_t$, where the process V is given by (3.11). Consequently, combining (3.6) with (3.13), we find that under the present assumptions the pre-default process associated with any defaultable claim (X, Z, τ) satisfies

$$\mathbb{E}_{\mathbb{P}^*} \big(B_\tau^{-1} \Delta V_\tau \mathbb{1}_{\{t<\tau\le T\}} \,\big|\, \mathcal{G}_t \big) = 0, \quad \forall\, t \in [0, T].$$

Remarks. Duffie and Singleton (1999) focus on the special case of fractional recovery of market value. They assume that: (i) there is a state-variables process Y that is Markovian under the spot martingale measure $\mathbb{P}^*$, (ii) the promised contingent claim is of the form $X = g(Y_T)$ for some function $g : \mathbb{R}^k \to \mathbb{R}$, (iii) the default-adjusted short-term rate $R_t = r_t + \lambda_t L_t = \rho(Y_t)$ for some function $\rho : \mathbb{R}^k \to \mathbb{R}$. Under (i)-(iii), we have

$$V_t = \mathbb{E}_{\mathbb{P}^*} \Big\{ \exp\Big(- \int_t^T \rho(Y_u)\, du \Big) g(Y_T) \,\Big|\, Y_t \Big\}. \tag{6.4}$$

Moreover, if Y follows a non-degenerate diffusion process, then $\Delta V_\tau = 0$ and thus $S_t = \mathbb{1}_{\{t<\tau\}} V_t$. Indeed, in this case the martingale N given by formula (3.21) is continuous. Consequently, in view of (3.14), the process V is also continuous.

6.1 Conditionally Markov Ratings Process

We shall now describe an extension – due to Lando (1998) – of the credit ratings model elaborated by Jarrow et al. (1997). As usual, we assume that the spot martingale measure $\mathbb{P}^*$, and risk-free term structure $B(t,T)$ are given. Lando (1998) modifies the Jarrow-Lando-Turnbull approach by introducing a conditionally Markov migration process, which accounts for both the presence of different rating classes, and for the postulated existence of the underlying state variables, as modelled by a process Y. It appears that this can be achieved by a suitable modification of the migration process C introduced in Section 5 (whenever possible, we preserve the notation introduced in Section 5).

We place ourselves in a continuous-time setup. The migration process C is now assumed to follow, under the spot martingale measure, a conditional Markov chain with the stochastic intensity matrix $\Lambda(Y_t) = [\lambda_{ij}(Y_t)]_{1\le i,j\le K}$ which is assumed to satisfy, for every $t \in [0,T^*]$ and $i = 1,\dots,K$,

$$\lambda_{ii}(Y_t) = -\sum_{j=1,\, j\neq i}^{K} \lambda_{ij}(Y_t), \quad \text{and} \quad \lambda_{K,i}(Y_t) = 0, \tag{6.5}$$

where $\lambda_{ij} : \mathbb{R}^k \to \mathbb{R}_+$ are non-negative functions. For any such a matrix, given the process Y and the initial rating i (at time 0, say), it is possible to construct a migration process C corresponding to the matrix $\Lambda(Y_t)$. More specifically, the migration process C is assumed to follow, conditionally on the path of the state-variables process Y, a Markov chain with finite state space $\{1,\dots,K\}$ and time-dependent (but deterministic) intensity matrix $\Lambda(Y_t)$. It follows from (6.5) that the K^{th} row of the matrix $\Lambda(Y_t)$ is assumed to vanish indentically, so that K is an absorbing state. As in Section 5, the absorbing state K represents the default event, and the default time is the first time the migration process C hits K. The construction of a process C with these properties is a straightforward generalization of the construction of a default time provided by formula (6.1) (though we need to deal with an infinite family of mutually independent exponentially distributed random variables).

Remarks. The migration process C can be seen as a generalization of the first jump process H introduced in Section 3. Recall that H was defined through the formula $H_t = \mathbb{1}_{\{t\ge\tau\}}$. If we put $C_t = 1 + H_t$ then the state space of C is $\{1,2\}$ with 2 being the absorbing state. In a general framework, the process $C_t = 1 + H_t$ is not necessarily a (conditionally) Markov process, however.

Due to the nature of the default time τ, the valuation of defaultable claims becomes more cumbersome. It is essential to note that the default time τ and short-term rate r are no longer mutually independent (as was postulated in Jarrow et al. (1997)). Therefore, no explicit valuation results, such as formula (5.26), are available in the present setup. Consequently, one is bound to employ the basic definition (3.6) of the price process of a defaultable claim. This observation applies also to the case of a zero-coupon bond, under the assumption that the recovery rate equals 0 (that is, when the recovery process Z vanishes identically). By definition, the price of such a bond equals (cf. (3.6) or (4.6))

$$D_i^0(t,T) = B_t\,\mathbb{E}_{\mathbb{P}^*}\big(B_T^{-1}\mathbb{1}_{\{T<\tau\}}\,\big|\,\mathcal{F}_t \vee \{C_t = i\}\big),$$

where we assume that at time t the bond belongs to the i^{th} rating class, for some $i < K$. Using a similar reasoning as in the proof of Proposition 6.1 (that is, conditioning first on the future evolution of the process Y), we find that

$$D_i^0(t,T) = B_t\,\mathbb{E}_{\mathbb{P}^*}\big(B_T^{-1}(1 - p_{iK}^Y(t,T))\,\big|\,\mathcal{F}_t\big), \tag{6.6}$$

where

$$p_{iK}^Y(t,T) = \mathbb{P}^*\big\{C_T = K \,\big|\, \{C_t = i\} \vee \sigma(Y_u \,:\, u \in [t,T])\big\}. \tag{6.7}$$

Notice that $p_{iK}^Y(t,T)$ is simply the conditional transition probability of the migration process C, over the time interval $[t,T]$, with conditioning on the future behaviour of the state-variables process Y. Evaluation of the conditional probability $p_{iK}^Y(t,T)$, given a particular sample path of the process

Y, would be thus a relatively simple task in the case of diagonal intensity matrix $\Lambda(Y_t)$. Indeed, we would be then able to separate variables in the corresponding system of Kolmogorov differential equations. A similar – but slightly less explicit – result holds provided that

$$\Lambda(Y_t) = B\,\Gamma(Y_t)B^{-1},$$

where $\Gamma(Y_t)$ is a diagonal matrix, and B is a $K \times K$ matrix whose columns are the eigenvectors of $\Lambda(Y_t)$. Under this rather restrictive condition, Lando (1998) derived a quasi-explicit valuation formula for a defaultable bond, and indeed for any (promised) European claim of the form $X = g(Y_T, C_T)$.

To conclude, the problem of valuation of defaultable debt is reduced to finding of a convenient representation of the right-hand side in (6.7), which would subsequently allow to evaluate the conditional expectation in (6.6). Generally speaking, this seems to be a rather difficult task, especially when restrictive regularity conditions are not imposed on the intensity matrix, or when we deal with a non-zero recovery rate. In any case, valuation of defaultable claims can be done through simulation techniques though.

7 Credit-spreads-based HJM Type Model

Results presented in this section are mainly due to Bielecki and Rutkowski (1999, 2000) (for related results, see Schönbucher (1998, 2000)). In contrast to the previous sections, we shall no longer assume that the default time of a T-maturity defaultable bond is prespecified. We postulate instead that we start with a given default-free and defaultable term structures, represented by a finite family of defaultable instantaneous forward rates. Our aim is thus to support an exogenously given defaultable term structure through an associated family of default times, defined on a suitable enlargement of the underlying probability space.

It should thus be stressed that in this section we are no longer concerned with the valuation of defaultable bonds for a given risk-free term structure and a given recovery rate. On the contrary, we assume that the 'pre-default' values of defaultable bonds are given a priori, and we search for an arbitrage-free bond market model that supports these values.

7.1 Single Credit Rating Case

In the first step, we focus on a defaultable bond from a given rating class and we assume that it cannot migrate to another class before default. We assume that the dynamics of defaultable instantaneous forward rates are given. Our goal is to explain these dynamics by introducing a judiciously chosen stopping time (on an enlarged probability space), which is interpreted as the bond's default time. Throughout this section the focus is on the case of fractional recovery of treasury value (that is, a fixed fraction of the nominal value is received at bond's maturity, if default occurs before or at maturity).

We make the following standing assumptions.

(B.1) We are given a d-dimensional standard Brownian motion W, defined on the underlying (real-world) filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$.
(B.2) For any fixed maturity $T \le T^*$, the default-free instantaneous forward rate $f(t,T)$ satisfies[8]

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T) \cdot dW_t, \tag{7.1}$$

where α and σ are adapted processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively.
(B.D) The defaultable instantaneous forward rate $g(t,T)$ satisfies

$$dg(t,T) = \tilde{\alpha}(t,T)\,dt + \tilde{\sigma}(t,T) \cdot dW_t, \tag{7.2}$$

[8]For technical conditions under which formulae (7.1)-(7.2) make sense, see Heath et al. (1992) or Chapter 13 in Musiela and Rutkowski (1997).

for some processes $\tilde{\alpha}$ and $\tilde{\sigma}$.

Conditions (B.1)-(B.2) are the standard hypotheses of the Heath-Jarrow-Morton approach to term structure modelling . By definition, the price at time t of a T-maturity default-free zero coupon bond thus equals

$$B(t,T) := \exp\Big(-\int_t^T f(t,u)\,du\Big). \tag{7.3}$$

The relevance of assumption (B.D) will be discussed later. For any $t \le T$, we set

$$\tilde{D}(t,T) := \exp\Big(-\int_t^T g(t,u)\,du\Big), \tag{7.4}$$

and we interpret $\tilde{D}(t,T)$ as the pre-default value of a T-maturity defaultable zero coupon bond with fractional recovery of par. In other words, we interpret $\tilde{D}(t,T)$ as the value of a T-maturity defaultable zero coupon bond conditioned on the fact the bond had not defaulted by the time t. To justify this heuristic interpretation, we need first to develop an arbitrage-free model for default-free and defaultable term structures. Our main goal will be then to show that the pre-default value $\tilde{D}(t,T)$ can be seen as the price before default of a T-maturity defaultable zero coupon bond in this framework. We assume, in addition, that the *credit spread* $g(t,T) - f(t,T)$ is strictly positive, so that $\tilde{D}(t,T) < B(t,T)$ (the case of $\delta = 1$ is thus excluded as trivial).

Default-free term structure. For the reader's convenience, we quote the following well-known result (see Heath et al. (1992)).

Lemma 7.1 *The dynamics of the default free bond price $B(t,T)$ are*

$$dB(t,T) = B(t,T)\big(a(t,T)\,dt + b(t,T)\cdot dW_t\big), \tag{7.5}$$

where

$$a(t,T) = f(t,t) - \alpha^*(t,T) + \tfrac{1}{2}\,|\sigma^*(t,T)|^2, \quad b(t,T) = -\sigma^*(t,T),$$

with $\alpha^*(t,T) = \int_t^T \alpha(t,u)\,du$ and $\sigma^*(t,T) = \int_t^T \sigma(t,u)\,du$.

An analogous result holds for $\tilde{D}(t,T)$, with an obvious change of notation. Namely,

$$d\tilde{D}(t,T) = \tilde{D}(t,T)\big(\tilde{a}(t,T)\,dt + \tilde{b}(t,T)\cdot dW_t\big) \tag{7.6}$$

with

$$\tilde{a}(t,T) = g(t,t) - \tilde{\alpha}^*(t,T) + \tfrac{1}{2}\,|\tilde{\sigma}^*(t,T)|^2, \quad \tilde{b}(t,T) = -\tilde{\sigma}^*(t,T). \tag{7.7}$$

We assume, as customary, that one may also invest in the risk-free savings account B, which corresponds to the short-term rate $r_t = f(t,t)$. In view of (7.5), the relative bond price $Z(t,T) = B_t^{-1}B(t,T)$ satisfies under $\mathbb{P}$

$$dZ(t,T) = Z(t,T)\Big(\big(\tfrac{1}{2}|b(t,T)|^2 - \alpha^*(t,T)\big)\,dt + b(t,T)\cdot dW_t\Big).$$

The following condition is known to exclude arbitrage across default-free bonds for all maturities $T \le T^*$, as well as between default-free bonds and the savings account.

Condition (M.1) There exists an adapted $\mathbb{R}^d$-valued process γ such that

$$\mathbb{E}_{\mathbb{P}}\Big\{\exp\Big(\int_0^{T^*} \gamma_u\cdot dW_u - \frac{1}{2}\int_0^{T^*} |\gamma_u|^2\,du\Big)\Big\} = 1$$

and, for any maturity $T \le T^*$, we have

$$\alpha^*(t,T) = \tfrac{1}{2}|\sigma^*(t,T)|^2 - \sigma^*(t,T)\cdot\gamma_t.$$

Let γ be some process satisfying Condition (M.1). Then the probability measure $\mathbb{P}^*$, given by the formula

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp\Big(\int_0^{T^*} \gamma_u \cdot dW_u - \frac{1}{2}\int_0^{T^*} |\gamma_u|^2\, du\Big), \quad \mathbb{P}\text{-a.s.,} \tag{7.8}$$

is a spot martingale measure for the default-free term structure. Moreover, if we define a Brownian motion W^* under $\mathbb{P}^*$ by setting

$$W_t^* = W_t - \int_0^t \gamma_u\, du, \quad \forall\, t \in [0, T^*],$$

then, for any fixed maturity $T \leq T^*$, the discounted price of risk-free bond satisfies under $\mathbb{P}^*$

$$dZ(t,T) = Z(t,T)b(t,T) \cdot dW_t^*. \tag{7.9}$$

We shall assume from now on that the process γ is uniquely determined, so that the default-free bonds market is complete.[9] Formally, this means that any default-free contingent claim can be priced through risk-neutral valuation formula. It should be stressed, however, that this remark does not apply to defaultable claims. After a recollection of the well-known facts about the Heath-Jarrow-Morton approach, we shall now focus on dynamics of the relative pre-default value of a defaultable bond. First, under $\mathbb{P}$ the process $\tilde{Z}(t,T) = B_t^{-1}\tilde{D}(t,T)$ satisfies

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big((\tilde{a}(t,T) - r_t)\, dt + \tilde{b}(t,T) \cdot dW_t\big). \tag{7.10}$$

Consequently, under the unique spot martingale measure $\mathbb{P}^*$, we have

$$d\tilde{Z}(t,T) = \tilde{Z}(t,T)\big(\lambda_t\, dt + \tilde{b}(t,T) \cdot dW_t^*\big), \tag{7.11}$$

where we set

$$\lambda_t := \tilde{a}(t,T) - r_t + \tilde{b}(t,T) \cdot \gamma_t, \quad \forall\, t \in [0,T]. \tag{7.12}$$

Notice that the process λ may depend on maturity T, in general. We shall however assume that λ does not depend on T. This assumption is satisfied, for instance, when $\sigma(t,T) = \tilde{\sigma}(t,T)$ (see footnote [11] below).

The no-arbitrage condition between a defaultable bond and savings account reads:[10] $\lambda_t = 0$ for $t \leq T$. It is easily seen that this condition is never satisfied, under the present assumptions. Indeed, were it true, $\tilde{Z}(t,T)$ would follow a martingale under $\mathbb{P}^*$, and we would have

$$\tilde{D}(t,T) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(- \int_t^T r_u\, du\Big) \,\Big|\, \mathcal{F}_t\Big\} = B(t,T), \quad \forall\, t \in [0,T].$$

The last formula clearly contradicts our assumption that $\tilde{D}(t,T) < B(t,T)$. Therefore, we shall assume from now on that the process λ does not vanish identically, for any maturity in question. From the property that credit spread $g(t,u) - f(t,u)$ is strictly positive, it is also possible to deduce that λ follows a strictly positive process.[11] In fact, first let us observe that the process

$$\tilde{Z}(t,T) \exp\Big(- \int_t^T \lambda_u\, du\Big)$$

is a $\mathbb{P}^*$-martingale. Put another way

$$\tilde{D}(t,T) = \mathbb{E}_{\mathbb{P}^*}\Big\{ \exp\Big(- \int_t^T (r_u + \lambda_u)\, du\Big) \,\Big|\, \mathcal{F}_t\Big\} \tag{7.13}$$

[9] Strictly speaking, this assumption is not required for our further development.

[10] More precisely, this would have been the no-arbitrage condition between defaultable bond and savings account, if we had have assumed that the process $\tilde{D}(t,T)$ represents the price (as opposed to the pre-default value) of a defaultable bond.

[11] This is obvious, if we assume, for instance, that $\sigma(t,T) = \tilde{\sigma}(t,T)$, since then $\lambda_t = g(t,t) - r_t$. Schönbucher (1998) derives the equality $\phi_t\lambda_t = g(t,t) - r_t$ for a strictly positive process ϕ, but he works in a somewhat different setup.

for every $t \in [0,T]$. Consequently, since we assume that $\tilde{D}(t,T) < B(t,T)$ for all $t \in [0,T)$ and for all maturities $T > 0$, it must hold that for every $s < t$

$$\int_s^t \lambda_u \, du > 0,$$

thereby implying that $\lambda_t > 0$ for almost all t and almost surely. Let us note that expression (7.13) jointly with the formula (7.21) below agree with the basic valuation formula (4.6) in the case of zero recovery.

Defaultable term structure. Let $\delta \in [0,1)$ be a fixed, but otherwise arbitrary, number. We introduce an auxiliary process $\lambda_{1,2}$ by setting

$$(\tilde{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}(t) = \tilde{Z}(t,T)\lambda_t, \quad \forall\, t \in [0,T]. \tag{7.14}$$

Notice that for $\delta = 0$ we have simply $\lambda_{1,2}(t) = \lambda_t$ for every $t \in [0,T]$. On the other hand, if we take $\delta > 0$ then the process $\lambda_{1,2}$ is strictly positive provided that $\tilde{D}(t,T) > \delta B(t,T)$ (recall that we have assumed that $\tilde{D}(t,T) < B(t,T)$).

Remarks. If the assumption $\tilde{D}(t,T) > \delta B(t,T)$ is relaxed, the process $\lambda_{1,2}$ is strictly positive provided that

$$\lambda_t(\tilde{Z}(t,T) - \delta Z(t,T)) > 0, \quad \forall\, t \in [0,T].$$

Notice also that $\lambda_{1,2}$ will depend both on the recovery rate δ and on maturity date T, in general. In what follows we shall be assuming that the process $\lambda_{1,2}$ is strictly positive.

We shall show that there exists a stopping time τ, such that the process (as before, $H_t = 1\!\!1_{\{t \geq \tau\}}$)

$$M_t = H_t - \int_0^t \lambda_{1,2}(u) 1\!\!1_{\{u < \tau\}} \, du, \quad \forall\, t \in [0,T], \tag{7.15}$$

follows a local martingale under $\mathbb{P}^*$ (or rather, under a suitable extension $\mathbb{Q}^*$ of $\mathbb{P}^*$, which we are now going to introduce). The existence of τ follows easily from standard results in the theory of stochastic processes, provided that we allow for a suitable enlargement of the underlying probability space. In fact, we cannot expect a stopping time τ with desired properties to exist on the original probability space $(\Omega, \mathbb{F}, \mathbb{P}^*)$, in general. For instance, if the underlying filtration is generated by a standard Brownian motion, which is the usual assumption imposed to ensure the uniqueness of the spot martingale measure $\mathbb{P}^*$, no stopping time with desired properties exists on the original space. Let us denote by $(\tilde{\Omega}, \mathbb{G}, \mathbb{Q}^*)$ the enlarged probability space, where $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T^*]}$. Our additional requirement is that W^* remains a standard Brownian motion when we switch from $\mathbb{P}^*$ to $\mathbb{Q}^*$. To satisfy all these requirements, it suffices to take a product space $(\Omega \times \hat{\Omega}, (\mathcal{F}_t \otimes \hat{\mathcal{F}})_{t \in [0,T^*]}, \mathbb{P}^* \otimes \hat{\mathbb{P}})$ where the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$ is large enough to support a unit exponential random variable, η say. Then we may put (cf. (6.1))

$$\tau = \inf \left\{ t \in \mathbb{R}_+ : \int_0^t \lambda_{1,2}(u) \, du \geq \eta \right\}. \tag{7.16}$$

As one might expect, we extend W^* (and all other previously introduced processes) to the enlarged space by setting $W_t^*(\omega, \hat{\omega}) = W_t^*(\omega)$, etc. Subsequently, we introduce the filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T^*]}$ generated by the random time τ, more precisely, $\mathcal{H}_t = \sigma(H_u : u \leq t)$, where $H_u = 1\!\!1_{\{u \geq \tau\}}$ is the jump process associated with τ. Finally, we set $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t = \sigma(\mathcal{F}_t, \mathcal{H}_t)$ for every t. Then, the desired properties are easily seen to hold under $\mathbb{Q}^* = \mathbb{P}^* \otimes \hat{\mathbb{P}}$. In particular, the process M given by (7.15) is a $\mathbb{G}$-local martingale under $\mathbb{Q}^*$, and W^* is a $\mathbb{G}$-Wiener process under $\mathbb{Q}^*$. It is worthwhile to notice that for obvious reasons we cannot require τ to be independent of W^*.

We are in a position to specify the price process of a T-maturity defaultable bond with fractional recovery of par. We first introduce an auxiliary process $\hat{Z}(t,T)$ by postulating that $\hat{Z}(t,T)$ solves the following SDE

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^* && (7.17)\\ &\quad + (\delta Z(t,T) - \hat{Z}(t-,T))\,dM_t && (7.18)\end{aligned}$$

with the initial condition $\hat{Z}(0,T) = \tilde{Z}(0,T)$. For obvious reasons, the process $\hat{Z}(t,T)$, if well defined, follows a local martingale under $\mathbb{Q}^*$. Combining (7.18) with (7.15), we obtain

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^*\\ &\quad + (\hat{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}(t)\mathbb{1}_{\{t<\tau\}}\,dt\\ &\quad + (\delta Z(t,T) - \hat{Z}(t-,T))\,dH_t.\end{aligned}$$

On the other hand, inserting (7.11) into (7.14), we find that $\tilde{Z}(t,T)$ solves

$$d\tilde{Z}(t,T) = (\tilde{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}(t)\,dt + \tilde{Z}(t,T)\tilde{b}(t,T)\cdot dW_t^*. \tag{7.19}$$

It is thus easily seen that $\hat{Z}(t,T) = \tilde{Z}(t,T)$ on $[0,\tau[$, and thus $\hat{Z}(t,T)$ satisfies also the following SDE

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^*\\ &\quad + \hat{Z}(t,T)\lambda_t\mathbb{1}_{\{t<\tau\}}\,dt + (\delta Z(t,T) - \hat{Z}(t-,T))\,dH_t.\end{aligned}$$

Next, from (7.9) we obtain (to check (7.20), it is enough to solve the SDE above first on the interval $[0,\tau[$ and subsequently on $[\tau,T]$)

$$\hat{Z}(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{Z}(t,T) + \delta\mathbb{1}_{\{t\geq\tau\}}Z(t,T) \tag{7.20}$$

for any $t\in[0,T]$. In view of the last equality, we may represent the differential of $\hat{Z}(t,T)$ in a still another way, namely,

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + \delta Z(t,T)b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^*\\ &\quad + \tilde{Z}(t,T)\lambda_t\mathbb{1}_{\{t<\tau\}}\,dt + (\delta Z(t,T) - \tilde{Z}(t-,T))\,dH_t.\end{aligned}$$

We are in a position to introduce the price process $D^\delta(t,T)$ of a T-maturity defaultable bond. For any $t\in[0,T]$, the process $D^\delta(t,T)$ is defined through the formula

$$D^\delta(t,T) := B_t\hat{Z}(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta\mathbb{1}_{\{t\geq\tau\}}B(t,T), \tag{7.21}$$

where the second equality is an immediate consequence of (7.20).

For $\delta = 0$, the process $\hat{Z}(t,T)$ vanishes on the stochastic interval $[\tau,T]$ and we have simply

$$d\hat{Z}(t,T) = \hat{Z}(t,T)\big(\lambda_t\,dt + \tilde{b}(t,T)\cdot dW_t^*\big) - \hat{Z}(t-,T)\,dH_t. \tag{7.22}$$

Remarks. It is interesting to notice that $\hat{Z}(t,T)$ satisfies also

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + \delta Z(t,T)b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^*\\ &\quad + (\tilde{Z}(t,T) - \delta Z(t,T))\lambda_{1,2}(t)\mathbb{1}_{\{t<\tau\}}\,dt\\ &\quad + (\delta Z(t,T) - \tilde{Z}(t,T))\,dH_t.\end{aligned}$$

This means that the process $\hat{Z}(t,T)$ can alternatively be introduced through the expression

$$\begin{aligned} d\hat{Z}(t,T) &= \big(\tilde{Z}(t,T)\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + \delta Z(t,T)b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big)\cdot dW_t^*\\ &\quad + (\delta Z(t,T) - \tilde{Z}(t,T))dM_t \end{aligned} \tag{7.23}$$

with $\hat{Z}(0,T) = \tilde{Z}(0,T)$. We shall use an analogous approach in the next section.

To simplify the exposition, we shall make throughout the following technical assumption, which will also be in force Section 7.3 (although the process $\hat{Z}(t,T)$ is defined differently in the next section).

Condition (M.D) The process $\hat{Z}(t,T)$, given by the stochastic differential equation (7.18) (or equivalently, by expression (7.23)), follows a $\mathbb{G}$-martingale (as opposed to a local martingale) under $\mathbb{Q}^*$.

Remarks. The necessity of enlarging the underlying probabilty space is closely related to the fact that it is not possible to replicate a defaultable bond using risk-free bonds. More exactly, the process $D^\delta(t,T)$ does not correspond to the wealth of a self-financing portfolio of risk-free bonds (i.e., it does not represent a redundant security in the risk-free bonds market). On the other hand, a defaultable bond $D^\delta(t,T)$ is redundant on the random set $[0,\tau[$, that is, before the default time. This is a rather weak statement, however, since the stopping time τ is not accessible.

Let us now focus on the migration process $C = (C^1, C^2)$. In the setting of this subsection, C lives on four states, since we have $K = 2$. We may and do assume that $C_0 = (C_0^1, C_0^2) = (1,1)$. Also, we assume that $C_t^2 = 1$ for every t.[12] Therefore, the only relevant states for the process C are $(1,1)$ and $(2,1)$. The state $(1,1)$ is the *pre-default state,* and the state $(2,1)$ is the absorbing *default state.* Since the component C^2 is described by the history of C^1, it is clear that it is enough to specify the dynamics of C^1. We postulate that the conditional intensity matrix for C^1 is given by the formula

$$\Lambda_t = \begin{pmatrix} -\lambda_{1,2}(t) & \lambda_{1,2}(t) \\ 0 & 0 \end{pmatrix}. \tag{7.24}$$

In the special case of $\delta = 0$ the matrix Λ takes the following simple form

$$\Lambda_t = \begin{pmatrix} -\lambda_t & \lambda_t \\ 0 & 0 \end{pmatrix}. \tag{7.25}$$

The default time τ is given by the formula

$$\tau = \inf\{ t \in \mathbb{R}_+ : C_t^1 = 2 \} = \inf\{ t \in \mathbb{R}_+ : C_t = (2,1) \}. \tag{7.26}$$

Using (7.21), we obtain for $t \in [0,T]$

$$\begin{aligned} D_{C_t}(t,T) &:= \mathbb{1}_{\{C_t^1 = 1\}} \tilde{D}(t,T) + \delta \mathbb{1}_{\{C_t^1 = 2\}} B(t,T) \\ &= \mathbb{1}_{\{t < \tau\}} \tilde{D}(t,T) + \delta \mathbb{1}_{\{t \geq \tau\}} B(t,T) = D^\delta(t,T) \end{aligned}$$

as expected. Notice that the component C^2 plays no essential role in the present setting. This will no longer be true in the case of multiple credit ratings.

Proposition 7.1 *Assume that the recovery rate $\delta = 0$. Let $D^0(t,T)$ be given by* (7.21), *that is, $D^0(t,T) = \mathbb{1}_{\{t<\tau\}} \tilde{D}(t,T)$. Then*

$$dD^0(t,T) = D^0(t,T)\Big(\big(\tilde{a}(t,T) + \tilde{b}(t,T)\cdot\gamma_t\big)\,dt + \tilde{b}(t,T)\cdot dW_t^*\Big) - D^0(t-,T)\,dH_t$$

under the martingale measure $\mathbb{Q}^*$. The risk-neutral valuation formula holds under $\mathbb{Q}^*$

$$D^0(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} \mathbb{1}_{\{T<\tau\}} \,|\, \mathcal{G}_t). \tag{7.27}$$

Equivalently,

$$D^0(t,T) = B(t,T)\, \mathbb{Q}_T\{ T < \tau \,|\, \mathcal{G}_t \}, \tag{7.28}$$

where $\mathbb{Q}_T$ is the T-forward measure associated with $\mathbb{Q}^*$, that is,

$$\frac{d\mathbb{Q}_T}{d\mathbb{Q}^*} = \frac{1}{B(0,T) B_T}, \quad \mathbb{Q}^* - \text{a.s.} \tag{7.29}$$

[12] The rationale for this convention will appear clear in the multiple credit ratings setup.

Proof. The first statement is an immediate consequence of definition (7.21), combined with (7.10) and (7.22)-(7.20). From (7.11), we get

$$d\tilde{D}(t,T) = \tilde{D}(t,T)\big((r_t + \lambda_t)\,dt + \tilde{b}(t,T)\cdot dW_t^*\big), \tag{7.30}$$

so that (recall that $\tilde{D}(T,T) = 1$)

$$\tilde{D}(t,T) = \tilde{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{F}_t) = \tilde{B}_t\,\mathbb{E}_{\mathbb{Q}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{G}_t) \tag{7.31}$$

with (cf. (3.12))

$$\tilde{B}_t = \exp\Big(\int_0^t (r_u + \lambda_u)\,du\Big). \tag{7.32}$$

This means that $\tilde{D}(t,T)$ corresponds to the process V introduced in Theorem 3.1 (with $Z = 0$ and $X = 1$). Since $\Delta V_\tau = 0$ (this holds since we know that the process $\tilde{D}(t,T)$ is continuous), using Corollary 3.1, we obtain

$$1\!\!1_{\{t<\tau\}}\tilde{D}(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1} 1\!\!1_{\{T<\tau\}}\,|\,\mathcal{G}_t).$$

In view of (7.21), this proves (7.27). □

The next result deals with the case of a general recovery rate. Notice that Proposition 7.2 covers also the case of zero recovery, therefore equality (7.27) can be seen as a special case of (7.35).

Proposition 7.2 *Assume that $\delta \in [0,1)$. The price process $D^\delta(t,T)$ of a defaultable bond satisfies*

$$D^\delta(t,T) = D_{C_t}(t,T) = 1\!\!1_{\{C_t^1=1\}}\exp\Big(-\int_t^T g(t,u)\,du\Big) \tag{7.33}$$

$$+\,\delta 1\!\!1_{\{C_t^1=2\}}\exp\Big(-\int_t^T f(t,u)\,du\Big). \tag{7.34}$$

Moreover, the risk-neutral valuation formula holds

$$D_{C_t}(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(\delta B_T^{-1} 1\!\!1_{\{T\geq\tau\}} + B_T^{-1} 1\!\!1_{\{T<\tau\}}\,\big|\,\mathcal{G}_t\big). \tag{7.35}$$

Furthermore

$$D_{C_t}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(\delta 1\!\!1_{\{T\geq\tau\}} + 1\!\!1_{\{T<\tau\}}\,|\,\mathcal{G}_t\big) \tag{7.36}$$

where $\mathbb{Q}_T$ is the T-forward measure associated with $\mathbb{Q}^*$.

Proof. Formula (7.34) is an immediate consequence of (7.3)-(7.4) combined with (7.21) and (7.26). In view of (7.21), it is also clear that $D^\delta(T,T) = \delta 1\!\!1_{\{T\geq\tau\}} + 1\!\!1_{\{T<\tau\}}$. It is thus enough to show that the discounted process $B_t^{-1}D^\delta(t,T)$ follows a martingale under $\mathbb{Q}^*$. This is obvious, however, since in view of equality (7.21) we have $B_t^{-1}D^\delta(t,T) = \hat{Z}(t,T)$. In view of (7.35), formula (7.36) is a consequence of the Bayes rule and the definition of the probability measure $\mathbb{Q}_T$. □

Remarks. The martingale property $B_t^{-1}D^\delta(t,T)$ can also be verified using the second equality in (7.21). Indeed, we may represent $D^\delta(t,T)$ as follows (recall that $H_t = 1\!\!1_{\{t\geq\tau\}}$)

$$D^\delta(t,T) = (1-H_t)\tilde{D}(t,T) + \delta H_t B(t,T).$$

Applying Itô's rule, we obtain

$$\begin{aligned} dD^\delta(t,T) &= (1-H_t)d\tilde{D}(t,T) - \tilde{D}(t,T)dH_t + \delta H_t dB(t,T) + \delta B(t,T)dH_t \\ &= (1-H_t)\tilde{D}(t,T)\big((r_t+\lambda_t)\,dt + \tilde{b}(t,T)\cdot dW_t^*\big) \\ &\quad - \tilde{D}(t,T)\big(dM_t + \lambda_{1,2}(t)(1-H_t)\,dt\big) \\ &\quad + \delta H_t B(t,T)\big(r_t\,dt + b(t,T)\,dW_t^*\big) \\ &\quad + \delta B(t,T)\big(dM_t + \lambda_{1,2}(t)(1-H_t)\,dt\big) \\ &= (1-H_t)\tilde{D}(t,T)\big(r_t + \lambda_t - \lambda_{1,2}(t)\big)\,dt \\ &\quad + \delta B(t,T)\big(r_t H_t + \lambda_{1,2}(t)(1-H_t)\big)\,dt + dN_t, \end{aligned}$$

where N denotes a $\mathbb{Q}^*$-martingale. Using (7.14), we get

$$dD^\delta(t,T) = r_t\big((1-H_t)\tilde{D}(t,T) + \delta H_t B(t,T)\big)\,dt + dN_t = r_t D^\delta(t,T)\,dt + dN_t,$$

and thus $d(B_t^{-1}D^\delta(t,T)) = B_t^{-1}dN_t$. Finally, one may check directly that $B_t^{-1}dN_t = d\hat{Z}(t,T)$.

Combining (7.31) with (7.21), we obtain

$$D^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{F}_t) + \delta\mathbb{1}_{\{t\ge\tau\}}B_t\,\mathbb{E}_{\mathbb{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t). \tag{7.37}$$

In view of (7.35), it is thus tempting to conjecture that

$$I_1(t) := B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1}\mathbb{1}_{\{T\ge\tau\}}\,|\,\mathcal{G}_t\big) = \mathbb{1}_{\{t\ge\tau\}}B_t\,\mathbb{E}_{\mathbb{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t)$$

and

$$I_2(t) := B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1}\mathbb{1}_{\{T<\tau\}}\,\big|\,\mathcal{G}_t\big) = \mathbb{1}_{\{t<\tau\}}\tilde{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\tilde{B}_T^{-1}\,|\,\mathcal{F}_t).$$

This conjecture is not true, however, as the following proposition shows.

Proposition 7.3 *For any $\delta\in[0,1)$, we have*

$$I_1(t) = B(t,T) - \mathbb{1}_{\{t<\tau\}}\bar{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\bar{B}_T^{-1}\,|\,\mathcal{F}_t), \tag{7.38}$$

and

$$I_2(t) = \mathbb{1}_{\{t<\tau\}}\bar{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\bar{B}_T^{-1}\,|\,\mathcal{F}_t), \tag{7.39}$$

where

$$\bar{B}_t = \exp\Big(\int_0^t \big(r_u + \lambda_{1,2}(u)\big)\,du\Big).$$

Furthermore

$$D^\delta(t,T) = \delta B(t,T) + (1-\delta)\mathbb{1}_{\{t<\tau\}}\bar{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\bar{B}_T^{-1}\,|\,\mathcal{F}_t), \tag{7.40}$$

or equivalently,

$$D^\delta(t,T) = B(t,T) - (1-\delta)\Big(B(t,T) - \mathbb{1}_{\{t<\tau\}}\bar{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\bar{B}_T^{-1}\,|\,\mathcal{F}_t)\Big). \tag{7.41}$$

Finally, we have

$$D_{C_t}(t,T) = B(t,T)\Big(\delta + (1-\delta)\mathbb{1}_{\{t<\tau\}}\,\mathbb{E}_{\mathbb{P}_T}\big(e^{-\int_t^T\lambda_{1,2}(u)\,du}\,|\,\mathcal{F}_t\big)\Big), \tag{7.42}$$

where $\mathbb{P}_T$ is the T-forward measure associated with $\mathbb{P}^*$.

Proof. Let us rewrite $I_1(t)$ as follows

$$I_1(t) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}H_T\,|\,\mathcal{G}_t) = B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}\,|\,\mathcal{G}_t) - B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}(1-H_T)\,|\,\mathcal{G}_t).$$

Reasoning similarly as in Lando (1998) (see also Lemma 13 and Corollary 14 in Wong (1998)) or as in the proof of Proposition 6.1, we obtain

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^*}(1-H_T\,|\,\mathcal{F}_T\vee\mathcal{H}_t) &= \mathbb{Q}^*\{\tau>T\,|\,\mathcal{F}_T\vee\mathcal{H}_t\} = \mathbb{1}_{\{t<\tau\}}e^{-\int_t^T\lambda_{1,2}(u)\,du}\\ &= (1-H_t)\,e^{-\int_t^T\lambda_{1,2}(u)\,du},\end{aligned}$$

where $\mathcal{H}_t = \sigma(H_u\,:\,u\le t)$. Combining the formulae above, we obtain

$$\begin{aligned}I_1(t) &= B_t\,\mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}\,|\,\mathcal{G}_t) - B_t\,\mathbb{E}_{\mathbb{Q}^*}\Big(B_T^{-1}(1-H_t)\,e^{-\int_t^T\lambda_{1,2}(u)\,du}\,\Big|\,\mathcal{G}_t\Big)\\ &= B_t\,\mathbb{E}_{\mathbb{P}^*}(B_T^{-1}\,|\,\mathcal{F}_t) - (1-H_t)\bar{B}_t\,\mathbb{E}_{\mathbb{Q}^*}(\bar{B}_T^{-1}\,|\,\mathcal{G}_t)\\ &= B(t,T) - (1-H_t)\bar{B}_t\,\mathbb{E}_{\mathbb{P}^*}(\bar{B}_T^{-1}\,|\,\mathcal{F}_t).\end{aligned}$$

Since for $I_2(t)$ we have

$$I_2(t) = B_t \, \mathbb{E}_{\mathbb{Q}^*}(B_T^{-1}(1 - H_T) \,|\, \mathcal{G}_t),$$

using the same arguments as for $I_1(t)$, we arrive at

$$I_2(t) = (1 - H_t)\bar{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\bar{B}_T^{-1} \,|\, \mathcal{G}_t).$$

Finally, $D^\delta(t,T) = \delta I_1(t) + I_2(t)$, and thus (7.40)-(7.41) are trivial consequences of (7.38)-(7.39). Formula (7.42) follows from (7.40) and the properties of the forward measure $\mathbb{P}_T$. □

Notice that for $\delta = 0$, we have $\bar{B} = \tilde{B}$, and thus formula (7.40) reduces to $D^0(t,T) = 1\!\!1_{\{t<\tau\}}\tilde{D}(t,T)$. On the other hand, for $\delta = 1$, we have, as expected, $D^1(t,T) = B(t,T)$. Finally, when $0 < \delta < 1$, expression (7.40) yields a decomposition of the price $D^\delta(t,T)$ of a defaultable bond into its predicted 'post-default value' $\delta B(t,T)$ and the 'pre-default premium' $D^\delta(t,T) - \delta B(t,T)$. Similarly, (7.41) represents $D^\delta(t,T)$ as the difference between its 'potential value' $B(t,T)$ and the 'expected loss in value' due to the credit risk. One might also look at (7.41) from the perspective of the buyer of a defaultable bond: the price $D^\delta(t,T)$ equals to the price of the default-free bond minus a compensation for credit risk.

Remarks. Let us denote

$$J(t) = 1\!\!1_{\{t<\tau\}}\bar{B}_t \, \mathbb{E}_{\mathbb{Q}^*}(\bar{B}_T^{-1} \,|\, \mathcal{G}_t) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\Big(B_T^{-1}(1 - H_t)e^{-\int_t^T \lambda_{1,2}(u)\,du} \,\Big|\, \mathcal{G}_t\Big).$$

From the proof of Proposition 7.3 we know that

$$(1 - H_t)\, e^{-\int_t^T \lambda_{1,2}(u)\,du} = \mathbb{Q}^*\{T < \tau \,|\, \mathcal{F}_T \vee \mathcal{H}_t\}$$

so that

$$J(t) = B_t \, \mathbb{E}_{\mathbb{Q}^*}\big(B_T^{-1}\mathbb{Q}^*\{T < \tau \,|\, \mathcal{F}_T \vee \mathcal{H}_t\} \,\big|\, \mathcal{F}_t\big).$$

As already mentioned, in the present setup the stopping time τ and the underlying Wiener process W^* (and consequently τ and B) usually are not mutually independent. Assume, on the contrary, that τ and B are mutually independent.[13] Under this – rather unplausible – assumption, $J(t)$ would read

$$J(t) = B(t,T)\mathbb{Q}^*\{T < \tau \,|\, \mathcal{H}_t\}.$$

Consequently, we would be able to rewrite the valuation formula (7.40) on the set $\{t < \tau\} = \{C_t^1 = 1\}$ in the following way

$$D^\delta(t,T) = \tilde{D}(t,T) = B(t,T)\big(\delta + (1 - \delta)\mathbb{Q}^*\{T < \tau \,|\, C_t^1 = 1\}\big). \tag{7.43}$$

The last formula corresponds to expression (5.30), obtained in a different setup by Jarrow et al. (1997). Let us recall that Jarrow et al. (1997) explicitly assume that the migrations process is independent of the underlying short-term rate process r. Needless to say that representation (7.40) is more general than (7.43) since it allows for the dependence between the migration process for defaultable bonds and the risk-free term structure.

7.2 Alternative Specifications of Recovery Payment

We have assumed so far that the recovery payment is fixed, and takes place at the maturity T of a defaultable bond. In this section, we shall assume instead that the constant (or random) payment is done at the default time rather than at the bond's maturity date. It appears that our approach can be easily extended to cover this case as well.

In what follows, we shall focus on two important special cases. First, let us observe that the constant payoff δ at time $t < T$ corresponds to the payoff $\delta B^{-1}(t,T)$ at the terminal date T. Similarly, the payoff $\delta\tilde{D}(t,T)$, which corresponds to the *fractional recovery of market value,* can be

[13] More precisely, we assume that the default time τ is independent of $\mathcal{F}_T$ and the process B is independent of $\mathcal{H}_t$.

represented by the payoff $\delta \tilde{D}(t,T)B^{-1}(t,T)$ at bond's maturity. We conclude that to cover typical cases when the recovery payment is done at time of default, it is enough to extend the construction above to the case of an $(\mathcal{F}_t)$-adapted stochastic process δ_t.

Let δ_t be given adapted process on the original probability space endowed with the filtration $\mathbb{F}$. Condition (7.14), which serves as a starting point in the specification of the default time τ now takes the following form

$$(\tilde{Z}(t,T) - \delta_t Z(t,T))\lambda_{1,2}(t) = \tilde{Z}(t,T)\lambda_t, \quad \forall\, t \in [0,T]. \tag{7.44}$$

We assume, as before, that the condition above defines a strictly positive adapted process $\lambda_{1,2}(t)$. We shall now show how to modify the basic equations (7.18)-(7.21).

We now introduce an auxiliary process $\hat{Z}(t,T)$ about which we postulate that it solves the SDE

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big) \cdot dW_t^* \\ &\quad + (\delta_t Z(t,T) - \hat{Z}(t-,T))\, dM_t \end{aligned}$$

with the initial condition $\hat{Z}(0,T) = \tilde{Z}(0,T)$. Notice that, as before, the process $\hat{Z}(t,T)$ follows a local martingale under $\mathbb{Q}^*$. Reasoning along the same lines as in the previous section, we find that $\hat{Z}(t,T)$ satisfies

$$\begin{aligned} d\hat{Z}(t,T) &= \hat{Z}(t,T)\big(\tilde{b}(t,T)\mathbb{1}_{\{t<\tau\}} + b(t,T)\mathbb{1}_{\{t\geq\tau\}}\big) \cdot dW_t^* \\ &\quad + \hat{Z}(t,T)\lambda_t \mathbb{1}_{\{t<\tau\}}\, dt + (\delta_t Z(t,T) - \hat{Z}(t-,T))\, dH_t, \end{aligned}$$

and thus

$$\hat{Z}(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{Z}(t,T) + \delta_\tau \mathbb{1}_{\{t\geq\tau\}} Z(t,T)$$

for any $t \in [0,T]$. The price process $\hat{D}^\delta(t,T)$ of a T-maturity defaultable bond is now given by the following expression

$$\hat{D}^\delta(t,T) := B_t \hat{Z}(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta_\tau \mathbb{1}_{\{t\geq\tau\}} B(t,T).$$

The payoff δ_τ at time τ corresponds to the random payoff $\delta^* = \delta_\tau B^{-1}(\tau,T)$ at time T. Therefore, arguing similarly as in the proof of Proposition 7.2, we may then show that

$$\hat{D}^\delta(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\delta^* B_T^{-1}\mathbb{1}_{\{T\geq\tau\}} + B_T^{-1}\mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big).$$

Fractional recovery of par. For $\delta_t = \delta B^{-1}(t,T)$, we obtain

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta B^{-1}(\tau,T)\mathbb{1}_{\{t\geq\tau\}} B(t,T).$$

This corresponds to the random payoff $\delta^* = \delta B^{-1}(\tau,T)$ at time T. Consequently, we obtain the following expression for the price process of a T-maturity defaultable bond

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta^* \mathbb{1}_{\{t\geq\tau\}} B(t,T).$$

Arguing similarly as in the proof of Proposition 7.2, we may then show that

$$\hat{D}^\delta(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\delta B^{-1}(\tau,T) B_T^{-1}\mathbb{1}_{\{T\geq\tau\}} + B_T^{-1}\mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big).$$

Fractional recovery of market value. Let us recall that this case was examined, in a slightly different setup, in Section 4.2. Let us assume that $\delta_t = \delta\tilde{D}(t,T)B^{-1}(t,T)$. Then

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta\tilde{D}(\tau,T)B^{-1}(\tau,T)\mathbb{1}_{\{t\geq\tau\}} B(t,T).$$

Consequently,

$$\hat{D}^\delta(t,T) = \mathbb{1}_{\{t<\tau\}}\tilde{D}(t,T) + \delta^* \mathbb{1}_{\{t\geq\tau\}} B(t,T),$$

where $\delta^* = \delta\tilde{D}(\tau,T)B^{-1}(\tau,T)$, and thus

$$\hat{D}^\delta(t,T) = B_t\, \mathbb{E}_{\mathbb{Q}^*}\big(\delta\tilde{D}(\tau,T)B^{-1}(\tau,T) B_T^{-1}\mathbb{1}_{\{T\geq\tau\}} + B_T^{-1}\mathbb{1}_{\{T<\tau\}} \,\big|\, \mathcal{G}_t\big).$$

7.3 Multiple Credit Ratings Case

We assume now that the set of rating classes is $\mathcal{K} = \{1, \dots, K\}$, where the class K corresponds to the default event. For any $i = 1, \dots, K$, we write $\delta_i \in [0, 1)$ to denote the corresponding recovery rate. By assumption, δ_i is the fraction of par paid at bond's maturity, if the bond which is currently in the i^{th} rating class defaults. In this section, we will consider a risk-free term structure (see Section 7.1), as well as $K-1$ different defaultable term structures (notice that the discussion in the previous section regarded the case where $K = 2$). We generalize condition (B.D) by making the following assumption.
(B.3) For any fixed maturity $T \le T^*$, the instantaneous forward rate $g_i(t, T)$, corresponding to the rating class $i = 1, \dots, K$ satisfies under $\mathbb{P}$

$$dg_i(t, T) = \alpha_i(t, T)\, dt + \sigma_i(t, T) \cdot dW_t, \tag{7.45}$$

where $\alpha_i(\cdot, T)$ and $\sigma_i(\cdot, T)$ are adapted stochastic processes with values in $\mathbb{R}$ and $\mathbb{R}^d$, respectively. In addition, we assume that

$$g_{K-1}(t, T) > g_{K-2}(t, T) > \ldots > g_1(t, T) > f(t, T). \tag{7.46}$$

As before, the price of a T-maturity default-free discount bond is denoted by $B(t, T)$ so that

$$B(t, T) = \exp\Big(-\int_t^T f(t, u)\, du\Big) \tag{7.47}$$

and we denote $Z(t, T) = B(t, T)/B_t$. We also set

$$D_i(t, T) := \exp\Big(-\int_t^T g_i(t, u)\, du\Big) \tag{7.48}$$

for $i = 1, \dots, K-1$. Formulae analogous to (7.5)-(7.7) hold for processes $B(t, T)$ and $D_i(t, T)$, $i = 1, \dots, K-1$, after a suitable change of notation. In particular, we now denote

$$a_i(t, T) = g_i(t, t) - \alpha_i^*(t, T) + \tfrac{1}{2}\, |\sigma_i^*(t, T)|^2, \quad b_i(t, T) = -\sigma_i^*(t, T), \tag{7.49}$$

where

$$\alpha_i^*(t, T) = \int_t^T \alpha_i(t, u)\, du, \quad \sigma_i^*(t, T) = \int_t^T \sigma_i(t, u)\, du.$$

As before, we assume that condition (M.1) is satisfied, with uniquely defined process γ.
Condition (M.2) For $i = 1, \dots, K-1$, the process λ_i, which is given by the formula

$$\lambda_i(t) := a_i(t, T) - f(t, t) + b_i(t, T) \cdot \gamma_t, \quad \forall\, t \in [0, T], \tag{7.50}$$

does not depend on the maturity T.

Remarks. If we assume, in addition, that

$$a_i(t, T) + b_i(t, T) \cdot \gamma_t = g_i(t, T)$$

then $\lambda_i(t) = g_i(t, t) - f(t, t)$ so that obviously $\lambda_i(t) > 0$ for $i = 1, \dots, K$. More generally, one can show, arguing along the same lines as in the preceding section, that processes λ_i are strictly positive (this is a consequence of (7.46)). It is worthwhile to stress, however, that neither the strict positivity of λ_i's nor their independence of maturity T are necessary requirements for our further developments.

From now on, we make standing assumptions (M.1)-(M.2). Proceeding as in Section 7.1, we construct a martingale measure $\mathbb{P}^*$ for the risk-free term structure. In particular, under $\mathbb{P}^*$ the process $Z(t, T) = B_t^{-1} B(t, T)$ satisfies

$$dZ(t, T) = Z(t, T) b(t, T) \cdot dW_t^*. \tag{7.51}$$

Similarly, if we define processes $Z_i(t,T) = B_t^{-1} D_i(t,T)$ for $i = 1, \dots, K-1,$ we obtain the following dynamics for $Z_i(t,T)$ under $\mathbb{P}^*$ (cf. (7.11))

$$dZ_i(t,T) = Z_i(t,T)\big(\lambda_i(t)\,dt + b_i(t,T)\cdot dW_t^*\big). \tag{7.52}$$

The next step is to introduce a conditionally Markov chain C^1 on the state space $\mathcal{K} = \{1, \dots, K\}$. To construct C^1 in a formal way, we shall typically need to enlarge the underlying probability space. Suitable extensions of $\mathcal{F}_t$ and $\mathbb{P}^*$ will be denoted by $\tilde{\mathcal{F}}_t$ and $\mathbb{Q}^*$, respectively, and they can be constructed in a way analogous to the one used in Section 7.1, although a countable number of independent unit exponential random variables will typically be needed for this construction (see the appendix below). The infinitesimal generator of C^1 at time t, given the σ-field $\mathcal{F}_t$, is

$$\Lambda_t = \begin{pmatrix} \lambda_{1,1}(t) & \dots & \lambda_{1,K}(t) \\ \cdot & \dots & \cdot \\ \lambda_{K-1,1}(t) & \dots & \lambda_{K-1,K}(t) \\ 0 & \dots & 0 \end{pmatrix}, \tag{7.53}$$

where $\lambda_{i,i}(t) = -\sum_{j\neq i} \lambda_{i,j}(t)$ for $i = 1, \dots, K-1,$ and where $\lambda_{i,j}$ are adapted, strictly positive processes. To provide our pricing model with an arbitrage free features, the processes $\lambda_{i,j}$ will be additionally assumed to satisfy the consistency condition (7.61) (or (7.58) if $K = 3$). We shall write $H_i(t) = 1\!\!1_{\{C_t^1 = i\}}$ for $i = 1, \dots, K$. Let us define

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(s) H_i(s)\,ds, \quad \forall\, t \in [0,T], \tag{7.54}$$

for $i = 1, \dots, K-1$ and $j \neq i,$ where $H_{i,j}(t)$ represents the number of transitions from i to j by C^1 over the time interval $(0,t]$. It can be shown (see the appendix) that $M_{i,j}(t)$ is a local martingale on the enlarged probability space $(\tilde{\Omega}, (\mathcal{G}_t)_{\,t\in[0,T^*]}, \mathbb{Q}^*)$. We set $C_t^2 = C^1_{u(t)-}$, where $u(t) = \sup\{u \le t : C_u^1 \neq C_t^1\}$ (by convention, $\sup\emptyset = 0$, therefore $C_t^2 = C_t^1$ if $C_u^1 = C_0^1$ for every $u \in [0,t]$). In words, $u(t)$ is the time of the last jump of C^1 before (and including) time t, so that C_t^2 represents the last state of C^1 before the current state C_t^1.

Case $K = 3$. For the reader's convenience, we shall first examine the case when $K = 3$. We assume that $(C_0^1, C_0^2) \in \{(1,1),(2,2)\}$, so that $H_1(0) + H_2(0) = 1\!\!1_{\{C_0^1=1\}} + 1\!\!1_{\{C_0^1=2\}} = 1$. We also observe that for $i,j = 1,2,\ i \neq j$, and for all $t \in [0,T]$ we have

$$H_i(t) = H_i(0) + H_{j,i}(t) - H_{i,j}(t) - H_{i,3}(t) \tag{7.55}$$

and

$$H_{i,3}(t) = 1\!\!1_{\{C_t^1=3,\, C_t^2=i\,\}}. \tag{7.56}$$

Next, we define an auxiliary process $\hat{Z}(t,T)$, which also follows a $\mathbb{G}$-local martingale under $\mathbb{Q}^*$, by setting (the formula below is a straightforward generalization of (7.23))

$$\begin{aligned} d\hat{Z}(t,T) := {} & \big(Z_2(t,T) - Z_1(t,T)\big)\,dM_{1,2}(t) + \big(Z_1(t,T) - Z_2(t,T)\big)\,dM_{2,1}(t) \\ & + \big(\delta_1 Z(t,T) - Z_1(t,T)\big)\,dM_{1,3}(t) + \big(\delta_2 Z(t,T) - Z_2(t,T)\big)\,dM_{2,3}(t) \\ & + \big(H_1(t)Z_1(t,T)b_1(t,T) + H_2(t)Z_2(t,T)b_2(t,T)\big)\cdot dW_t^* \\ & + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)Z(t,T)b(t,T)\cdot dW_t^* \end{aligned}$$

with the initial condition

$$\hat{Z}(0,T) = H_1(0)Z_1(0,T) + H_2(0)Z_2(0,T). \tag{7.57}$$

Using (7.54), we arrive at the following representation for the dynamics of $\hat{Z}(t,T)$

$$d\hat{Z}(t,T) = Z_1(t)\big(dH_{2,1}(t) - dH_{1,2}(t) - dH_{1,3}(t)\big) + H_1(t)\,dZ_1(t)$$

$$
\begin{aligned}
&+ Z_2(t)\big(dH_{1,2}(t) - dH_{2,1}(t) - dH_{2,3}(t)\big) + H_2(t)\,dZ_2(t)\\
&+ Z(t)\big(\delta_1\,dH_{1,3}(t) + \delta_2\,dH_{2,3}(t)\big) + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)\,dZ(t)\\
&- \big[\lambda_{1,2}(t)\big(Z_2(t) - Z_1(t)\big) + \lambda_{1,3}(t)\big(\delta_1 Z(t) - Z_1(t)\big) + \lambda_1(t)Z_1(t)\big]\,H_1(t)\,dt\\
&- \big[\lambda_{2,1}(t)\big(Z_1(t) - Z_2(t)\big) + \lambda_{2,3}(t)\big(\delta_2 Z(t) - Z_2(t)\big) + \lambda_2(t)Z_2(t)\big]\,H_2(t)\,dt,
\end{aligned}
$$

where $Z_i(t) = Z_i(t,T)$ and $Z(t) = Z(t,T)$. To construct a consistent model of the term structure, it is indispensable to specify the matrix Λ in a judicious way. We postulate that the entries of Λ are chosen in such a way that the equalities

$$
\begin{cases}
\lambda_{1,2}(t)\big(Z_2(t) - Z_1(t)\big) + \lambda_{1,3}(t)\big(\delta_1 Z(t) - Z_1(t)\big) + \lambda_1(t)Z_1(t) = 0,\\
\lambda_{2,1}(t)\big(Z_1(t) - Z_2(t)\big) + \lambda_{2,3}(t)\big(\delta_2 Z(t) - Z_2(t)\big) + \lambda_2(t)Z_2(t) = 0
\end{cases}
\tag{7.58}
$$

are satisfied for all $t \in [0,T]$.

Remarks. Suppose first that $\delta_1 = \delta_2 = 0$. In this case, we postulate that the entries of Λ satisfy

$$
\begin{cases}
\lambda_{1,2}(t)(1 - D_{21}(t)) + \lambda_{1,3}(t) = \lambda_1(t),\\
\lambda_{2,1}(t)(1 - D_{12}(t)) + \lambda_{2,3}(t) = \lambda_2(t),
\end{cases}
$$

where we set $D_{ij}(t) = Z_i(t,T)/Z_j(t,T) = D_i(t,T)/D_j(t,T)$. Notice that the coefficients $\lambda_{i,j}(t)$ are not uniquely determined. We may take, for instance, $\lambda_{1,2}(t) = \lambda_{2,1}(t) = 0$ (no migration between classes 1 and 2) to obtain $\lambda_{1,3}(t) = \lambda_1(t)$ and $\lambda_{2,3}(t) = \lambda_2(t)$, but other choices are also possible. Notice also that we cannot set $\lambda_{1,3}(t) = \lambda_{2,3}(t) = 0$ (no default possible) since we would then have either $\lambda_{1,2}(t) < 0$ or $\lambda_{2,1}(t) < 0$. Suppose, on the contrary, that $\delta_1 + \delta_2 > 0$. In this case, we have

$$
\begin{cases}
\lambda_{1,2}(t)(1 - D_{21}(t)) + \lambda_{1,3}(t)(1 - \delta_1 d_{31}(t)) = \lambda_1(t),\\
\lambda_{2,1}(t)(1 - D_{12}(t)) + \lambda_{2,3}(t)(1 - \delta_2 d_{32}(t)) = \lambda_2(t),
\end{cases}
$$

where $d_{ij}(t) = Z(t,T)/Z_j(t,T) = B(t,T)/D_j(t,T)$.

Let us return to the analysis of the process $\hat{Z}(t,T)$. Under (7.58), $\hat{Z}(t,T)$ satisfies

$$
\begin{aligned}
d\hat{Z}(t,T) := &\big(Z_2(t,T) - Z_1(t,T)\big)\,dH_{1,2}(t) + \big(Z_1(t,T) - Z_2(t,T)\big)\,dH_{2,1}(t)\\
&+ \big(\delta_1 Z(t,T) - Z_1(t,T)\big)\,dH_{1,3}(t) + \big(\delta_2 Z(t,T) - Z_2(t,T)\big)\,dH_{2,3}(t)\\
&+ H_1(t)\,dZ_1(t,T) + H_2(t)\,dZ_2(t,T) + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)\,dZ(t,T)
\end{aligned}
$$

with the initial condition (7.57). The above representation of the process $\hat{Z}(t,T)$, combined with (7.55) and (7.56), results in the following important formula

$$
\hat{Z}(t,T) = \mathbb{1}_{\{C^1_t=1\}}\,Z_1(t,T) + \mathbb{1}_{\{C^1_t=2\}}\,Z_2(t,T) + \big(\delta_1 H_{1,3}(t) + \delta_2 H_{2,3}(t)\big)Z(t,T).
$$

Put another way

$$
\hat{Z}(t,T) = \mathbb{1}_{\{C^1_t\neq 3\}}\,Z_{C^1_t}(t,T) + \delta_{C^2_t}\mathbb{1}_{\{C^1_t=3\}}\,Z(t,T).
\tag{7.59}
$$

Finally, we introduce the price process of a T-maturity defaultable bond by setting

$$
D_{C_t}(t,T) := B_t\hat{Z}(t,T) = \mathbb{1}_{\{C^1_t\neq 3\}}\,D_{C^1_t}(t,T) + \delta_{C^2_t}\mathbb{1}_{\{C^1_t=3\}}\,B(t,T).
\tag{7.60}
$$

Remarks. Under the present assumptions the process $\hat{Z}(t) := \hat{Z}(t,T)$, given by (7.59), can also be defined as the unique solution of the following SDE (cf. (7.18))

$$
\begin{aligned}
d\hat{Z}(t) = &\big(Z_2(t) - H_1(t)\hat{Z}(t-)\big)\,dM_{1,2}(t) + \big(Z_1(t) - H_2(t)\hat{Z}(t-)\big)\,dM_{2,1}(t)\\
&+ \big(\delta_1 Z(t) - H_1(t)\hat{Z}(t-)\big)\,dM_{1,3}(t) + \big(\delta_2 Z(t) - H_2(t)\hat{Z}(t-)\big)\,dM_{2,3}(t)\\
&+ \big(H_1(t)\hat{Z}(t)b_1(t,T) + H_2(t)\hat{Z}(t)b_2(t,T) + H_3(t)\hat{Z}(t)b(t,T)\big)\cdot dW^*_t
\end{aligned}
$$

with the initial condition (7.57). Indeed, since $H_3(t) = 1 - H_1(t) - H_2(t) = H_{13}(t) + H_{23}(t)$, we may rewrite this SDE as follows

$$\begin{aligned} d\hat{Z}(t) = & \big(Z_2(t) - H_1(t)\hat{Z}(t-)\big)\, dH_{1,2}(t) + H_1(t)\hat{Z}(t)\big(\lambda_1(t)\, dt + b_1(t,T)\big) \cdot dW^*_t \\ & + \big(Z_1(t) - H_2(t)\hat{Z}(t-)\big)\, dH_{2,1}(t) + H_2(t)\hat{Z}(t)\big(\lambda_2(t)\, dt + b_2(t,T)\big) \cdot dW^*_t \\ & + \big(\delta_1 Z(t) - H_1(t)\hat{Z}(t-)\big)\, dH_{1,3}(t) + \big(\delta_2 Z(t) - H_2(t)\hat{Z}(t-)\big)\, dH_{2,3}(t) \\ & + \big(H_{1,3}(t) + H_{2,3}(t)\big)\hat{Z}(t) b(t,T) \cdot dW^*_t \\ & - H_1(t)\big[\lambda_{1,2}(t)\big(Z_2(t) - \hat{Z}(t)\big) + \lambda_{1,3}(t)\big(\delta_1 Z(t) - \hat{Z}(t)\big) + \lambda_1(t)\hat{Z}(t)\big]\, dt \\ & - H_2(t)\big[\lambda_{2,1}(t)\big(Z_1(t) - \hat{Z}(t)\big) + \lambda_{2,3}(t)\big(\delta_2 Z(t) - \hat{Z}(t)\big) + \lambda_2(t)\hat{Z}(t)\big]\, dt. \end{aligned}$$

In view of (7.51)-(7.52) and (7.58), it is not difficult to check that the unique solution $\hat{Z}(t,T)$ to the SDE above coincides with the process given by the right-hand side of (7.59).

General case. We are in a position to examine the general case. For any $K \geq 3$, we define the process $\hat{Z}(t,T)$ by setting

$$\begin{aligned} d\hat{Z}(t,T) \;:=\; & \sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dM_{i,j}(t) \\ & + \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - Z_i(t,T)\big)\, dM_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t) Z_i(t,T) b_i(t,T) \cdot dW^*_t \\ & + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t) Z(t,T) b(t,T) \cdot dW^*_t \end{aligned}$$

with the initial condition

$$\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0) Z_i(0,T).$$

We shall now generalize the consistency condition (7.58). We write $Z_i(t) = Z_i(t,T)$.

Condition (M.3) The following equalities are satisfied for each $i = 1, \ldots, K-1$, and for every $t \in [0,T]$,

$$\sum_{j=1,\, j\neq i}^{K-1} \lambda_{i,j}(t)\big(Z_j(t) - Z_i(t)\big) + \lambda_{i,K}(t)\big(\delta_i Z(t) - Z_i(t)\big) + \lambda_i(t) Z_i(t) = 0. \tag{7.61}$$

Under the assumption above, the process $\hat{Z}(t,T)$ is easily seen to satisfy

$$\begin{aligned} d\hat{Z}(t,T) \;=\; & \sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - Z_i(t,T)\big)\, dH_{i,j}(t) \\ & + \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - Z_i(t,T)\big)\, dH_{i,K}(t) \\ & + \sum_{i=1}^{K-1} H_i(t)\, dZ_i(t,T) + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)\, dZ(t,T). \end{aligned}$$

The following lemma can be proved along the similar lines as in the case of $K = 3$, therefore its proof is omitted.

Lemma 7.2 *Under* (7.61), *the process* $\hat{Z}(t,T)$ *satisfies*

$$\hat{Z}(t,T) = \sum_{i=1}^{K-1} (H_i(t)Z_i(t,T) + \delta_i H_{i,K}(t)Z(t,T)),$$

or equivalently

$$\hat{Z}(t,T) = 1\!\!1_{\{C_t^1 \neq K\}}\, Z_{C_t^1}(t,T) + \delta_{C_t^2} 1\!\!1_{\{C_t^1 = K\}}\, Z(t,T). \tag{7.62}$$

Moreover, the process $\hat{Z}(t,T)$ *is the unique solution to the SDE*

$$\begin{aligned} d\hat{Z}(t,T) &= \sum_{i,j=1,\, i\neq j}^{K-1} \big(Z_j(t,T) - H_i(t)\hat{Z}(t-,T)\big)\, dM_{i,j}(t) \\ &\quad + \sum_{i=1}^{K-1} \big(\delta_i Z(t,T) - H_i(t)\hat{Z}(t-,T)\big)\, dM_{i,K}(t) \\ &\quad + \sum_{i=1}^{K-1} H_i(t)\hat{Z}(t,T) b_i(t,T)\cdot dW_t^* + H_K(t)\hat{Z}(t,T)b(t,T)\cdot dW_t^* \end{aligned}$$

with the initial condition $\hat{Z}(0,T) = \sum_{i=1}^{K-1} H_i(0)Z_i(0,T)$.

As expected, to define the price of a T-maturity defaultable bond we set

$$D_{C_t}(t,T) := B_t\hat{Z}(t,T) = 1\!\!1_{\{C_t^1 \neq K\}}\, D_{C_t^1}(t,T) + \delta_{C_t^2} 1\!\!1_{\{C_t^1 = K\}}\, B(t,T). \tag{7.63}$$

The following result is thus an immediate consequence of the properties of the auxiliary process $\hat{Z}(t,T)$.

Proposition 7.4 *The dynamics of the price process* $D_{C_t}(t,T)$ *under the risk-neutral probability* $\mathbb{Q}^*$ *are*

$$\begin{aligned} dD_{C_t}(t,T) &= \sum_{i,j=1,\, i\neq j}^{K-1} \big(D_j(t,T) - D_i(t,T)\big)\, dH_{i,j}(t) \\ &\quad + \sum_{i=1}^{K-1} \big(\delta_i B(t,T) - D_i(t,T)\big)\, dH_{i,K}(t) + \sum_{i=1}^{K-1} H_i(t)\, dD_i(t,T) \\ &\quad + \sum_{i=1}^{K-1} \delta_i H_{i,K}(t)\, dB(t,T) + r_t D_{C_t}(t,T)\, dt, \end{aligned}$$

where the differentials $dB(t,T)$ *and* $dD_i(t,T)$ *are given by the formulae*

$$dB(t,T) = B(t,T)\big(r_t\, dt + b(t,T)\cdot dW_t^*\big)$$

and

$$dD_i(t,T) = D_i(t,T)\big((r_t + \lambda_i(t))\, dt + b_i(t,T)\cdot dW_t^*\big).$$

The next proposition shows that the process $D_{C_t}(t,T)$, formally introduced through (7.63), can be given an intuitive interpretation in terms of default time and recovery rate. To this end, we make the following technical assumption (cf. condition (M.D) of Section 7.1).

Condition (M.4) The process $\hat{Z}(t,T)$, given by formula (7.62), follows a $\mathbb{G}$-martingale (as opposed to a local martingale) under $\mathbb{Q}^*$.

The main result of this section holds under assumptions (B.1)-(B.3) and (M.1)-(M.4).

Theorem 7.1 *For any $i = 1, \dots, K-1$, let $\delta_i \in [0,1)$ be the recovery rate for a defaultable bond which belongs to the i^{th} rating class at time of default. The price process $D_{C_t}(t,T)$ of a T-maturity defaultable bond equals, for any $t \in [0,T]$,*

$$D_{C_t}(t,T) = \mathbb{1}_{\{C_t^1 \neq K\}} e^{-\int_t^T g_{C_t^1}(t,u)\,du} + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}} e^{-\int_t^T f(t,u)\,du}, \tag{7.64}$$

or equivalently,

$$D_{C_t}(t,T) = B(t,T)\Big(\mathbb{1}_{\{C_t^1 \neq K\}} e^{-\int_t^T \gamma_{C_t^1}(t,u)\,du} + \delta_{C_t^2} \mathbb{1}_{\{C_t^1 = K\}}\Big), \tag{7.65}$$

where $\gamma_i(t,u) = g_i(t,u) - f(t,u)$ is the i^{th} credit spread. Moreover, $D_{C_t}(t,T)$ satisfies the following version of the risk-neutral valuation formula

$$D_{C_t}(t,T) = B_t\,\mathbb{E}_{\mathbb{Q}^*}\big(\delta_{C_T^2} B_T^{-1} \mathbb{1}_{\{T \geq \tau\}} + B_T^{-1} \mathbb{1}_{\{T < \tau\}} \,|\, \mathcal{G}_t\big), \tag{7.66}$$

where τ is the default time, i.e., $\tau = \inf\{t \in \mathbb{R}_+ : C_t^1 = K\}$. The last formula can also be rewritten as follows

$$D_{C_t}(t,T) = B(t,T)\,\mathbb{E}_{\mathbb{Q}_T}\big(\delta_{C_T^2} \mathbb{1}_{\{T \geq \tau\}} + \mathbb{1}_{\{T < \tau\}} \,|\, \mathcal{G}_t\big), \tag{7.67}$$

where $\mathbb{Q}_T$ is the T-forward measure associated with $\mathbb{Q}^*$ through (7.29).

Proof. The first formula is an immediate consequence of (7.63) combined with (7.47)-(7.48). For the second, notice first that in view of the second equality in (7.63) and the definition of τ, the process $D_{C_t}(t,T)$ satisfies the terminal condition

$$D_{C_T}(T,T) = \delta_{C_T^2} \mathbb{1}_{\{T \geq \tau\}} + \mathbb{1}_{\{T < \tau\}}.$$

Furthermore, using the first equality in (7.63), we deduce the discounted process $B_t^{-1} D_{C_t}(t,T)$ equals $\hat{Z}(t,T)$, so that it follows a $\mathbb{Q}^*$-martingale. Equality (7.66) is thus obvious. ☐

Defaultable coupon bonds. Consider a default-risky coupon bond with the face value F that matures at time T and promises to pay coupons c_i at times T_i ($T_i < T$), $i = 1, 2, \dots, n$. The coupon payments are only made prior to default. For simplicity we also assume that the recovery payment is made at maturity T, in case the bond defaults before or at the maturity. Arbitrage valuation of such a bond is a straightforward consequence of the results obtained earlier in this section. As we have noted before, the intensity matrix of the migration process C_t may depend on both the maturity T and on the recovery rates δ_i, $i \in \mathcal{I} := \{1, 2, \dots, K-1\}$. We shall emphasize this [possible] dependence by writing $C_t(T, \delta_{\mathcal{I}})$. In case of zero recovery we shall write $C_t(T,0)$. Similarly, we find it convenient to emphasize the dependence of the defaultable bond's value on the recovery rates by writing $D^{\delta_{\mathcal{I}}}_{C_t(T,\delta_{\mathcal{I}})}(t,T)$ (or $D^0_{C_t(T,0)}(t,T)$, in case of zero recovery).

We postulate that the arbitrage price $B_c(t,T)$ of the coupon bond considered here is given by

$$B_c(t,T) = \sum_{i=1}^{n} c_i D^0_{C_t(T_i,0)}(t,T_i) + F D^{\delta_{\mathcal{I}}}_{C_t(T,\delta_{\mathcal{I}})}(t,T), \tag{7.68}$$

with the usual convention that $D^0_{C_t(T_i,0)}(t,T_i) = 0$ for $t > T_i$. Notice the defaultable bond covenants described above do not necessarily hold (unless a certain monotonicity of default times is imposed). Also, each zero coupon component of a defaultable coupon bond has its own ratings process.

This means that a defaultable coupon bond is treated as a portfolio of defaultable zero coupon bonds. An alternative way would be to consider a particular defaultable coupon bond as a non-divisible asset, and to introduce its own ratings process.

7.4 Market Prices of Interest Rate and Credit Risk

Let us fix a horizon date T^*. We shall now change, using a suitable generalization of Girsanov's theorem, the measure $\mathbb{Q}^*$ to the equivalent probability measure $\mathbb{Q}$. In financial interpretation, the probability measure $\mathbb{Q}$ plays the role of the real-world probability in our model. For this reason, we postulate that the restriction of $\mathbb{Q}$ to the original probability space Ω necessarily coincide with the underlying probability $\mathbb{P}$. Recall that (cf. (7.8))

$$\frac{d\mathbb{P}^*}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \tilde{L}_t, \quad \mathbb{P}\text{-a.s.},$$

where the process $\tilde{L}$ satisfies

$$d\tilde{L}_t = \tilde{L}_t \gamma_t \cdot dW_t, \quad \tilde{L}_0 = 1.$$

We now set

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^*}\Big|_{\mathcal{G}_t} = L_t, \quad \mathbb{Q}^*\text{-a.s.}, \tag{7.69}$$

where the $\mathbb{Q}^*$-local positive martingale L is given by the formula

$$dL_t = -L_t \gamma_t \cdot dW_t^* + L_{t-}\, dM_t, \quad L_0 = 1, \tag{7.70}$$

where in turn the $\mathbb{Q}^*$-local martingale M equals

$$dM_t = \sum_{i \neq j} (\phi_{i,j}(t) - 1)\, dM_{i,j}(t) = \sum_{i \neq j} (\phi_{i,j}(t) - 1) \big(dH_{i,j}(t) - \lambda_{i,j}(t) H_i(t)\, dt \big),$$

and, for any $i \neq j$, we denote by $\phi_{i,j}$ an arbitrary nonnegative $\mathbb{F}$-predictable process such that

$$\int_0^{T^*} \phi_{i,j}(t) \lambda_{i,j}(t)\, dt < \infty, \quad \mathbb{Q}^*\text{-a.s.}$$

We assume that $\mathbb{E}_{\mathbb{Q}^*}(L_{T^*}) = 1$, so that the probability measure $\mathbb{Q}$ is well defined on $(\tilde{\Omega}, \mathcal{G}_{T^*})$.

Remarks. The process γ ($\phi_{i,j}$, resp.) is referred to as the market price of interest rate risk (market prices of credit risk, resp.)

To analyse the bahaviour of the migration process C under $\mathbb{Q}^*$, we find it convenient to introduce the following point process $\bar{Z}$ (for the definition of the sequence $\tilde{C}_k$, see Section A.1)

$$\bar{Z} := \big(\tau_k, (\tilde{C}_k, \tilde{C}_{k-1})\big), \quad k = 1, 2, \ldots,$$

and the associated 'transition' process

$$\bar{\Phi}(t, i, l) := \sum_{k=1}^{\infty} \mathbb{1}_{\{\tau_k \le t, \tilde{C}_k = i, \tilde{C}_{k-1} = l\}}.$$

Now, let us define processes $\bar{\lambda}_{(i,k),(j,l)}(t)$ for $i, j \in \{1, 2, \ldots, K\}$, $k, l \in \{1, 2, \ldots, K-1\}$ by

$$\bar{\lambda}_{(i,k),(j,l)}(t) = \begin{cases} \lambda_{i,j}(t), & \text{if } l = i \neq K, \\ 0, & \text{if } l = i \text{ or } i = K. \end{cases}$$

It is thus clear that the processes $\bar{\lambda}_{(i,k),(j,l)}$ are the conditional intensities of our migration process $C = (C^1, C^2)$ on the state space $\mathcal{K} \times (\mathcal{K} \backslash K)$. Next, let us define

$$\bar{\nu}_{\mathbb{Q}^*}(t, i, l) := \int_0^t \bar{\lambda}_{(C^1_{s-}, C^2_{s-}),(i,l)}(s)\, ds.$$

It is useful to observe that we have the following identities: $\bar{\Phi}(t,i,l) = H_{l,i}(t)$, and

$$\bar{\nu}_{\mathbb{Q}^*}(t,i,l) = \int_0^t \lambda_{(C^1_{s-},i)}(s) H_l(s-)\, ds = \int_0^t \lambda_{(l,i)}(s) H_l(s)\, ds.$$

Thus, in view of the Corollary A.2 the process $\bar{\nu}_{\mathbb{Q}^*}(t,i,l)$ is the $\mathbb{Q}^*$-compensator for $\bar{\Phi}(t,i,l)$. From the Girsanov theorem for local martingales[14] it thus follows that the process

$$\bar{\nu}_{\mathbb{Q}}(t,i,l) := \int_0^t \phi_{(l,i)}(s) \bar{\lambda}_{(C^1_{s-},C^2_{s-}),(i,l)}(s)\, ds$$

is the $\mathbb{Q}$-compensator for $\bar{\Phi}(t,i,l)$. Now observe that

$$\Phi(t,i) = \sum_{l\in\mathcal{K}} \bar{\Phi}(t,i,l).$$

Thus, the $\mathbb{Q}$-compensator for $\Phi(t,i)$ is

$$\begin{aligned}
\nu_{\mathbb{Q}}(t,i) &= \sum_{l\in\mathcal{K}} \bar{\nu}_{\mathbb{Q}}(t,i,l) \\
&= \sum_{l\in\mathcal{K}} \int_0^t \phi_{(l,i)}(s)\bar{\lambda}_{(C^1_{s-},C^2_{s-}),(i,l)}(s)\, ds \\
&= \sum_{l\in\mathcal{K}} \int_0^t \phi_{(C^1_{s-},i)}(s)\lambda_{(C^1_{s-},i)}(s) H_l(s-)\, ds \\
&= \int_0^t \phi_{(C^1_{s-},i)}(s)\lambda_{(C^1_{s-},i)}(s)\, ds.
\end{aligned}$$

Let us now define processes $\lambda^{\mathbb{Q}}_{(i,j)}$ by setting

$$\lambda^{\mathbb{Q}}_{(i,j)}(t) = \phi_{(i,j)}(t)\lambda_{(i,j)}(t), \quad i \neq j,$$

and

$$\lambda^{\mathbb{Q}}_{(i,i)}(t) = -\sum_{j\neq i} \lambda^{\mathbb{Q}}_{(i,j)}(t).$$

Repeating the construction of Section A.1, we can now construct a conditionally Markov chain $C^1_{\mathbb{Q}}(t)$, on the state space $\mathcal{K}$, whose conditional intensity matrix under the probability measure $\mathbb{Q}$ is

$$\Lambda^{\mathbb{Q}}_t = \begin{pmatrix} \lambda^{\mathbb{Q}}_{1,1}(t) & \dots & \lambda^{\mathbb{Q}}_{1,K}(t) \\ \cdot & \dots & \cdot \\ \lambda^{\mathbb{Q}}_{K-1,1}(t) & \dots & \lambda^{\mathbb{Q}}_{K-1,K}(t) \\ 0 & \dots & 0 \end{pmatrix}.$$

Moreover, all the results of Sections A.2 and A.3 hold for the process $C^1_{\mathbb{Q}}(t)$. Thus, we have the following result.

Proposition 7.5 *The finite-dimensional laws of the processes C^1 and $C^1_{\mathbb{Q}}$ are the same under the probability measure $\mathbb{Q}$.*

Consequently, under the probability measure $\mathbb{Q}$ given by (7.69)-(7.70) the migration process C^1 is still a conditionally Markov process, and it has under $\mathbb{Q}$ the conditional infinitesimal generator $\Lambda^{\mathbb{Q}}_t$ for every $t \in [0,T^*]$.

Remarks. In particular, if the market price for credit risk depends only on the current rating i (and not on the rating j after jump) so that $\phi_{i,j} = \phi_{i,i} =: \phi_i$ for every j, the relationship between the intensity matrices under $\mathbb{Q}$ and $\mathbb{Q}^*$ is the following: $\Lambda^{\mathbb{Q}}_t = \Phi\Lambda_t$, where $\Phi = \operatorname{diag}[\phi_i]$ is the diagonal matrix. It is worthwhile to notice that such a relationship has been postulated, for instance, in Jarrow et al. (1997).

[14]See, e.g., Theorem III.3.11 in Jacod and Shiryaev (1987).

7.5 Model Parameters

For several reasons, the parameter specification is the most difficult task in any attempt to measure and to value the credit risk. First, a credit risk model usually involves a relatively large number of parameters, when compared with any standard model of market risk. Second, frequently the volume of available empirical data related to credit-sensitive assets is insufficient for statistical studies (the scarcity of data makes problematic even the possiblity of reliable estimation of the credit-spread curve). Before discussing the question of specifying model parameters, let us emphasize that the notion of a credit rating should not be understood literally, but rather in a wider sense. Indeed, by a credit rating we mean here any 'reasonable' grouping of credit-sensitive assets, as opposed to 'official' credit ratings provided by any of the widely accepted ratings agencies.

Default probabilities. The notion of a credit event involves a number of various situations related to the credit quality of the reference asset. It is thus worthwhile to mention that in most empirical studies undertaken before 1990 by a *default probability* researchers have meant a probability of defaulting on either interest or principal payment. In more recent studies, it is common to adopt a less stringent definition of default, which can be more adequately referred to as *credit distress.* In this context, let us observe that though the different debt of the same firm encounter credit distress at the same time, it may well happen that senior debt obligations are satisfied in full during bankruptcy procedures, while subordinated debt is paid of only partially. This feature is accounted for in the specification of differing recovery rates to different debt of the same firm, according to the debt seniority. Let us stress that observed default frequencies correspond to the actual probabilities of default, as opposed to the risk-neutral probabilities which are used to value derivative securities. In an arbitrage-free setup, the risk-neutral default probabilities should be seen as by-products obtained within the model, rather then the model inputs.

Recovery rates. It is commonly known that, in the case of default, the likely residual value net of recoveries heavily depends on the seniority class of the debt. To accomodate for this feature, we may assume that the value of a recovery rate reflects not only on the bond credit quality, but also on the seniority classification of the bond (from senior secured to junior unsecured). It is debatable whether it should be represented as a constant or as a random variable. For simplicity, a random recovery rate can be assumed to be independent of other random quantities involved in model's construction.

Credit spreads. The knowledge of credit spreads represents a salient ingredient of the approach presented in Section 7. To be more specific, we need to examine beforehand not only the *credit-spread curves,* but also *credit-spread volatilities,* and, if several distinct assets are modelled simultaneously, the *credit-spread correlations.* Due to the relative scarcity od data, the estimation of the credit-spread curve is more problematic than the estimation of the risk-free yield curve. This is especially difficult to overcome when one deals with the debt issued by a particular firm. In such a case, one might use the rating-specific credit-spread curve as a proxy for the unobservable firm-specific credit-spread curve (see Fridson and Jónsson (1995)).

On the positive side, there is a good chance that the difficulty in collecting of sufficient empirical data will lessen in the future, with the further development of the sector of credit derivatives. The same remarks apply to the estimation of credit-spread volatilities, which in principle can be statistically inferred from the observed variations of the credit-spread yield curve (see, e.g., Fons (1987, 1994) or Foss (1995)). An alternative, and perhaps more promising, approach would be to focus instead on volatilities implicit in market prices of the most actively traded option-like credit derivatives.

Let us finally mention that the valuation of complex credit derivatives requires also to take into account correlations between the behaviour of several credit-sensitive assets (cf. Zhou (1997) or Duffie and Singleton (1998b)).

In view of the discussion above, it is apparent that our model relies on the strong belief that credit risk inherent in credit-sensitive securities is fully explained by the credit-spread curve and its volatility. Such an approach parallels the common belief that the market risk of interest-rate securities is entirely determined through the behaviour of the default-free yield curve and its volatility. This statement should not be misunderstood; it does not mean that several relevant quantities which are typically present in credit-risk considerations should be totally neglected in our setup. On the contrary, all other quantities commonly used in most econometric models of credit risk (that is: default probabilities, migration matrix, recovery rates, as well as correlations) are also used. Since econometric models of credit risk are not discussed in the present work, we refer the interested reader to Altman and Bencivenga (1995), Altman and Kishore (1996), Duffie and Singleton (1997), Monkkonen (1997), Wilson (1997), Duffee (1998) or Kiesel et al. (1999a, 1999b).

7.6 Valuation of Credit Derivatives

We shall only discuss here valuation issues for the two most common credit derivatives: a basic default swap and a total rate of return swap.

Default swaps. Consider first a *basic default swap*, as described, for instance, in Duffie (1999). The contingent payment X is triggered by the default event $\{C^1_t = K\}$. It is settled at time τ, and equals

$$X = \big(1 - \delta_{C^2_T}\, B(\tau,T)\big)\, \mathbb{1}_{\{\tau \le T\}}.$$

Notice the dependence of the payment X on the initial rating C^1_0 through default time τ and recovery rate $\delta_{C^2_T}$. We consider two cases. Either (i) the buyer pays a lump sum at contract's inception (such a contract is referred to as the *default option*), or (ii) the buyer pays an annuity at the fixed time instants t_i, $i = 1,2,\dots,m$ (*default swap*). In case (i), the value at time 0 of a default option is given by the risk-neutral valuation formula

$$\pi_0(X) = \mathbb{E}_{\mathbb{Q}^*}\Big(B^{-1}_\tau \big(1 - \delta_{C^2_T}\, B(\tau,T)\big)\, \mathbb{1}_{\{\tau \le T\}}\Big).$$

In case (ii), the annuity κ satisfies

$$\pi_0(X) = \kappa\, \mathbb{E}_{\mathbb{Q}^*}\Big(\sum_{i=1}^m B^{-1}_{t_i}\, \mathbb{1}_{\{t_i < \tau\}}\Big).$$

Both the price $\pi_0(X)$ and the annuity κ depend on the initial rating C^1_0 of the underlying bond.

Total rate of return swaps. Next consider a *total rate of return swap* as described, for instance, in Das (1998a). We take as a *reference asset* the coupon bond described with the promised cash flows c_i at times T_i. We assume that its price process is described by equality (7.68). We assume that the contract maturity is $\tilde{T} \le T$, where T is the maturity date of the underlying coupon-bond. In addition, suppose that the *reference rate* payments [the annuity payments] are made by the investor at fixed scheduled times $t_i \le \tilde{T}$, $i = 1,2,\dots,m$. As explained in Section 2.1, the owner of a total rate of return swap is entitled not only to all coupon payments during the life of the contract, but also to the change in the value of the underlying bond paid as a lump sum at the contract's termination. Then, the reference rate ρ to be paid by the investor should be computed from

$$\begin{aligned}\rho\, \mathbb{E}_{\mathbb{Q}^*}\Big(\sum_{i=1}^m B^{-1}_{t_i}\, \mathbb{1}_{\{C^1_{t_i}(T,\delta_{\mathcal{I}}) \neq K\}}\Big) \;&=\; \sum_{i=1}^n c_i D^0_{C_0(T_i,0)}(0,T_i)\, \mathbb{1}_{\{T_i \le \tilde{T}\}} \\ &\quad + \mathbb{E}_{\mathbb{Q}^*}\Big(B^{-1}_{\tilde{\tau}}\, \big(B_c(\tilde{\tau},T) - B_c(0,T)\big)\Big),\end{aligned}$$

where $\tilde{\tau} = \tau \wedge \tilde{T}$, and

$$\tau = \inf\{t \ge 0 : C^1_t(T,\delta_{\mathcal{I}}) = K\}.$$

For simplicity, in the left-hand side of the valuation formula above, as well as in the second term in the right-hand side, the default time of the underlying coupon bond was assumed to be represented by the default time of its face value component.

In view of the incompleteness of the model, the important issue of hedging strategies for credit derivatives should be dealt with caution; typically, only an approximate hedge is possible (see Arvanitis and Laurent (1999) and Lotz (1998, 2000) in this regard).

A Credit Migration Process: Construction and Properties

In this appendix we shall provide a formal construction of the migration process C. In addition, we shall construct important (local) martingales associated to the process C. Our construction is inspired by Davis (1993) (Chapter 2), Last and Brandt (1995), and Yin and Zhang (1997) (Sections 2.3-2.4).

Let us first introduce some notation: If $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ are two measurable spaces then we write $\mathcal{X} \otimes \mathcal{Y} = \sigma\{A \times B \,:\, A \in \mathcal{X},\, B \in \mathcal{Y}\}$ and $\mathcal{X} \vee \mathcal{Y} = \sigma\{\mathcal{X} \otimes \{\emptyset, Y\} \cup \{\emptyset, X\} \otimes \mathcal{Y}\}$. If $\mathcal{X}_1$ is a sub-σ-field of $\mathcal{X}$ and if $\mathcal{Z}$ is a sub-σ-field of $\mathcal{X} \otimes \mathcal{Y}$ then $\mathcal{X}_1 \vee \mathcal{Z} = \sigma\{\mathcal{X}_1 \otimes \{\emptyset, Y\} \cup \mathcal{Z}\}$.

A.1 Construction of the Process

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be the underlying probability space, with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ satisfying the usual conditions. Let also $U_{1,k},\, U_{2,k},\, k = 1, 2, \ldots,$ denote two sequences of mutually independent random variables uniformly distributed on $[0, 1]$, defined on a Hilbert cube $(\Omega^U, \mathcal{F}^U, \mathbb{P}^U)$ (see Davis (1993), Section 23). The generic elements of Ω, Ω^U and the set $\mathcal{K}$ will be denoted by ω, by $\omega^U = (\omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \omega^U_{2,2}, \omega^U_{1,3}, \ldots)$ and by i, respectively. Next, let $\mu \in \mathcal{P}(\mathcal{K})$, where $\mathcal{P}(\mathcal{K})$ stands for the set of all probability distributions on $\bar{\Omega} = \mathcal{K}$. In addition, let $\tilde{C}_0 : \bar{\Omega} \to \mathcal{K}$ be a random variable distributed according to μ. We may and do assume that $\tilde{C}_0(i) = i$ (since the generic element of $\bar{\Omega}$ is denoted by $\bar{\omega}$, we shall also write $\tilde{C}_0(\bar{\omega}) = \bar{\omega}$).

We consider a $K \times K$ matrix $\Lambda(t)$ of $\mathbb{F}$-adapted, non-decreasing processes $\Lambda_{ij}(t)$, with $\Lambda_{ij}(0) = 0$,

$$\Lambda(t) = \begin{pmatrix} \Lambda_{1,1}(t) & \ldots & \Lambda_{1,K-1}(t) & \Lambda_{1,K}(t) \\ \cdot & \ldots & \cdot & \cdot \\ \Lambda_{K-1,1}(t) & \ldots & \Lambda_{K-1,K-1}(t) & \Lambda_{K-1,K}(t) \\ 0 & \ldots & 0 & 0 \end{pmatrix}$$

In addition, we assume about the processes $\Lambda_{ij}(t)$ that $\Lambda_{i,i}(t) = -\sum_{j \neq i} \Lambda_{i,j}(t)$ for $i = 1, \ldots, K-1$, and that $\Lambda_{i,i}(\infty) = -\infty$.

We shall analyze in detail the case where the processes $\lambda_{i,j}(t)$ are absolutely continuous with respect to the Lebesgue measure on $[0, \infty)$, that is, there exist $\mathbb{F}$-adapted processes $\lambda_{i,j}(t)$ so that

$$\Lambda_{ij}(t) = \int_0^t \lambda_{i,j}(u)\, du.$$

We shall denote

$$\Lambda_t = \begin{pmatrix} \lambda_{1,1}(t) & \ldots & \lambda_{1,K-1}(t) & \lambda_{1,K}(t) \\ \cdot & \ldots & \cdot & \cdot \\ \lambda_{K-1,1}(t) & \ldots & \lambda_{K-1,K-1}(t) & \lambda_{K-1,K}(t) \\ 0 & \ldots & 0 & 0 \end{pmatrix}$$

The following notation will be used throughout for the survival functions of the jump times of the process that we are constructing here:

$$F(t, i, \omega) = e^{\Lambda_{ii}(t,\omega)} = e^{\int_0^t \lambda_{i,i}(v,\omega)\, dv},$$

where the second equality holds in the absolutely continuous case. We can now define an application $\mathbb{T} : \mathcal{K} \times [0, \infty) \times [0, 1] \times \Omega \to [0, \infty]$ by setting

$$\mathbb{T}(i, s, u, \omega) := \inf \Big\{ t \geq 0 \,:\, \frac{F(t+s, i, \omega)}{F(s, i, \omega)} \leq u \Big\} = \inf \Big\{ t \geq 0 \,: e^{\int_s^{t+s} \lambda_{i,i}(v,\omega)\, dv} \leq u \Big\},$$

where, as usually, the $\inf \emptyset = \infty$. Finally, let $\mathbb{C} :\, [0, 1] \times \mathcal{K} \times [0, \infty) \times \Omega \to \mathcal{K}$ be any (random) function such that

$$\ell\big(\{u \in [0, 1] : \mathbb{C}(u, i, t, \omega) = j\}\big) = \begin{cases} -\frac{\lambda_{i,j}(t,\omega)}{\lambda_{i,i}(t,\omega)}, & \text{if } \lambda_{i,i}(t, \omega) > 0, \\ 0, & \text{otherwise,} \end{cases}$$

for $j \in \mathcal{K},\, j \neq i,$ where $\ell(A)$ stands for the Lebesgue measure of the set A.

Step 1: Construction of the 1st jump time for C^1. Let $\tau_0 := 0$. We define (for brevity, we shall frequently write $\tilde{\omega}$ instead of $(\omega, \omega^U, \bar{\omega})$)

$$\eta_1(\tilde{\omega}) = \eta_1(\omega, \omega^U_{1,1}, \bar{\omega}) := \mathbb{T}(\bar{\omega}, 0, U_{1,1}(\omega^U_{1,1}), \omega),$$

that is,

$$\eta_1 = \inf \Big\{ t \geq 0 \, : e^{\int_0^t \lambda_{\tilde{C}_0, \tilde{C}_0}(v)\, dv} \leq U_{1,1} \Big\},$$

and we set $\tau_1 := \tau_0 + \eta_1$ so that $\tau_1 = \tau_1(\omega, \omega^U_{1,1}, \bar{\omega})$. Observe that η_1 could be equivalently defined as

$$\eta_1(\tilde{\omega}) := \inf \Big\{ t \geq 0 \, : \, -\int_0^t \lambda_{\tilde{C}_0, \tilde{C}_0}(v)\, dv \geq \tilde{e}_{1,1} \Big\},$$

where $\tilde{e}_{1,1} := -\ln U_{1,1}$ is a unit exponential random variable. Clearly, τ_1 is a random variable on the product probability space (notice that $\mathbb{P}$ and $\mathbb{Q}$ correspond to $\mathbb{P}^*$ and $\mathbb{Q}^*$ in Section 7)

$$(\tilde{\Omega}, (\hat{\mathcal{F}}_t)_{t\in[0,\infty)}, \mathbb{Q}) = (\Omega \otimes \Omega^U \otimes \bar{\Omega}, (\mathcal{F}_t \otimes \mathcal{F}^U \otimes 2^{\mathcal{K}})_{t\in[0,\infty)}, \mathbb{P} \otimes \mathbb{P}^U \otimes \mu).$$

In fact, τ_1 only depends on ω, $\omega^U_{1,1}$ and $\bar{\omega}$. It is easily verified that for every $t > 0$

$$\mathbb{Q}\left\{\tau_1 > t \,\middle|\, \mathcal{F}_t \vee \sigma(\tilde{C}_0)\right\}(\tilde{\omega}) = \frac{F(t, \tilde{C}_0, \omega)}{F(0, \tilde{C}_0, \omega)} = e^{\int_0^t \lambda_{\tilde{C}_0, \tilde{C}_0}(v,\omega)\, dv},$$

and that

$$\mathbb{Q}\{\tau_1 > t\} = \mathbb{E}_{\mathbb{Q}}\big(F(t, \tilde{C}_0, \omega)\big) = \mathbb{E}_{\mathbb{Q}}\Big(e^{\int_0^t \lambda_{\tilde{C}_0, \tilde{C}_0}(v,\omega)\, dv}\Big),$$

where $\mathbb{E}_{\mathbb{Q}}$ is the expectation with respect to $\mathbb{Q}$. We also see that under uniform boundedness assumptions on the processes $\lambda_{i,j}$ we have $\mathbb{Q}\{\tau_1 = 0\} = 0$. Also, since by assumption $\int_0^\infty \lambda_{i,i}(t)\, dt = -\infty$ for any $i = 1, \ldots, K-1$, we have $\mathbb{Q}\{\tau_1 < \infty\} = 1$.

We shall now check that $\mathbb{Q}\{\lambda_{\tilde{C}_0, \tilde{C}_0}(\tau_1) = 0\} = 0$, or equivalently, that $\mathbb{Q}\{\lambda_{i,i}(\tau_1) = 0, \, \tilde{C}_0 = i\} = 0$ for every $i = 1, \ldots, K-1$. From the construction of τ_1 it can be deduced that for any bounded $\mathbb{F}$-adapted process Z we have[15]

$$\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tilde{C}_0 = i\}} Z_{\tau_1(\tilde{\omega})}(\omega)\big) = -\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{\tilde{C}_0 = i\}} \int_0^\infty Z_t \lambda_{i,i}(t)\, e^{\int_0^t \lambda_{i,i}(s)\, ds}\, dt\Big). \tag{A.1}$$

This in turn implies that

$$\mathbb{Q}\{\lambda_{i,i}(\tau_1) = 0, \, \tilde{C}_0 = i\} = \mathbb{Q}\{(\tau_1(\tilde{\omega}), \omega) \in \tilde{B}, \, \tilde{C}_0 = i\} = 0.$$

where we denote $\tilde{B} = \{(t,\omega) \, : \, \lambda_{i,i}(t,\omega) = 0\}$. Indeed, to check the last equality it is enough to apply formula (A.1) to the bounded $\mathbb{F}$-adapted process $Z_t = \mathbb{1}_{\tilde{B}}(t)$.

Step 2: Construction of the 1st jump for C^1. For any $\tilde{\omega} = (\omega, \omega^U, \bar{\omega})$, we set

$$\tilde{C}_1(\tilde{\omega}) = \tilde{C}_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar{\omega}) := \mathbb{C}(U_{2,1}(\omega^U_{2,1}), \tilde{C}_0(\bar{\omega}), \tau_1(\omega, \omega^U_{1,1}, \bar{\omega}), \omega)$$

so that $\tilde{C}_1$ is a random variable on the probability space $(\tilde{\Omega}, \hat{\mathbb{F}}, \mathbb{Q})$, where, as expected, we set $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t\in[0,\infty)}$. It is also apparent that $\tilde{C}_1$ only depends on ω, $\omega^U_{1,1}$, $\omega^U_{2,1}$ and $\bar{\omega}$. Moreover, it holds that

$$\mathbb{Q}\{\tilde{C}_1 = j | \mathcal{G}^{1,0}_{\tau_1}\}(\omega, \omega^U_{1,1}, \bar{\omega}) = -\frac{\lambda_{\tilde{C}_0, j}(\tau_1(\omega, \omega^U_{1,1}, \bar{\omega}), \omega)}{\lambda_{\tilde{C}_0, \tilde{C}_0}(\tau_1(\omega, \omega^U_{1,1}, \bar{\omega}), \omega)},$$

where $\mathcal{G}^{1,0}_t := \mathcal{F}_t \vee \mathcal{H}^1_t \vee \sigma(\tilde{C}_0)$ with $\mathcal{H}^1_t := \sigma\left(\mathbb{1}_{\{\tau_1 \leq s\}} : 0 \leq s \leq t\right)$ (notice that $\sigma(\tau_1) \subset \mathcal{H}^1_{\tau_1}$).

[15] We follow here the reasoning from Page 19 in Yin and Zhang (1997), who deal with the piecewise deterministic process, however.

Step 3: Construction of the 2nd jump time for C^1. We set

$$\eta_2(\tilde\omega) = \eta_2(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \bar\omega) := \mathbb{T}(\tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \tau_1(\omega, \omega^U_{1,1}, \bar\omega), U_{1,2}(\omega^U_{1,2}), \omega),$$

or briefly,

$$\eta_2 = \inf\Big\{ t \geq 0 : e^{\int_{\tau_1}^{\tau_1+t} \lambda_{\tilde C_1, \tilde C_1}(v)\,dv} \leq U_{1,2} \Big\} = \inf\Big\{ t \geq 0 : -\int_0^t \lambda_{\tilde C_1, \tilde C_1}(v)\,dv \geq \tilde e_{1,2} \Big\},$$

where $\tilde e_{1,2} := -\ln U_{1,2}$. As expected, we set $\tau_2 := \tau_1 + \eta_2$. The random variable τ_2 on $(\tilde\Omega, \hat{\mathbb{F}}, \mathbb{Q})$ depends only on ω, $\omega^U_{1,1}$, $\omega^U_{2,1}$, $\omega^U_{1,2}$, and $\bar\omega$. Again, it is easily verified that

$$\mathbb{Q}\{\eta_2 > t \,|\, \mathcal{F}_{t+\tau_1} \vee \mathcal{H}^1_{\tau_1} \vee \sigma(\tilde C_1))(\tilde\omega) = \frac{F(t + \tau_1(\omega, \omega^U_{1,1}, \bar\omega), \tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \omega)}{F(\tau_1(\omega, \omega^U_{1,1}, \bar\omega), \tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \omega)},$$

and thus

$$\mathbb{Q}\{\eta_2 > t\} = \mathbb{E}_{\mathbb{Q}}\Big(\frac{F(t + \tau_1(\omega, \omega^U_{1,1}, \bar\omega), \tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \omega)}{F(\tau_1(\omega, \omega^U_{1,1}, \bar\omega), \tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \omega)} \Big).$$

Similarly as in Step 1, we see that $\mathbb{Q}\{\eta_2 = 0\} = 0$, $\mathbb{Q}\{\eta_2 < \infty\} = 1$, and $\mathbb{Q}\{\lambda_{\tilde C_1, \tilde C_1}(\tau_2) = 0\} = 0$.

Step 4: Construction of the 2nd jump for C^1. The random variable $\tilde C_2$ is defined through the formula

$$\tilde C_2(\tilde\omega) := \mathbb{C}(U_{2,2}(\omega^U_{2,2}), \tilde C_1(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \bar\omega), \eta_2(\omega, \omega^U_{1,1}, \omega^U_{2,1}, \omega^U_{1,2}, \bar\omega), \omega).$$

Similarly as in Step 2, it holds that

$$\mathbb{Q}\{\tilde C_2 = j \,|\mathcal{G}^{2,1}_{\tau_2}\} = -\frac{\lambda_{\tilde C_1, j}(\tau_2)}{\lambda_{\tilde C_1, \tilde C_1}(\tau_2)},$$

where we set $\mathcal{G}^{2,1}_t = \mathcal{F}_t \vee \mathcal{H}^2_t \vee \sigma(\tilde C_1)$, and $\mathcal{H}^2_t := \sigma\,(\mathbb{1}_{\{\tau_2 \leq s\}} : 0 \leq s \leq t)$ (observe that $\sigma(\tau_2) \subset \mathcal{H}^2_{\tau_1}$).

Step 5: Construction of the k-th jump time and the k-th jump for C^1 ($k > 2$) In a similar way as in previous steps, we may construct the k-th jump time $\tau_k = \tau_{k-1} + \eta_k$, as well as the k-th jump $\tilde C_k$ for C^1. We have the general formulae

$$\mathbb{Q}\big\{\eta_k > t \,\big|\, \mathcal{F}_{t+\tau_{k-1}} \vee \mathcal{H}^{k-1}_{\tau_{k-1}} \vee \sigma(\tilde C_{k-1})\big\} = \frac{F(t + \tau_{k-1}, \tilde C_{k-1})}{F(\tau_{k-1}, \tilde C_{k-1})},$$

for every $t > 0$, and

$$\mathbb{Q}\big\{\tilde C_k = j \,\big|\, \mathcal{G}^{k,k-1}_{\tau_k}\big\} = -\frac{\lambda_{\tilde C_{k-1}, j}(\tau_k)}{\lambda_{\tilde C_{k-1}, \tilde C_{k-1}}(\tau_k)},$$

where we write $\mathcal{G}^{k,k-1}_t = \mathcal{F}_t \vee \mathcal{H}^k_t \vee \sigma(\tilde C_{k-1})$ and $\mathcal{H}^k_t := \sigma\,(\mathbb{1}_{\{\tau_k \leq s\}} : 0 \leq s \leq t)$. Observe that, under uniform boundedness conditions imposed on the processes $\lambda_{i,j}(t)$ we have, with probability 1, $\tau_k \to \infty$ as $k \to \infty$.

Step 6: Construction of C. We are in a position to formally define the process C. We set $C^1_t := \tilde C_{k-1}$ if $t \in [\tau_{k-1}, \tau_k)$ for $k = 1, 2, \ldots$. This defines the first component of C. The second component is defined as

$$C^2_t = \begin{cases} \tilde C_0, & \text{if } t \in [0, \tau_2) \\ \tilde C_{k-1}, & \text{if } t = [\tau_k, \tau_{k+1}),\ k = 2, 3, \ldots . \end{cases}$$

This completes our construction of the two-dimensional process $C_t = (C^1_t, C^2_t)$.

A.2 Conditional Markov Property

Our next goal is to verify that C has indeed a conditional Markov property with respect to some filtration. Let $\mathcal{F}^C_t = \sigma(C_s : 0 \le s \le t)$, denote the natural filtration generated by the process C on $(\tilde\Omega, \mathbb{G}, \mathbb{Q})$, where $\tilde\Omega = \Omega \otimes \Omega^U \otimes \mathcal{K}$, $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}^C_t$, and $\mathbb{Q} = \mathbb{P} \otimes P^U \otimes \mu$.

Lemma A.1 *For any $k = 0, 1, \dots,$ and for any $t \ge 0$, we have*

$$\mathbb{Q}\big\{C_{\tau_{k+1}} = (j,i) \,\big|\, \mathcal{F}_t \vee \mathcal{F}^C_t\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} = \mathbb{Q}\big\{C_{\tau_{k+1}} = (j,i) \,\big|\, \mathcal{F}_t \vee \sigma(C_t)\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}.$$

Proof. Notice first that since by construction the process C^1 (and thus also the component C^2) jumps only at times τ_k for $k = 1, 2, \dots,$ it is clear that

$$J := \mathbb{Q}\big\{C_{\tau_{k+1}} = (j,i) \,\big|\, \mathcal{F}_t \vee \mathcal{F}^C_t\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} = \mathbb{Q}\big\{C_{\tau_{k+1}} = (j, C^1_{\tau_k}) \,\big|\, \mathcal{F}_t \vee \mathcal{F}^C_{\tau_k}\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}$$

for every $k = 0, 1, \dots$ Therefore,

$$\begin{aligned}
J &= \mathbb{Q}\big\{C_{\tau_{k+1}} = (j, C^1_{\tau_k}) \,\big|\, \mathcal{F}_t \vee \mathcal{F}^C_{\tau_k}\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} \\
&= \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{Q}\big\{C_{\tau_{k+1}} = (j, C^1_{\tau_k}) \,\big|\, \mathcal{G}^{k+1,k}_{\tau_{k+1}} \vee \mathcal{F}^C_{\tau_k}\big\} \,\Big|\, \mathcal{F}_t \vee \mathcal{F}^C_{\tau_k}\Big)\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} \\
&= \mathbb{E}_{\mathbb{Q}}\Big(-\frac{\lambda_{\tilde C_k, j}(\tau_{k+1})}{\lambda_{\tilde C_k, \tilde C_k}(\tau_{k+1})} \,\Big|\, \mathcal{F}_t \vee \mathcal{F}^C_{\tau_k}\Big)\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} \\
&= \mathbb{E}_{\mathbb{Q}}\Big(-\frac{\lambda_{\tilde C_k, j}(\tau_{k+1})}{\lambda_{\tilde C_k, \tilde C_k}(\tau_{k+1})} \,\Big|\, \mathcal{F}_t \vee \sigma(C_{\tau_k})\Big)\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}.
\end{aligned}$$

since $\mathcal{G}^{k+1,k}_{\tau_{k+1}} := \mathcal{F}_{\tau_{k+1}} \vee \mathcal{H}^{k+1}_{\tau_{k+1}} \vee \sigma(\tilde C_k)$ and $C^1_t = C^1_{\tau_k} = \tilde C_k$ on the random interval $[\tau_k, \tau_{k+1})$. On the other hand, reasoning similarly to the above, we obtain

$$\mathbb{Q}\big\{C_{\tau_{k+1}} = (j,i) \,\big|\, \mathcal{F}_t \vee \sigma(C_t)\big\}\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}} = \mathbb{E}_{\mathbb{Q}}\Big(-\frac{\lambda_{\tilde C_k, j}(\tau_{k+1})}{\lambda_{\tilde C_k, \tilde C_k}(\tau_{k+1})} \,\Big|\, \mathcal{F}_t \vee \sigma(C_{\tau_k})\Big)\, \mathbb{1}_{\{\tau_k \le t < \tau_{k+1}\}}.$$

This proves the lemma. □

For any $t \ge 0$, we set $\tilde\tau(t) := \inf\{s \ge t \,:\, C_s \ne C_{s-}\}$. Then the above lemma implies that

$$\mathbb{Q}\big\{C_{\tilde\tau(t)} = (j,i) \,\big|\, \mathcal{G}_t\big\} = \mathbb{Q}\big\{C_{\tilde\tau(t)} = (j,i) \,\big|\, \mathcal{F}_t \vee \mathcal{F}^C_t\big\} = \mathbb{Q}\big\{C_{\tilde\tau(t)} = (j,i) \,\big|\, \mathcal{F}_t \vee \sigma(C_t)\big\}$$

for all $t \ge 0$. The equality above is referred to as the *conditional Markov property* of C with respect to the filtration $\mathbb{F}$.

A.3 Local Martingales Associated to the CMP

We need to introduce some notation. First, we introduce a point process $Z := \big\{(\tau_k, \tilde C_k),\, k = 0, 1, 2, \dots\big\}$. Next, for every $j \in \mathcal{K}$ and $t \ge 0$ we define

$$\Phi(t, j) = \sum_{k=1}^{\infty} \mathbb{1}_{\{\tau_k \le t \,:\, \tilde C_k = j\}}$$

and

$$\nu(t, j) = \int_0^t \lambda_{C^1_{s-}, j}(s)\, ds.$$

Thus, the process $\Phi(\cdot, j)$ is the counting process for Z. Finally, we write $q(t, j) = \Phi(t, j) - \nu(t, j)$. Then have the following auxiliary result.

Lemma A.2 *For any $j \in \mathcal{K}$, the process $\nu(\cdot, j)$ is the predictable compensator for $\Phi(\cdot, j)$; that is, the process $q(t \wedge \tau_k, j)$ is a $\mathbb{G}$-martingale for every k and every $j \in \mathcal{K}$.*

Proof. We shall first verify that $q(t\wedge\tau_1,j)$ is a $\mathbb{G}$-martingale. Fix $0\le s<t$. Let us denote

$$J_s(\Phi)=\mathbb{E}_{\mathbb{Q}}\Big(\Phi(t\wedge\tau_1,j)-\Phi(s\wedge\tau_1,j)\,\Big|\,\mathcal{G}_s\Big)$$

and

$$J_s(\nu)=\mathbb{E}_{\mathbb{Q}}\Big(\nu(t\wedge\tau_1,j)-\nu(s\wedge\tau_1,j)\,\Big|\,\mathcal{G}_s\Big)$$

For $J_s(\Phi)$, we get

$$\begin{aligned}
J_s(\Phi) &= \mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t\ge\tau_1\}}\mathbb{1}_{\{\tilde C_1=j\}}-\mathbb{1}_{\{s\ge\tau_1\}}\mathbb{1}_{\{\tilde C_1=j\}}\,\Big|\,\mathcal{G}_s\Big)\\
&= \mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t\ge\tau_1\}}\mathbb{1}_{\{\tilde C_1=j\}}\,\Big|\,\mathcal{G}_s\Big)\\
&= -\mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\frac{\lambda_{i,j}(\tau_1)}{\lambda_{i,i}(\tau_1)}\,\Big|\,\mathcal{G}_s\Big)+\mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\mathbb{1}_{\{t<\tau_1\}}\frac{\lambda_{i,j}(\tau_1)}{\lambda_{i,i}(\tau_1)}\,\Big|\,\mathcal{G}_s\Big)\\
&= \mathbb{1}_{\{s<\tau_1\}}\Big\{\mathbb{E}_{\mathbb{Q}}\Big(\int_s^\infty\lambda_{i,j}(r)e^{\int_s^r\lambda_{i,j}(u)\,du}\,dr\,\Big|\,\mathcal{G}_s\Big)-\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\infty\lambda_{i,j}(r)e^{\int_s^r\lambda_{i,j}(u)\,du}\,dr\,\Big|\,\mathcal{G}_s\Big)\Big\}\\
&= \mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_s^t\lambda_{i,j}(r)e^{\int_s^r\lambda_{i,j}(u)\,du}\,dr\,\Big|\,\mathcal{G}_s\Big).
\end{aligned}$$

On the other hand, for $J_s(\nu)$ we obtain

$$\begin{aligned}
J_s(\nu) &= \mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_s^{t\wedge\tau_1}\lambda_{i,j}(r)\,dr\,\Big|\,\mathcal{G}_s\Big)\\
&= -\mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_s^\infty(\int_s^{t\wedge u}\lambda_{i,j}(r)\,dr)\,\lambda_{i,i}(u)e^{\int_s^u\lambda_{i,j}(v)\,dv}\,du\,\Big|\,\mathcal{G}_s\Big)\\
&= -\mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_s^t(\int_s^{u}\lambda_{i,j}(r)\,dr)\,\lambda_{i,i}(u)e^{\int_s^u\lambda_{i,j}(v)\,dv}\,du\,\Big|\,\mathcal{G}_s\Big)\\
&\quad-\mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_t^\infty(\int_s^{t}\lambda_{i,j}(r)\,dr)\,\lambda_{i,i}(u)e^{\int_s^u\lambda_{i,j}(v)\,dv}\,du\,\Big|\,\mathcal{G}_s\Big)\\
&= \mathbb{1}_{\{s<\tau_1\}}\mathbb{E}_{\mathbb{Q}}\Big(\int_s^t\lambda_{i,j}(r)e^{\int_s^r\lambda_{i,j}(u)\,du}\,dr\,\Big|\,\mathcal{G}_s\Big).
\end{aligned}$$

We conclude that the process $q(t\wedge\tau_1,j)$ follows a $\mathbb{G}$-martingale under $\mathbb{Q}$. Using a reasoning analogous to the above, combined with the conditional Markov property and the concatenation argument, the martingale property is proved for each $q(t\wedge\tau_k,j)$, $k>1$. □

Next, we consider a bounded, measurable function $\phi:\mathcal{K}\times[0,\infty)\times\tilde\Omega\to\mathbb{R}$. We assume that for any $i\in\mathcal{K}$, the process $\phi(i,\cdot,\cdot)$ is $\mathbb{G}$-predictable. Let us define the associated adapted process M^ϕ by setting

$$M_t^\phi(\tilde\omega)=\sum_{i\in\mathcal{K}}\int_0^\infty\phi(i,s,\tilde\omega)\mathbb{1}_{\{s\le t\}}\,q(ds,i)(\tilde\omega)\,.$$

Using Lemma A.2, we immediately obtain the following corollary.

Corollary A.1 *The process M^ϕ is a $\mathbb{G}$-martingale under $\mathbb{Q}$.*

We are in a position to show that the process Λ_t is a conditional intensity process (conditional infinitesimal generator process) for the first component of the CMP C. For a real valued function f on $\mathcal{K}$ we denote

$$\Lambda_t f(i)=\sum_{j\in\mathcal{K}}\lambda_{i,j}(t)f(j).$$

Proposition A.1 *For every real valued function f on $\mathcal{K}$ we have that the process $\mathcal{M}^f$, given by the formula*

$$\mathcal{M}^f_t = f(C^1_t) - \int_0^t \Lambda_s f(C^1_s)\, ds, \quad \forall\, t \in \mathbb{R}_+,$$

is a $\mathbb{G}$-martingale under $\mathbb{Q}$.

Proof. It is enough to apply Corollary A.1 to the function $\phi(i,t,\tilde\omega) = f(i) - f\big(C^1_{t-}(\tilde\omega)\big)$. □

The following result is another important consequence of Lemma A.2 (compare also Corollary 7.5.3 in Last and Brandt (1997)). Let us define (see (7.54))

$$M_{i,j}(t) := H_{i,j}(t) - \int_0^t \lambda_{i,j}(s) H_i(s)\, ds, \quad \forall\, t \in [0,T],$$

where $H_i(t) = 1\!\!1_{\{C^1_t = i\}}$, and $H_{i,j}(t)$ represents the number of transitions from i to j by C^1 over the time interval $(0,t]$.

Corollary A.2 (i) *Let h be a real valued function on $\mathcal{K}\times\mathcal{K}$. Then the process N^h defined as*

$$N^h_t = \sum_{s\le t,\, C^1_{s-}\neq C^1_s} - \int_0^t \sum_{k\neq C^1_s} \lambda_{C^1_s,k}(s) h(C^1_s,k)\, ds$$

is a $\mathbb{G}$-martingale. (ii) *For any $i,\, j\in\mathcal{K}$, $i\neq j$, the process $M_{i,j}$ follows a $\mathbb{G}$-martingale.*

Proof. To prove (i), it is enough to apply Corollary A.1 to the function: $\phi(k,t,\tilde\omega) = h(C^1_{t-}(\tilde\omega),k)$. The second statement follows from the first, if we set $h(c,c') = 1\!\!1_{\{i\}}(c) 1\!\!1_{\{j\}}(c')$. □

A.4 Forward Kolmogorov Equation

Let $\hat P(s,t)$ be the transition probability matrix $\hat P(s,t)$ for the process C^1 – that is,

$$\hat P(s,t) := \big[p_{i,j}(s,t)\big]_{(i,j)\in\mathcal{K}\times\mathcal{K}},$$

where

$$p_{i,j}(s,t) := \mathbb{Q}\big\{C^1_t = j \,\big|\, \mathcal{F}_s \vee \sigma(\{C^1_s = i\})\big\}. \tag{A.2}$$

Observe that the martingale property of $\mathcal{M}^f$ (cf. Proposition A.1) implies that for $s\le t$

$$\mathbb{E}_{\mathbb{Q}}\Big(f(C^1_t) - \int_s^t \Lambda_u f(C^1_u)\, du \,\big|\, \mathcal{G}_s\Big) = f(C^1_s).$$

In particular, letting $f = 1\!\!1_{\{j\}}$ in the last formula, we obtain

$$\mathbb{Q}\big\{C^1_t = j \,\big|\, \mathcal{G}_s\big\} = 1\!\!1_{\{j\}}(C^1_s) - \int_s^t \mathbb{E}_{\mathbb{Q}}\big(\lambda_{C^1_u,j}(u) \,\big|\, \mathcal{G}_s\big)\, du\,.$$

We conclude that for every $i\in\mathcal{K}$

$$p_{i,j}(s,t) = 1\!\!1_{\{j\}}(i) - \int_s^t \mathbb{E}_{\mathbb{Q}}\Big(\sum_{k\in\mathcal{K}} \lambda_{k,j}(u) p_{i,k}(s,u) \,\Big|\, \mathcal{F}_s\vee\sigma(\{C^1_s = i\})\Big)\, du.$$

From this we obtain the following result.

Corollary A.3 *For any fixed s, the transition probability matrix $\hat P(s,t)$, $t\ge s$, satisfies the forward Kolmogorov equation*

$$\frac{d\hat P(s,t)}{dt} = \hat P(s,t)\mathbb{E}_{\mathbb{Q}}\big(\Lambda_t \,\big|\, \mathcal{F}_s\vee\sigma(\{C^1_s = i\})\big)$$

with the initial condition $\hat P(s,s) = \mathrm{Id}$.

References

[1] AB Altman, E.I. and Bencivenga, J.C. (1995) "A yield premium model for the high-yield debt market." *Financial Analysts Journal* 51(5), 49–56.

[2] AKi Altman, E.I. and Kishore, V.M. (1996) "Almost everything you wanted to know about recoveries on defaulted bonds." *Financial Analysts Journal* 52(6), 57–64.

[3] Am Ammann, M. (1999) *Pricing Derivative Credit Risk.* Lecture Notes in Economics and Mathematical Systems 470, Springer, Berlin Heidelberg New York.

[4] An Antonelli, F. (1993) "Backward–forward stochastic differential equations." *Annals of Applied Probability* 3, 777–793.

[5] AnS Anderson, R. and Sundaresan, S. (2000) "A comparative study of structural models of corporate bond yields: an exploratory investigation." *Journal of Banking and Finance* 24, 255–269.

[6] Artzner, P. and Delbaen, F. (1995) "Default risk insurance and incomplete markets." *Mathematical Finance* 5, 187–195.

[7] AGL Arvanitis, A., Gregory, J. and Laurent, J.-P. (1999) "Building models for credit spreads." *Journal of Derivatives* 6(3), 27–43.

[8] AL Arvanitis, A. and Laurent, J.-P. (1999) "On the edge of completeness." *Risk,* October.

[9] BeSaw BeSaw, J. (1997) "Pricing credit derivatives." *Derivatives Week*, September 8, 6–7.

[10] BiRut Bielecki, T. and Rutkowski, M. (1999) "Defaultable term structure: conditionally Markov approach." Working paper.

[11] BiRu Bielecki, T. and Rutkowski, M. (2000) "Multiple ratings model of defaultable term structure." *Mathematical Finance* 10, 125–139.

[12] BC Black, F. and Cox, J.C. (1976) "Valuing corporate securities: some effects of bond indenture provisions." *Journal of Finance* 31, 351–367.

[13] BJ Blanchet-Scalliet, C. and Jeanblanc, M. (2000) "Dynamics of defaultable zero-coupon." Working paper, Université d'Evry.

[14] BR Brémaud, P. (1981) *Point Processes and Queues. Martingale Dynamics.* Springer, Heidelberg Berlin New York.

[15] BS1 Brennan, M. and Schwartz, E. (1977) "Convertible bonds: valuation and optimal strategies for call and conversion." *Journal of Finance* 32, 1699–1715.

[16] Brennan, M. and Schwartz, E. (1980) "Analyzing convertible bonds." *Journal of Financial and Quantitative Analysis* 15, 907–929.

[17] Briys, E. and de Varenne, F. (1997) "Valuing risky fixed rate debt: an extension." *Journal of Financial and Quantitative Analysis* 32, 239–248.

[18] *CreditMetrics: Technical Document.* J.P.Morgan, New York, 1997.

[19] *CreditRisk$^+$: Technical Document.* Credit Suisse Financial Products, 1997.

[20] CGM Crouhy, M., Galai, D. and Mark, R. (1998) "Credit risk revisited." *Risk – Credit Risk Supplement,* March, 40–44.

[21] CGM1 Crouhy, M., Galai, D. and Mark, R. (2000) "A comparative analysis of current credit risk models." *Journal of Banking and Finance* 24, 59–117.

[22] Da1 Das, S. (1998a) "Credit derivatives - instruments." In: *Credit Derivatives: Trading and Management of Credit and Default Risk,* S.Das, ed., J.Wiley, Singapore, pp.7–77.

[23] Da2 Das, S. (1998b) "Valuation and pricing of credit derivatives." In: *Credit Derivatives: Trading and Management of Credit and Default Risk,* S.Das, ed., J.Wiley, Singapore, pp.173–231.

[24] Davis, M.H.A. (1993) *Markov Models and Optimization.* Chapman & Hall, London.

[25] DM Dellacherie, C. and Meyer, P.A. (1975) *Probabilités et potentiel.* Hermann, Paris.

[26] DF1 Duffee, G. (1998) "The relation between Treasury yields and corporate bond yield spreads." Forthcoming in *Journal of Finance.*

[27] D98a Duffie, D. (1998a) "First-to-default valuation." Working paper, Stanford University.

[28] D98b Duffie, D. (1998b) "Defaultable term structure models with fractional recovery of par." Working paper, Stanford University.

[29] D98 Duffie, D. (1999) "Credit swap valuation." *Financial Analysts Journal* 55(1), 73–87.

[30] Duffie, D. and Lando, D. (1998) "The term structure of credit spreads with incomplete accounting data." Working paper, Stanford University and University of Copenhagen.

[31] DSS1 Duffie, D., Schroder, M. and Skiadas, C. (1996) "Recursive valuation of defaultable securities and the timing of resolution of uncertainty." *Annals of Applied Probability* 6, 1075–1090.

[32] DS2 Duffie, D. and Singleton, K. (1997) "An econometric model of the term structure of interest rate swap yields." *Journal of Finance* 52, 1287–1321.

[33] Duffie, D. and Singleton, K. (1998a) "Ratings-based term structures of credit spreads." Working paper, Stanford University.

[34] Duffie, D. and Singleton, K. (1998b) "Simulating correlated defaults." Working

[35] Duffie, D. and Singleton, K. (1999) "Modelling term structures of defaultable bonds." *Review of Financial Studies* 12, 687–720.

[36] EQ El Karoui, N. and Quenez, M.C. (1997a) "Nonlinear pricing theory and backward stochastic differential equations." In: *Financial Mathematics, Bressanone, 1996,* W.Runggaldier, ed. *Lecture Notes in Math.* 1656, Springer, Berlin Heidelberg New York, pp. 191–246.

[37] ELQ El Karoui, N. and Quenez, M.C. (1997b) "Imperfect markets and backward stochastic differential equations." In: *Numerical Methods in Finance,* L.C.G.Rogers, D.Talay, eds. Cambridge University Press, Cambridge, pp. 181–214.

[38] EPQ El Karoui, N., Peng, S. and Quenez, M.C. (1997) "Backward stochastic differential equations in finance." *Mathematical Finance* 7, 1–72.

[39] Elliott, R.J., Jeanblanc, M. and Yor, M. (2000) "On models of default risk." *Mathematical Finance* 10, 179–195.

[40] Fo1 Fons, J.S. (1987) "The default premium and corporate bond experience." *Journal of Finance* 42, 81–97.

[41] Fo2 Fons, J.S. (1994) "Using default rates to model the term structure of credit risk." *Financial Analysts Journal* 50(5), 25–32.

[42] Fos Foss, G.W. (1995) "Quantifying risk in the corporate bond markets." *Financial Analysts Journal* 51(2), 29–34.

[43] FJ Fridson, M.S. and Jónsson, J.G. (1995) "Spread versus Treasuries and the riskiness of high-yield bonds." *Journal of Fixed Income* 5(3), 79–88.

[44] Ge Geske, R. (1977) "The valuation of corporate liabilities as compound options." *Journal of Financial and Quantitative Analysis* 12, 541–552.

[45] Ges Geske, R. (1979) "The valuation of compound options." *Journal of Financial Economics* 7, 63–81.

[46] HJM Heath, D., Jarrow, R. and Morton, A. (1992) "Bond pricing and the term structure of interest rates: a new methodology for contingent claim valuation." *Econometrica* 60, 77–105.

[47] Huge, B. and Lando, D. (1998) "Swap pricing with two-sided default risk in a rating-based model." Working paper, University of Copenhagen.

[48] HW1 Hull, J.C. and White, A. (1995) "The impact of default risk on the prices of options and other derivative securities." *Journal of Banking and Finance* 19, 299–322.

[49] JS Jacod, J. and Shiryaev, A.N. (1987) *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin.

[50] JLT Jarrow, R.A., Lando, D. and Turnbull, S.M. (1997) "A Markov model for the term structure of credit risk spreads." *Review of Financial Studies* 10, 481–523.

[51] JT Jarrow, R.A. and Turnbull, S.M. (1995) "Pricing derivatives on financial securities subject to credit risk." *Journal of Finance* 50, 53–85.

[52] Jarrow, R.A. and Turnbull, S.M. (2000) "The intersection of market and credit risk." *Journal of Banking and Finance* 24, 271–299.

[53] JR Jeanblanc, M. and Rutkowski, M. (2000a) "Modelling of default risk: an overview." In: *Mathematical Finance: Theory and Practice,* Higher Education Press, Beijing, pp. 171–269.

[54] JR1 Jeanblanc, M. and Rutkowski, M. (2000b) "Modelling of default risk: mathematical tools." Working paper.

[55] KPT1 Kiesel, R., Perraudin, W., Taylor, A. (1999a) "Credit and interest rate risk." Working paper, Birbeck College.

[56] KPT2 Kiesel, R., Perraudin, W., Taylor, A. (1999b) "The structure of credit risk." Working paper, Birbeck College.

[57] Kijima, M. (1998) "Monotonicity in a Markov chain model for valuing coupon bond subject to credit risk." *Mathematical Finance* 8, 229–247.

[58] KRS Kim, I.J., Ramaswamy, K. and Sundaresan, S. (1993) "Does default risk in coupons affect the valuation of corporate bonds?" *Financial Management* 22, 117–131.

[59] Ku Kusuoka, S. (1999) "A remark on default risk models." *Advances in Mathematical Economics* 1, 69–82.

[60] La1 Lando, D. (1997) "Modelling bonds and derivatives with credit risk." In: *Mathematics of Derivative Securities,* M.Dempster, S.Pliska, eds., Cambridge University Press, Cambridge, pp. 369–393.

[61] La2 Lando, D. (1998) "On Cox processes and credit-risky securities." *Review of Derivatives Research* 2, 99–120.

[62] Last, G. and Brandt, A. (1995) *Marked Point Processes on the Real Line. The Dynamic Approach.* Springer, New York Berlin Heidelberg.

[63] Le Leland, H.E. (1994) "Corporate debt value, bond covenants, and optimal capital structure." *Journal of Finance* 49, 1213–1252.

[64] LeT Leland, H.E. and Toft, K. (1996) "Optimal capital structure, endogenous bankruptcy, and the term structure of credit spreads." *Journal of Finance* 51, 987–1019.

[65] Li, H. (1998) "Pricing of swaps with default risk." *Review of Derivatives Research* 2, 231–250.

[66] LI Litterman, R. and Iben, T. (1991) "Corporate bond valuation and the term structure of credit spreads." *Journal of Portfolio Management* 17(3), 52–64.

[67] LS1 Longstaff, F.A. and Schwartz, E.S. (1995) "A simple approach to valuing risky fixed and floating rate debt." *Journal of Finance* 50, 789–819.

[68] Lo1 Lotz, C. (1998) "Locally risk minimizing the credit risk." Working paper, London School of Economics.

[69] Lo2 Lotz, C. (1999) "Optimal shortfall hedging of credit risk." Working paper, University of Bonn.

[70] LoS Lotz, C. and Schlögl, L. (2000) "Default risk in a market model." *Journal of Banking and Finance* 24, 301–327.

[71] MU Madan, D.B. and Unal, H. (1998a) "Pricing the risk of default." *Review of Derivatives Research* 2, 121–160.

[72] MU1 Madan, D.B. and Unal, H. (1998b) "A two-factor hazard-rate model for pricing risky debt and the term structure of credit spreads." Working paper, University of Maryland.

[73] Ml Mella-Barral, P. and Tychon, P. (1996) "Default risk in asset pricing." Working paper, London School of Economics and Université Catholique de Louvain.

[74] Me Merton, R.C. (1974) "On the pricing of corporate debt: the risk structure of interest rates." *Journal of Finance* 29, 449–470.

[75] Monkkonen, H. (1997) "Modelling default risk: theory and empirical evidence." Ph.D. thesis, Queen's University.

[76] MR Musiela, M. and Rutkowski, M. (1997) *Martingale Methods in Financial Modelling.* Springer, Heidelberg Berlin New York.

[77] NSRSC Nielsen, T.N., Saà-Requejo, J. and Santa-Clara, P. (1993) "Default risk and interest rate risk: the term structure of default spreads." Working paper, INSEAD.

[78] Rendleman, R.J. (1992) "How risks are shared in interest rate swaps." *Journal of Financial Services Research* 5–34.

[79] Ru Rutkowski, M. (1999) "On models of default risk: by R.Elliott, M.Jeanblanc and M.Yor." Working paper, Warsaw University of Technology.

[80] SC1 Schönbucher, P.J. (1996) "The term structure of defaultable bond prices." Working paper, University of Bonn.

[81] SC2 Schönbucher, P.J. (1998) "Term structure modelling of defaultable bonds." *Review of Derivatives Research* 2, 161–192.

[82] SC3 Schönbucher, P.J. (2000) "Credit risk modelling and credit derivatives." Ph.D. thesis, University of Bonn.

[83] Sch Schlögl, L. (1998) "An exposition of intensity-based models of securities and derivatives with default risk." Working paper, University of Bonn.

[84] Shirakawa, H. (1999) "Evaluation of yield spread for credit risk." *Advances in Mathematical Economics* 1, 83–97.

[85] Ta Tavakoli, J.M. (1998) *Credit Derivatives: A Guide to Instruments and Applications.* J.Wiley, New York.

[86] Thomas, L.C., Allen, D.E. and Morkel-Kingsbury, N. (1998) "A hidden Markov chain model for the term structure of bond credit risk spreads." Working paper, Edith Cowan University.

[87] Wi Wilson, T. (1997) "Portfolio credit risk." *Risk* 10(9), 111–117, 10(10) 56–61.

[88] Wong Wong, D. (1998) "A unifying credit model." Working paper, Scotia Capital Markets.

[89] Yin, G.G. and Zhang, Q. (1997) *Continuous-Time Markov Chains and Applications. A Singular Perturbation Approach.* Springer, New York Berlin Heidelberg.

[90] Z Zhou, C. (1997a) "A jump diffusion approach to modelling credit risk and valuing defaultable securities." Working paper, Federal Reserve Board.

[91] Zh Zhou, C. (1997b) "Default correlation: an analytical result." Working paper, Federal Reserve Board.